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ABSTRACT

The study of conformal group symmetries within the framework of non-linear realizations is extended and re-expressed in terms of metric tensors and connections on space-time. The standard Vierbein formalism of general relativity is then re-interpreted in terms of non-linear realizations of the group GL(4,$\mathbb{R}$). Throughout we emphasise the connection between massless Goldstone bosons and the preferred fields of non-linear realizations.

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I. INTRODUCTION

It is well known that Lorentz-covariant systems of equations can be made covariant under general co-ordinate transformations by introducing into them the appropriate gravitational couplings. In this development the gravitational field itself is to be regarded as a dynamical variable and must be represented by a suitable term in the action integral.

A similar technique has recently come into prominence in strong interaction theory, where it is known as the method of non-linear realizations [1). More particularly, a system which begins by possessing only the isospin symmetry SU(2) is modified by the introduction of certain couplings to a (massless) pion field, so as to obtain the chiral SU(2) x SU(2) symmetry. In this intrinsically interacting scheme the pion plays the role of the graviton.

Our aim in this paper is to extend this qualitative analogy between the geometric structure of general relativity and the group-theoretic techniques of non-linear realizations to a quantitative one. We do this in two parts. In the first (Sec. II), the recent treatment [2), using non-linear realizations, of the conformal group is re-examined and re-interpreted in terms of metric tensors and connections on space-time. In the second part (Sec. III), the classical theory of general relativity, especially in relation to its Vierbein content, is reformulated within the framework of non-linear realizations. One does not expect any new development in the notoriously difficult problem of quantizing gravity to result from this modified point of view. However, some insight may be gained by regarding general covariance as a spontaneously violated symmetry and, correspondingly, regarding gravitons as Goldstone bosons. The point here is that a non-linear realization scheme, with its intrinsic c-number displacement group action on certain fields (known as "preferred" fields), is a natural vehicle for the manifestation of this phenomenon.

In the section on the conformal group these "preferred" fields are a scalar particle σ and a 4-vector, ϕ_a. [3) The group action on these fields is very similar to a gauge transformation and, indeed, this partially motivates the re-expression of the theory in terms of connections and tensors. We show in Section II, by the usual current field commutator method [4], that the theory contains a massless scalar particle (a Goldstone boson) which we shall associate with the preferred σ-field. This is in spite of the non-local nature of the dilatation current operator.

We will start by giving a short summary of the relevant features of non-linear realizations. Let G and H denote, respectively, the group whose realizations are required and some subgroup whose linear representations are known. For example, G could be chiral SU2 x SU2 and H the isospin subgroup, the latter group serving to classify the particle multiplets. The basic group action [5), 6)] is that of G on the quotient space G/H, taking one coset into another. That is, if g_0H is a coset in G/H, then a group element g maps it into the coset gg_0H, thus inducing a non-linear realization on the various quantities of physical interest which are used as parameters or co-ordinates on this quotient space. When these parameters are fields they are known as preferred fields and in the chiral SU2 x SU2 example are interpreted as the triplet of pions. In the case of induced representations [7), 8)] of the Poincaré group, in which such group actions on coset spaces also occur, the co-ordinates are associated either with Minkowski space-time or with a mass hyperbola in momentum space. When G is the conformal group we take the homogeneous Lorentz group as H and the nine parameters on G/H are identified as four (flat) space-time co-ordinates and the scalar and vector fields, σ and ϕ_a, previously mentioned. In the gravitational case G is GL(4, $\mathbb{R}$) while H is O(3, 1) with the ten parameters now being related to a set of "Vierbein" fields.

This group action of an arbitrary element g of G on the cosets may be written in the form [2), 5)]

$$g: \; L_\pi \to L_{\pi'} = g L_\pi h^{-1} \tag{1.1}$$

where L_π denotes a matrix of G, parametrized by the coset space coordinates π^i, which represents one of the cosets in G/H, and where $h \equiv h(\pi, g)$ is an element of the subgroup H and is a calculable function of π^i and g.

The formula (1.1) defines implicitly the basic non-linear realization $\pi \to \pi'$ but it also defines the group element $h(\pi, g)$ which belongs to H and which governs the behaviour of all the other fields ψ under transformations of G. That is,

$$g : \psi \to \psi' = D(h(\pi, g))\psi \qquad (1.2)$$

where D(h), defined for all $h \in H$, denotes one of the linear representations of H and is assumed known.

Having specified the transformation laws for π^i and ψ, we must now define "covariant derivatives", by which is meant combinations of the fields and their ordinary derivatives, which transform according to the law in eq. (1.2). All of the relevant information is contained in the matrix $L_\pi^{-1} \partial_\mu L_\pi$ which evidently belongs to the infinitesimal algebra of G. Under the transformation (1.1) we find

$$g : L_\pi^{-1} \partial_\mu L_\pi \to L_{\pi'}^{-1}\partial_\mu L_{\pi'} = h(L_\pi^{-1} \partial_\mu L_\pi) h^{-1} + h \partial_\mu h^{-1} . \qquad (1.3)$$

Thus, the matrix $L_\pi^{-1} \partial_\mu L_\pi$ decomposes into at least two pieces, one of which transforms according to a non-linear realization of the type (1.2) while the other, belonging to the subalgebra H, involves the inhomogeneous term $h \partial_\mu h^{-1}$ in its transformation law. The latter part is to be interpreted as a type of connection [9] and used in the construction of the covariant derivative $D_\mu \psi$. The former may be interpreted as the covariant derivative of π^i itself. Thus, if we write the matrix

$$(L_\pi^{-1} \partial_\mu L_\pi)_\alpha{}^\beta = D_\mu \pi^i (A^i)_\alpha{}^\beta + \Gamma_\mu^a (V^a)_\alpha{}^\beta$$

where the generators of H are denoted by V^a and the remainder by A^i, then $D_\mu \pi^i$ transforms covariantly as does

$$D_\mu \psi \equiv \partial_\mu \psi - \Gamma^a_\mu(\pi)\, T^a \psi$$

where the matrices T^a generate the infinitesimal form of D(h).

To construct Lagrangians which are invariant under G is now straightforward: take any Lagrangian which is invariant under H and replace the ordinary derivatives $\partial_\mu \psi$ everywhere by the covariant $D_\mu \psi$. Add the term $\frac{1}{2} D_\mu \pi^i\, D_\mu \pi^i$ in order that π^i shall propagate like the other dynamical variables. The resulting Lagrangian does indeed possess the enlarged symmetry, although one cannot infer that this symmetry will be reflected in the solutions. Rather, one can show that, as mentioned before, the transformation law $\pi^i \to \pi'^i$ contained in (1.1) is generally inconsistent with the assumption of a unique invariant ground state. The whole technique appears to be geared to the description of spontaneously broken symmetries with the preferred fields playing the role of the well-known Goldstone bosons [4].

