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ABSTRACT

It is suggested that the use of K-matrix formalism may remove the difficulty of non-Froissart-like behaviour encountered in calculations of finite order in non-polynomial Lagrangian theories.

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One of the difficulties of non-polynomial interactions treated according to the techniques developed by Okubo, Efimov, Fradkin and others [1)] is the unsatisfactory asymptotic behaviour of the momentum space amplitudes as computed in second order. On the other hand, it has recently been argued, in an axiomatic context, that the so-called non-localizable field theories yield the usual spin-statistics, TCP [2)] and cluster decomposition [3), 4)] theorems as well as a polynomially bounded S-matrix [3)]. No-one has yet suggested a computational procedure, however, which exhibits the last feature. In the present note we propose a reinterpretation of the programme of Ref. 1 which may guarantee an acceptable high-energy behaviour, thus perhaps removing the most serious blemish from which these theories have suffered.

The techniques of Ref. 1 are applicable to interactions such as $GV(f\phi)$ where V denotes a local, non-polynomial function of the scalar field $\phi(x)$. Prescriptions are given for summing the (divergent) perturbation series to finite order in G and all orders in f. The p-space amplitudes obtained by these methods are defined in the Symanzik region, and an essential ingredient of the programme has been the requirement that the physical region amplitudes be obtained by analytic continuation. The imaginary parts of the resulting functions (in the second order) are found to explode in energy like $\exp(f^2p^2)$. The origin of this behaviour is not difficult to trace. It lies in the burgeoning effect of unlimited particle numbers in the intermediate states of finite order (in G) graphs without a compensating damping from the vertices. Such vertex damping could possibly arise if <u>all higher order</u> contributions in G were included. Thus a realistic computation should either include terms of all orders in G or else in some way suppress the phase-space effects. We wish to suggest that the summation prescriptions in non-polynomial theories should be combined with a modified analytic continuation so as to yield the matrix elements of the reaction operator K, rather than the scattering operator S. <u>The crucial point to note is that these matrix elements are identical in the Symanzik region.</u>

The off-mass-shell continuations of the K-matrix elements can be defined perturbatively in terms of those of S by substituting the power series $S = 1 + i \Sigma G^n T_n$ into the formula

$$\frac{i}{2} K = \frac{S-1}{S+1} . \qquad (1)$$

On comparing this term by term with the series $K = \Sigma G^n K_n$ one finds [5)]

$$\left.\begin{aligned} K_1 &= T_1 \\ K_2 &= T_2 - \frac{i}{2} T_1^2 \\ K_3 &= T_3 - \frac{i}{2} (T_1 T_2 + T_2 T_1) - \frac{1}{4} T_1^3 \end{aligned}\right\} \qquad (2)$$

etc., which expressions can be interpreted as follows. In the Symanzik region we have $K_n = T_n$ because none of the product terms, $T_1 T_2$, etc., contributes. Thus K_n is a real quantity. As the external momenta are continued into timelike regions and physical thresholds are passed, the matrix element T_n develops an imaginary part while, simultaneously, new terms appear in the expression for K_n which exactly cancel the imaginary part of T_n. In other words, the Landau-Cutkosky contributions to T_n (due to real intermediate states) which are represented by terms like $\frac{i}{2} T_1 T_{n-1}, \ldots,$ are explicitly removed so that K_n remains real[6)] in the physical regions. <u>Our conjecture is that the unacceptable asymptotic behaviour is removed along with the imaginary parts</u>. We can verify this conjecture in the following particularly simple example.

