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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

NON-POLYNOMIAL LAGRANGIAN THEORIES

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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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I. INTRODUCTION

Barring lepton electrodynamics, most Lagrangians of physical interest are "non-renormalizable", the apparent non-renormalizability arising either from their non-polynomial nature or from higher spins. Typical non-polynomial cases are the chiral SU(2) x SU(2) Lagrangian

$$\mathcal{L} = \frac{(\partial_\mu \phi)^2}{(1 + f \underline{\phi}^2)^2} \tag{1.1}$$

in Weinberg's representation or the gravitational Lagrangian

$$\mathcal{L} = \frac{1}{K^2} \sqrt{-g}\; g^{\mu\nu}\left(\Gamma^\lambda_{\mu\rho}\,\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\,\Gamma^\rho_{\lambda\rho}\right) \tag{1.2}$$

where

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \;.$$

The components $g_{\mu\nu}$ which enter the expression for $g = \det g_{\alpha\beta}$ are a ratio of two polynomials in $g^{\mu\nu}$. A typical example of a higher spin case is the intermediate-boson mediated weak Lagrangian, e.g., the neutral vector W_μ interacting with quarks Q,

$$\mathcal{L}_{int} = f\, \overline{Q}\, \gamma_\mu\, (1 + \gamma_5)\, Q\, W_\mu \;. \tag{1.3}$$

So far as non-renormalizability is concerned, this is manifested most simply by transforming (1.3) into a non-polynomial form. In Stückelberg variables ($Q' = e^{-i\gamma_5 f \frac{B}{K}} Q$, $W_\mu = A_\mu + \frac{1}{K}\partial_\mu B$) an equivalent interaction is given by

$$\mathcal{L}'_{int} = f\, \overline{Q}'\gamma_\mu(1 + \gamma_5)\, Q'\, A_\mu + m\,\overline{Q}'(e^{i\gamma_5 \frac{f}{K} B} - 1)\, Q' \;. \tag{1.4}$$

It is clear therefore that if Lagrangian theory is to play any direct role in particle physics beyond that for electrodynamics, methods must be developed to extract numbers from non-polynomial theories. Basically any such methods must ensure the resolution of the two distinct difficulties of non-renormalizable theories, i.e., an infinite number of distinct infinity types and a high-energy behaviour which violates Froissart-like bounds.

Problems with conventional treatment of non-renormalizable theories

1) An infinite number of infinity types:

Ignoring derivatives for the moment, one may write $\mathcal{L}_{int}$ in the typical form

$$\mathcal{L}_{int} = G \sum \frac{v(n)}{n!} (\phi)^n$$

where v(n) contain powers of f (typically $v(n) \propto f^n$). (We shall call f the minor coupling constant.) A perturbation expansion may be written to any given order N in the major coupling constant G and to any desired n order in the minor coupling f . In this linearized form all contributions of $f^n \phi^n$ interactions with $n > 4$ give rise to non-renormalizable infinities. To remove these in the conventional manner, one would need more and more counter-terms in each order, reducing very considerably the predictive power of the theory.

2) Unacceptable high-energy behaviour

The high-energy dependence of individual graphs in all theories with $\mathcal{L} \propto f^n \phi^n$ $(n > 4)$ increases (unacceptably) as the order increases and is not polynomially bounded. (One aspect of this is that the counter-terms needed to cancel infinities must contain arbitrarily high-order derivatives of field variables, making the counter-Lagrangians non-local.)

To my knowledge the first acceptable treatment of problem 1) was given by S. Okubo[1] as early as 1954 in a paper which was apparently overlooked by others who subsequently worked on different aspects of this problem. These include Arnowitt and Deser, Fradkin, Efimov, Feinberg and Pais, Güttinger, Volkov, Fried, Lee and Zumino, Fivel and Mitter in addition to Delbourgo, Strathdee, Boyce and Sultoon, and Koller, Hunt and Shafi[2]. I shall review the earlier results and also state some new ones particularly relating to renormalization constants. These are joint work of Trieste and London groups.

The basic idea in dealing with problem 1) is that for a fixed order in the major coupling constant G^N one can Borel-sum the entire perturbation series to all orders in f^n . Formally this is an asymptotic series with each term given by an infinite expression. These Borel sums have the remarkable property that the summation automatically quenches most of the infinities. (This is perhaps not too unexpected a result when one considers that the Lagrangians of the type

$$\frac{1}{1+f\,\phi^2}$$

visibly appear to possess a built-in damping factor for higher frequencies.) For some Lagrangians this quenching is so strong that all matrix elements are rendered finite, offering thus the possibility of computing even self-masses and self-charges. For others some few infinities still survive and these need renormalizing.

There are a number of different formulations of the summation procedure - several variants - which fall basically into two classes: the x-space methods and the p-space methods. The results obtained using either method are equivalent. The chief problem is to ensure that the Borel sums i) possess the requisite analyticity properties in p-space, ii) satisfy unitarity and iii) are unique. Since good reviews[3)] of these methods exist, I shall not attempt to make this report comprehensive; I shall confine myself to a statement of results. In respect of problem 1), these are: i) the requirements of analyticity and unitarity are most likely met by these asymptotic sums, though uniqueness seems to need additional criteria; ii) for a large class of non-polynomial Lagrangians, a consistent renormalization programme can be devised where all infinities can be incorporated into acceptable counter-Lagrangians.

Regarding problem 2), which concerns the high-energy behaviour of Borel sums in the minor coupling constant, we obtain a perfectly acceptable behaviour for space-like momenta. For time-like momenta the cross-sections computed to order G^2 in the major coupling constant increase unacceptably fast with energy. It appears, however, that a further summation, this time in the major coupling constant G, of sets of chain-graphs alters this, just as is the case in conventional theory where, for example, a summation of ladder type perturbation diagrams produces Regge asymptotic behaviour.

II. A RAPID EXPOSÉ OF THE METHODS

The basic ideas of the summation methods can perhaps be rapidly illustrated by considering [4)]

$\mathcal{L}_{int}(\phi) = G \frac{1}{1+f\phi}$.

a) The formal series expansion for amplitudes

Formally an expectation value like

$$F(\Delta) = \langle L_{int}(\phi(x_1)), L_{int}(\phi(x_2)) \rangle$$

equals the asymptotic series:

$$G^2 \sum_{n=0}^{\infty} n! \ f^{2n} \Delta_F^n(x_1 - x_2) \quad . \tag{2.1}$$

Each term is infinite. Indeed as n increases,the singularity of $\Delta^n(x) \propto (1/x^2)^n$ gets worse and worse. We shall use the Borel method to sum the series. Ultimately we are interested in the Fourier transform of this sum:

$$\tilde{F}(p^2) = \int F(\Delta) e^{ipx} d^4x \quad . \tag{2.2}$$

The criterion for an acceptable summation technique is that $\tilde{F}(p^2)$ should exhibit conventional p-space analyticity.