II. THE CONFORMAL GROUP

The use of non-linear realizations to deal with the conformal group of space-time was considered by two of the authors in Ref. 2 (referred to as I and II in the following). In the first part of the present paper this work is extended and some of the concepts are clarified by reformulating the theory in a more intuitive geometrical language.

The conformal group is a fifteen-parameter Lie group of transformations of the flat space-time of special relativity. It contains the Poincaré group as a subgroup, the remaining transformations being

$$x^a \to \bar{x}^a = \frac{x^a + \beta^a x^2}{1 + 2\beta\cdot x + \beta^2 x^2} \qquad (2.1)$$

and

$$x^a \to \bar{x}^a = e^\lambda x^a \qquad (2.2)$$

where β^a and λ are the special conformal and dilatation group parameters, respectively. The dot products in these formulae refer to the usual Minkowski metric which we will consistently denote as $\eta_{ab} \equiv \mathrm{diag}(+1, -1, -1, -1)$ with inverse written as η^{ab}. These expressions may be inverted to give

$$x^a = \frac{\bar{x}^a - \beta^a \bar{x}^2}{1 - 2\beta\cdot\bar{x} + \beta^2\bar{x}^2} \tag{2.3}$$

and

$$x^a = e^{-\lambda}\,\bar{x}^a \quad . \tag{2.4}$$

For later convenience we define the quantity

$$J(x,\beta) \equiv 1 + 2\beta\cdot x + \beta^2 x^2 = \left|\det \frac{\partial \bar{x}}{\partial x}\right|^{-1/4} \tag{2.5}$$

which is related to the determinant of the Jacobian of the special conformal transformations as indicated and in terms of which eq. (2.1) may be succinctly written as

$$\bar{x}^a = \frac{1}{2}\frac{\partial}{\partial\beta_a} \log J(x,\beta) \quad . \tag{2.6}$$

The Lie algebra of the conformal group is that of the non-compact group SO(4, 2) and indeed, the transformations above may be derived from the natural action of this orthogonal group on the "light cone" of a six-dimensional pseudo-Euclidean space equipped with the metric (+1, -1, -1, -1, -1, +1). Minkowskian space-time is then regarded as a subspace of the five-dimensional projective space with which the Euclidean space and its light cone are canonically associated. We shall not adopt this viewpoint here but will take eqs. (2.1) and (2.2) as our definitive starting point.

The basic question which now naturally arises is, how should particle fields transform under the action of the conformal group? Two powerful, closely related, methods that have been employed are those of induced representations [10] and non-linear realizations [2]. Both techniques construct a representation of the full conformal group from some known representation of one of its subgroups. In II the latter approach is used in which the group G (the conformal group) acts in the standard manner, summarized in the introduction, on the quotient space G/H (where H is the homogeneous Lorentz subgroup) taking one coset into another. We write for an element of G/H

$$e^{ix\cdot P} \quad e^{iK\cdot\phi(x)} \quad e^{-iD\,\sigma(x)}\cdot H \tag{2.7}$$

in which $\phi^a(x)$ and $\sigma(x)$ are the preferred fields and where P_a, K_a and D are the translation, special conformal and dilatation generators, respectively [10]. Note that the space-time co-ordinates themselves form the remaining four parameters. This is possible since the group action induced on the "x" in the $e^{ix\cdot P}$ part of eq. (2.7) is precisely that of eqs. (2.1) and (2.2).

The resulting special conformal transformations of the preferred fields were derived in II as

$$\phi'_a(\bar{x}) = (1 + 2\beta\cdot x + \beta^2 x^2)\,\phi_a(x) + (1 + 2x\cdot\phi(x))\,(1 + 2\beta\cdot x - 2x^2\beta\cdot\phi(x))\,\beta_a - $$
$$- (2\beta\cdot\phi(x) + \beta^2(1 + 2x\cdot\phi(x))\,x_a \tag{2.8}$$

$$\sigma'(\bar{x}) = \sigma(x) - \log(1 + 2\beta\cdot x + \beta^2 x^2) \tag{2.9}$$

while the dilatation actions are simply $\phi'_a(\bar{x}) = e^{-\lambda}\,\phi_a(x)$ and $\sigma'(\bar{x}) = \sigma(x) + \lambda$. The characteristic feature of these transformations is the appearance of an inhomogeneous term corresponding to a c-number displacement of the quantum fields and resulting in a possible manifestation of the Goldstone phenomenon. Under the action of the Poincaré subgroup the induced transformations of x^a, $\phi_a(x)$ and $\sigma(x)$ are the usual ones of space-time, a Lorentz vector and a Lorentz scalar, respectively.

The action of the conformal group on any other field may be constructed in what is by now standard non-linear realization fashion. The result, which is the same as that derived using induced representations, is given in infinitesimal form in II as

$$\delta\psi(x) = -\beta_a(x^2\eta^{ab} - 2x^a x^b)\,\partial_b\,\psi(x) - 2\ell\beta\cdot x\;\psi(x) - 2i\beta^a x^b\,S_{ab}\,\psi(x) \tag{2.10}$$

$$\delta\psi(x) = -\lambda\,x^a\partial_a\,\psi(x) + \lambda\,\ell\,\psi(x) \tag{2.11}$$

where S_{ab} are the homogeneous Lorentz group generators appropriate to the arbitrary quantum field $\psi(x)$. The constant ℓ is known as the "ℓ-value" of the field $\psi(x)$ and is essentially defined by eq. (2.11), which describes the actions of space-time dilatations on $\psi(x)$. It is not an a priori determined quantity and is initially, at least as far as the group theory is concerned, open to some choice. The "covariant derivatives" of all fields may now be constructed within the non-linear realization framework as sketched in the introduction and we refer to Ref. 2 for further details.

As a first extension of these results we will express the "matter" field transformations of eqs. (2.10) and (2.11) in a compact form which is also valid for finite group actions. Let $A_{ab}(x) = \frac{\partial\bar{x}^a}{\partial x^b}$ denote the Jacobian of the special conformal transformations. Then the following key identity may be readily verified:

$$A(x)\,\eta\,A^t(x) = \frac{\eta}{(J(x))^2} \quad \text{or} \quad \sum_{\substack{b,d\\=0}}^{3} \frac{\partial\bar{x}^a}{\partial x^b}\,\eta^{bd}\,\frac{\partial\bar{x}^c}{\partial x^d} = \eta^{ac}\left|\det\left(\frac{\partial\bar{x}}{\partial x}\right)\right|^{1/2} \tag{2.12}$$

where the function $J(x)$ was defined in eq. (2.5), showing that the matrix $J(x)\;A_{ab}(x)$ is an element of the Lorentz group. Therefore if $\psi(x)$ is an arbitrary field with Lorentz group representation matrices D, then the transformation

$$\psi(x) \rightarrow \psi'(\bar{x}) = J^{-\ell}(x)\,D(J(x)\,A(x))\;\psi(x) \tag{2.13}$$

is a well defined action on this field with ℓ an arbitrary constant. Similarly for space-time dilatations we get the transformation

$$\psi(x) \to \psi'(\bar{x}) = e^{\ell\lambda} \psi(x) \qquad (2.14)$$

and these two expressions define what is, in fact, a linear representation of the full conformal group. That it actually is a representation follows immediately from the fact that $J(x)$, being related to the determinant of the Jacobian, itself defines a one-dimensional representation. The infinitesimal forms of eqs. (2.13) and (2.14) are exactly eqs. (2.10) and (2.11), respectively, which demonstrates the equivalence of this global representation with that of II.

