IC/70/8
INTERNAL REPORT

INTERNATIONAL ATOMIC ENERGY AGENCY

and

UNITED NATIONS EDUCATIONAL SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON EQUIVALENT FORMULATIONS OF MASSIVE VECTOR FIELD THEORIES *

ABDUS SALAM **

and

J. STRATHDEE

ABSTRACT

The renormalization problem for vector meson theories is reviewed. The application of path-integral methods to the formulation of modified perturbation developments is considered. These methods, while particularly suited to the Yang-Mills theory of charged vector mesons, can, at least in principle, be applied more generally. The virtue of these modified perturbation series is that - while being unrenormalizable in the usual sense - they may lend themselves to treatment by the recently developed non-polynomial techniques.

MIRAMARE - TRIESTE

March 1970

* To be submitted for publication.

** On leave of absence from Imperial College, London, England.

I. INTRODUCTION

One of the troublesome features of vector field theories - and, in fact, of all higher spin field theories - is the appearance in the free-particle propagator of terms more singular than $1/x^2$ as $x \to 0$. Such terms arise from the spin-zero daughter of the spin-one particle and, as is well known, must necessarily accompany any local field description of the spin-one particles. When considering the high-energy behaviour of the S-matrix, or the renormalizability of its series development, it is advantageous to isolate the contributions of the more singular part of the propagator and study them separately. One of the best known procedures to accomplish this is by means of the Stückelberg split of the vector field and a corresponding transformation of those fields which interact with it. In this paper we wish to review the problems which arise when such a redefinition is made. While the conventional Stückelberg split and the canonical formalism which employs it may be of doubtful utility for any but the simplest case of neutral vector fields, we wish to advocate a non-linear generalization of it which, as will be shown in detail in the accompanying paper [1], is better suited for dealing with charged vector meson theories. Our tool is the path-integral formalism [2] of field theory. This powerful and flexible formulation has been used to analyse similar problems to the ones we discuss by Feynman [3], Faddeev and Popov [4], by Boulware [5] and by Fradkin and Tyutin [6] in their treatments of Yang-Mills theory. Our main purpose in reviewing these techniques is to show that they can be applied with advantage to systems more general than the Yang-Mills field including all cases of charged vector mesons. In particular we shall show that the S-matrix elements can be developed in series which, while not being renormalizable in the conventional sense, are of the Efimov-normal type [7]. This paper is therefore somewhat in the nature of a review but what we wish very strongly to emphasise is that with these methods it becomes clear that the path-integral formulation of field theory can with facility provide results which are derived with great labour using the canonical formalism.

II. THE STÜCKELBERG FORMALISM

Consider a set of massive vector meson fields U^i_μ in interaction with themselves and with other fields ψ. Provided that the interaction Lagrangian does not contain second-order derivatives [8], the Lagrangian describing pure spin-one particles corresponding to the fields U^i_μ is given by

$$L(U, \psi) = L_f(U) + L_f(\psi) + L_{int}(U, \psi) \tag{2.1}$$

where the free meson Lagrangian $L_f(U)$ is given by

$$L_f(U) = -\frac{1}{4} (\partial_\mu U^i_\nu - \partial_\nu U^i_\mu)^2 + \frac{1}{2} m^2 (U^i_\mu)^2 \quad . \tag{2.2}$$

Corresponding to this the chronological pairing for the field U^i_μ is given by

$$\begin{aligned} \overbrace{U^i_\mu(x)\ U^j_\nu}(y) &= \delta^{ij}\ \Delta_{\mu\nu}\ (x-y\,;\ m) \\ &= -\delta^{ij} \left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2} \right) \Delta_F(x-y\,;\ m) \end{aligned} \tag{2.3}$$

where Δ_F denotes the casual function satisfying the equation

$$(\partial^2 + m^2)\ \Delta_F\,(x-y\,;\ m) = -i\ \delta(x-y)\,.$$

We use the metric: $g_{\mu\nu}$ = diag (+ − − −).

In momentum space the absorptive part of the propagator is proportional to $\delta(p^2-m^2)\ (g_{\mu\nu} - p_\mu p_\nu / p^2)$ which evidently describes pure spin-one propagation.

One can split $\Delta_{\mu\nu}$ into two parts:

$$\Delta_{\mu\nu} = g_{\mu\nu}\,\Delta_F + \frac{\partial_\mu \partial_\nu}{m^2}\ \Delta_F \tag{2.4}$$

and formally reflect this separation in a split of the field U^i_μ into two parts

$$U^i_\mu = A^i_\mu + \frac{1}{m} \partial_\mu B^i \tag{2.5}$$

where the fields A^i_μ and B^i are assigned the chronological pairings

$$\overline{A^i_\mu(x) A^j_\nu}(y) = - \delta^{ij} g_{\mu\nu} \Delta_F(x-y; m)$$

$$\overline{A^i_\mu(x) B^j}(y) = 0$$

$$\overline{B^i(x) B^j}(y) = \delta^{ij} \Delta_F(x-y; m) \quad . \tag{2.6}$$

Insofar as internal lines are concerned,the split (2.4) is completely equivalent to substituting the expression (2.5) into $L_{int}(U, \psi)$ and using the pairings (2.6). For external lines one continues to use the purely transverse wave functions, $(-g_{\mu\nu} + p_\mu p_\nu/p^2) \; \delta(p^2-m^2)$.