b) The euclidicity postulate

To guarantee this consider the Symanzik region in p-space $(p^2 < 0)$. (When more than one external momentum p_i is involved, the Symanzik region is the region for which $p_i^2 \leqslant 0$, $p_i p_j \leqslant 0$. Certain other restrictions on momenta are also placed but the heart of the matter is that all momenta can be simultaneously chosen such that $p_{i0} = 0$.) For $p^2 < 0$, choose the frame where $p_0 = 0$. Clearly we may make a Wick rotation $x_0 \to ix_4$ without altering the value of $\tilde{F}$. Thus for the Symanzik region of p-space one needs to consider $\Delta^n(x)$ for euclidean vectors x^2 only. (For a zero-mass field $\Delta(x) = -1/4\pi^2 x^2$, where $x^2 = -x_4^2 - \underline{x}^2$ and is real and positive.) For p-space regions outside the Symanzik region we analytically continue (2.2). (It cannot be emphasised strongly enough that for divergent series of the type (2.1) one is not starting by "proving" the validity of the Wick rotation. Rather, euclidicity is a basic postulate - part of the process of defining the theory. One accepts it for the Symanzik region in p-space; outside this region one makes an analytic continuation in the momenta.)

c) Borel summation

To give meaning to the divergent sum $F(\Delta)$, use Borel transforms and write:

$$F(\Delta) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\zeta} (f^2 \zeta \Delta)^n \quad . \qquad (2.3)$$

using the identity:

$$n! = \int_0^{\infty} \zeta^n e^{-\zeta} d\zeta \quad .$$

d) The x-space method

The x-space method consists of inverting integration and summation in (2.3) and writing it as:

$$F(\Delta) = \int_0^{\infty} d\zeta e^{-\zeta} (1 - \zeta f^2 \Delta)^{-1} \quad . \qquad (2.4)$$

The expression (2.4) defines the amplitude $F(\Delta)$. For zero-mass particles $(m = 0)$ this equals:

$$\int_0^{\infty} \frac{r^2}{r^2 - \zeta f^2} e^{-\zeta} d\zeta \quad .$$

Notice that as $r \to 0$, $F(\Delta)$ is perfectly well behaved. The Borel summation has quenched the ultraviolet infinities. (Fradkin and Efimov have given explicit expressions of the type (2.4) for Borel sums to all orders G^N where this quenching effect can be explicitly seen.)

e) At this stage we encounter our first problem in the x-space method: the integrand has a pole on the integration path at

$$\zeta = \frac{1}{f^2\Delta} \quad \text{which equals} \quad \frac{4\pi^2 r^2}{f^2} \qquad \text{when} \qquad m = 0 \;\; (r^2 = \underline{x}^2 + x_4^2) \;.$$

We must define how to go round this singularity, the final objective being that the Fourier transform should be an analytic function which when continued to positive p^2 (outside the Symanzik region) has the unitarity cut from $p^2 = 0$ to ∞ .

One answer is: take the principal value. This is because, from (2.3), $F(\Delta)$ must be real. The p.v. prescription for the integral representation (2.4) of $F(\Delta)$ will guarantee this.*) The Fourier transform of (2.4) when $m = 0$ can be explicitly evaluated and a continuation to time-like values of p^2 carried out to demonstrate explicitly that $\tilde{F}(p^2)$ possesses the correct analyticity structure in the p^2-plane. The asymptotic behaviour of $\tilde{F}(p^2)$ is:

$$\begin{aligned} \tilde{F}(s) &\longrightarrow \frac{1}{(f^2 s)^3} & s &\longrightarrow -\infty \\ &\longrightarrow \pm\, i\pi \exp(f^2 s) & s &\longrightarrow +\infty \pm\, i0 \end{aligned} \tag{2.5}$$

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*) Footnote 1:

Ambiguities arise if instead of the p.v. we consider the more general real combination

$$(\tfrac{1}{2} + ib)F(\Delta, f^2 + i\epsilon) + (\tfrac{1}{2} - ib)F(\Delta, f^2 - i\epsilon) \;.$$

The result differs from the principal value integral by a purely real term of the form $b \exp(1/f^2\Delta)$ which possesses everywhere a zero expansion around $\Delta = 0$, and which, when added to the p.v., does not affect its perturbation representation:

$$\sum_{n=0}^{\infty} n!(f^2\Delta)^n \;.$$

The Fourier transform of this additional term is analytic in the entire p^2-plane so that in this order unitarity places no restriction on it. Higher-order unitarity, however, does seem to restrict such ambiguous terms. In Ref.3 it is argued that once the constant b is defined in the second-order super-propagator (see also footnote 3) the same constant or its multiples appear in all higher orders.

where $s = p^2$.*)

f) The p-space method

One can use an alternative method which works directly in p-space. It depends on Volkov's observation of the power of the Gel'fand-Shilov investigation of the Fourier transform of the generalized function $(\Delta(m = 0))^z = r^{-2z}$ in the range $0 < \mathrm{Re}\, z < 2$.

The crucial formula is

$$\Delta^z(x) = \frac{1}{(2\pi)^4} \int d^4p\, e^{-ipx} \frac{(-p^2)^{z-2}\; \pi(4\pi)^{2-2z}}{\sin \pi z\, \Gamma(z)\, \Gamma(z-1)} \tag{2.6}$$

$$0 < \mathrm{Re}\, z < 2 \quad .$$

To use this formula go back to the Borel sum (2.3) and employ a Sommerfeld-Watson transformation to convert the series into a formal integral of the form

$$F(\Delta) = \frac{i}{2} \int_\Gamma \frac{dz}{\sin \pi z} \int d\zeta\, e^{-\zeta} (-\zeta f^2 \Delta)^z \tag{2.7}$$

with the contour Γ enclosing the positive real axis in the z-plane. Straighten the contour to lie along the imaginary axis with $\mathrm{Re}\, z$ constrained to lie in the range $0 < \mathrm{Re}\, z < 2$. Using Gel'fand-Shilov's formula we obtain:

$$\tilde{F}(p^2) = \frac{i}{2} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{dz}{\sin \pi z} \frac{(-f^2)^z (-p^2)^{z-2}\, \Gamma(z+1)}{\sin \pi z\, \Gamma(z)\, \Gamma(z-1)} + (2\pi)^4 \delta(p) \tag{2.8}$$

---

* Footnote 2:

The basic reason why the Fourier transforms of the Borel sums possess the correct unitarity cuts has been spelled out by Lee and Zumino. While the infinities come from small r values of $\left(\Delta(r) \underset{r\to 0}{\approx} 1/r^2\right)$ of the propagators, the unitarity (singularity and threshold) structure arises from large values of r $\left(\Delta(r) \underset{r\to\infty}{\approx} e^{-mr}/r^{3/2}\right)$. Once it can be shown that (2.4) is an asymptotic representation of the perturbation expression (2.3) for large r , the correct unitarity behaviour of (2.4) is guaranteed.

where $0 < \alpha < 2$. (The term $\delta(p)$ corresponds to a graph which contains no internal line.)

f) Formula (2.8) is the master formula. By closing the contour along the left, one can immediately obtain the asymptotic behaviour of $\tilde{F}(p^2)$ for $p^2 \to -\infty$ and the result (2.5). As in Regge pole theory, the right-most pole of the integrand gives the leading contribution to the asymptotic behaviour; in this case the right-most pole lies *) at $z = -1$, giving the asymptotic expression $\approx \frac{1}{(f^2 p^2)^3}$ as before in (2.5).

III. HIGHER ORDERS

a) Super-graphs

Consider $\mathcal{L}_{int}(\phi) = G \sum_{n=D_0}^{\infty} \frac{v(n)}{n!} (\phi)^n$ ($v(n)$ contains the minor coupling parameter f^n).