For integer spin fields $\psi(x)$, these finite transformations take on the interesting form of those of a tensor density. For example, let $\psi_a(x)$ be a Lorentz vector transforming under the homogeneous Lorentz group contragrediently to the fundamental representation

$$\psi_a(x) \to \psi'_a(x') = (\Lambda^{t^{-1}})_a{}^b \psi_b(x) \quad ; \quad x'^a = \Lambda_{ab} x^b \quad ; \quad \Lambda \in SO(3,1) \ . \qquad (2.15)$$

Then under a conformal transformation $x^a \to \bar{x}^a$ we have

$$D(JA)_a{}^b = J^{-1}(x) \frac{\partial x^b}{\partial \bar{x}^a}$$

and eq. (2.13) reads

$$\psi_a(x) \to \psi'_a(\bar{x}) = J(x)^{-(\ell+1)} \frac{\partial x^b}{\partial \bar{x}^a} \psi_b(x) = \left|\det\left(\frac{\partial \bar{x}}{\partial x}\right)\right|^{\frac{\ell+1}{4}} \frac{\partial x^b}{\partial \bar{x}^a} \psi_b(x) \qquad (2.16)$$

showing that $\psi_a(x)$ does indeed transform under conformal co-ordinate changes as a genuine tensor density of weight $\frac{-(\ell+1)}{4}$.

This observation motivates the introduction on space-time of a metric tensor g^{ab} transforming correctly under a conformal group action as a rank-two tensor, so that a quantity such as $\psi_a\psi_b g^{ab}$ is manifestly a scalar density. The weight of any density may be altered by multiplying it by suitable powers of $\sqrt{-g}$ where $g \equiv \det(g_{ab})$. This will, in general, be necessary since the requirement that the action integral

$$\mathcal{A} = \int d^4x \, \mathcal{L}(x)$$

be a group invariant implies that $\mathcal{L}(x)$ must transform as a scalar density of weight one, owing to the non-invariance of the volume element d^4x under special conformal and dilatation group actions.

The incentive to re-interpret the group theoretic structure of conformal invariance in this geometric fashion is further increased by the observation that the conformal group transformations of the preferred fields, given in eqs. (2.8) and (2.9), may be compactly rewritten as

$$\phi'_a(\bar{x}) = \frac{\partial x^b}{\partial \bar{x}^a} \left\{ \phi_b(x) + \frac{1}{2} \left(\log \left| \det \left(\frac{\partial \bar{x}}{\partial x} \right) \right|^{-1/4} \right)_{,b} \right\} \qquad (2.17)$$

$$\sigma'(\bar{x}) = \sigma(x) - \log \left| \det \left(\frac{\partial \bar{x}}{\partial x} \right) \right|^{-1/4} \qquad (2.18)$$

where $,b$ means $\dfrac{\partial}{\partial x^b}$. These are significantly reminiscent of gauge transformations and forcibly suggest that the fields ϕ_a and σ could be used to construct affine connections on space-time.

The first step is the construction of a suitable metric tensor which, since we do not wish to introduce additional dynamical variables, must be expressible in terms of the fields ϕ_a and σ already at our disposal. The simplest such object is

$$g_{ab}(x) = \eta_{ab} \, e^{-2\sigma(x)} \qquad (2.19)$$

with the inverse

$$g^{ab}(x) = \eta^{ab} \, e^{2\sigma(x)} \quad . \qquad (2.20)$$

The correct tensorial transformations of these quantities follows at once from eqs. (2.12) and (2.18). We shall adopt this definition for the metric tensor on space-time and will use it in the usual manner to raise and lower indices. The only exceptions to this rule are the upper indices on η^{ab} in eq. (2.20) which, as before, merely imply that η^{ab} is the inverse matrix of $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$ to which it is, of course, numerically equal. The compatibility of these definitions with the Poincaré transformations is immediate since these transformations are already in the form of eqs. (2.17) and (2.18) with $\left|\det\left(\frac{\partial \bar{x}}{\partial x}\right)\right| \equiv 1$.

The natural connection, constructed from this metric tensor, which may be imposed on space-time is the standard Christoffel symbol

$$\left\{ {a \atop b\;c} \right\} = \frac{1}{2}\, g^{ad}\, (g_{bd,c} + g_{cd,b} - g_{bc,d}) \tag{2.21}$$

$$= -\,(\delta^a_b\, \sigma_{,c} + \delta^a_c\, \sigma_{,b} - \eta_{bc}\, \eta^{ad}\, \sigma_{,d}) \tag{2.22}$$

where, in deriving the second line, eqs. (2.19) and (2.20) have been used. We may now construct the usual covariant derivatives of tensors in terms of this connection and ask how they compare with the group theoretic "covariant derivatives" derived in a completely different fashion in II. The answer is, as shown below, that they are identical provided the preferred field ϕ_a in II is identified with $-\frac{1}{2}\,\frac{\partial\sigma(x)}{\partial x^a}$. This is clearly possible because, as can be seen from eqs. (2.17) and (2.18), the fields $\phi_a(x)$ and $-\frac{1}{2}\,\sigma(x)_{,a}$ have exactly the same group transformation properties. This remark is relevant outside the context of the present paper and implies that to achieve full conformal symmetry only one extra scalar field $\sigma(x)$ is strictly necessary.