Consider the second-order vacuum graphs corresponding to the interaction $GV(f\phi) = G(1 + f\phi)^{-1}$ which, in the case that ϕ describes a massless scalar particle, can be shown [7] to yield the amplitude

$$F(s) = G^2 \left(s \frac{\partial^3}{\partial s^3} + 2 \frac{\partial^2}{\partial s^2} \right) e^{f^2 s} E^*(-f^2 s) \qquad (3)$$

where $s = p^2$ and $E^*(z)$ denotes one of the exponential integral functions [8]. The behaviour of $E^*(z)$ for $z \to -\infty \pm i0$ is such as to yield

$$\mathrm{Re}\, F(s) \sim \frac{1}{s^3} \quad \text{and} \quad \mathrm{Im}\, F(s) \sim \pm e^{f^2 s} \quad \text{for} \quad s \to \infty \pm i0 \qquad (4)$$

which clearly exhibits the radically different behaviour of $T_2 = F$ and $K_2 = \mathrm{Re}\, F$. Since, according to (2), $T_2 = K_2 + \frac{i}{2} K_1^2$, it appears that the unacceptable part of T_2 can be associated with K_1^2 which arises in its turn from the representation of the matrix $T = K(1 - \frac{i}{2} K)^{-1}$ by a power series [9]. The representation of T by a series in powers of K must be avoided.

Our prescription is therefore as follows. Compute the matrices $K_n = T_n$ in the Symanzik region. Define K_n in the physical regions by the appropriate analytic continuation or by means of (2). If our conjecture is correct this matrix is energy bounded. Now use the expression $T = K(1 - \frac{i}{2} K)^{-1}$ to compute T but do not use an expansion in powers of K. Alternatively and equivalently, since $T^{-1} = K^{-1} - \frac{i}{2}$ we are advising a non-power-series inversion of the K-matrix. This is a problem of very considerable subtlety and has been solved for a partial-wave analysed K-matrix in two- and three-body processes. A possible approach might be to discard all matrix elements which correspond to transitions involving a change in the total particle number in excess of some value[10]. We do not know, of course, if the problem of diagonalization and inversion of an infinite matrix is easier in practice than the summation of an infinite series in the major coupling constant G. The main point of our paper is that whereas the use of a K-matrix formalism may be a luxury for polynomial theories, it is perhaps a necessity for non-polynomial interactions.

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This author stresses the possible superiority of perturbative computation of K as against S.

6) In a single-channel case in second order, remarking that iT_1^2 is the Cutkosky discontinuity of T_2, then $K_2 = T_2 - \frac{i}{2} T_1^2$ gives
$K_2(s) = T_2(s + i\epsilon) - \frac{1}{2}(T_2(s_1 + i\epsilon) - T_2(s_1 + i\epsilon)) = \frac{1}{2}(T_2(s + i\epsilon) + T_2(s - i\epsilon))$.
For the fourth-order box diagram the corresponding conjecture would be that the relevant Cutkosky discontinuities and discontinuities of

discontinuities, $-\frac{i}{2}T_2^2$, $-\frac{i}{8}T_1^4$, etc., all combine to give $K_4(s,t) = (1/4)\,[T_4(s+i\epsilon, t+i\epsilon) + T_4(s+i\epsilon, t-i\epsilon) + T_4(s-i\epsilon, t+i\epsilon) + T_4(s-i\epsilon, t-i\epsilon)\,]$. Note the somewhat transparent interpretation of the K-matrix from eq. (2) as a discontinuity-subtracted T-matrix.

7) Abdus Salam and J. Strathdee, Ref. 2, Sec. 4.

8) "Higher Transcendental Functions", Vol. II, Ed. A. Erdelyi, W. Magnus and F. Oberhettinger (McGraw Hill Publishing Co.) p.143.

9) A series expansion of $(1 - \frac{i}{2}K)^{-1}$ can be justified only if, in some sense, $\left|\frac{1}{2}K\right| < 1$. Such is clearly not the case here since $K_1^2 \sim \exp(f^2 p^2)$, which is precisely the source of the unbounded behaviour of non-polynomial theories.

10) This unitarity-preserving amputation may introduce bound-state poles into the S-matrix corresponding to zeros of $1 - \frac{i}{2}K$. The existence of these poles and their physical interpretation is of course a problem well known in the case of polynomial interactions and is not specific to non-polynomial theories.