This formal procedure of splitting propagators can be embedded in a canonical theory if one uses, instead of (2.1), the following Lagrangian:

$$L(A, B, \psi) = L_f(A, B) + L_f(\psi) + L_{int}(A + \frac{1}{m} \partial B, \psi) \tag{2.7}$$

where

$$L_f(A, B) = -\frac{1}{2} (\partial_\mu A^i_\nu)^2 + \frac{1}{2} m^2 (A^i_\mu)^2 + \frac{1}{2} (\partial_\mu B^i)^2 - \frac{1}{2} m^2 (B^i)^2 \tag{2.8}$$

which can also be expressed in the form

$$L_f(A, B) = L_f(A + \frac{1}{m} \partial B) - \frac{1}{2} (\partial_\mu A^i_\mu - m B^i)^2 - m \partial_\mu (A^i_\mu B^i) \; .$$

The advantage of starting with (2.7) is that one may use canonical methods treating A_μ and B as independent variables. From (2.8) one obtains the pairings (2.6). The disadvantage is that for each value of the index i there are now five fields A_μ and B instead of the original four. The Fock space generated by A_μ and B is larger than the space of physical state vectors and it contains zero-spin particles of negative metric. The physical subspace consists of those vectors which are annihilated by the positive frequency part of the operator $(\partial_\mu A^i_\mu - mB^i)$. Since L_{int} contains only the combination $A_\mu + (1/m) \partial_\mu B$ it follows from the equations of motion that

$$(\partial^2 + m^2) (\partial_\mu A^i_\mu - mB^i) = 0$$

so that the prescription for projecting out physical states is consistent with the equations of motion. The theory indeed describes only spin-one particles. Up to this point the Stückelberg field theoretic procedure, though cumbersome, is mathematically correct, consistent and fully acceptable.

The weakness of this formulation and particularly of the subsidiary condition

$$(\partial_\mu A^i_\mu - mB^i)^{(+)} \mid \rangle = 0 \tag{2.9}$$

become apparent when one attempts to make changes of the field variables. Such variable changes are a potent tool in showing, for example, that the infinities the S-matrix on the mass shell caused by the derivative couplings of the field $B^i(x)$ are in many cases less virulent than one might naively imagine. A classic example - and one in which the range of variables does not affect the subsidiary condition - is the interaction of fermions with a neutral vector meson

$$L = L_f(A, B) + L_f(\psi) + g\bar{\psi}\gamma_\mu \psi (A_\mu + \frac{1}{m}\partial_\mu B) \tag{2.10}$$

where

$$L_f(\psi) = \frac{i}{2}(\bar{\psi}\gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma_\mu \psi) - \kappa\bar{\psi}\psi \quad .$$

In terms of a modified fermion field

$$\psi'(x) = \exp\left(-i\frac{g}{m}B(x)\right)\psi(x) \tag{2.11}$$

the Lagrangian (2.10) takes the form

$$L = L_f(A, B) + L_f(\psi') + g\bar{\psi}'\gamma_\mu \psi' A_\mu \tag{2.12}$$

in which the field B(x) is decoupled. This was the formulation first used by Matthews to establish the renormalizability of massive electrodynamics. Another example is the neutral pseudovector theory with

$$L_{int} = g\bar{\psi} i\gamma_\mu \gamma_5 \psi (A_\mu + \frac{1}{m}\partial_\mu B) \quad .$$

Defining the new fermion field

$$\psi'(x) = \exp\left(-\frac{g}{m}\gamma_5 B(x)\right)\psi(x) \quad ,$$

one finds for the corresponding Lagrangian the expression

$$L = L_f(A, B) + L_f(\psi') + g\bar{\psi}'_i \gamma_\mu \gamma_5 \psi' A_\mu - \kappa \bar{\psi} \left[\exp\left(\frac{2g}{m} \gamma_5 B \right) - 1 \right] \psi \quad .$$

While this Lagrangian yields an unrenormalizable expansion in powers of g it can by a suitable summation technique be shown to yield finite results [9].

These are examples of neutral fields A_μ and B where the variable changes do not affect A_μ itself, and the subsidiary condition remains unchanged. Consider now the case of isovector Stückelberg fields A^i_μ and B^i with, for example, the interaction

$$L_{int} = g \bar{\psi} \gamma_\mu \tau^i \psi \left(A^i_\mu + \frac{1}{m} \partial_\mu B^i \right) \quad . \tag{2.13}$$

An obvious generalization of the transformation (2.10) would appear to be

$$\psi'(x) = \Omega(B)\ \psi(x) = \exp\left(-i \frac{g}{m} B^i(x)\ \tau^i \right) \psi(x) \tag{2.14}$$

which, accompanied by the transformation

$$A^{i'}_\mu \tau^i = \Omega\ \tau^i\ \Omega^{-1} \left(A^i_\mu + \frac{1}{m} \partial_\mu B^i \right) - \frac{1}{ig}\ \Omega \partial_\mu \Omega^{-1} \tag{2.15}$$

and, allowing for a contribution from $L_f(\psi)$, takes the interaction (2.13) into the form

$$L^{(1)}_{int} = g\ \bar{\psi}'\ \gamma_\mu\ \tau^i\ \psi'\ A^{i'}_\mu \quad . \tag{2.16}$$

This does not constitute the full interaction, however. Substitution of the transformation into $L_f(A, B)$ yields further contributions which are very complicated (although the quadratic terms maintain their form).