The vector field ϕ_a can, of course, be retained if desired, as a separate entity and must, as such, be slotted into the geometric framework that we are building up. This can be done by defining an alternative connection (not the Christoffel symbol) on space-time as

$$\Gamma_{bc}{}^{a} = C(\delta^a_b\, \phi_c + \delta^a_c\, \phi_b - g_{bc}\, \phi^a) \tag{2.23}$$

where C is some constant yet to be determined. It must be emphasised that we are still regarding the space-time manifold as being equipped with the metric tensor of eqs. (2.19) and (2.20) and with which indices are raised and lowered. For the object defined above to be a connection it is necessary and sufficient that it should transform under a conformal group action as

$$\Gamma_{bc}{}^{a} \longrightarrow \Gamma'_{bc}{}^{a} = \frac{\partial x^i}{\partial \bar{x}^b} \frac{\partial x^j}{\partial \bar{x}^c} \frac{\partial \bar{x}^a}{\partial x^k} \Gamma_{ij}{}^{k} - \frac{\partial^2 \bar{x}^a}{\partial x^i \partial x^j} \frac{\partial x^i}{\partial \bar{x}^b} \frac{\partial x^j}{\partial \bar{x}^c} \qquad (2.24)$$

which, from the known transformation properties of ϕ_a (eq. (2.17)), fixes the value of C as +2. Thus

$$\Gamma_{bc}{}^{a} = 2(\delta^a_b \phi_c + \delta^a_c \phi_b - g_{bc} \phi^a) \equiv 2(\delta^a_b \phi_c + \delta^a_c \phi_b - \eta_{bc} \eta^{ad} \phi_d) \qquad (2.25)$$

which on comparison with eq. (2.22) shows clearly that one "degenerate" possibility is $\phi_a = -\frac{1}{2}\sigma_{,a}$ in which case $\Gamma_{bc}{}^{a}$ becomes the normal Christoffel symbol. In general, however, this will not be so, and as a result the geometry defined by Γ is non-Riemannian. The covariant derivative of the metric tensor g_{ab} defined with respect to this connection will not vanish (as it does in the Riemannian case) resulting in the non-invariance of lengths of vectors under parallel transport.

Whenever two connections are available, such as, for example, $\Gamma_{bc}{}^{a}$ and $\left\{ {a \atop b\ c} \right\}$, the affine sum $f\Gamma_{bc}{}^{a} + (1-f)\left\{ {a \atop b\ c} \right\}$ defines a new connection where f may be any real number or indeed any scalar function. In our case this sum is

$$\left\{ {a \atop b\ c} \right\} + f\left(\Gamma_{bc}{}^{a} - \left\{ {a \atop b\ c} \right\}\right) = \left\{ {a \atop b\ c} \right\} + 2f\left(\delta^a_b(\phi_c + \tfrac{1}{2}\sigma_{,c}) + \delta^a_c(\phi_b + \tfrac{1}{2}\sigma_{,b}) - g_{bc}\, g^{ad}(\phi_d + \tfrac{1}{2}\sigma_{,d})\right) \qquad (2.26)$$

The second term in this expression is in the form of eq. (2.23) but with ϕ_a replaced by the combination $(\phi_a + \frac{1}{2}\sigma_{,a})$. As is clear from the transformation laws of the preferred fields, this particular combination (denoted $D_a\sigma$ in II) transforms linearly, without any inhomogeneous terms, as a covariant vector. Thus the second term in eq. (2.26) transforms as a genuine tensor and could, as far as the dictates of the group theory are concerned, be dropped leaving just the Christoffel symbol as the connection, thereby re-affirming our previous remark that $-\frac{1}{2}\sigma_{,a}$ is an adequate form for the vector field ϕ_a. For the time being, we will continue to keep the field ϕ_a as an independent dynamical variable, bearing in mind that the particular "solution" $\phi_a = -\frac{1}{2}\sigma_{,a}$ may always be inserted into any results that follow.

We will therefore now construct the geometric covariant derivatives associated with the connection of eq. (2.25) and show that they agree exactly with the group-theoretic ones of II. In the field of general relativity, where the main concern is with invariance under the infinite-dimensional gauge group of general co-ordinate transformations, the covariant derivatives of an arbitrary tensor T may be written as

$$\nabla_\mu T = T_{,\mu} - iF^\nu{}_\gamma \Gamma_{\mu\nu}{}^\gamma T \qquad (2.27)$$

where $\Gamma_{\mu\nu}{}^\gamma$ is the general gravitational affine connection. The finite-dimensional matrices $F^\nu{}_\gamma$ define the representation of $GL(4,\mathbb{R})$ to which the tensor T belongs and satisfy the usual commutation rules

$$\frac{1}{i}[F^\alpha{}_\beta, F^\gamma{}_\delta] = \delta^\alpha_\delta F^\gamma{}_\beta - \delta^\gamma{}_\beta F^\alpha{}_\delta \quad . \qquad (2.28)$$

Fields with half-integer spin are incorporated either by using the Vierbein formalism or by adopting the alternative scheme proposed in the second part of this paper.

We should like to write down an expression similar to eq. (2.27) for our simpler case in which the invariance group is the finite-dimensional conformal group. The first observation is that, as shown in eq. (2.12), only the subset of F matrices corresponding to Lorentz and dilatation (determinant changing) subgroups of $GL(4,\mathbb{R})$ ever occur in the transformation laws for any tensor. Secondly, the number ℓ occurring in the field transformations of eqs. (2.13) and (2.14) is the eigenvalue of the finite-dimensional matrix $-iF^{c}{}_{c}$ in the irreducible representation of the field $\psi(x)$. This may be demonstrated by considering a simple space-time dilatation

$$x^a \rightarrow \bar{x}^a = e^{\lambda} x^a \quad ; \quad \frac{\partial \bar{x}^a}{\partial x^b} = \delta^a_b \, e^{\lambda} \tag{2.29}$$

which may be written in terms of the self-representation of the F matrices

$$\left(F^c{}_d\right)^e{}_f = i\delta^c_f \delta^e_d \quad , \quad \left(F^c{}_c\right)^e{}_f = i\delta^e_f \tag{2.30}$$

in the form

$$\frac{\partial \bar{x}^a}{\partial x^b} = \left(\exp[-i\lambda F^c{}_c]\right)^a{}_b \quad . \tag{2.31}$$

This enables the representation of the dilatation group on the field $\psi(x)$ (eq. (2.14)) to be written as

$$\psi(x) \rightarrow \psi'(\bar{x}) = e^{\lambda \ell} \psi(x) = D\left(\frac{\partial \bar{x}}{\partial x}\right) \psi(x)$$

$$= e^{-i\lambda \tilde{D}(F^c{}_c)} \psi(x) \tag{2.32}$$

where D and $\tilde{D}$ denote the Lie group and Lie algebra representation matrices, respectively. Evidently $\tilde{D}(F^c{}_c)$ is simply $-i\ell$, in keeping with Schur's lemma, since the dilatation and Lorentz subalgebras of the conformal group algebra commute. Naturally, if ψ is a reducible Lorentz field then $\tilde{D}(F^c{}_c)$ need not be a multiple of the unit matrix although it must restrict to one on any Lorentz irreducible subspace.

