IC/70/106

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON RENORMALIZATION CONSTANTS AND INTER-RELATION OF FUNDAMENTAL FORCES

ABDUS SALAM

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1970 MIRAMARE-TRIESTE

IC/70/106

INTERNATIONAL ATOMIC ENERGY AGENCY
and
UNITED NATIONS EDUCATIONAL SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON RENORMALIZATION CONSTANTS AND INTER-RELATION OF FUNDAMENTAL FORCES

ABDUS SALAM

International Centre for Theoretical Physics, Trieste, Italy,
and
Imperial College, London, England.

Presented at the 15th International Conference on High-Energy Physics,
Kiev, 26 August - 4 September 1970.

MIRAMARE - TRIESTE
September 1970

I. INTRODUCTION

One of the important recent advances[1] in field theory is the recognition that non-polynomial Lagrangians of the transcendental (e.g. $g e^{\kappa\phi}$) or rational (e.g. $g(1+\kappa\phi)^{-1}$) variety offer the prospect of infinity-free and (if some additional criteria are used) unambiguous [2] computation of all matrix elements including the <u>finite</u> computation of the traditionally infinite renormalization constants. It also appears possible to extend the methods used to more frequently encountered mixed Lagrangians of the type:

$$g(\bar{\psi}\psi A)(1+\kappa\phi)^{-1} \qquad \text{or} \qquad g(\bar{\psi}\psi A)\,\exp(\kappa\phi) \ .$$

The most serious problem as far as this finite computation of renormalization constants is concerned is the problem of ambiguities. Present-day mathematics appears to offer no unique prescription in this respect and additional physical or mathematical criteria appear to be needed to solve this problem. The approach in this note will essentially be an experimental one in the sense that we shall try to guess from experiment what the dependence of the renormalization constants on physical coupling constants should be and then formulate a mathematical procedure to resolve the ambiguities accordingly. In arriving at the suggestions made in this paper there is the important circumstance that the above Lagrangians possess a simple analytic dependence on the variable $\kappa\phi$ where ϕ is the field and κ is the so-called minor coupling constant of the theory with dimensions of inverse mass (g is the traditional <u>major</u> coupling parameter). Although it is only in a vague manner that the idea is taking shape at present, it appears that one may be able to formulate, corresponding to a given Lagrangian, a "maximum analyticity principle" in the κ plane, with some of the singularities at $\kappa = 0$ being associated with the conventional infinities of residual Lagrangians like $g\bar{\psi}\psi A = \lim_{\kappa\to 0} \dfrac{g\bar{\psi}\psi A}{1+\kappa\phi}$. This principle may then provide one specific recipe for the definition of the renormalization constants.

In the Appendix to this note, which reports on work done together with R. Delbourgo, C. J. Isham and J. Strathdee, we shall review the present status of the ambiguity problem. The Appendix also gives the procedure for the computation of the renormalization constants. In the text of the note we shall attempt to draw some qualitative and preliminary conclusions about the magnitudes of the finite renormalization constants within the context of this procedure which the present theory gives. The conclusions rest on the following observations:

1) The matrix elements in non-polynomial theories exhibit a characteristic [3] dependence on the powers of $\log(\kappa^2 p^2)$ where p^2 is $(\text{momentum})^2$.

2) The traditional renormalization infinities of Lagrangians like $\bar{\psi}\psi A$ manifest themselves as singularities in the minor coupling constant κ plane when instead of $g\bar{\psi}\psi A$ we start with the Lagrangian $(g(\bar{\psi}\psi A))/(1+\kappa\varphi)$. Thus the traditional logarithmic infinities now make their appearance in the form $(\log(\kappa^2 m^2))^n$, the quadratic infinities in the form $1/\kappa^2$ and quartic infinities as $1/\kappa^4$. As $\kappa \to 0$ one recovers the old infinities. For conventional non-renormalizable theories the κ plane singularities are of the form $1/(\kappa^2)^n$ with n arbitrarily large. The magnitude of renormalization constants in any theory is therefore connected with the (inverse) magnitude of the minor coupling parameter.

3) We appear to have the paradoxical situation of the weaker the minor coupling parameter, the stronger its influence on the magnitudes of the renormalization constants. This apparent paradox becomes comprehensible if one remembers that κ is proportional to an inverse mass so that small values of κ are associated with small radii of particles or large inbuilt cut-off masses.

4) Among the accepted non-polynomial Lagrangians of physics are:

i) The strong chiral Lagrangians of Gürsey-Weinberg variety:

$$\mathcal{L} = \frac{(\partial\varphi)^2}{(1+\lambda^2\varphi^2)^2}$$

with the minor constant $\lambda \approx m_\pi^{-1}$.

ii) Weak Lagrangians with the Fermi constant:

$$G_F \approx 10^{-5}\, m_N^{-2} \quad .$$

Here, as we shall see in Sec. III, G_F acts both as the minor and major constant.

iii) Einstein's gravitational Lagrangian with the newtonian constant :

$$G_N = 8\pi \kappa_g^2$$

$$\kappa_g = 2.2 \times 10^{-22}\ m_e^{-1} \qquad (m_e \text{ is electron mass}).$$

Here also κ_g acts as minor as well as major constant (see Sec. IV). These constants then appear to define a hierarchy of inbuilt cut-offs, the least important (for the magnitudes of the renormalization constants) being the strong cut-off defined by λ_π and the most potent the one defined by κ_g. In the next section we wish to consider - in a purely qualitative manner - the interplay of these constants and the possibility that (with some further assumptions) there might emerge connecting relations among them. As an example we shall advance plausibility arguments within the theory for the order of magnitude relation:

$$\frac{e^2}{4\pi} \log(G_N m_e^2) \approx 1 \tag{1}$$

connecting electrodynamics and gravity. (The present value of the left-hand side is $\approx 100/137$.)

II. STRONG INTERACTIONS

As stated before, the SU(2) × SU(2) chiral theories with

$$\mathcal{L}_{\pi} = \mathrm{Tr}\ \partial S\ \partial S^{+}$$
$$\mathcal{L}_{\pi N}^{inter} = m_N \bar{\psi} S'^2 \psi \tag{2}$$

are intrinsically non-polynomial in form. Here S and S' (in Weinberg's formulation) are given by

$$S(\pi) = \frac{1 + i\lambda\, \underline{\tau}\cdot\underline{\pi}}{1 - i\lambda\, \underline{\tau}\cdot\underline{\pi}}$$

and

$$S'(\pi) = \frac{1 + i\lambda\, \gamma_5\, \underline{\tau}\cdot\underline{\pi}}{1 - i\lambda\, \gamma_5\, \underline{\tau}\cdot\underline{\pi}} \quad . \tag{3}$$

With the non-polynomial methods developed[1)], these Lagrangians would give rise to finite matrix elements with an inbuilt cut-off at $\lambda^{-1} \approx 2m_N/g_{\pi N} \approx m_\pi$. Unhappily, in addition to pions (kaons and η's) there are other strongly interacting particles - notably the gauge 1^- and 1^+ particles - and the question arises: what can be done to compute the renormalization constants surviving in these theories?