The important point about the transformation of the field A^i_μ is that

$$A^{i'}_\mu = A^i_\mu + O(g)$$

so that $\langle 0 | A^{i'}_\mu | p \rangle = \langle 0 | A^i_\mu | p \rangle$ to zeroth order in g where $| p \rangle$ denotes the one-meson state. Thus, from Borchers' theorem - if this theorem can indeed be applied to such highly non-linear transformations - the on-mass-shell S-matrix computed using A' must equal the corresponding S-matrix computed from A.

So much is true. But consider what happens to the subsidiary condition (2.9). All statements previously made about the effectiveness of the subsidiary condition in eliminating the unwanted zero-spin components of the field A_μ^i may perhaps remain true, but the conventional proofs, which depend on the properties of the solutions of second-order Cauchy equations, cannot immediately be seen to apply. One needs different, more powerful and more reliable methods of changing variables. Just such methods are available in the path-integral formulation as has been shown recently by Faddeev and Popov [4]. These authors used the method (to be described in Sec. 3) in a treatment of the Yang-Mills theory of massless vector particles. This work was extended by Boulware [5] to cover the massive counterpart. We shall argue that the method is more generally applicable - even to the interactions of only two charged mesons where there is no isospin symmetry. The method is useful not only in Yang-Mills theories but wherever non-linear field transformations can be applied with advantage. Taken in conjuction with the recently developed techniques for computing with non-polynomial Lagrangians [7], such non-linear transformations should find increasing scope for application.

III. PATH-INTEGRAL REPRESENTATIONS

If the Green's functions of a theory are represented by path integrals it becomes possible to view field transformations as straightforward changes of integration variables. In passing from one set of variables to another, one needs at most to compute a Jacobian determinant, but even this is often unnecessary. To see this, consider the "canonical" representation

$$Z(I, J) = \int (d\phi\, d\pi)\; e^{i\int dx(\pi\dot{\phi} - H(\phi,\nabla\phi,\pi) + I\phi + J\pi)} \qquad (3.1)$$

where ϕ denotes a collection of field variables and π their associated canonical momenta. The symbol $(d\phi\, d\pi)$ is meant to indicate the functional volume element, a simple product of differentials *

$$(d\phi d\pi) = \prod_x d\phi(x)\ d\pi(x) \tag{3.2}$$

and the integration range in (3.1) must include all functions $\phi(x)$ and $\pi(x)$ which vanish asymptotically. The complex number $Z(I, J)$ then represents the amplitude for a vacuum-vacuum transition in the presence of external sources $I(x)$ and $J(x)$. The Green's functions are defined as usual by functional derivatives with respect to I and J taken at $I = J = 0$. No generality is lost by normalizing the volume element to give

$$Z(0, 0) = 1 . \tag{3.3}$$

One can set up functional differential equations for Z which take the form of Hamilton's equations

$$\left[\dot{\phi}(x) - \frac{\partial H}{\partial \pi(x)}\right] Z(I, J) = - J(x)\ Z(I, J)$$

$$\left[\dot{\pi}(x) + \frac{\partial H}{\partial \phi(x)} - \nabla \left(\frac{\partial H}{\partial \nabla \phi(x)}\right)\right] Z(I, J) = I(x)\ Z(I, J) \quad ,$$

where, on the left-hand side, $\phi(x)$ and $\pi(x)$ are represented by the functional derivatives $-i\delta/\delta I(x)$ and $-i\delta/\delta J(x)$, respectively.

Since the volume element (3.2) has the form of a phase space measure it is clearly invariant under canonical transformations. Hence, for this very large group of transformations there is no need to compute a Jacobian determinant. Suppose the transformation

$$\left.\begin{aligned} \phi &\rightarrow \phi' = f(\phi, \pi) \\ \pi &\rightarrow \pi' = g(\phi, \pi) \end{aligned}\right\} \tag{3.4}$$

$$H(\phi, \nabla\phi, \pi) \rightarrow H'(\phi', \nabla\phi', \pi') \tag{3.5}$$

is canonical. Then one can contemplate using the Hamiltonian, H', in (3.1) to compute a modified amplitude Z' , i.e.,

* These considerations are purely formal. We make no attempt to specify precisely the nature of the summation. For example, factors independent of I and J , even infinite factors, are absorbed in the volume element which will always be adjusted to give $Z(0, 0) = 1$.

$$Z'(I, J) = \int (d\phi\, d\pi)\; e^{i \int dx (\pi\dot{\phi} - H'(\phi, \nabla\phi, \pi) + I\phi + J\pi)} \quad . \qquad (3.6)$$

Applying the transformation (3.4) to the integration variables takes (3.6) into the form

$$Z'(I, J) = \int (d\phi\, d\pi)\; e^{i \int dx (\pi\dot{\phi} - H(\phi, \nabla\phi, \pi) + I f(\phi, \pi)\, J g(\phi, \pi))} \qquad (3.7)$$

which constitutes the basic equivalence theorem for Green's functions. From this, together with the assumption that $f(\phi, \pi)$ and $g(\phi, \pi)$ connect the vacuum to the same set of one-particle states as do ϕ and π, respectively, it follows that the on-mass-shell S-matrix computed using H' is equal to that obtained from H, i.e., that the S-matrix is a canonical invariant.