It is easy to verify that the G^N contribution to an amplitude $F(x_1, \ldots, x_N)$ with E external line can be written as a sum of contributions from a set of super-graphs constructed as follows:

a) Take N points $x_1, x_2, \ldots, x_N$.

b) Join all points pair-wise with just one super-line joining two distinct points (x_i, x_j) ; associate with this line a positive integer n_{ij} .

*) Footnote 3:

The principal value ambiguity of the x-space method noted in Footnote 1 has a counterpart when we take into account the appearance of the (-)ive sign in front of Δ in $(-\Delta)^z$ in the Sommerfeld-Watson transform. To see this more explicitly, introduce a multiplier λ in front of Δ ; thus,

$$F(\lambda\Delta) = \frac{i}{2} \int \frac{dz}{\sin\pi z} \int d\zeta\, e^{-\zeta} (-\zeta\lambda f^2 \Delta)^z .$$

We must interpret the limit $\lambda \to +1$ by a real average of the values $(-\lambda)^z = e^{i\pi z}$ and $(-\lambda)^z = e^{-i\pi z}$, obtaining in general:

$$F(\Delta) = \int dz \left(\frac{1}{\tan\pi z} + b\right) \Gamma(z+1) (f^2\Delta)^z$$

with b an arbitrary real constant. This ambiguity of the constant b parallels the ambiguity noted in Footnote 1. As noted in Footnote 1, from unitarity one can show that all ambiguous constants arising in higher orders are multiples of this second-order b .

c) For each line write the factor $\frac{1}{n_{ij}!} [\Delta_F(x_i - x_j)]^{n_{ij}}$.

d) For each point x_i write a vertex factor $v(\sum_j n_{ij} + m_i)$. Here

m_i is the number of external lines impinging on the point x_i .

e) The contribution of the super-graph to the amplitude equals

$$F_{m_1 m_2 \dots}(x_1, \dots, x_N) = G \sum_{n_{ij}} \prod_i v(\sum_j n_{ij} + m_i) \prod_{i<j} \frac{(\Delta_F(x_i - x_j))^{n_{ij}}}{n_{ij}!} \quad . \tag{3.1}$$

The limits of the n_{ij} are given by

$$L_i = \sum_j n_{ij} + m_i \geqslant D_0 \quad .$$

f) To get the total contribution in order G^N, sum over all configurations of the external lines with the m_i lines at the i-th vertex distributed over the various vertices, such that

$$\sum m_i = E \quad .$$

b) <u>Super-graphs in momentum space</u>

The great beauty of the p-space method lies in the similarity of the p-space expressions for super-graphs and normal Feynman diagrams.

One can introduce Feynman's auxiliary parameters and carry out the loop integrations. The result is an elegant expression for the super-graph contribution as a weighted average integral of contributions of conventional graphs. The utility of such an expression is two-fold.

i) The sums of super-graphs in different orders of G closely resemble the sums for conventional graphs and the methods previously discussed by Polkinghorne, Federbush [5] and others for carrying through the summation can be taken over.

ii) The discontinuity formulae of Cutkosky - and the proof of the unitary relations using such formulae - follow the conventional lines.

For the zero mass case, the integral expressions for the N-th order super-graph is the following: (We consider here the simple case $D_0 = 0$.)

Associate with each super-line a four-momentum vector q_{ij}. The Sommerfeld-Watson transform of (3.1) in p-space equals:

$$\tilde{F}(p_i) = G^N \prod_{i<j} \int dz_{ij}\, \rho(z_{ij}) \int d_4 q_{ij}\, (-q_{ij}^2)^{z_{ij}-2}\, \delta_4(\Sigma p_i + \Sigma q_{ij}) \quad . \qquad (3.2)$$

Here $\rho(z_{ij})$ is the product of the vertex factors $v(\sum_{i \neq j} z_{ij} + m_i)$, the factors $\frac{1}{\sin \pi z_{ij}}$ (or more generally $\frac{1 + b \cos \pi z_{ij}}{\sin \pi z_{ij}}$) and the factors $\frac{1}{\sin \pi z_{ij}\, \Gamma(z_{ij})\, \Gamma(z_{ij} - 1)}$ for each super-line. The p_i's are the momenta carried by the external lines at the i-th vertex and the δ-functions express conservation of energy and momentum. The contour in each z_{ij}-plane for the case $D_0 = 0$ liesalong the imaginary axis for each z_{ij}. (We consider later the location of these contours when $D_0 \neq 0$. The problem of any surviving infinities in the theory is bound up with the location of these contours.)

Introduce Feynman's auxiliary parameters, using the integral representation *)

$$(-q^2)^{z-2} = \frac{1}{\pi \Gamma(2-z)} \int_0^{\infty} d\alpha\, \alpha^{1-z}\, e^{\alpha q^2} \quad . \qquad (3.3)$$

One may now carry through the $d^4 q$ integrations in the subsidiary integral I defined by

$$I(p_i, \alpha_{ij}) = \int (\exp \sum \alpha_{ij} q_{ij}^2)\, [\delta_4 \sum p_i + \sum q_{ij}]^N \prod d^4 q_{ij} \quad . \qquad (3.4)$$

* Footnote 4:
For $z = 1$ we recover Feynman's formula for normal propagators.

The result is identical to the case of conventional Feynman graphs with $F = (N(N-1))/2$ internal lines. (This is because $I(p_i, \alpha_{ij})$ is not z_{ij}-dependent.) The evaluation of $I(p_i, \alpha_{ij})$ can easily be carried through using the methods of Chisholm[5]; the final expression for the amplitude $F(p_i)$ reads:

$$\tilde{F}(p_i) = \prod_{i<j} \int dz_{ij} \; \rho'(z_{ij}) \int d\alpha_{ij} \, \alpha_{ij}^{1-z_{ij}} \, I(p_i, \alpha_{ij}) \qquad (3.5)$$

where ρ' differs from ρ by the factors $\prod_{i,j} \frac{1}{\pi\Gamma(2-z_{ij})}$.

The result for the N-point function evaluated in order G^N can therefore be stated thus:

Draw a Feynman graph with internal lines joining all the N-points pair-wise. Introduce Feynman parameters; the result of performing loop integrations is the standard Chisholm expression $I(p_i, \alpha_{ij})$. Multiply this by the factors $(\alpha_{ij})^{1-z_{ij}}$ and the weight function $\rho'(z_{ij})$; integrate over Feynman parameters ρ_{ij} and the Sommerfeld-Watson parameters z_{ij} . This gives the super-graph contribution.

IV. SUPER-GRAPHS

Infinities and renormalization

1. Using super-graphs one can investigate quite simply the possible infinities of non-polynomial theories. Among these are theories with no infinities whatsoever. The physically interesting cases, however, are of mixed theories where polynomial and non-polynomial Lagrangians both occur together. Such, for example, is the case for chiral Lagrangians (with nucleons interacting with pions for example) or weak Lagrangians (where the Stückelberg B-field occurs non-polynomially while the A-field interacts polynomially (see (1.4))). Not all these mixed theories are renormalizable. By renormalizable we shall mean theories where all infinities can be absorbed in a finite set of counter-terms. (Naturally the counter-terms must NOT contain arbitrarily high-order derivatives of field variables

$\partial_\mu \phi$, for example in non-polynomial combinations like $1/(1+f(\partial\phi)^2)$, otherwise the counter-terms would represent non-local additions to the original Lagrangian.)