Now in Ref. 1 it was shown that there does exist a part-non-polynomial formulation of gauge theories of massive spin-one particles which renders some - though not all - renormalization constants finite. This is the formulation due to Boulware[4)] and we shall describe it here to point out precisely what one may achieve and what problems are still left if one limits oneself to strong gauge theories alone.

Consider a triplet of Yang-Mills fields described by

$$L_{YM} = \hat{\underline{W}}_{\mu\nu} \cdot \hat{\underline{W}}_{\mu\nu} + m^2\, \underline{W}_\mu^2 \tag{4}$$

where

$$\hat{\underline{W}}_{\mu\nu} = (\partial_\nu\, \underline{W}_\mu - \partial_\mu \underline{W}_\nu + 2i\, f\, \underline{W}_\mu \times \underline{W}_\nu) \quad . \tag{5}$$

The propagator for the W fields $(W_\mu, W_\nu)_+ = \left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right) \Delta(x)$ is highly singular and the theory as it stands is non-renormalizable.

Following Boulware, let us now make a non-linear Stückelberg-like transformation on the field variables[5] $\underline{W}_\mu$. Write $W_\mu = \underline{W}_\mu \cdot \underline{\tau}$ and introduce two sets of fields A_μ and B, defined by the relation

$$W_\mu = S(B)\, A_\mu S^{-1}(B) + i/f\, S(B)\, \partial_\mu S^{-1}(B) \quad . \tag{6}$$

Here S(B) is a unitary matrix which could be taken in the Weinberg form[6] as $\frac{1 + i\,(f/m)\,B}{1 - i\,(f/m)\,B}$. Write $\mathcal{J}_\mu = \frac{im}{f}\, S \partial_\mu S^{-1}$. The net effect of (6) is to transform $\mathcal{L}_{YM}$ to a part-polynomial ($\mathcal{L}_{YM}(A)$) and a part-non-polynomial form:

$$\mathcal{L}_{YM}^{(W)} = (\hat{\underline{A}}_{\mu\nu} \cdot \hat{\underline{A}}_{\mu\nu} + m^2 \underline{A}_\mu^2) + \left(2m \quad \underline{A}_\mu \cdot \underline{\mathcal{J}}_\mu \quad + \quad \underline{\mathcal{J}}_\mu \cdot \underline{\mathcal{J}}_\mu\right) \quad . \tag{7}$$

Now comes the important point. Boulware has shown that the two Stückelberg fields A_μ and B in terms of which W_μ has been re-expressed can be assigned normal propagators $(A_\mu, A_\nu)_+ = g_{\mu\nu}\,\Delta$ and $(B(x), B(0))_+ = \Delta(x)$ provided the conventional rules for writing the S-matrix corresponding to the Lagrangian (7) are supplemented by adding to (7) a term of the form $(i f\, \underline{F}^+ \times \partial_\mu \underline{F} \cdot \underline{A}_\mu)$. Here the triplet of $\underline{F}$-particles represents "fictitious" bosons of Fermi statistics first introduced into the theory by Feynman who showed that the introduction of these bosons is needed to preserve unitarity of the S-matrix.

Consider now the final effective Lagrangian for the Yang-Mills field. It can be written in two parts, $\mathcal{L}_{YM} = \mathcal{L}_{YM}^{(1)} + \mathcal{L}_{YM}^{(2)}$.

$$\mathcal{L}_{YM}^{(1)} = \mathcal{L}_{YM}(A) + i f\, \underline{F}^+ \times \partial_\mu \underline{F} \cdot \underline{A}_\mu + \partial_\mu \underline{F}^+ \cdot \partial_\mu \underline{F} \tag{8}$$

$$\mathcal{L}_{YM}^{(2)} = 2i\, m \quad \underline{\mathcal{J}}_\mu \cdot \underline{A}_\mu \quad + \quad \underline{\mathcal{J}}_\mu \cdot \underline{\mathcal{J}}_\mu \quad , \tag{9}$$

$\mathcal{L}_{YM}^{(1)}$ is polynomial in form. $\mathcal{L}_{YM}^{(2)}$ is non-polynomial with an inbuilt cut-off at about m/f . The contributions to the traditional Z-factors, the self-mass, the self-charge and the meson-meson scattering length, arising from $\mathcal{L}_{YM}^{(2)}$, are finite using the methods of the Appendix and proportional to $\log f$ or m^2/f^2, etc. The contributions arising from the polynomial part of the Lagrangian $\mathcal{L}_{YM}^{(1)}$ are still, however, ultraviolet infinite in the traditional manner. To make them finite would need further realistic non-polynomiality to be built into the theory. In the next section we come back to a "realistic" provision of such non-polynomiality using, for example, strong gravity theory. Without this, the traditional infinities arising from $\mathcal{L}_{YM}^{(1)}$ would survive.

To summarize: Chiral Lagrangians possess inbuilt cut-off factors. Parts of Yang-Mills Lagrangians also possess such factors but there are other parts which are obstinately polynomial in form and give rise, if no further modification of these Lagrangians is made, to the traditional ultraviolet infinities. If an ad hoc procedure is adopted to regularize such infinities, there is no known way - as, in contrast, there probably is for non-polynomial Lagrangians - to remove ambiguities.

III. WEAK INTERACTIONS

The discussion of the Yang-Mills field provides us with a model for weak interactions mediated by intermediate bosons $W^{\pm}$. Since nothing essential in the mathematics is altered even if we assume that the W mesons form a gauge triplet $W_\mu^{\pm}$ and W_μ^0 , we shall do so. (The physics is of course altered because of this introduction of neutral currents, but at this stage we are concerned with mathematical difficulties.) As is well known (see Fig. 2), the Fermi constant and the constants f and m are related through the formula

$$G_F \approx \frac{f^2}{m^2} \quad . \tag{10}$$

As before, make the Stückelberg split of the W field into normal fields A and B , using the transformation matrix $S(B)$, given by

$$S(B) = \frac{1 + i\sqrt{G_F}\,B}{1 - i\sqrt{G_F}\,B} \quad .$$

From the non-polynomial part of the Lagrangian (analogous to $\mathcal{L}_{YM}^{(2)}$ of (9)) one can compute finite contributions to the renormalization constants. As explained before, these depend on $G_F^{-\frac{1}{2}}$. As an example consider the computation of δm^2 up to the second order in f^2. The non-polynomial part of the Lagrangian $\mathcal{L}_{YM}^{(2)}$ will give rise to the super-graph

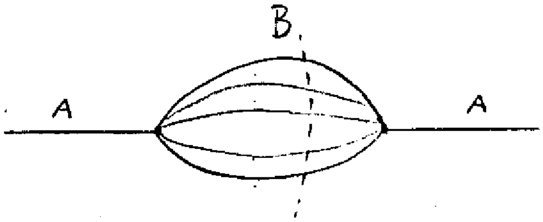


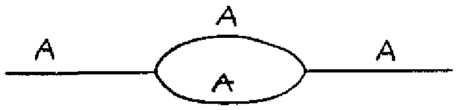
Fig. 1

with millions of B-lines going across. The order of magnitude contribution of this graph can be shown to be

$$\delta m^2 \propto f^2/G_F \quad . \tag{11}$$

From the relation $G_F \propto f^2/m^2$ we would thus obtain, if the present computational procedure for renormalization constants is correct, as an order of magnitude relation $\delta m^2 \approx m^2$, i.e. nearly all mass of the particle is self-mass.