The required covariant derivatives may therefore be written in the form of eq. (2.27) using only the subset of the GL(4, $\mathbb{R}$) matrices that corresponds to the dilatation and Lorentz subgroups, the generator of the former being $-iF^c{}_c$ with eigenvalue ℓ. As is shown below, the form of $\Gamma_{bc}{}^a$, defined by eq. (2.5) is such as to project out exactly this particular seven-element subset of the sixteen GL(4, $\mathbb{R}$) generators.

The resulting expression for the covariant derivatives is immediately applicable to half-integer as well as to integer spin fields. This is because, as already stated, only Lorentz group and dilatation group generators occur and any spinor representation of the former may always be extended to include the latter by simply defining the field transformation under this abelian subgroup as $\psi(x) \rightarrow e^{\lambda\ell}\,\psi(x)$. The representation cannot, of course, be extended further to include the full $GL(4, \mathbb{R})$ group, a step which necessitates the introduction of Vierbein fields or some similar technique.

The covariant derivative of an arbitrary field ψ may therefore be written, using eq. (2.25), as

$$\nabla_a \psi = \partial_a \psi - i F^b{}_c\, \Gamma_{ab}{}^c \psi$$

$$= \partial_a \psi - 2i F^b{}_c (\delta^c_a \phi_b + \delta^c_b \phi_a - g_{ab}\phi^c)\psi$$

$$= \partial_a \psi - 2i F^c{}_c\, \phi_a \psi - 2i (F^b{}_a \phi_b - F^b{}_c\, g_{ab}\, g^{cd}\phi_d)\psi . \quad (2.33)$$

The generators of the homogeneous Lorentz group are defined in terms of the $GL(4, \mathbb{R})$ generators as

$$S_{ca} = \eta_{ad} F^d{}_c - \eta_{cd} F^d{}_a \quad (2.34)$$

and so

$$\eta^{bc} S_{ca} = \eta^{bc} \eta_{ad} F^d{}_c - F^b{}_a \quad . \quad (2.35)$$

Thus, remembering that $g_{ad} = \eta_{ad}\, e^{-2\sigma}$ and hence that $g_{ad}\, g^{cb} \equiv \eta_{ad}\, \eta^{cb}$, we finally obtain the following form for the covariant derivative of any field ψ:

$$\nabla_a \psi = \partial_a \psi + 2\ell\, \phi_a \psi - 2i S_{ab}\, \eta^{bc} \phi_c\, \psi \quad (2.36)$$

which is precisely the expression obtained in II using the non-linear realization techniques.

The next stage is the construction of "covariant derivatives" for the preferred fields σ and ϕ_a. Due to the non-tensorial nature of σ and ϕ_a, the term "covariant derivative" does not refer to the standard geometric object constructed above but rather to combinations of these fields which contain their first derivatives and transform as genuine tensors. For the σ-field a suitable combination is $\sigma_{,a} + 2\phi_a$ which, as already mentioned, transforms as a covariant vector.

For the case of an independent ϕ_a field, the key point is to observe from eq. (2.25) that this Lorentz vector can be reconstructed from the connection $\Gamma_{ab}{}^c$ by means of the identity $\phi_a = \frac{1}{8}\Gamma_{ab}{}^b$. This implies that the rank-two tensors for which we are looking will very probably be expressible in terms of the connection and its first derivatives. The fundamental geometric object of this type is the rank-four curvature tensor defined conventionally as

$$R_{bcd}{}^a = \Gamma_{cd,b}{}^a - \Gamma_{bd,c}{}^a + \Gamma_{cd}{}^e \Gamma_{be}{}^a - \Gamma_{bd}{}^e \Gamma_{ce}{}^a \quad . \tag{2.37}$$

The associated Ricci tensor is defined as

$$R_{bd} = R_{bcd}{}^c \tag{2.38}$$

while the curvature scalar is

$$R = g^{bd} R_{bd} \quad . \tag{2.39}$$

There are three linearly independent rank-two tensors containing the first derivatives of ϕ_a which may be constructed from the above expressions. They are $R_{(ab)}$, $R_{[ab]}$ and $g_{ab} R$, where the first two are defined as

$$R_{(ab)} = \frac{1}{2}(R_{ab} + R_{ba}) \tag{2.40}$$

$$R_{[ab]} = \frac{1}{2}(R_{ab} - R_{ba}) \quad . \tag{2.41}$$

Inserting the explicit form for the connection we obtain

$$\frac{1}{2} R_{(ab)} = \phi_{a,b} + \phi_{b,a} - 4\phi_a \phi_b + 4 g_{ab} (\phi^c \phi_c + \frac{1}{4} \phi^c{}_{,c}) \qquad (2.42)$$

$$\frac{1}{4} R_{[ab]} = \phi_{a,b} - \phi_{b,a} \qquad (2.43)$$

$$\frac{1}{12} g_{ab} R = g_{ab} (2\phi^c \phi_c + \phi^c{}_{,c}) \quad . \qquad (2.44)$$

The non-vanishing of the antisymmetric part of the Ricci tensor is a direct result of the non-Riemannian nature of the geometry defined by the connection Γ. In the degenerate case when $\varphi_a = -\frac{1}{2}\sigma_{,a}$, the connection becomes the Christoffel symbol and this antisymmetric tensor vanishes identically. This is in accord with the general principle that a Riemannian geometry has a symmetric Ricci tensor.

Any combination of $R_{(ab)}$, $R_{[ab]}$ and $g_{ab}R$ may be used in the construction of conformally invariant terms for a Lagrangian. A useful one is the symmetric tensor E_{ab} defined as

$$\frac{1}{12} E_{ab} = \frac{1}{2} R_{(ab)} - \frac{1}{12} g_{ab} R = \phi_{a,b} + \phi_{b,a} - 4\phi_a \phi_b + 2 g_{ab} \phi^c \phi_c \qquad (2.45)$$

in which the divergence $\phi^a{}_{,a}$ has dropped out. A covariant derivative for the ϕ field may be constructed from this expression by covariantly subtracting off the $\phi_{b,a}$ term. Thus we define

$$D_a \phi_b = \frac{1}{8} (2R_{(ab)} - \frac{1}{3} g_{ab} R - R_{[ab]}) = \phi_{a,b} - 2\phi_a \phi_b + g_{ab} \phi^c \phi_c \qquad (2.46)$$

which is indeed the covariant derivative of the preferred field ϕ_a as computed in II (remembering again the numerical identity $g_{ab} g^{cd} \equiv \eta_{ab} \eta^{cd}$). Conformally invariant Lagrangians may now be easily constructed using covariant derivatives and the curvature tensors, with indices saturated by the metric tensor. Notice in this context that bilinears constructed, for example, from Dirac spinors transform as densities since from eq. (2.13) we obtain

$$\bar{\psi}\,\gamma_a\,\psi \;\rightarrow\; J^{-(\bar{\ell}+\ell)}\;\bar{\psi} D^{-1}(JA)\,\gamma_a\,D(JA)\,\psi$$

$$= J^{-(\bar{\ell}+\ell)}\left((JA)^{-1t}\right)_a^{\;b}\;\bar{\psi}\,\gamma_b\,\psi$$

$$= \left|\det\left(\frac{\partial \bar{x}}{\partial x}\right)\right|^{\frac{\ell+\bar{\ell}+1}{4}}\;\frac{\partial x^b}{\partial \bar{x}^a}\,\bar{\psi}\,\gamma_b\,\psi \quad . \tag{2.47}$$

In this expression γ_a, with a lower index, refers to the usual constant Dirac matrices, which are also used in the definition of the adjoint field $\bar{\psi}$; the constant $\bar{\ell}$ is simply the complex conjugate of ℓ. Appropriate factors of $\sqrt{-g}$ may be used to adjust the weight of the final Lagrangian density to +1 or, equivalently, to an ℓ value of -4.