The weakness of the canonical representation (3.1) is, of course, its lack of manifest covariance. However, it is a simple matter to recover this property by taking $J = 0$ and carrying out the integration over $\pi(x)$. The integral over $\pi(x)$ can best be performed by first translating the variable. Write

$$\pi(x) = \pi_0(\phi, \nabla\phi, \dot{\phi}) + u(x) \qquad (3.8)$$

where the function π_0 is obtained by solving the first of Hamilton's equations

$$\dot{\phi} = \frac{\partial H(\phi, \nabla\phi, \pi_0)}{\partial \pi_0} \quad . \qquad (3.9)$$

Substitution of π_0 into the canonical form $\pi\dot{\phi} - H(\phi, \nabla\phi, \pi)$ yields the Lagrangian $L(\phi, \nabla\phi, \dot{\phi})$. Therefore one can write

$$\int (d\pi)\; e^{i \int dx (\pi\dot{\phi} - H(\phi, \nabla\phi, \pi))} = M(\phi)\; e^{i \int dx\; L(\phi, \nabla\phi, \dot{\phi})}$$

where the functional $M(\phi)$ is defined by the functional integral

$$M(\phi) = \int (du)\ e^{-i\int dx\,(H(\phi,\nabla\phi,\pi_0+u) - H(\phi,\nabla\phi,\pi_0) - u\frac{\partial H(\phi,\nabla\phi,\pi_0)}{\partial\pi_0})} \quad . \tag{3.10}$$

The generating functional $Z(I) = Z(I,0)$ can therefore be represented by the Feynman integral

$$Z(I) = \int (d\phi)\ M(\phi)\ e^{i\int dx(L(\phi) + I\phi)} \tag{3.11}$$

where $L(\phi) = L(\phi, \nabla\phi, \dot{\phi})$ denotes the usual Lagrangian density. For a large class of variables this will be a manifestly Lorentz-invariant function. The functional $M(\phi)$ which must, in general, be present in the representation (3.11) may be looked upon as a correction factor which renders the representation unitary. It is clear from the expression (3.10) that this correction factor reduces to a constant if the coefficients of $\pi^2, \pi^3, \ldots,$ in an expansion of H are independent of ϕ and $\nabla\phi$ or, in other words, if the interaction Lagrangian contains no more than one derivative. This class includes most of the usual Lagrangians of field theory.

To illustrate the computation of $M(\phi)$ in a non-trivial case of physical interest consider the chiral SU(2) x SU(2) invariant Lagrangian for pions. This is given by Isham[10)] in the covariant form

$$L = \frac{1}{2}\ g_{ij}(\phi)\ \partial_\mu\phi^i\,\partial_\mu\phi^j \tag{3.12}$$

where $\phi^i(x)$ denotes the pion triplet and $g_{ij}(\phi)$ plays the role of a metric tensor. The corresponding Hamiltonian is given by

$$H = \frac{1}{2}\ g^{ij}(\phi)\ \pi_i\pi_j + \frac{1}{2}\ g_{ij}(\phi)\ \nabla\phi^i\,\nabla\phi^j \tag{3.13}$$

where g^{ij} denotes the reciprocal of g_{ij}. The integral (3.6) for $M(\phi)$ now takes the form

$$M(\phi) = \int (d_3u)\ e^{-\frac{i}{2}\int dx\, g^{ij}(\phi)\, u_i\, u_j} \tag{3.14}$$

which can be evaluated explicitly since it is Gaussian. The result is

$$M(\phi) = \left|\text{Det } g_{ij}\right|^{1/2}$$

$$= \exp\left(\frac{1}{2}\,\delta(0)\int dx\,\ell n\,\det g_{ij}(\phi)\right) \qquad (3.15)$$

where Det g_{ij} means determinant in the functional sense while det g_{ij} means determinant in the sense of 3 x 3 matrices. Because of the presence of the factor $\delta(0)$ this expression is highly singular. In fact its purpose is to cancel off all contributions to the Green's functions which are proportional to $\delta(0)$. That such a cancellation must take place is clear since these contributions never appear in the canonical representation (3.1). It is interesting to remark that the functional (3.15) is precisely the factor needed to make the integration measure chiral invariant,

$$(d\phi)\;M(\phi) = \prod_x \left(d\phi^1(x)\,d\phi^2(x)\,d\phi^3(x)\,\sqrt{\det g_{ij}(\phi)}\right) \quad .$$

This phenomenon is simply a reflection of the fact that chiral transformations are canonical and leave the Hamiltonian (3.13) invariant.

In general, field transformations $\phi \rightarrow \phi'$ are accommodated in the representation (3.11) by regarding the Lagrangian as a scalar

$$L'(\phi') = L(\phi)$$

and the measure functional $M(\phi)$ as a scalar density

$$M'(\phi') = M(\phi)\;\text{Det}\left(\frac{\delta\phi}{\delta\phi'}\right) \quad .$$

The Jacobian determinant $\text{Det}(\delta\phi/\delta\phi')$ can be evaluated explicitly for local transformations

$$\phi(x) = f(\phi'(x))$$

in which case it takes the form

$$\text{Det}\,(\delta\phi/\delta\phi') = \exp\left\{ \delta(0)\int dx\, \ln\left(\frac{\partial f}{\partial \phi'}\right)\right\} \quad .$$

It can be represented diagrammatically by graphs in which the ϕ' lines emerge from structureless (point) vertices. Since all of these vertices are accompanied by the factor $\delta(0)$ they must be cancelled by corresponding terms generated by derivatives in the interaction Lagrangian.