The contribution from the polynomial part $\mathcal{L}_{YM}^{(1)}$, represented by



is, however, still unregularized and - as in the case of strong Yang-Mills theory, explained in Sec. II - will, in accordance with the ideas of this note, need further (realistic) damping if we desire to assign to this contribution a well-defined number.

Let us neglect for the present this infinite contribution and examine the relation (11). We know that the relation $G_F^2 \approx f^2/m^2$ correctly represents the approximate inter-relation of the three constants f, m and G_F; we normally obtain the relation by considering the exchange graph

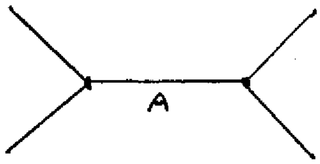


Fig. 2

That the same relation should emerge from the self-mass graph

Fig. 3

is a welcome confirmation of the thesis of this note - which states that in a non-polynomial theory the inbuilt cut-off is determined by the minor coupling constant (in this case $G_F^{-\frac{1}{2}}$) and the renormalization constants of the theory are essentially expressed as functions of its inverse.

IV. GRAVITATIONAL INTERACTION

Einstein's gravitational Lagrangian is the non-polynomial Lagrangian par excellence. Its form is given by:

$$L = \frac{1}{\sqrt{\det g^{\mu\nu}}} \left[\kappa_g^{-2} R(g) + L(\text{matter}) \right] .$$

Here

$$R = g^{\mu\nu}(\Gamma^{\lambda}_{\mu\rho} \Gamma^{\rho}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu} \Gamma^{\rho}_{\lambda\rho})$$

with

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}\left[(\partial_\nu (g^{-1})_{\beta\mu} + \partial_\mu (g^{-1})_{\beta\nu} - \partial_\beta (g^{-1})_{\mu\nu}\right] .$$

Both Γ (through its dependence on $(g^{-1})_{\alpha\beta}$ and $(\det g)^{-\frac{1}{2}}$ are the sources of the non-polynomiality. The important point to note is that the latter factor universally multiplies L(matter) ; in addition, all the derivatives occurring in L(matter) (the covariant derivatives $[(\partial_\mu - \Gamma_\mu)]$) give rise to non-polynomial expressions through the occurrence of Γ_μ .

In the vierbein gravity scheme, $g^{\mu\nu}$ equals the product of vierbein fields $L^{\mu a} L^{\nu}_{a}$ with $L^{\mu a} = \eta^{\mu a} + \kappa_g h^{\mu a}$. $(\eta^{\mu a} = 1, -1, -1, -1)$. Thus

$(\det g^{\mu\nu})^{+\frac{1}{2}} = [\det (\eta^{\mu a} + \kappa_g h^{\mu a})]$ is a polynomial of fourth order in κ_g. In two previous papers [7] the reality of the inbuilt cut-off at κ_g^{-1} was demonstrated in a calculation of electron's self-mass and self-charge. It was shown, for example, that the traditional logarithmic infinities get regularized to the form $\log(\kappa_g^2 m_e^2)$ with

$$\frac{\delta m}{m} \approx \frac{3}{4\pi} \cdot \alpha \log\left(\frac{4\pi}{\kappa m}\right)^2 + \text{terms of order } \alpha\kappa^n \log\kappa \text{ and } \alpha\kappa^n (\log\kappa)^2 .$$

As was remarked in the first paper, the crucial part of the result - and one which gives faith in the basic soundness of the ideas - is the appearance in the matrix elements of the characteristic logarithmic dependence on the minor coupling constant ($\log(\kappa^2 m^2)$ factors). For gravity theory these factors are of the real essence. This is because there is no question but that the gravitational constant $\kappa m_e \approx 10^{-22}$ represents an amazingly out-of-line magnitude - out of line with the other constants for the other forces. However, the logarithm of the newtonian constant $\log(G_N m_e^2) \approx 100$ $(G_N = 8\pi\kappa_g^2)$ is of the order of α^{-1}. The natural and characteristic appearance of the combination $\alpha \log(G_N m_e^2) \approx 1$ for non-polynomial gravity-modified electrodynamics appears to us far from being an accident. It is important to remark that Gm_e^2 is the Schwarzschild radius of the electron in natural units and this inbuilt cut-off has come at this magnitude.

From this point of view it is also encouraging that already in the lowest order in α, $\delta m/m$ is of a reasonable order of magnitude $(\delta m/m \approx 2/11)$. Since we expect on general grounds that, for higher orders in α, the effective constant will indeed be $\left[\alpha \log(G_N m^2)\right]$, one may start with the ansatz that all electron self-mass may have its origin in gravity-modified electrodynamics $(\delta m/m \approx 1)$ and then compute $\alpha \log(G_N m^2)$ in reverse, from the series

$$\frac{\delta m}{m} = 1 \approx \sum a_n (\alpha \log (G_N m^2))^n .$$

Consider now the prospects of a universal gravity-modified field theory in a general manner. The renormalized electrodynamics of leptons and photons exhibits no infinities higher than logarithmic. (The ratio g_A/g_V for muon-decay is not even logarithmically infinite in the lowest order while the potentially quadratically infinite photon self-mass should be zero from gauge invariance.) When we consider strong interaction physics, however, (and also weak non-leptonic physics) the situation alters. This is because here one encounters quadratic self-mass infinities for bosons which when computed as $1/\kappa_g^2$ by our methods would unacceptably give large masses to particles. We do need a universal, all-embracing non-polynomiality for interactions other than lepton-electrodynamics, but not the one provided by Einstein's gravity theory with its very small (minor) coupling constant κ_g. Fortunately, the model for a universal hadronic force, with the same characteristics as Einstein's gravity (except for the coupling strength), already exists, and this we exploit in the next section.