A suitable kinetic term for the preferred field ϕ_a is, in the non-degenerate case

$$\frac{1}{16}\sqrt{-g}\;R_{[ab]}\;R_{[cd]}\;g^{ac}\,g^{bd} = \frac{1}{4}\left(\phi_{a,b}-\phi_{b,a}\right)\left(\phi_{c,d}-\phi_{d,c}\right)\eta^{ac}\,\eta^{bd} \; . \tag{2.48}$$

A mass term for this field can be constructed from the covariant combination $D_a\sigma = \sigma_{,a} + 2\phi_a$ as

$$\sqrt{-g}\;g^{ab}\,D_a\sigma\,D_b\sigma = e^{-2\sigma}\left(\sigma_{,a}\,\sigma_{,b} + 4\,\phi_a\,\phi_b + 4\,\sigma_{,a}\,\phi_b\right)\eta^{ab} \tag{2.49}$$

or may be taken from the scalar curvature in the form

$$\sqrt{-g}R = e^{-2\sigma}\left(\phi_a\,\phi_b + \phi_{a,b}\right)\eta^{ab} \quad . \tag{2.50}$$

Eq. (2.49) also contains a kinetic term for the σ-particle although in fact the expression

$$\eta^{ab}\,\partial_a\,e^{-\sigma}\,\partial_b\,e^{-\sigma} = \eta^{ab}\,e^{-2\sigma}\,\partial_a\sigma\,\partial_b\sigma \tag{2.51}$$

is sufficient since under a conformal group action it transforms into itself plus a divergence. The σ-field may easily be given a bare mass via the density $\sqrt{-g} = e^{-4\sigma}$, in marked contrast to the preferred field associated with a non-linear realization of an internal symmetry. This is an interesting distinction between the "Goldstone bosons" of the two theories and is a direct result of the non-invariance of the volume element d^4x in the action integral under a space-time transformation. It does not however imply that the physical mass of the σ is non-zero. To illustrate this consider the dilatation current which is defined as [10)]

$$\mathcal{D}_\nu(x) = x^\lambda T_{\nu\lambda}(x) + \mathcal{D}_\nu^{loc}(x) \tag{2.52}$$

where $T_{\nu\lambda}$ is the usual energy-momentum tensor. The second term in this expression is a local field whose space-time displacements are generated, as usual, by the momentum operator in the theory. This property is not enjoyed by the first term because of its explicit x-dependence (a manifestation of the non-invariance of d^4x) and results in the displacement law

$$e^{-iP\cdot x}\, \mathcal{D}_\nu(x)\, e^{iP\cdot x} = \mathcal{D}_\nu(0) + x^\lambda\, T_{\nu\lambda}(0) \; . \tag{2.53}$$

The existence of Goldstone bosons is usually demonstrated by examining the spectral sum of the vacuum expectation value of the commutator of the relevant currents and particle fields. In the case above, the use of eq. (2.53) to obtain the spectral resolution results in the appearance of an extra, explicitly x-dependent, term which has to be dealt with before deriving the usual conclusions. Thus we write the two spectral resolutions:

$$\langle [\mathcal{D}_\mu^{loc}(x), \sigma(0)] \rangle = i \int d^4k\; \epsilon(k)\, e^{ik\cdot x}\, k_\mu A(k^2)$$

$$\langle [T_{\mu\nu}(x) x^\nu, \sigma(0)] \rangle = \int d^4k\; \epsilon(k)\, e^{ik\cdot x}\, x^\nu\, B(k^2)\, (k^2 \eta_{\mu\nu} - k_\mu k_\nu)$$

$$= -\, 3i \int d^4k\; \epsilon(k)\, e^{ik\cdot x}\, k_\mu B(k^2)$$

leading to

$$\langle [\mathfrak{D}_\mu(x), \sigma(0)] \rangle = i \int d^4k \ \epsilon(k) \ e^{ik \cdot x} \ k_\mu \ C(k^2)$$

where $C(k^2) \equiv A(k^2) - 3B(k^2)$. Taking the derivative with respect to x on both sides of this equation leads to the conclusion that $C(k^2) = 0$ or $\delta(k^2)$. The former is excluded by the c-number displacement characterizing the σ-field transformation induced by the dilatation current, and thus we are left with the usual prediction of zero-mass particles in the theory.

There are therefore two immediate physical interpretations for the σ-particle. It may be taken as massless, in which case the theory is evidently one of scalar gravity[11)]. This possibility is strengthened by the suggestive coupling of the σ field to the symmetric energy momentum tensor of all the fields (including the σ itself) in the theory, which an application of general canonical field theory reveals as

$$\Box \left(e^{-2\sigma}\right) = -2 \ \theta_{\mu\nu} \ \eta^{\mu\nu} \quad . \tag{2.54}$$

This argument is by no means conclusive, however, being really only a restatement of the universal coupling of the σ-field to all other matter, a fact which does not in itself exclude a mass. In the alternative of a massive particle an obvious candidate is the ubiquitous scalar, isoscalar, two-pion resonance with a mass around 700 MeV. With this interpretation some form of explicit symmetry breaking is necessary in order to avoid the appearance of a Goldstone particle with physical mass zero. This will be discussed at greater length in a forthcoming article. Similar considerations apply to the (optional) ϕ_μ field which could correspond to electromagnetism in the massless case and to one of the various strongly interacting neutral vector particles in the massive situation. These various possibilities are discussed at greater length in another paper[12)], to which the reader is referred.

Finally we must remark that for notational ease we have not yet normalized the preferred fields. This is easily performed by re-defining them as $f\phi_\mu$ and $k\sigma$, resulting in a trivial re-expression of the various formulae. For example, the covariant derivative of a matter field is

$$\nabla_a \psi = \partial_a \psi + 2\ell f\, \phi_a \psi - 2if\, S_{ab}\, \eta^{bc}\, \phi_c\, \psi \tag{2.55}$$

while the new metric tensor is

$$g_{\mu\nu} = \eta_{\mu\nu}\, e^{-2k\sigma} \quad . \tag{2.56}$$

Here f and k are two coupling constants.