2. Before we proceed, it is important to remark that, for non-polynomial Lagrangians with multitudes of external lines coming out of single vertices, the familiar statement of graphs getting less and less singular as the number of their external lines increases needs revision. The worst offenders in this respect are graphs with only two vertices. Here we have the surprising result: $S_{m,0} \approx S_{0,0}$. To see this, consider the simple case

$$\mathcal{L}_{int}(\phi) = \sum_{n\geq 1} v(n) \frac{\phi^n}{n!}$$

$$S_{m,0} = \sum_{n\geq 1} \frac{v(n+m)\ v(n)}{n!} \Delta^n(x) \quad .$$

In momentum space

$$S_{m0}(p^2) = \int dz \frac{1}{\sin\pi z} \frac{v(z+m)v(z)}{\Gamma(z+1)} \frac{(p^2)^{z-2}\,\Gamma(1-z)}{\Gamma(z-1)} \quad ,$$

where the contour lies parallel to the imaginary axis along $\mathrm{Re}\, z = 1$. For $p^2 < 0$ the high-energy behaviour is given by the first pole of the integrand on the left of $\mathrm{Re}\, z = 1$. Clearly this lies to the left of $z = 0$ (it would come from the factor $v(z)$) irrespective of what the value of m is. (Barring special cases, the corresponding singularity of $v(m+z)$ is still further left since $m > 0$.)

One can easily prove the following results which give the precise connection between the singularities of graphs with different number of external lines.

Theorem: If N is the total number of vertices:

For $N = 2$

$$S_{m_1, m_2}(\Delta_{12}(x)) = \left(\frac{\partial}{\partial\Delta_{12}}\right)^{m_2} S_{m_1-m_2,\,0}(\Delta_{12}) \ , \qquad m_1 > m_2 \tag{4.1}$$

For $N = 3$ or greater,

$$S_{2,0,0}(\Delta_{12}, \Delta_{23}, \Delta_{13}) = \left(\int_{-\infty}^{\Delta_{23}} d\Delta_{23} \frac{\partial}{\partial \Delta_{12}} \frac{\partial}{\partial \Delta_{13}} \right) S_{0,0,0} \quad . \quad (4.2)$$

Thus, for $N \geqslant 3$, all $S_{m_1, m_2, m_3, \ldots}$ can be related by repeated differentiations (A) or by operations of the type (B) above to the amplitude $S_{0,0,0,\ldots}$, or at worst to $S_{1,0,0,\ldots}$. Roughly this states that if in momentum space $\tilde{S}_{0,0,0,\ldots}(p_i)$ behaves like $\underset{M \to \infty}{M^{\alpha}}$, as external momenta p_i go to infinity $\tilde{S}_{m_1, m_2, \ldots}$ behaves like $M^{\alpha - \Sigma m}$.

3. A rough estimate for finiteness of super-graphs may be stated at this stage. The total number of super-lines F in a super-graph where all vertices are connected to each other is given by $F = \frac{N(N-1)}{2}$ while the number of loops equals $(F - N + 1)$. Thus the convergence of an integral

$$\tilde{S} = \int \frac{(d^4k) \text{ loops}}{(k^{\alpha})^F}$$

requires that the factor associated with each super-line $1/k^{\alpha}$ must be such that

$$k^{(4-\alpha)F - 4N + 4} \quad (4.3)$$

does not increase with N. Clearly $\alpha \geqslant 4$ is sufficient to ensure this. Roughly, then, each super-propagator should behave for large k like a dipole $1/k^4$ for finiteness. Later we make this criterion more precise.

4. Although the considerations of this section are really more general, to simplify discussion consider interactions of the type:

$$\mathcal{L} = \frac{\phi^{D_0}}{1 + f\phi^{D_0 - D}}$$

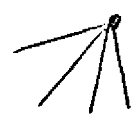
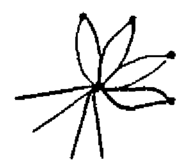
where D_0 and D are integers. Note that

$$\mathcal{L}_{\phi \to \infty} \longrightarrow \phi^D \quad .$$

The index D is the Dyson index which for conventional polynomial theories determines the possible infinities of the theory and if the theory is renormalizable. For example, for theories with $D = 3$, the second-order vacuum graph (with no external lines $E = 0$) is quadratically infinite $\left(\underset{M \to \infty}{\approx} M^2 \right)$ while second-order self-energy $(E = 2)$ is logarithmically infinite ($\approx$ log M). For $D = 4$, all vacuum graphs $(E = 0)$ behave like M^4 , self-energy graphs $(E = 2)$ like M^2 and scattering graphs $(E = 4)$ like log M .

For non-polynomial cases the infinities and renormalizability again depend on the index D but in a more subtle manner. Our tentative results are:

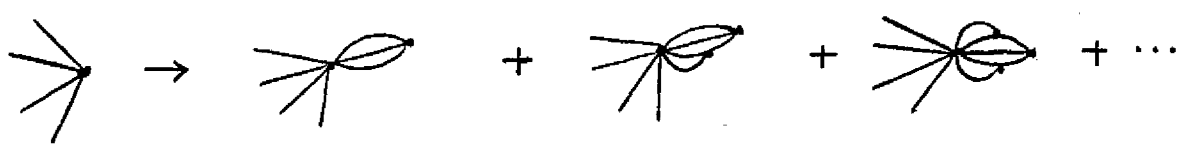
A) When $D < 2$ there are no infinities.

B) For $D = 2$, the only graphs possibly infinite are the star-fish graphs which modify the fundamental super-vertex  to 

(with arbitrary numbers of external lines and arbitrary numbers of stars). Counter-terms (independent of field-derivatives) can be introduced to absorb these.

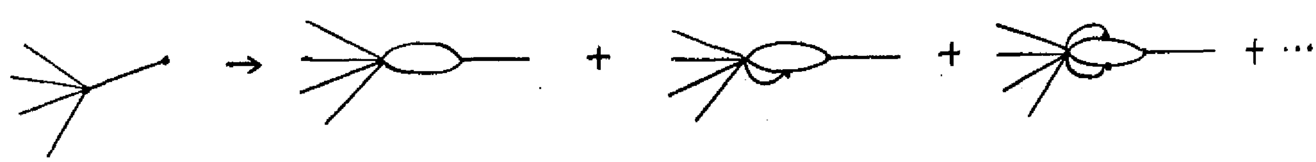
C) For $D = 3$, the only infinities again come from modifications of the fundamental super-vertex. These are of two types:

1) Tadpole modifications:



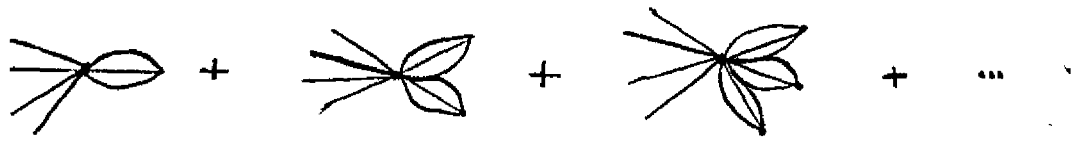
(4.4)