V. F-MESON DOMINATED GRAVITY

According to our present ideas, one of the fundamental forces of nature, electrodynamics, is mediated through two different mechanisms depending on whether we are considering leptons or hadrons. For lepton electrodynamics the present picture is that of a Dirac equation with photons interacting directly with muons and electrons. For hadrons, no such direct interaction is postulated. Instead the photon is pictured as inter-converting into a (prescribed) mixture of the known 1^- strongly-interacting particles $(\rho^0, \phi^0, \omega^0)$, which themselves couple strongly to hadronic electric charge and which for this reason may be called "strong photons".

Now nature has been prodigal in exactly the same manner with 2^+ particles. In addition to the massless graviton (with its obvious analogy with the massless photon), we know of at least three 2^+ massive strongly interacting particles f^0, $f^{0'}$ and A_2^0. It seems very natural that the analogy should carry further and that while leptons may interact directly with gravitons, so far as hadrons are concerned, it may be a mixture generically called F^0 of f^0, $f^{0'}$ (and perhaps other 2^+ objects which may be discovered) which provides the agency mediating gravity.*) For this to happen, it is mandatory that the F-meson in its strong interaction should couple to the hadronic stress tensor just as the graviton does to the lepton stress tensor.

We have constructed[8)] a generally covariant theory of a universal strong coupling of F-mesons to hadronic stress tensor (with a coupling parameter $\kappa_f \approx m_F^{-1} \approx 1$ BeV) and of the mixing of these particles to gravitons, on an analogy with ρ-γ mixing in electrodynamics. The formalism is elegant - as indeed everything where general relativistic invariance is concerned should be. The form of the final Lagrangian is simple; it consists of three pieces :

$$L^{(1)} = (\det g)^{-\frac{1}{2}} \left[R(g) + L(\text{leptons}) \right]$$

$$L^{(2)} = (\det f)^{-\frac{1}{2}} \left[R(f) + L(\text{hadrons}) \right]$$

$$L^{(3)} = \frac{M_f^2}{4\kappa_f^2} \quad (\det f)^{-\frac{1}{2}} \begin{bmatrix} \operatorname{Tr}\,(fg^{-1})^2 - (\operatorname{Tr} fg^{-1})^2 \\ + 6 \operatorname{Tr} fg^{-1} - 12 \end{bmatrix} .$$

Notice the symmetry of $L^{(1)}$ and $L^{(2)}$ so far as f and g tensors are concerned.**) The lack of symmetry in $L^{(3)}$ (which is a sort of cosmological term) is a reflection of the physical lack of symmetry - in that the f-field represents particles of mass M_f while the g-field represents massless gravitons. (In the vierbein formalism, where

$$\left.\begin{aligned} g^{\mu\nu} &= L^{\mu a} L^{\nu}_{\ a} , & L^{\mu a} &= \eta^{\mu a} + \kappa_g h^{\mu a} \\ f^{\mu\nu} &= \Phi^{\mu a} \Phi^{\nu}_{\ a} , & \Phi^{\mu a} &= \eta^{\mu a} + \kappa_f F^{\mu a} \end{aligned}\right\} \qquad (12)$$

,

the physical fields are $h^{\mu a}$ and $F^{\mu a}$.)

---

*) We discuss the problem presented by W mesons and to which particle, g or F they should directly interact, later. The idea of F dominance of gravity has been expressed by numerous authors, e.g., P. G. O. Freund, J. Schwinger, R. Delbourgo, Abdus Salam and J. Strathdee, K. Raman and, in a form essentially identical to the above, by J. Wess and B. Zumino.

**) In Eq.(12) we define the relation of the tensor $f^{\mu\nu}$ to the physical F field.

It is clear what this design will achieve. Strong interaction **physics will now have a** <u>universal</u> **inbuilt cut-off from the non-polynomiality** of $(\det f)^{-\frac{1}{2}}$ at about $(\kappa_f)^{-1}$. Lepton physics, or those parts of it included in $L^{(1)}$, will exhibit the inbuilt cut-off at $(\kappa_g)^{-1}$ as before.

Let us reflect on the work of Secs. II and III in the light of f and g gravitons and their interactions. Recall that in Sec. II a part of the Yang-Mills strong Lagrangian, even after the Stückelberg transformation, still remained obstinately polynomial in character. This part will now be multiplied by the factor $(\det f)^{-\frac{1}{2}}$. In Sec. III the same thing happened for W mesons. Now if these are treated on a par with <u>leptons</u>, and their free (and self-interacting) Yang-Mills Lagrangian is added to $L^{(1)}$, their self-mass will be proportional to $(f^2/\kappa_g^2) + (f^2/G_{Fermi})$, i.e., these particles would each weigh some 10^{-5} gms. It would seem more reasonable - if W mesons exist at all - to class them with <u>hadrons</u>. Notice we are making the definite physical statement that W mesons interact <u>strongly</u> with F particles. Paradoxically this is in aid of making them <u>light</u> $(\delta m^2 \approx f^2/G_{Fermi})$ rather than <u>too massive</u> $(\delta m^2 \approx f^2/\kappa_g^2)$.

Of course there is no question that the photon free Lagrangian must belong to $L^{(1)}$ and so $\overset{\text{must}}{\wedge}$ the weak $(J_\mu^{lep}\ W_\mu)$ terms. On the other hand, terms of the form $m^2(\rho_\mu^0 - A_\mu)^2$ giving the mixing of photons with the strong $\rho^0 - \phi^0 - \omega^0$ complex would belong to $L^{(2)}$ and so would the term $J_\mu^{had}\ W_\mu$. The mixed leptonic-hadronic weak processes thus acquire different cut-offs, depending on the company they keep. Clearly the interplay of the hierarchy of the various inbuilt cut-offs λ_π^{-1}, $G_{Fermi}^{-\frac{1}{2}}$, κ_g^{-1} and κ_f^{-1} in prediction of reasonable magnitudes for the renormalization constants when worked out fully will provide severe and non-trivial*) tests of the ideas here expressed.**)

---

*) There is of course another and more primitive level on which the notion of f-g mixing can be tested. The uncompromising statement of this theory that F-mesons couple with the strong stress tensor implies that F-mesons (like the ω and ϕ particles interacting respectively with baryon and hypercharge) interact more strongly with ordinary matter, the more massive it is. Even if the F meson of the present theory is identified with a mixture of the existing f^0 and $f^{0'}$ particles, there appears at present to be no experimental contradiction to this statement.

**) If one does not believe in strong gravity, and the only universal cut-off is assumed to lie at $(\kappa_g)^{-1}$, we may possibly understand the empirical relation $m_N/m_e \approx g_{\pi N}^2/e^2$ (Abdus Salam and J. Tiomno, Nucl. Phys. 9, 585 (1958)) as arising from $\delta m_e/m_0 = 6\alpha/4\pi \log(\kappa_g m_0)$ and $\delta m_p/m_0 = 1/4\pi\ G^2 \log(\kappa_g m_0)$ where G^2 is the (3D+2F) combination of octet coupling constants $(\approx 4.5\ g_{\pi N}^2)$ and m_0 is assumed the same for nucleons and electrons. The origin of muon mass remains a mystery even among such disreputable (second order) derivations of mass formulae.