III. GENERAL RELATIVITY

The first part of this paper was concerned with the reformulation of the group-theoretic approach to conformal symmetries in terms of a geometric structure imposed on physical space-time. In this second part we wish to reverse the procedure and show that the normal geometric treatment of general relativity, particularly as it concerns the Vierbein formalism, may be partially and usefully re-interpreted in terms of non-linear realizations.

The invariance group of general relativity is the infinite-dimensional pseudogroup of differentiable redefinitions of co-ordinate systems. Under such a co-ordinate transformation $x^{\mu} \rightarrow \bar{x}^{\mu}(x)$, the Jacobian matrices

$$A_{\mu}{}^{\nu}(x) = \frac{\partial x^{\nu}(\bar{x})}{\partial \bar{x}^{\mu}} \tag{3.1}$$

being real and non-singular, belong to the group $GL(4, \mathbb{R})$. The usual transformations of tensors constructed from these matrices and their inverses constitute representations of this general invariance group. Our aim is to obtain non-linear realizations of the group $GL(4, \mathbb{R})$ and to connect these with the conventional structures of general relativity.

The first step is to choose the subgroup H of G (= $GL(4, \mathbb{R})$) from which the quotient space G/H will be formed. This subgroup must be one whose representations are already known. We will choose the Lorentz group, thus enabling us to define group actions of $GL(4, \mathbb{R})$ on half-integer as well as integer spin fields.

The dimensions of $GL(4, \mathbb{R})$ and $O(3,1)$ are sixteen and six, respectively, implying the need for $16 - 6 = 10$ parameters/co-ordinates to specify an element of the quotient space $GL(4, \mathbb{R})/O(3,1)$. The assignment of these parameters, which are the preferred fields of the present theory, is facilitated by the following remark. Any element of $GL(4, \mathbb{R})$ admits [13] a unique polar decomposition as the product of a positive symmetric matrix (which is itself the exponential of a symmetric matrix) with an element of the maximal compact subgroup O(4). A similar local result applies if the Lorentz O(3,1) subgroup is used. Thus if $L_\mu{}^b$ is any 4 x 4 invertible matrix, in some suitable neighbourhood of the unit matrix, it may be written as

$$L_\mu{}^b = \left\{ L \right\}_\mu{}^a \left[L \right]_a{}^b \tag{3.2}$$

where $\left[L \right]_a{}^b$ is an element of the Lorentz group, so that

$$\left[L \right]_a{}^b \left[L \right]_c{}^d \eta_{bd} = \eta_{ac} \tag{3.3}$$

and $\left\{ L \right\}$ is pseudo-symmetric (with ten independent elements) in the sense that

$$\left\{ L \right\}_\mu{}^a \eta_{a\nu} = \left\{ L \right\}_\nu{}^a \eta_{a\mu} \tag{3.4}$$

where η is the Minkowski metric. As is implied above by the word "local", this decomposition is no longer unique. It is, however, well defined on some finite neighbourhood of the identity element of $GL(4,\mathbb{R})$ and this is sufficient for our purposes. Eq. (3.2) mimics the local isomorphism $G \simeq G/H \times H$, thus enabling the matrices $\left\{ L \right\}_\mu{}^a$ to form a local co-ordinate system on the quotient space G/H. These matrices, being the exponentials of pseudo-symmetric matrices, may be written as

$$\left\{ L \right\}_\mu{}^a = \left(\exp[\, i\, J_\alpha{}^\beta\, V_\beta{}^\alpha(x) \,] \right)_\mu{}^a \tag{3.5}$$

where $J_\alpha{}^\beta$ are the appropriate subset of the $GL(4,\mathbb{R})$ Lie algebra generators and the ten preferred fields $V_\beta{}^\alpha(x)$ constitute our parametrization of the coset space $GL(4,\mathbb{R})/O(3,1)$. We will now fix our attention on a matrix L which is already pseudo-symmetric in the form of eq. (3.5), and we will for notational ease drop the curly brackets round it.

The non-linear realization induced on these preferred fields by a group element $A_{\mu}^{\ \nu} = \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}}$ is obtained from eq. (3.5) by multiplying L on the left by A and re performing the polar decomposition

$$L_{\mu}^{\ b} \rightarrow \left(AL\right)_{\mu}^{\ b} = \left\{AL\right\}_{\mu}^{\ a} \left[AL\right]_{a}^{\ b} \quad . \tag{3.6}$$

The action on the preferred fields, which constitutes the GL(4, ℝ) realization, is derived from the transformation

$$L_{\mu}^{\ b}(x) \longrightarrow L'^{\ b}_{\mu}(x) = \left\{AL\right\}_{\mu}^{\ b}(x) \quad . \tag{3.7}$$

For convenience we will denote the Lorentz matrix $\left[AL\right]_{a}^{\ b}$ as $h_{a}^{\ b}$, in terms of which eq. (3.7) may be rewritten as (cf. eq.(1.1))

$$L'^{\ b}_{\mu}(\bar{x}) = \left(AL\right)_{\mu}^{\ a} \left([AL]^{-1}\right)_{a}^{\ b}$$

$$= \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}} \ L_{\nu}^{\ a} \left(h^{-1}\right)_{a}^{\ b} \quad . \tag{3.8}$$

This is the basic equation for the non-linear realization description of the general invariance group of co-ordinate transformations. Evidently the matrix L transforms in a hybrid manner, linearly on the greek indices and non-linearly on the latin indices.

We may now define the non-linear realization of GL(4, ℝ) on any field (including those with half-integer spins) which carries a representation D of the Lorentz group:

$$\psi(x) \rightarrow \psi'(\bar{x}) = D(h)\ \psi(x) \quad . \tag{3.9}$$

Hybrid fields carrying representations $D^{(1)}$ and $D^{(2)}$ of GL(4, ℝ) and the Lorentz group, respectively, may also be incorporated by defining the transformation

$$\psi(x) \rightarrow \psi'(\bar{x}) = D^{(1)}\left(\frac{\partial x}{\partial \bar{x}}\right) D^{(2)}(h)\ \psi(x) \qquad (3.10)$$

of which eq. (3.8) is clearly the prototype.