Transformations which involve derivatives of the fields, i.e., which are not canonical, can yield a Jacobian factor whose structure is non-trivial. For example, corresponding to the transformation

$$\phi'(x) = g(\phi(x), \partial_\mu \phi(x))$$

we can write, formally,

$$\text{Det}\,(\delta\phi/\delta\phi') = \exp \int dx \left\{ \ln\left(\frac{\partial g}{\partial \phi} + \frac{\partial g}{\partial \phi_{,\mu}}\, \frac{\partial}{\partial x_\mu}\right) \delta(x-y) \right\}_{y=x} .$$

The presence of the derivative makes this an essentially non-local functional. It will be represented diagrammatically by a set of vertices which are not structureless and not negligible. In the following section we shall consider an example where such a structured Jacobian factor can be evaluated. It will be found there that the structure takes the form of closed loops of a fictitious massless particle to which the other particles of the theory are coupled.

IV. VECTOR MESON THEORIES

The formalism described in Sec. III can be applied with advantage to the computation of vector meson Green's functions. Let us first illustrate the method by reformulating the Stückelberg technique of Sec. II in functional notation. The generating functional which corresponds to the Lagrangian (2.1) is given by

$$Z(I,\eta) = \int (dU\,d\psi)\, e^{i\int dx(L(U,\psi) + I_\mu U_\mu + \eta\psi)}$$

$$= e^{i\int dx\, L_{int}(-i\frac{\delta}{\delta I}, -i\frac{\delta}{\delta\eta})}\; Z_0(I,\eta) \qquad (4.1)$$

where Z_0 corresponds to the free Lagrangian $L_f(U) + L_f(\psi)$ and can be evaluated explicitly since it is represented by a Gaussian integral:

$$Z_0(I,\eta) = e^{\frac{1}{2}\int dx\,dy(I_\mu(x)\,\Delta_{\mu\nu}(x-y\,;\,m)\,I_\nu(y) + \eta^T(x)\;S(x-y)\,\eta(y))} \qquad (4.2)$$

where $\Delta_{\mu\nu}$ denotes the free meson propagator (2.3) and S(x-y), similarly, the free propagator of the fields ψ. The functional differential operator in (4.1) is interpreted by expanding the exponential in powers of L_{int}. The terms of this series, which operate upon the explicitly given functional (4.2), yield the usual perturbation series for $Z(I,\eta)$. If $L_{int}(U,\psi)$ is not a polynomial in U_μ and ψ, but rather some rational or algebraic function, then it will be necessary to interpret the expression $L_{int}(-i\delta/\delta I, -i\delta/\delta\eta)$ as a functional integral operator employing, for example, the methods of Efimov and Fradkin.

Into the representation (4.1) it is possible to introduce a new variable B by taking advantage of the identity

$$\int (dB)\; e^{-\frac{i}{2}\int dx(\partial_\mu U_\mu + \frac{1}{m}(\partial^2 + m^2)B)^2} = \text{const.}\;, \qquad (4.3)$$

which can be proved by making a simple translation of the integration variable $B \rightarrow B + B_0(U)$ where $B_0(U)$ denotes a solution of the equation $(\partial^2 + m^2)\,B_0 = -\partial_\mu U_\mu$. The translated integrand no longer depends on U_μ and, if it is required in addition that the limits of integration be invariant under translations, the result (4.3) then follows. The constant on the right-hand side can be set equal to unity without loss of generality since the integration measure (dB) is itself defined only up to a constant factor.*) Now insert the expression (4.3) into (4.1) and make the change of integration variable

$$U_\mu \rightarrow A_\mu = U_\mu + \frac{1}{m}\,\partial_\mu B \quad . \qquad (4.4)$$

*) These considerations are purely formal. We make no attempt to specify precisely the nature of the summation. For example, factors independent of I and J, even infinite factors, are absorbed in the volume element which will always be adjusted to give Z(0, 0) = 1 .

One obtains the form

$$Z(I,\eta) = \int (dA\, dB\, d\psi)\; e^{i\int dx\left(L(A - \frac{1}{m}\partial B, \psi) - \frac{1}{2}(\partial A + mB)^2 + I_\mu (A_\mu - \frac{1}{m}\partial_\mu B) + \eta\psi\right)}$$

$$= \int (dA\, dB\, d\psi)\; e^{i\int dx\left(L_f(\psi) + \frac{1}{2}A_\mu(\partial^2 + m^2)A_\mu - \frac{1}{2}B(\partial^2 + m^2)B + L_{int} + I(A - \frac{1}{m}\partial B) + \eta\psi\right)} \tag{4.5}$$

the perturbation development of which proceeds analogously to (4.1) and (4.2) except that A_μ and B lines are now represented by the propagators (2.6) and the interaction involves the combination $A_\mu - 1/m\, \partial_\mu B$. The integral representation (4.5) exactly parallels the Stückelberg development of Sec. II.