APPENDIX I

A. THE NON-POLYNOMIAL METHOD AND ASSOCIATED AMBIGUITIES

No-one would question the thesis[9] that the highly singular lagrangian operators in field theories are at best symbolic entities, and that to get meaningful and unambiguous numbers from them is an art to be justified, at the present stage of our mathematics, post hoc on the twin criteria of internal self-consistency of any prescriptions used and the agreement of the computed numbers with experiment. Two examples of the practice of this art are the defining of the (highly singular) photon self-mass integral as zero in deference to gauge invariance and the invention of Feynman-DeWitt-Faddeev particles in the singular (zero mass) Yang-Mills theory (see Sec. II) to ensure conservation of unitarity. But the most spectacular and most successful example of supplementation of lagrangian theory with extra rules is the invention of the renormalization procedure in electrodynamics designed to calculate unambiguously all mass-shell quantities except two - self-mass and self-charge. To excuse the inability of the procedure as originally formulated to compute these two (singular) numbers in polynomial lagrangian theories, Dyson[10] put forward the wonderful thought that these two were in any case intrinsically unmeasurable quantities and no heartbreaks need occur if they cannot be computed. . Unfortunately, while this may have been true of electrodynamics considered in isolation, the two magnitudes (defined, as they are, on the physical mass-shell) $\delta m/m = (Z_1 m_0 - m)/m$ and $\delta e/e = 1 - Z_3^{1/2}$ are definitely measurable in a symmetry theory; for example in a theory where the electron and the neutrino are treated as members of the same doublet. Nothing in the mathematics is altered, but the numerical value of m_0 can now be read off from neutrino mass. A better example is $\pi^+ - \pi^0$ mass difference[11] which in strict renormalization theory would be called unmeasurable.

It is clear why in Dyson's procedure Zm_0 and $Z_3^{1/2}$ could not be computed. In a polynomial lagrangian theory - like $\mathcal{L}_{int} = g\phi^3$ - the constant Zm_0^2 is related to the Fourier transform of the chronological product distribution:

$$g^2 \langle T\,\phi^2(x)\,\phi^2(0)\rangle_+ = g^2 \quad \Delta_F^2(x) \quad .$$

Now Gel'fand and Shilov[12], for example, do define this distribution and assign a value to its Fourier transform

$$-\pi^2 g^2 \left[\log\left(-\frac{p^2}{(4\pi)^2}\right) - \psi(2) - \psi(1)\right] \quad ;$$

it is, however, well known[13] that the distribution itself is ambiguous *) at $x = 0$ up to an arbitrary multiple of $\delta^4(x)$ and likewise for its Fourier transform. The same applies to distributions like $1/n!\ \langle T\,\phi^n(x)\,\phi^n(0)\rangle$ which are ambiguous at $x = 0$ to the extent of $(\partial^2)^{n-2}\,\delta^4(x)$.

To put it very crudely, the so-called infinite constants are uncalculable not because they are (in naive physicists' mathematics) infinite; they are uncalculable because (even though sophisticated mathematics computes them as finite) they are ambiguous. The ambiguity problem is therefore the heart of the problem of computing self-mass, self-charge and other renormalization constants. The problem is bad enough for polynomial renormalizable Lagrangians with but a few matrix elements ambiguous. For non-renormalizable Lagrangians of polynomial variety considered in the past, its resolution appears to be nearly impossible.

Now, paradoxical though it seems, it appears that at least one consistent resolution of this problem can be formulated for (the seemingly non-renormalizable) non-polynomial Lagrangians - specifically for Lagrangians like $\mathcal{L}_{int} = g(\kappa\phi)^r\, e^{\kappa\phi}$. Here the Lagrangian is an entire function of the variable $\kappa\phi$. It might, on general grounds, be expected that a super-propagator like $\langle T\, e^{\kappa\phi(x)}\, e^{\kappa\phi(0)}\rangle$ would be ambiguous up to $\sum_{n=0}^{\infty} a_n(\partial^2)^n\,\delta(x)$ with a_n real and $\sum a_n z^n$ an entire function **) of order $< \frac{1}{2}$. (In momentum space the corresponding ambiguity is that of the entire function $\sum a_n(-p^2)^n$.)

*) In fact, as we shall see later, the definition we finally adopt differs from the above by a $\log g^2$ term.

**) This condition has its origin in the Jaffe localizability of the operator $e^{\kappa\phi(x)}$.

Lehmann and Pohlmeyer[2] have, however, shown that for such cases **there exists a unique minimally singular super-propagator.** The definition of this least singular super-propagator coincides with the one heuristically proposed earlier by Volkov, Filippov, Salam and Strathdee[14] (VFSS) prescription for all cases of what the last authors called super-normal, finite, non-polynomial Lagrangians (i.e., when r in $\mathcal{L}_{int}$ is less than or equal to 2). The important point which Lehmann and Pohlmeyer and also Blomer, Constantinescu and Mitter[15] make is that from their point of view the very existence of such a super-propagator is intimately connected with the fact that, in contrast to the polynomial Lagrangian case, the distribution $\exp(i\kappa^2 \Delta_F)$ for non-polynomial situations can be defined as a limiting value of an analytic function. We come back to this point later.

The minimally singular ansatz itself is very simple to state. Consider the super-propagator $\exp(i\kappa^2 \Delta_F)$. The existence of the ansatz is based on the observation that the VFSS prescription for defining this super-propagator gives rise to a function the real part of whose Fourier transform (being the Hankel-transform of an infinitely differentiable function) decreases strongly either for $p^2 \to +\infty$ or $p^2 \to -\infty$, the sign $\pm\infty$ depending on the sign of the exponent in $\exp(\pm i\kappa^2 \Delta_F(x))$. The ambiguous terms, on the other hand, possess the form $\left[\sum_{n=0}^{\infty} a_n(-p^2)^n\right]$ and define an entire function of order $< \frac{1}{2}$. These terms do not vanish in any direction in the p^2 plane. The VFSS prescription therefore selects (out of all ambiguous choices possible for the definition of the super-propagator) the one which is minimally*) singular for the appropriate limit $p^2 \to +\infty$ or $-\infty$.

Lehmann and Pohlmeyer made their suggestions for purely exponential Lagrangians, considering only second orders in the major coupling constant. We shall tentatively accept their suggestion and also hopefully assume that the analyticity ansatz in the variable $\kappa^2 \Delta(x)$ can be extended to functions of many complex variables $\kappa_{ij}^2 \Delta(x_i - x_j)$ and similar uniqueness statements can be made for higher-order super-propagators. Here we wish to extend the criterion in two other directions:

---

* A more physical formulation of the same ansatz is due to Filippov who states it in the form of the requirement that the ratio of the real part to the imaginary of the Fourier transform of the super-propagator should vanish when $p^2 \to +\infty$ (i.e. along the physical cut in p^2 plane).