2) Proper self-energy-like modifications:



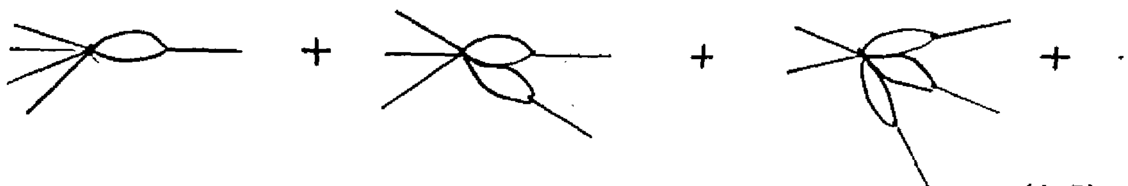
(4.5)

Plus what we shall call repetitions of these patterns; for example tadpole repetitions



(4.6)

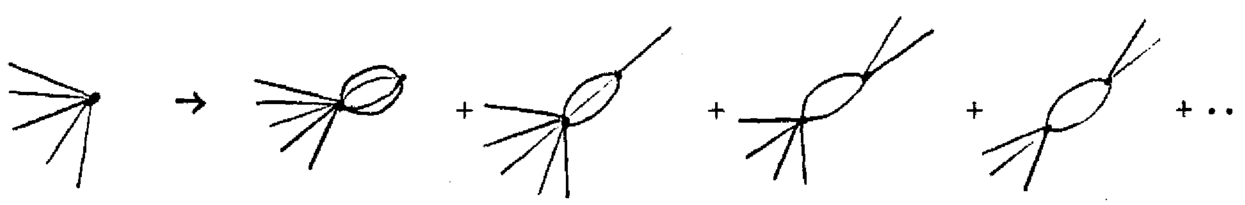
Similarly the self-energy-like repetitions:



(4.7)

These infinities can be absorbed in non-derivative counter-terms which renormalize the basic super-vertex. (These counter-terms are exhibited in the next section.)

D) For $D = 4$, the infinities come from the star-fish modifications of the basic super-vertex



Type I Type II Type III Type IV

One may write counter-terms to absorb these, but this time there are an infinity of distinct types of counter-terms and these also contain derivatives of field-variables to arbitrary high orders. Even if this fundamental super-vertex is made finite, new infinities arise when graphs with two and more super-vertices are considered. Thus a non-polynomial theory like $\phi^5/(1+f\phi)$ is non-renormalizable.

E) To complete the statement of renormalizable theories, it appears that one may introduce as many as four derivatives without affecting renormalizability, though this conclusion is as yet tentative. Symbolically, for $\mathcal{L} = \partial^\alpha \phi^D$ with $D \leqslant 3$, $D + \alpha \leqslant 4$, it is likely that no new problems arise, but this needs further examination.

5. To prove these results, consider the basic x-space expression:

$$\sum_{n_{ij}} \cdots \; v(m_i + \sum_j n_{ij}) \; \cdots \; \frac{\Delta^{n_{ij}}(x_i - x_j)}{n_{ij}!} \cdots$$

where all $n_{ij} \geqslant 0$ and are subject to the restrictions

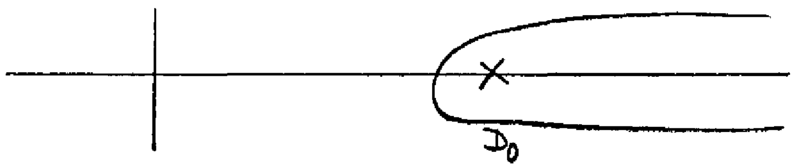
$$L_i = \sum_j n_{ij} + m_i \geqslant D_0 \quad .$$

(L_i is the total number of lines at the vertex i.)

A Sommerfeld-Watson complexification of n_{ij} gives

$$\int \frac{1}{\sin \pi z_{ij}} \cdots \frac{v(m_i + \sum_j z_{ij})}{\Gamma(z_{ij} + 1)} \cdots (-\Delta)^{z_{ij}} \cdots$$

with the contours in z_{ij} planes encircling the real axes



subject to the restrictions:

$$\mathrm{Re}\, L_i = \sum_j \mathrm{Re}\, z_{ij} + m_i \geqslant D_0 \quad . \qquad \mathrm{Re}\, z_{ij} \geqslant 0 \; .$$

Now, from the Gel'fand-Shilov theorem we know that the Fourier transform of Δ^z exists provided $0 < \mathrm{Re}\, z < 2$. Our first task is to shift the contours so that $\mathrm{Re}\, z < 2$; in the process we shall pick up infinite modifications of super-vertices which will need renormalizing. A second minor task will be to shift the contours still further down to $0 < \mathrm{Re}\, z < 1$ to get all superpropagators to behave like $1/k^4$ for space-like k^2. This will give rise to some completely harmless tadpoles, the finiteness of the theory being maintained.

This double shifting task is facilitated by expressing the limits

$$\sum_{L_i \geq D} \quad \text{in the form} \quad \sum_{\substack{L_i \geq 0 \\ \text{all } i}} - \sum_{\substack{L_i \geq 0 \\ \text{one } L_i = 0,1,2,\ldots,D_0}} - \sum_{\substack{L_i \geq 0 \\ \text{two } L_i = 0,1,2,\ldots,D_0}}$$

$$\cdots - \sum_{\text{all } L_i = 0,1,2,\ldots,D_0} . \tag{4.8}$$

We shall call these subtracted terms the "ghost terms". There is a very simple graphical representation of these. Write:

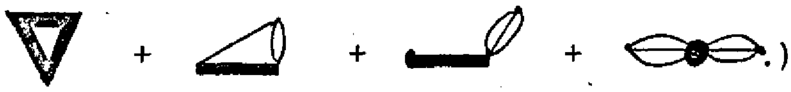
$$L = \frac{\phi^{D_0}}{1 + f\,\phi^{D_0 - D}} = \sum_0^D a_r\,\phi^r + \sum_i \frac{\alpha_i}{1 + \beta_i\,\phi}$$

$$= L_P + L_{N\,P} \quad , \tag{4.9}$$

i.e., as a sum of a polynomial Lagrangian L_P and non-polynomial Lagrangians $L_{N\,P}$. For the latter the relevant index $D_0 = 0$. The important point to stress is that in the expansion (4.9) the highest polynomial term has the index D and not D_0. We shall assume henceforth that $D \leqslant 4$ - if we do not make this assumption the polynomial part of L is non-renormalizable from the start. In terms of this split of the Lagrangian (4.9), the expansion (4.8) of the matrix element has the following meaning. The first term in the sum $\sum_{L_i \geqslant 0}$ corresponds to the contribution from $(L_{N\,P})^N$, the first ghost term to $(L_{N\,P})^{N-1}(L_P)$ and so on, the last ghost term corresponding to $(L_P)^N$. (As an illustration consider

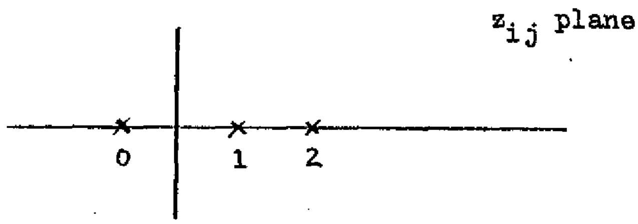
$$L = \frac{\lambda + \phi^3 + f\,\phi^4}{1 + f\,\phi} = \phi^3 + \frac{\lambda}{1 + f\,\phi} \; .$$

The third-order vacuum graphs are



Thick lines represent super-lines. Thin lines are ordinary lines with propagators $\lambda\,\Delta_F$.