An equivalent, although different, formulation results if, instead of (4.3), one uses the identity

$$\int (dB)\; \delta(\partial_\mu U_\mu + \frac{1}{m}\partial^2 B) = 1 \tag{4.6}$$

where the integrand is meant to signify a "δ-functional" which might be represented by the Fourier-type integral

$$\int (dC)\; e^{i\int dx\, C(x)\left(\partial U + \frac{1}{m}\partial^2 B\right)} = \delta(\partial U + \frac{1}{m}\partial^2 B) \quad . \tag{4.7}$$

Substitution of (4.6) into (4.1) followed by the change of integration variable (4.4) yields (after suppressing the non-essential ψ and η) the expression

$$Z(I) = \int (dA\, dB)\; \delta(\partial_\mu A_\mu)\; e^{i\int dx\left(L(A - \frac{1}{m}\partial B) + I(A - \frac{1}{m}\partial B)\right)}$$

$$= \int (dA\, dB\, dC)\; e^{i\int dx\left(L(A - \frac{1}{m}\partial B) + C\partial A + I(A - \frac{1}{m}\partial B)\right)} \quad . \tag{4.8}$$

It is clear from the latter form that C plays the role of a Lagrange multiplier. The free part of $L(A - \frac{1}{m}\partial B)$ together with the term $C\partial A$ yields the chronological pairings

$$\overbrace{A_\mu(x)\, A_\nu(y)} = -\left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\right) \Delta_F\,(x-y\,;\, m)$$

$$\overbrace{A_\mu(x)\, B(y)} = 0$$

$$\overbrace{B(x)\, B(y)} = \Delta_F\,(x-y\,;\, 0) \tag{4.9}$$

which are to be used in developing the perturbation series. Note that

$$\overbrace{(A_\mu - \frac{1}{m}\,\partial_\mu B)\ (A_\nu - \frac{1}{m}\,\partial_\nu B)} = \overbrace{A_\mu A_\nu} - \frac{1}{m}\,\partial_\mu \overbrace{B A_\nu} - \frac{1}{m}\,\overbrace{A_\mu \partial_\nu B} + \frac{1}{m^2}\,\partial_\mu \overbrace{B \partial_\nu B}$$

$$= -\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right) \Delta_F\,(x-y\,;\, m)$$

which shows that the representation (4.8) is trivially equivalent to the original one.

So far we have only transcribed into functional notation the well-known Stückelberg representation and one of its modifications. A non-trivial generalization can be developed for systems of charged vector mesons. To illustrate this consider an isotriplet U_μ^i of vector fields. (The presence of isosymmetry is not an essential requirement for the method but leads to great simplifications. We shall indicate below what modifications would be needed if, for example, (the neutral component) U_μ^0 were not present.) The analogue of the representation (4.8) in which the transverse and longitudinal components of U_μ appear as A_μ and $\frac{1}{m}\,\partial_\mu B$, respectively, would involve an equivalence of the type

$$\int (dU) = \int (dA\, d\Omega)\ \delta(\partial_\mu A_\mu)\ M(A) \tag{4.10}$$

where A_μ and Ω represent the transverse and longitudinal parts of U_μ as defined by the formulae

$$U_\mu^i\, \tau^i = A_\mu^i\, \Omega\, \tau^i\, \Omega^{-1} - \frac{1}{ig}\, \Omega\, \partial_\mu \Omega^{-1} \tag{4.11}$$

and

$$\partial_\mu\, A_\mu^i = 0 \tag{4.12}$$

where $\Omega(x)$ is expressed in the form of a 2 x 2 unitary matrix. The formula (4.11) constitutes a non-Abelian generalization of (4.4) looked upon as a gauge transformation. It is a non-linear and, in fact, a non-canonical transformation. The Jacobian determinant $M(A)$ will be found to have a non-local structure.

The change of variables (4.10) assumes its simplest form if the integration measure $(d\Omega)$ is required to be the invariant group measure. A good way to represent this integration is in terms of a set of constrained variables. Write

$$\Omega(x) = \frac{g}{m}\left(\sigma(x) - i\underline{\tau}\cdot\underline{B}(x)\right) \quad . \tag{4.13}$$

This 2 x 2 matrix will be unitary provided

$$\sigma^2 + \underline{B}^2 = \frac{m^2}{g^2}$$

and the group-invariant functional integral over all unitary matrices $\Omega(x)$ can be given in the form

$$\int (d\Omega) = \int (d\sigma\, d\underline{B})\ \delta(\sigma^2 + \underline{B}^2 - m^2/g^2) \tag{4.14}$$

using the δ-functional defined above. By $(d\underline{B})$ is meant the ordinary product measure.

An integral representation for the Jacobian factor $M(A)$ is given by

$$\frac{1}{M(A)} = \int (d\Omega)\ \delta(\partial_\mu A_\mu^\Omega) \tag{4.15}$$

where A^Ω stands for the right-hand side of (4.11). The invariance of $(d\Omega)$ implies the gauge invariance of $M(A)$

$$M(A^\Omega) = M(A) \quad .$$

To verify (4.10) one can follow closely the steps which led up to (4.8), i.e.,

$$\int (dU) = \int (dU\, d\Omega)\ M(U)\ \delta(\partial_\mu U^\Omega_\mu)$$

$$= \int (dU\, d\Omega\, dA)\ M(U)\ \delta(\partial_\mu U^\Omega_\mu)\ \delta(A - U^\Omega)$$

$$= \int (dA\, d\Omega)\ M(A)\ \delta(\partial_\mu A_\mu)\ \int (dU)\ \delta(A - U^\Omega)$$

from which (4.10) follows since (dU) is invariant under the transformation $U \to U' = \Omega U \Omega^{-1}$.