1) We wish to consider not just the super-renormalizable Lagrangians of the type $ge^{\kappa\phi}$ but also the renormalizable Lagrangians $g\phi^2 e^{\kappa\phi}$, $g\phi^3 e^{\kappa\phi}$ and mixed varieties $g(\bar{\psi}\psi A)\, e^{\kappa\phi}$ in order to gain a similarly well-defined computation of the still surviving infinities.

2) We further desire to consider rational Lagrangians of the super-renormalizable $\left((g\phi)/(1+\kappa\phi) \text{ or } (g\phi^2)/(1+\kappa\phi)\right)$ and renormalizable varieties $\left((g\phi^3)/(1+\kappa\phi),\ (g\phi^4)/(1+\kappa\phi) \text{ and } (g\bar{\psi}\psi A)/(1+\kappa\phi)\right)$ and show that here too the renormalization constants can be unambiguously defined and that their dependence on the variable κ is as stated in the text, viz., $\log(\kappa m)$ for the traditional logarithmic, $1/\kappa^2$ for the traditional quadratic and $1/\kappa^4$ for the quartic infinities (where we have assumed that all particles are massless in order to simplify discussion).

B. TREATMENT OF SURVIVING AMBIGUITIES IN RENORMALIZABLE NON-POLYNOMIAL THEORIES

Let us briefly consider how we wish to extend the minimality principle to give a meaning to the old infinite renormalization constants. Consider two Lagrangians, one super-normal and the other renormalizable:

$$L_I = g(e^{\kappa\phi} - 1 - \kappa\phi)$$

$$L_{II} = g\left(e^{\kappa\phi} - 1 - \kappa\phi - \frac{\kappa^2\phi^2}{2}\right) .$$

The minimality ansatz for the super-renormalizable (finite) case L_I suggests setting all ambiguity constants $a_i = 0$ in the super-propagator $S = \langle TL(x)\, L(0)\rangle$ right away. We wish to show that this is not enough for (infinity-containing) renormalizable Lagrangian L_{II} and we need an extension of the VFSS method. To see this, set $a_i = 0$ in the super-propagator for both L_I and L_{II}, so that

$$S_I = g^2 \sum_{n=2} \frac{1}{n!}\, (\kappa^2 \Delta)^n \qquad \text{for } L = L_I \tag{A. 1a}$$

$$S_{II} = g^2 \sum_{n=3} \frac{1}{n!}\, (\kappa^2 \Delta)^n \qquad \text{for } L = L_{II} . \tag{A. 1b}$$

The VFSS method starts by converting S_I into a Sommerfeld-Watson integral, rotating the contour and then taking the Fourier transform; thus

$$S_I = \frac{g^2}{2\pi i} \int_{Re\, z<2} \frac{(\kappa^2 \Delta)^z}{\Gamma(z+1)\tan\pi z}\, dz \quad . \tag{A. 2}$$

Since for this super-renormalizable theory the z-contour lies along $0 < \mathrm{Re}\, z < 2$ we can use the well-known expression for the Fourier transform of $\Delta^z(x)$ (for zero-mass case)

$$i\, \tilde{\Delta}(p, z) = \frac{\Gamma(2-z)}{\Gamma(z)} \frac{(16\pi^2)^{1-z}}{(-p^2)^{2-z}} \quad . \tag{A. 3}$$

Thus (ignoring factors of 4π) and on the basis of no more than classical mathematics, once we start with (A. 2) it leads on to the result:

$$\tilde{S}_I(p^2) = g^2 \kappa^4 \int_{Re\, z<2} \frac{\Gamma(2-z)(-\kappa^2 p^2)^{z-2}}{\tan\pi z\, \Gamma(z+1)\, \Gamma(z)} \quad . \tag{A. 4}$$

Consider $S_{II}(x)$ now. Here the Sommerfeld-Watson-rotated contour lies between $2 < \mathrm{Re}\, z < 3$ and does not satisfy $\mathrm{Re}\, z < 2$. The "classical" Fourier transform formula (A. 3) can no longer be used. The old VFSS prescription for this case involved writing

$$S_{II} = S_I - \frac{g^2}{2} (\kappa^2 \Delta)^2 \quad . \tag{A. 5}$$

Now, while S_I can be Fourier transformed as stated above, the second term in this split represents the familiar (logarithmic) infinity. Such terms in popular parlance have been called "sore thumbs (S. T.'s)" since they stick out as residual infinities when we shift the z-plane contour from $2 < \mathrm{Re}\, z < 3$ to the region $\mathrm{Re}\, z < 2$.*) It was suggested in the earlier papers that the sore thumbs - which, as we said, represent surviving infinities - should be left to be treated using Dyson's subtraction formalism[16].

*) Quite generally in x-space S.T.'s are the residues of those z-plane poles which lie between $2 < \mathrm{Re}\, z < n_0$ where n_0 is the first term of the series expansion $L(\phi) = \sum_{n \geq n_0} \frac{v(n)}{n!} \phi^n$.

From the present point of view, it is clear that leaving sore thumbs thus sticking out is wrong. One should never make a split of the type (A. 4); simply define:

$$\widetilde{S}_{II}(p^2) = g^2 \kappa^4 \int_{Re\, z < 3} \frac{\Gamma(2-z)\,(-\kappa^2 p^2)^{z-2}}{\tan\pi z\ \Gamma(z+1)\ \Gamma(z)} \quad . \tag{A. 6}$$

Here the Dyson term $(-g^2/2\ (\kappa^2\ \Delta)^2)$ is not being Fourier transformed in isolation; its transform is uniquely determined as the appropriate residue at $z = 2$ of an analytic continuation of the minimally singular object $\widetilde{S}_I$ in the $\kappa^2 p^2$ plane, defining thereby one special value for the renormalization constant in question.

To summarize, what we are saying is the following. If we accept the Lehman-Pohlmeyer criterion for selecting from among the many possible definitions of the super-propagator the minimally singular one for super-renormalizable theories, a simple extension of the VFSS procedure will give an equally specific definition linked to the Lehmann-Pohlmeyer ansatz for the renormalizable cases, and thereby define a unique and distinguished value for the old Dyson renormalization constants. It is this value which formed the basis of all discussion in the present note.

C. RATIONAL LAGRANGIANS

The field-theoretic distinction between rational lagrangian operators of the type $L_R = g\ (\phi^n/(1+\kappa\phi))$ and transcendental lagrangian operators

$L_T = g\phi^n\ e^{\kappa\phi}$ is well known. The transcendental variety $e^{\kappa\phi}$ belong to the Jaffe class of localizable operators - at least when the behaviour of Jaffe's indicatrix function $\rho(p^2)$ is considered to second order in the major coupling constant g^2 $\left[\rho_T(p^2) \approx g^2 \exp|\ (\kappa^2 p^2)^{1/3}\right]$. This is not the case for the

rational Lagrangians $(\rho_R(p^2) \approx g^2 \exp|\ (\kappa^2 p^2)|$ when ϕ represents zero-mass

particles and, as shown by Efimov, $\rho_R \approx g^2 \exp\left|\left(\sqrt{\kappa^2 |p|^2}\ \log\sqrt{\kappa^2 |p|^2}\right)\right|$ when $m \neq 0$.