Consider the graph which consists of super-lines only. There is no difficulty in going over to Fourier space; this is because $L_i \geq 0$ implies $n_{ij} \geq 0$, so that the z_{ij} contours can be Sommerfeld-Watson rotated to lie between $0 < \operatorname{Re} z_{ij} < 1$,

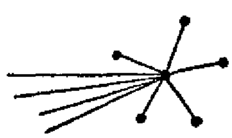


To estimate the high-energy behaviour of these graphs we need knowledge of the left-most singularity of the integrand. Barring some exceptional cases, this will lie to the left of $z_{ij} = 0$, giving at least a factor $(k_{ij}^2)^{z_{ij}-2} = (k_{ij}^2)^{-2}$ for each super-line. This guarantees the finiteness of all such graphs.

Consider now the cases D = 1, 2, 3, 4 individually for singularities of ghost graphs. The sub-graphs which are entirely made from the polynomial Lagrangian ϕ^r , $r \leq D$ may have their own singularities; these will need the conventional counter-terms and we shall assume that these have been introduced. We have only to consider mixed graphs and, in particular, star-fish modifications of super-vertices.

a) <u>D = 1</u>

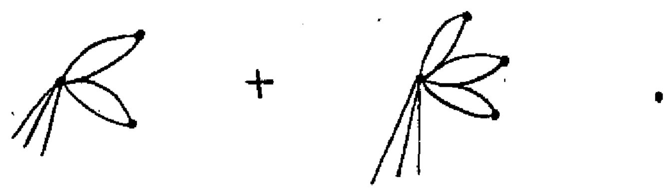
The star-fish consists of spokes with a factor $\Delta(p = 0, m^2) = (1/m^2)$. These tadpoles are harmless so far as infinities are concerned.



b) D = 2

The star-fish consists of the basic star

plus repetitions:



The full contribution of these diagrams is

$$\left(c_0 \left(\frac{\partial}{\partial\varphi}\right)^2 + c_0' \left(\frac{\partial}{\partial\varphi}\right)^4 + c_0'' \left(\frac{\partial}{\partial\varphi}\right)^6 + \cdots\right) L_{N.P}(\phi) \tag{4.10}$$

where

$$c_0 = \int \Delta^2(x)\, d^4x$$

$$c_0' = \frac{1}{2!}\, c_0^2$$

$$c_0'' = \frac{1}{3!}\, c_0^3 \quad \cdots \quad .$$

Clearly (and not unexpectedly) the series (4.10) sums to an exponential

$$\left(\left[\exp\left(c_0 \frac{\partial^2}{\partial\varphi^2}\right)\right] - 1\right) L_{N.P} \quad .$$

If we had started with the Lagrangian

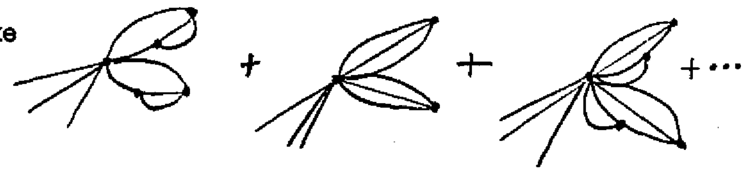
$$(L + \delta L)_{np} = \exp\left[-c_0\left(\frac{\partial^2}{\partial\phi^2}\right)\right] L_{NP}(\phi)$$

instead of L_{NP} there would be no star-fish infinities.

c) D = 3

In this case there are two types of modifications :

Type I - tadpole-like modifications:

+ repetitions of these, like 

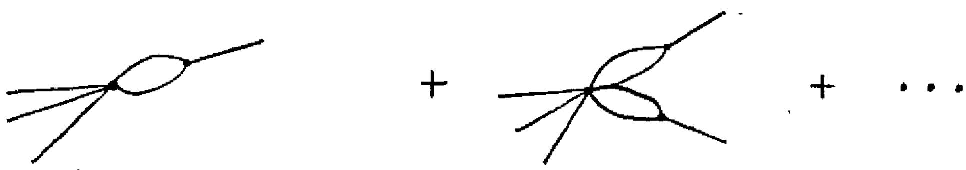
The modified Lagrangian reads:

$$\exp\left(-\left[c_1\left(\frac{\partial^2}{\partial\varphi^2}\right) + c_2\left(\frac{\partial^3}{\partial\varphi^3}\right) + c_3\left(\frac{\partial^4}{\partial\varphi^4}\right) + \cdots\right]\right) L_{NP}(\phi) \;.$$

C_1, C_2, C_3, ..., are the (infinite) contributions from the basic graphs. The repetitions are all taken care of by the exponential.

Type II

The second category of infinities arises from self-energy-like graphs



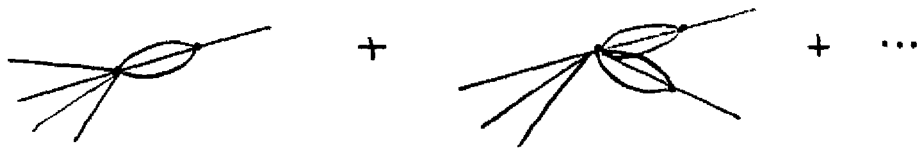
These are taken care of by starting the theory with the modified Lagrangian

$$\exp\left(-d_1 \frac{\partial^2}{\partial\phi^2}\,\phi\right) L_{NP} \;.$$

One final modification. Replace L_{NP} (ϕ) in the formulae above by $L(\phi)$. This takes care now of the self-energy infinity  as well.

d) D = 4

The tadpole-like infinities present no difficulty, but the infinities of Type II,

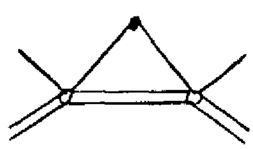


being quadratic (M^2), quartic (M^4), ..., now need a modified Lagrangian:

$$\exp\left[\left(d_0\,\phi + d_0'\,\partial^2\phi\right)\left(\frac{\partial}{\partial\phi}\right)^3 + \cdots\right] L_{NP}(\phi)$$

i.e., the modified Lagrangian contains derivatives of field variables to any arbitrarily high order. Clearly, this - according to the criterion stated earlier - is a non-renormalizable situation.