An alternative scheme, one which is a closer analogue of the Stückelberg representation (4.5), is obtained by defining

$$\frac{1}{M_1(A)} = \int (d\Omega)\ \exp\left(-\frac{i}{2}\int dx\ (\partial_\mu A^\Omega_\mu + mB)^2\right), \qquad (4.16)$$

a functional which is <u>not</u> gauge invariant. Following the same steps as before we find

$$\int (dU) = \int (dU\, d\Omega)\ M_1(U)\ e^{-\frac{i}{2}\int dx(\partial U^\Omega + mB)^2} =$$

$$= \int (dA\, d\Omega)\ M_1(A^\Omega)\ e^{-\frac{i}{2}\int dx(\partial A + mB)^2} \qquad (4.17)$$

(after the replacement $\Omega^{-1} \to \Omega$).

At this stage we have three equivalent integral representations for the Green's functions of a charged meson theory:

$$Z(I) = \int (dU)\ e^{i\int dx(L(U) + IU)} \qquad (4.18a)$$

$$= \int (dA\, d\Omega)\ \delta(\partial_\mu A_\mu)\ M(A)\ e^{i\int dx(L(A^\Omega) + IA^\Omega)} \qquad (4.18b)$$

$$= \int (dA\, d\Omega)\ M_1(A^\Omega)\ e^{i\int dx(L(A^\Omega) - \frac{1}{2}(\partial A + mB)^2 + IA^\Omega)} \qquad (4.18c)$$

where A^{Ω} denotes the combination of A and Ω given on the right-hand side of (4.11) and Ω is expressed in terms of B through (4.13). The chronological pairings which must be used in the perturbation developments of these functionals are, respectively,

$$\overbrace{U^i_\mu U^j_\nu} = -\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right) \Delta(x-y;\, m)\ \delta^{ij} \tag{4.19a}$$

$$\overbrace{A^i_\mu A^j_\nu} = -\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\partial^2}\right) \Delta(x-y,\, m)\ \delta^{ij}\ ,\quad \overbrace{A^i_\mu B^j} = 0\ ,\quad \overbrace{B^i B^j} = \Delta(x-y, 0)\ \delta^{ij} \tag{4.19b}$$

$$\overbrace{A^i_\mu A^j_\nu} = -\ g_{\mu\nu}\ \Delta(x-y;\, m)\ \delta^{ij}\ ,\quad \overbrace{A^i_\mu B^j} = 0\ ,\quad \overbrace{B^i B^j} = \Delta(x-y,\, m)\ \delta^{ij}\ . \tag{4.19c}$$

The Jacobian factors M(A) and $M_1(A)$ as defined by the functional integrals (4.15) and (4.16) can also be developed in perturbation series the terms of which can be interpreted graphically. To this end it is convenient to define the currents ℓ^i_μ by

$$\ell^i_\mu\, \tau^i = -\frac{m}{ig}\ \Omega^{-1} \partial_\mu \Omega \quad . \tag{4.20}$$

The factor m/g is chosen to normalize the leading term in a series expansion in powers of $\underline{B}(x)$, the three-component field which parametrizes Ω. That is,

$$\ell^i_\mu = \partial_\mu B^i + \ldots \quad . \tag{4.21}$$

Since the Jacobian factor M(A) is needed only on the subspace $\partial_\mu A_\mu = 0$ according to (4.18b), it is sufficient to evaluate the integral (4.15) on that surface where it takes the form

$$\begin{aligned} \frac{1}{M(A)} &= \int (d\Omega)\ \delta(\partial_\mu \ell^i_\mu + 2g\, \epsilon^{ijk} A^j_\mu \ell^k_\mu) \\ &= \int (d\Omega\, dC)\ e^{-i\int dx\, C^i(\partial_\mu \ell^i_\mu + 2g\, \epsilon^{ijk} A^j_\mu \ell^k_\mu)} \\ &= \int (dB\, dC)\ e^{i\int dx\, \partial_\mu C^i(\partial_\mu B^i + 2g\, \epsilon^{ijk} A^j_\mu B^k + \ldots)} \end{aligned} \tag{4.22}$$

where, in the last step, only the terms linear in B are shown explicitly. The bilinear term $\partial B \cdot \partial C$ can be looked upon as a "free Lagrangian" which yields the chronological pairings

$$\overbrace{BB} = 0 \quad , \quad \overbrace{BC} = \Delta(x-y;0) \quad , \quad \overbrace{CC} = 0 \quad . \tag{4.23}$$

The integral over B and C is to be evaluated by expanding everything except the bilinear term in a power series and substituting the pairings (4.23) in all possible combinations in the usual way. This procedure will yield the perturbation development of $1/M(A)$. Because of the peculiar structure of the pairings (4.23) it is necessary to retain only those terms in the expansion where the number of B-fields is equal to the number of C-fields. Since the exponent in (4.22) is linear in C it is therefore necessary to keep only the term which is linear in B, i.e., the one shown explicitly. The terms of the perturbation series must form themselves into closed loops of a massless scalar particle with A-lines emerging from the vertices. A compact expression for the functional M(A) is in the form of a determinant

$$\begin{aligned} M(A) &= \mathrm{Det}\,(1-K)^{-1} \\ &= \exp \mathrm{Tr}\, \ln (1-K)^{-1} \\ &= \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\,(K^n) \end{aligned} \tag{4.24}$$

where K denotes an integral operator which is characterized by the kernel

$$K^{ij}(x,y) = i \; \frac{\partial \Delta(x-y;0)}{\partial x_\mu} \; 2g\epsilon^{ikj} \; A_\mu^k(y) \tag{4.25}$$

and the traces are defined by

$$\mathrm{Tr}(K^n) = \int dx_1 \ldots dx_n \; K^{i_1 i_2}(x_1,x_2) \; K^{i_2 i_3}(x_2,x_3) \ldots K^{i_n i_1}(x_n,x_1) \; .$$