To belong to the Jaffe class, $\rho(p^2)$ should not increase faster than $\exp\left(\sqrt{|p^2|}/(\log p^2)^2\right)$. If rational Lagrangians do not represent Jaffe localizable fields - and doubts on this could be entertained on the score that the hitherto tested second-order behaviour of the indicatrix function may not be a true index of its exact behaviour - then local commutativity in the Jaffe sense does not automatically hold. In a recent preprint, Taylor[17)] has examined this problem and attempted to define local commutativity for rational lagrangian operators by means of a limiting procedure for the Wightman functions (super-propagators) of the theory. Taylor demonstrates for these Lagrangians the existence of TCP operator, the cluster property, the existence of the asymptotic limit, LSZ reduction formulae and forward dispersion relations.

Considering the ambiguity problem for such Lagrangians, one prescription, related to the one used for transcendental Lagrangians for defining super-propagators, is the following: Write $L_R(\phi) = \int_{-\infty}^{\infty} \tilde{L}_R(\lambda)\, e^{i\lambda\phi} d\lambda$ and define

$$\langle T\, L_R(\phi(x))\; L_R(\phi(0)) \rangle \quad \text{as} \quad \int \tilde{L}(\lambda_1)\, \tilde{L}(\lambda_2)\, \langle e^{\mp\lambda_1\lambda_2\Delta(x)} \rangle_{d\lambda_1\, d\lambda_2}$$

where on the right, under the integral sign, appears the Lehmann-Pohlmeyer minimally singular super-propagator. This is unambiguous but it is a tricky problem to show that this definition coincides exactly with the one based on the Mellin transform method used in our earlier paper[14]. We believe (without having computed the formal proof) that this is indeed the case. (A proper proof would involve tricky changes of orders of integrations and summations.) Accepting this conjecture tentatively, however, we may consider the problem of estimating the κ-plane behaviour of matrix elements in (rational) renormalizable Lagrangians of the type $g\left((\bar{\psi}\psi A)/(1+\kappa\phi)\right)$ or $g\left(\phi^n/(1+\kappa\phi)\right)$, $n \leq 4$. What we wish to show is that so far as the renormalization constants are concerned, their dependence on (inverse) powers of κ is simple and just what one might expect when the naive limit $\kappa \to 0$ is taken.

According to the traditional analysis of Ref.16 for self-interacting situations of Lagrangians like $\varphi^n/(1+\kappa\varphi)$, the cases $n < 3$ are super-normalizable with all matrix elements finite; while $n = 3$ and 4 represent renormalizable situations, and $n = 5$ and higher are non-renormalizable. For mixed cases no complete analysis exists; one may be certain, however, that theories of the type $(\bar{\psi}\psi A)/(1+\kappa\phi)$ are renormalizable.

Let us take the two cases $L_1 = \varphi^4/(1+\kappa\varphi)$ and $L_2 = (\bar{\psi}\psi A)/(1+\kappa\phi)$ as typical of renormalizable theories. Both are characterized by the fact that in the limit $\kappa \to 0$, the Lagrangians are singular like $\underset{M\to\infty}{M^4}$. (We assign singularity behaviour $\varphi \sim M$, $A \sim M$, $\psi \sim M^{3/2}$ to the fields from the knowledge of how their propagators behave at $x = 0$ and throughout consider all fields massless.)

It is convenient to rewrite L_1 and L_2 in the form

$$L_1 = \frac{1}{\kappa^4}\,\frac{(\kappa\phi)^4}{1+\kappa\phi} \quad , \qquad L_2 = \frac{1}{\kappa^4}\,\frac{(\kappa^{3/2}\bar{\psi})(\kappa A)(\kappa^{3/2}\psi)}{(1+\kappa\phi)} \quad .$$

From the result

$$S_n = (-i)^n \int [L(\phi)]^n \,(d^4x)^{n-1}$$

$$= (-i)^n \,\frac{1}{\kappa^4} \int [L(\kappa\phi)]^n \,[d^4(x/\kappa)]^{n-1}$$

it is clear (assuming zero masses) that $\tilde{S}_n(\kappa, p)$ for a process with E_b external Bose and E_f external Fermi lines would in general behave like

$$\tilde{S}_n(\kappa, p) \approx \frac{1}{\kappa^{4-E_b-\frac{3}{2}E_f}} \qquad F(\kappa^2 p^2) \quad .$$

Now the "sore thumb" contributions - which are related to the renormalization constants - are always of the form $(\kappa^2p^2)^s\,[\log(\kappa^2p^2)]^r$. Thus the maximum κ-plane singularity (at $\kappa = 0$) exhibited by these terms is $\approx (\log\kappa^2p^2)^r/\kappa^{4-E_b-\frac{3}{2}E_f}$, thus proving the result stated in paragraph 2 of the Introduction. It is interesting to remark that since κ^2 always appears multiplied by p^2 (apart from the overall factor $1/\kappa^{4-E_b-\frac{3}{2}E_f}$) the singularity structure in κ^2 plane and in p^2 plane are the same for the super-propagator. Maximal analyticity in κ^2 plane is synonymous with maximal analyticity in p^2 plane. One can clearly generalize this to functions of many variables $1/\kappa^{4-E_b-\frac{3}{2}E_f} \cdot F(\kappa^2 p_1^2, \kappa^2 p_2^2, \kappa^2 p_3^2 \cdots)$.

APPENDIX II

We wish to evaluate the super-propagator

$$D^{\mu a, \nu b}(x) = \left\langle 0 \right| T \; \frac{L^{\mu a}(x)}{\det L(x)} \; , \; \frac{\det L^{\nu b}(0)}{\det L(0)} \left| 0 \right\rangle$$

where

$$L^{\mu a} = \eta^{\mu a} + \kappa h^{\mu a}$$

and the vierbein graviton propagator is given by

$$\left\langle 0 \right| T\, h^{\mu a}(x) \, , \, h^{\nu b}(0) \left| 0 \right\rangle = \tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab} + \eta^{\mu b}\eta^{\nu a} - \eta^{\mu a}\eta^{\nu b})\, D(x)$$

in a suitable gauge. The heart of the derivation lies in noticing that $\frac{1}{\sqrt{\det L}}$ is proportional to $\int d^4 n \exp(L^{\mu a} n_\mu n_a)$ while $\frac{1}{\det L}$ can be similarly expressed as an eightfold integral. Since formulae for super-propagator involving exponentials are easy to write (e. g. $\langle e^{\phi(x)} e^{\phi(x')} \rangle_+ = e^{\Delta(x-x')}$) such parametric representations of $(\det L)^{-1}$ simplify the work. More precisely, write

$$D^{\kappa\lambda, \mu\nu} = \eta^{\kappa\lambda}\eta^{\mu\nu} D^{(0)} + \tfrac{1}{2}(\eta^{\kappa\mu}\eta^{\lambda\nu} + \eta^{\kappa\nu}\eta^{\lambda\mu} - \eta^{\kappa\lambda}\eta^{\mu\nu}) D^{(1)} \; .$$