Having taken care of vertex modifications of Type I and Type II for $D = 1, 2, 3$ cases, we now need to see if there is any possibility of new infinities arising from joining pure L_P sub-graphs with pure L_{NP} graphs. For $D = 2$, the proof that none so arise is trivial. One can get, at worst, situations like



which are finite if one remembers that the super-line gives a factor like $1/k^4$. A slightly more complicated argument is necessary for $D = 3$. Basically, the proof needs the result (4.2) stated earlier, viz., a super-graph $S_{m_1, m_2, \ldots}(k_i)$, with $\sum m_i$ external lines, decreases faster by a factor $1/(k^{\Sigma m_i - 1})$ than $S_{1,0,0\ldots}$. Consider a super-graph connected with m internal lines with a graph made of ϕ^3 vertices only. The worst case for infinities is when neither the super-graph nor the pure ϕ^3 graph has any external lines. From the well-known Dyson count, the pure ϕ^3 graph contributes a factor

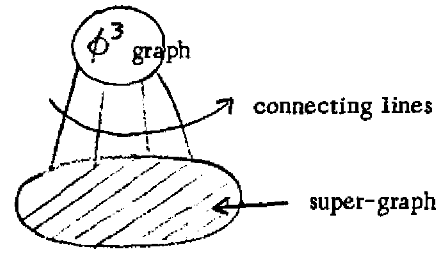
$$\frac{S_{0,0,0\cdots}(k)}{k^{\Sigma m}} \approx \frac{1}{k^{\Sigma m - 2}} .$$

From (4.2) the super-graph contributes

$$\frac{S'_{0,0,0\cdots}(k)}{k^{\Sigma m-1}} \quad ;$$

the m-lines give $1/k^{2\Sigma m}$. There are at most $(\sum m-1)$ new loops made by these m connecting lines, so that the over-all behaviour of the mixed graph is finite and given by:

$$\lim_{M\to\infty} \int k^{-4\Sigma m+3} \left(d^4k\right)^{\Sigma m-1} \approx \frac{1}{M}$$



(Here we have assumed (see (4.3)) that $S'_{0,0,0\cdots}(k) \approx \dfrac{k^{4F_s-4N_s+4}}{k^{4F_s}} \approx k^{-4N_s+4}$.)

Note that our final result regarding high-energy behaviour for the total vacuum contribution of a non-polynomial ϕ^D Lagrangian is

$$\tilde{S} \approx \log M \text{ for } D=2$$
$$\tilde{S} \approx M^2 \text{ for } D=3 \; .$$

For D = 3, for example, although the pure super-graphs gave a finite contribution, the pure polynomial ϕ^3 graphs give the well-known behaviour M^2 of the ϕ^3-theory.

6. This is perhaps the stage at which one might remark on derivative couplings. Notice that when we power-count for super-graphs, in the naive count $\int \dfrac{k^{4F_s-4N_s+4}}{k^{4F_s}}$ there appears the factor k^{-4N_s}. If each super-vertex carried derivatives up to fourth, the extra contribution would not exceed k^{4N_s}. Thus, provided that all sub-graph infinities (like those of star-fish graphs) could be consistently removed, up to fourth-order derivatives might be acceptable (with Lagrangians of the type $\partial^\alpha \phi^D$, $\alpha + D \leqslant 4$.

7. The ghost-diagrams which have played such an essential role in the above analysis can always be associated with ghost-Lagrangians whenever we can write $\mathcal{L}$ in the form $\sum a_r \phi^r + \frac{\alpha_r}{1+\beta_r \phi}$. When I-spin or unitary spin is present and terms like $(\partial_\mu \varphi \cdot \varphi)$ are involved this is clearly impossible. We believe one can still carry through the ideas of ghost-graphs without writing corresponding ghost-Lagrangians; we hope to discuss this in detail elsewhere.

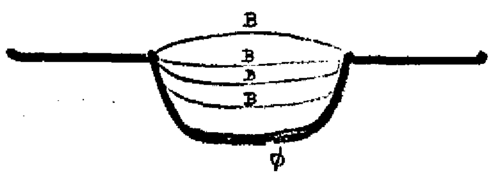
8. Let us now finally turn to weak interactions. This is the case of mixed fields and, as we shall see, here new types of infinities will arise and will need renormalizing. But before considering this difficult case, take a simple example of a mixed theory with two fields ϕ and B with $\mathcal{L}_{int}$ of the type:

$$\mathcal{L}_{int} = B^p \frac{\phi^{D_0}}{1 + f\phi^{D_0 - D}} \quad .$$

At each of the N vertices there are p, B-lines and (N-1) super-lines. A <u>necessary</u> condition for renormalizability is clearly $p + D \leqslant 4$. (This is easily seen by writing

$$\frac{\phi^{D_0}}{1 + f\phi^{D_0 - D}} = \sum a_r \phi^r + \frac{\alpha_i}{1 + f_i \phi} \quad .$$

In what follows we concentrate on the case $D = 0$, $p \leqslant 4$. All infinities come from the vacuum, self-energy and scattering graphs of the B-field, embedded inside super-graphs. For example, for $\mathcal{L} = B^4 f(\phi)$ where $f(\phi) \approx \phi^0$,



one may expect from the graph shown in the figure an over-all singularity at worst of the type:

$$\int (k^4) \times \frac{d^4k}{k^4} \approx M^4 \quad .$$

(The factor k^4 is the contribution of the B-field vacuum graph; the factor $1/k^4$ is for the ϕ super-propagator.) This infinity needs a new variety of counter-term of the form:

$$\delta\mathcal{L}(x) = \int f(\phi(y)) d^4y \left[c_0 \delta(x-y) + c_1 \partial^2 \delta(x-y) + c_2 \partial^{-4} \delta(x-y)\right] f(\phi(x))$$

where C_0, C_1, C_2 are infinite constants of order M^4, M^2, $\log M$, respectively. The important point is that the counter-Lagrangian which has the form

$$\delta\mathcal{L} = d_1(\phi^0) + d_2\, \partial^2(\phi^0) + d_3\, \partial^4(\phi^0)$$

may itself produce singularities in its turn and, to absorb these, an exponential form of counter-term discussed before will be needed, but any counter-terms needed at any stage appear to fall within what we have called the renormalizable class.

The situation above is typically the weak interaction situation. Using a formalism involving the intermediate boson W, write the minimal weak Lagrangian:

$$\mathcal{L}_W = J^+_\mu W^-_\mu + \text{h.c.}$$

$$\mathcal{L}_{EM} = \left[J^{EM}_\mu + J^{EM}_\mu(W)\right] A^0_\mu \quad .$$

Here $J^\pm_\mu$ are the charged weak currents; $W^\pm_\mu$ are charged intermediate bosons. For fixing ideas one may assume that $J^\pm_\mu$ and J^{EM}_μ are currents made up of quarks (Q) and lepton fields (ℓ) and are of order M^3.