The Jacobian factor $M_1(A)$, needed for the representation (4.18c), has a much more complicated structure and is also non-gauge-invariant. According to (4.16)

$$\frac{1}{M_1(A)} = \int (d\Omega)\, e^{-\frac{1}{2m^2}\int dx\,(\partial_\mu \ell^i_\mu + m^2 B^i + 2g\,\epsilon^{ijk} A^j_\mu \ell^k_\mu + m\,\partial_\mu A^i_\mu)^2}$$

$$= \int (d\Omega\, dC)\, e^{i\int dx\,\{\frac{1}{2} m^2 C^2 - C^i(\partial_\mu \ell^i_\mu + m^2 B^i + 2g(A_\mu \times \ell_\mu)^i + m\partial_\mu A^i_\mu)\}} \quad . \tag{4.26}$$

The bilinear terms give the chronological pairings

$$\overline{CC} = 0 \quad , \quad \overline{BC} = \Delta(x-y;\, m) \quad , \quad \overline{BB} = m^2 \mathbf{\Delta}(x-y;\, m) \tag{4.27}$$

where the function $\mathbf{\Delta}$ corresponds to dipole propagation, i.e.,

$$(\partial^2 + m^2)\, \mathbf{\Delta}(x-y;\, m) = \Delta(x-y;\, m) \quad .$$

Because of the non-vanishing of $\overline{BB}$ it is not possible to neglect the higher-order terms as was done with M(A). It is therefore not possible to give a compact expression of the type (4.24) and (4.25) for the functional $M_1(A)$. The perturbation expansion for $1/M_1$ can, however, be developed in the usual way by separating the bilinear terms in the exponent of (4.26) and expanding the rest in a power series. The individual terms of this series contain powers of the current $\ell^i_\mu(x)$ which is itself a non-polynomial function of (g/m) B(x) . One could employ the method of Efimov-Fradkin to compute these terms or one could expand $\ell^i_\mu(x)$ in powers to each of which the usual computing method can be applied. In the latter case one would, of course, be dealing with a highly unrenormalizable series.

In conclusion it may be remarked that the transformations considered in this section, while particularly suited to systems with a Yang-Mills symmetry, $L(A^\Omega) = L(A)$, can be applied to any system containing charged vector mesons. For example, if the neutral component U^0_μ is not present then the above formulae must be adjusted by making, everywhere, the replacement

$$(dU) \rightarrow (dU)\ \delta(U^0_\mu) \quad .$$

If this is done then formulae (4.18b) and (4.18c) will receive the factor

$$\int (dU)\ \delta(U_\mu^0)\ \delta(A_\mu - U_\mu^\Omega) = \delta((A_\mu^\Omega)^0)$$

$$= \int (dD)\ e^{i\int dx\, D_\mu (A_\mu^\Omega)^0}$$

where D_μ is to be looked upon as a Lagrange multiplier field. The chronological pairings (4.19b) and (4.19c) must be modified accordingly. Thus, for example, the neutral components of the set (4.19b) should be replaced by

$$\overbracket{A_\mu^0 A_\nu^0} = 0 \quad , \quad \overbracket{A_\mu^0 B^0} = 0 \quad , \quad \overbracket{B^0 B^0} = 0$$

$$\overbracket{A_\mu^0 D_\nu} = (-g_{\mu\nu}\partial^2 + \partial_\mu \partial_\nu)\ \Delta(x-y;0) \quad , \quad \overbracket{B^0 D_\nu} = m\partial_\nu\ \Delta(x-y;0)$$

$$\overbracket{D_\mu D_\nu} = (m^2 g_{\mu\nu} + (g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu))\ \delta(x-y)$$

while the charged components are unaffected.

REFERENCES AND FOOTNOTES

1) R. Delbourgo, Abdus Salam and J. Strathdee, Renormalization of weak interaction Lagrangians (in preparation).

2) R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York 1965);
P.T. Matthews and Abdus Salam, Nuovo Cimento 2, 120 (1955);
E.S. Fradkin, Dokl. Akad. Nauk 98, 47 (1954).

3) R.P. Feynman, Acta Phys. Pol. 24, 697 (1963).

4) L.D. Faddeev and V.N. Popov, Phys. Letters 25B, 29 (1967).

5) D.G. Boulware, Ann. Phys. (NY) 56, 140 (1970).

6) E.S. Fradkin and I.V. Tyutin, ICTP, Trieste, preprint IC/70/1.

7) G.V. Efimov, Soviet Phys.-JETP 17, 1417 (1963);
E.S. Fradkin, Nucl. Phys. 49, 624 (1963);
R. Delbourgo, Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/69/17 (to be published in Phys. Rev.)

8) N. Kemmer, Helv. Phys. Acta 33, 829 (1960).

9) S. Okubo, Progr. Theoret. Phys. (Kyoto) 11, 80 (1954).

10) C.J. Isham, Nuovo Cimento 60A, 188 (1969).