One can show that $D^{(0)}$ and $D^{(1)}$ can be expressed in terms of a function $\mathcal{D}(\alpha, \beta, \gamma)$ and its derivatives as follows:

$$\mathcal{D}^{(0)} = \left[1 + \kappa^2 D(x) \left(-\frac{1}{4}\frac{\partial}{\partial\alpha} + \frac{1}{36}\frac{\partial}{\partial\beta} - \frac{1}{9}\frac{\partial}{\partial\gamma} \right) \right] \mathcal{D}(\alpha, \beta, \gamma) \Big|_{\alpha=\beta=\gamma=1}$$

$$\mathcal{D}^{(1)} = \kappa^2 D(x) \left[1 + \frac{1}{9}\frac{\partial}{\partial\beta} + \frac{1}{18}\frac{\partial}{\partial\gamma} \right] D(\alpha, \beta, \gamma) \Big|_{\alpha=\beta=\gamma=1} \quad ,$$

where $D(\alpha, \beta, \gamma)$ can be expressed as an eightfold integral over two four-vectors, which eventually simplifies to the form:

$$D(\alpha, \beta, \gamma) = \frac{8}{(\kappa^2 D)^4} \frac{1}{\beta^4} \int_{\frac{\gamma-\beta}{\beta}}^{\infty} \frac{d\sigma}{1+\sigma^2}$$

$$\left[(\sigma - \sigma_0) \exp\left[-\frac{1}{\kappa^2 D(x)} \frac{2\alpha^2/\beta}{\sigma - \sigma_0} \right] \times \frac{1}{2} \int_0^1 dv^2 \frac{\sqrt{1-v^2}}{\sigma^2 - v^2} \right] ,$$

where $\sigma_0 = \frac{\gamma-\beta}{\beta}$.

This integral can be further simplified and expressed in terms of hypergeometric functions. We shall not set down these forms, since for most practical applications the form given above is adequate.

REFERENCES

1) For references see the review paper Abdus Salam "Non-polynomial lagrangian theories", report to the Miami Conference 1970, Coral Gables.

2) Ref. 1 and H. Lehmann and K. Pohlmeyer, DESY preprint 70/26, June 1970.

3) This was first pointed out by M. K. Volkov, Ann. Phys. (N. Y.) 49, 202 (1966). It is possible that summations over the major coupling constant g may also give $(\log g^2)$-like terms. See, in this connection T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962) and T. D. Lee, Phys. Rev. 128, 889 (1962), where they compute quadrupole moment for charged vector mesons and obtain results proportional to $\log\alpha$. We could treat this case using methods of Sec. II in which case α would act both as the major and (as part of) the minor constant.

4) D. Boulware, Ann. Phys. (N. Y.) 56, 140 (1970).

5) See Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/70/8 "On equivalent formulations of massive vector field theories" (to appear in Phys. Rev.).

6) This has been emphasised also by Prof. V. F. Weisskopf to whom we are deeply indebted for inspiring discussions.

7) Abdus Salam and J. Strathdee, Lettere al Nuovo Cimento IV, 101 (1970); R. Delbourgo, C. J. Isham, Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/70/131.
Whereas in the first paper calculations were simplified through replacing the tensor field $h^{\mu a}$ by a scalar field $\eta^{\mu a}\phi$, in the second paper full tensor expressions were used to carry through the non-polynomial calculations. Surprisingly, there was no essential change in results. (The factor 4π inside the logarithm gets replaced by 2π.) The closed expressions we have obtained for the super-propagators

$$\left\langle \frac{\eta^{\mu a} + \kappa h^{\mu a}}{\det(\eta^{\mu a} + \kappa h^{\mu a})} , \frac{\eta^{\nu b} + \kappa h^{\nu b}}{\det(\eta^{\nu b} + \kappa h^{\nu b})} \right\rangle$$ are exhibited in

Appendix II. We have worked throughout with the quantity $L^{\mu a}$ as the fundamental and $L_{\mu a}$ and L_{μ}^{a} as derived quantities, expressed in terms of $L^{\mu a}$. The equivalence of different quantized versions of this theory, differing from each other by the use of these different co-ordinates, is a problem (in non-localizable situations) on which no general agreement seems to exist (see for example J. G. Taylor, Ref. 17). Presumably, if equivalence theorems on the mass shell exist, it should not matter which formulation of gravity theory one adopts as the starting point. If it is convenient to use one formulation - say with $L^{\mu a}$ as fundamental fields - and certain results on the mass shell obtained, the same results must be obtainable (by a suitable summation of diagrams) in any other formulation.

8) C. J. Isham, Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/70/108.

9) Particularly after the recent work of A. Jaffe and J. Glimm, Commun. Math. Phys. 11, 9 (1968) on renormalization constants in two-dimensional polynomial field theories.

10) F. J. Dyson, Phys. Rev. 75, 1736 (1949).

11) This has been emphasised strongly by T. D. Lee, Proceedings of the CERN Conference on Weak Interactions, CERN, Geneva (1969).

12) I. M. Gel'fand and G. E. Shilov, "Les Distributions" (Dunod, Paris 1962); W. Güttinger, Fortschr. Phys. 14, 483 (1966); W. Güttinger and E. Pfaffelhüber, Nuovo Cimento 52, 389 (1967); G. C. Bollini, J. J. Giambiagi and A. Gonzales Domingues, Nuovo Cimento 31, 550 (1964).

13) See for example N. N. Bogolubov and V. D. Shirkov, "Introduction to Quantum Field Theory" (Interscience Publishers, New York 1954).

14) M. K. Volkov, Commun. Math. Phys. 7, 289 (1968) and 15, 69 (1969);
A. T. Filippov, Dubna preprint (1968).
Abdus Salam and J. Strathdee, Phys. Rev. D1, 3296 (1970).

15) R. Blomer and F. Constantinescu, Munich preprint (July 1970);
P. K. Mitter, Oxford preprint 18/70.

16) See also R. Delbourgo, K. Koller and Abdus Salam, Imperial College, London, preprint ICTP/69/10 (March 1970) (to appear in Ann. Phys. (N. Y.));
J. G. Taylor and B. W. Keck, Southampton University preprint (1970).

17) J. G. Taylor, Southampton University preprint, July 1970.