For the W-fields themselves it is not essential, but it makes things very much easier, if we consider not just the two charged fields but a self-interacting triplet of Yang-Mills fields

$$\mathcal{L}_{YM} = \hat{\underline{W}}_{\mu\nu} \cdot \hat{\underline{W}}_{\mu\nu} + m^2 \underline{W}^2_\mu$$

where

$$\hat{\underline{W}}_{\mu\nu} = (\partial_\nu W_\mu - \partial_\mu W_\nu + if\, \underline{W}_\mu \times \underline{W}_\nu) \quad .$$

We now make a non-linear Stückelberg transformation on the $\underline{W}_\mu$ field variables; write $W_\mu = \underline{W}_\mu \cdot \underline{\tau}$ and introduce two fields A_μ and B by the relation

$$W_\mu = \Omega(B) A_\mu \Omega^{-1}(B) + \frac{i}{f} \Omega(B) \partial_\mu \Omega^{-1}(B) \quad .$$

Here $\Omega(B)$ is a unitary matrix; (in Weinberg's representation one may write $\Omega(B)$ in the form $\frac{1 - if\underline{\tau} \cdot \underline{B}}{1 + if\underline{\tau} \cdot \underline{B}}$; $\Omega(B) \approx M^0$). Writing

$$\frac{im}{f} \Omega(B) \partial_\mu \Omega^{-1}(B) = X_\mu \quad ,$$

the net effect on $\mathcal{L}_{YM}$ is to transform it to the form

$$\mathcal{L}_{YM} = \hat{\underline{A}}_{\mu\nu} \cdot \hat{A}_{\mu\nu} + m^2 \underline{A}_\mu^2 + 2m \underline{A}_\mu \cdot \underline{X}_\mu + \underline{X}_\mu \cdot \underline{X}_\mu \quad .$$

There are corresponding changes in the interaction Lagrangians for both weak and E. M. cases. For example the new weak Lagrangian has the form

$$\mathcal{L} = F^{(1)}(Q, \ell, A_\mu) f^{(1)}(B) + F^{(2)}_\mu(Q, \ell, A_\mu) f^{(2)}_\mu(B) + X_\mu(B) X_\mu(B) \quad ,$$

$F^{(1)}$ is at most of order M^4 with $f^{(1)}(B) \approx B^0$ while $f^{(2)}_\mu(B)$ is at most of order $\partial_\mu(B)$ with $F^{(2)}_\mu f^{(2)}_\mu(B) \approx M^4$.

Now comes the important point. Boulware[6] has shown that this non-linear Stückelberg analysis gives, for the A and B fields, propagators which are perfectly <u>normal</u> (i.e. are no more singular than $\Delta(x)$) and the S-matrix is unitary provided the Lagrangian is supplemented by an additional term of the form $\vec{\underline{F}} \times \partial_\mu \underline{F} \cdot A_\mu$ where the triplets of $\underline{F}$ represent "fictitious" particles first introduced by Feynman. From the point of view of renormalizability all we need to know is that in our power counts, $A \sim M$ and $B \sim M$, while $\Omega \approx M^0$ and $X_\mu \approx \partial_\mu(M^{-1})$. <u>Clearly $\mathcal{L}_W$ falls within the category of renormalizable interactions tabulated in this discussion</u>, and so does the additional Lagrangian for the fictitious particles.

We have not written out in detail all the counter-terms, nor is it interesting for anyone undertaking any practical calculations. As in most renormalization theory, what is important is the existence theorem - the statement that it can be done. We expect the practical rules for writing S- matrix elements to be:

1) Replace W_μ by the Stückelberg field A_μ.

2) For closed loops of W_μ fields, introduce Feynman's fictitious particles to preserve unitarity.

3) Add to the contributions above, super-graph contributions involving B-particles. These will need renormalizing. (In practice, knowing that these super-graphs, after renormalization, are finite, one may as a first approximation neglect these B-particle contributions. One may be certain that unitarity is preserved with just the contributions 1) and 2).)

All this was on the assumption that we do not wish to modify the basic weak Lagrangian but wish to start with what we have called the minimal Lagrangian. There is no reason why one may not modify the weak Lagrangian itself such that its Dyson index is less than two and it produces no infinities whatsoever. This is what Mitter and Fivel have done.[2)]

V. SUMMARY

We summarize the situation regarding non-polynomial Lagrangians: I should make the qualification that an enormous amount of verification is needed before the problems of renormalizability are all sorted out, but one may tentatively state:

1) All matrix elements are finite for theories where the Dyson index D is less than two.

2) For the cases when $D = 2$ or 3, counter-terms have been explicitly written which absorb all infinities and the theories are renormalizable.

3) Mixed theories of polynomial and non-polynomial fields appear to be renormalizable provided the Dyson indices separately and jointly fulfill renormalizability criteria. We believe that weak interactions, chiral Lagrangians and Yang-Mills theory fall into this class though detailed proofs have not yet been constructed.

4) It seems likely that to each order in the major coupling (and to all orders in the minor coupling) the S-matrix elements, as computed by methods outlined, satisfy the necessary unitarity and analyticity requirements.

5) The real parts of the physical amplitudes (the parts not restricted by unitarity) are non-unique. This appears to be similar to the type of non-uniqueness one meets in conventional renormalization theory for polynomial cases, i.e. arbitrariness up to finite renormalizations. Unitarity requirements restrict this lack of uniqueness though they do not completely eliminate it. If one imposes on the theory the criterion that all such extra terms must be represented by (finite) modifications to the starting Lagrangian - and with no derivatives of arbitrarily high order appearing - no arbitrariness remains.

6) Non-polynomial theories give perfectly acceptable high-energy behaviour in the Symanzik region and where external momenta are space-like or on the mass shells. For time-like momenta, however, the lowest order in the major coupling constant gives cross-sections increasing arbitrarily fast with energy. If now a simple chain diagram is summed, or a Regge ladder summation carried out, the results alter drastically. Alternatively and perhaps equivalently, if in the Symanzik region one computed K-matrix elements and used the expression $S = (1 - iK)/(1 + iK)$ to continue to time-like momenta, the exponential growth would not survive. (It is important to realize that in the Symanzik region T_N (the T-matrix element to order G^N in the major coupling constant) equals K_N . This very powerful method will be elaborated on elsewhere.) This appears in line with a general result recently claimed by Fradkin and Feinberg (unpublished) where they assert axiomatic CPT, spin and statistics and polynomial boundedness in energy for theories of the type we have considered [7]. If this

result holds and if a reliable summation technique in the major coupling can be devised, the last major objection to these theories would disappear. This is because if one extrapolates the results of Jaffe, Glimm, Hepp and others for polynomial Lagrangians in two dimensions to those in four dimensions there is no hope of obtaining finite self-masses and finite self-charges. We must turn to Lagrangians described in this paper if we are ever to compute these constants finitely and to make acceptable statements regarding phenomena like Goldstone bosons and symmetry breaking through the graphs for vacuum expectation values of scalar fields.

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K. Koller, A.P. Hunt and Q. Shafi, "Self-masses in chiral Lagrangian theories", Imperial College (in preparation).

3) G.V. Efimov, CERN, preprint 1087, Oct. 1969;
Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/69/120, to appear in Phys. Rev.

4) S. Fells (UCLA preprint) has shown that the vacuum self-energy of ϕ^3-like non-polynomial Lagrangians which are odd in powers of ϕ (e.g., $\phi/(1+f^2\phi^2)$) do not possess a lower bound. Presumably ϕ^4-like even theories do not suffer from this objection. Our use of a mixed power theory $1/(1+f\phi)$ is purely to illustrate the necessary techniques. Physical theories (like the chiral theory) are even in field variables.

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J.S.R. Chisholm, Proc. Cambridge Phil. Soc. 48, 300 (1952).

6) D. Boulware, Seattle preprint (1969).

7) While this was being printed, Dr. O.V. Steinmann (Zürich) has sent a preprint which shows that, for non-polynomial theories, the scattering matrix exists in an axiomatic formulation and also that the LSZ reduction procedure can be carried through even though, for rational non-polynomial theories, strict local commutativity may possibly not hold. It appears that the quiet revolution which has been taking place with non-polynomial physics may acquire respectability yet.