From the pseudo-symmetric matrix $L_\mu{}^a$ we can construct a genuine rank-two tensor, which transforms linearly, by defining $g_{\mu\nu}$ as

$$g_{\mu\nu} = L_\mu{}^a\, L_\nu{}^b\, \eta_{ab} \quad . \qquad (3.11)$$

Under a co-ordinate transformation we obtain, from eq. (3.8),

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(\bar{x}) = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\, \frac{\partial x^\beta}{\partial \bar{x}^\nu}\, L_\alpha{}^a\, L_\beta{}^b \left(h^{-1}\right)_a{}^c \left(h^{-1}\right)_b{}^d\, \eta_{cd} \qquad (3.12)$$

$$= \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\, \frac{\partial x^\beta}{\partial \bar{x}^\nu}\, g_{\alpha\beta}(x) \quad . \qquad (3.13)$$

This enables us to identify $g_{\mu\nu}$ as the metric tensor whose components form the basic dynamical potentials of general relativity. It is related via eq. (3.5) to the preferred fields $V_\alpha{}^\beta(x)$ of the non-linear realizations. The contragredient tensor $g^{\mu\nu}$ is defined as usual as the inverse matrix of $g_{\mu\nu}$ and these two tensors may be used in the standard manner to raise and lower greek indices. It is convenient to regard the latin indices (which carry the non-linear realization) on $L_\mu{}^a$ as being raised and lowered with the usual Minkowski metric tensor. This convention may be consistently extended to include the latin indices on the Lorentz matrix h of eq. (3.8) and indeed this motivated our original assignment of the various indices from the two alphabets. With this notation we may derive the expressions

$$L^{\mu a}\, L_{\nu a} = \delta^\mu_\nu \quad ; \quad L^{\mu a}\, L_{\mu b} = \delta^a_b \quad ; \quad L^{\mu a}\, L^\nu{}_a = g^{\mu\nu}, \ L^{\mu a}\, L_\mu{}^b = \eta^{ab} \qquad (3.14)$$

which demonstrates the duality between the two types of index.

There is obviously a very close connection between our non-linear realization scheme and the Vierbein formalism [14)] which takes eq. (3.11) as its starting point. In the latter case, however, the matrix $L_\mu{}^a$ is only determined up to an arbitrary Lorentz transformation on its latin indices. This group of space-time dependent Lorentz transformations, known as the Vierbein gauge group, acts upon spinor fields which are, however, taken as scalars under general co-ordinate transformations. The requirement that a Lagrangian be invariant under the general co-ordinate group and also under an arbitrary redefinition of $L_\mu{}^a$ (which is equivalent to a Vierbein gauge group action) necessitates the introduction of a spinor connection onto the space-time manifold with associated covariant derivatives.

We have proceeded differently by observing that out of all possible choices (which are interrelated by Vierbein transformations) for the matrices $L_\mu{}^a$ in eq. (3.11), one unique choice exists which is pseudo-symmetric. It might be thought that by using this special form "for all time" we have lost the invariance under the Vierbein group and the equivalence with the standard formalism. This is not so because our spinor fields are not scalars under co-ordinate transformations but transform in the manner prescribed by eq. (3.19). Any arbitrary Lorentz transformation H, say, may be imparted to the spinor field $\psi(x)$ by means of the co-ordinate transformation $x^\mu \rightarrow \bar{x}^\mu(x)$ where x^μ is arranged such that

$$A_\mu{}^\nu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} = \left(HL^{-1}\right)_\mu{}^\nu \tag{3.15}$$

which, since

$$h_a{}^b \equiv \left[AL\right]_a{}^b = H_a{}^b \tag{3.16}$$

leads, by eq. (3.9), to the desired result. This particular co-ordinate transformation has a natural interpretation since, from eq. (3.11), L^{-1} transforms the system to a co-ordinate chart in which the metric tensor is (locally) Minkowskian and this is then followed by the desired Lorentz transformation H.

The final equivalence between these two approaches is demonstrated by constructing the non-linear realization "covariant derivatives" for the fields transforming as in eq. (3.10). These turn out to be identical with the geometric covariant derivatives constructed from the spin connection of the Vierbein scheme, showing that the two formalisms really are physically equivalent. Essentially, we have exchanged the Vierbein gauge group for the "non-linearly" transforming spinor fields and it is hoped that this throws an interesting light on the whole Vierbein structure (or vice versa). The non-linear approach does of course carry the bonus of potentially incorporating the Goldstone phenomenon which in this case is manifested by the symmetric Vierbein fields.

Before constructing the covariant derivatives an important technical distinction must be made between the non-linear spinor representations occurring above and those of, for example, chiral symmetries. In the latter case the non-linear realization of the group G on a matter field (by which is meant all fields except the preferred ones) may be embedded in a higher-dimensional linear representation of the group. More precisely, if D denotes a linear representation of G on a field Ψ, then the field ψ defined by

$$\psi = D(L)^{-1}\Psi \tag{3.17}$$

transforms in the standard non-linear way. Conversely, given any non-linear realization of G on a field ψ, it may be embedded in a linear representation D on a field Ψ provided that D, when restricted to the fundamental subgroup H, contains that finite-dimensional representation of H which is carried by the field ψ. If G is a chiral group this can always be done. This will not, however, always be possible when G is $GL(4, \mathbb{R})$ because no finite-dimensional linear representation of this group contains half-integer spin representations of the homogeneous Lorentz subgroup H. The technique employed in I and II for constructing covariant derivatives relied to some extent on the existence of such an embedding and is also algebraically complicated in the present case. Therefore we will use here a different, more geometric, method.

To orientate our sign conventions we will write out the infinitesimal form of the hybrid transformation of eq. (3.10) as

$$\psi'(\bar{x}) = \left(1 - i\epsilon^{\nu}{}_{\mu} F^{\mu}{}_{\nu} - i\delta h_{ab} S^{ab}\right) \psi(x) \tag{3.18}$$

where $F^{\mu}{}_{\nu}$ and S^{ab} are the Lie algebra representation matrices of $D^{(1)}$ and $D^{(2)}$, respectively. We define the covariant derivative of $\psi(x)$ as

$$\psi_{;\mu} = \psi_{,\mu} - i\left(\Gamma_{\mu\nu}{}^{\lambda} F^{\nu}{}_{\lambda} + B_{\mu ab} S^{ab}\right) \psi \tag{3.19}$$

where the connections Γ and B are required to transform as

$$\bar{\Gamma}_{\mu\nu}{}^{\lambda} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \left(\frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial \bar{x}^{\lambda}}{\partial x^{\gamma}} \Gamma_{\alpha\beta}{}^{\gamma} - \frac{\partial^2 \bar{x}^{\lambda}}{\partial x^{\alpha} \partial x^{\beta}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \right) \tag{3.20}$$

$$\bar{B}_{\mu ab} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \left(h_a{}^c h_b{}^d B_{\alpha cd} - h_{a,\alpha}{}^{c} h_b{}^d \eta_{cd} \right) \tag{3.21}$$

and are at the moment two independent quantities. If we require (as is customary) that $\Gamma_{\mu\nu}{}^{\lambda} = \Gamma_{\nu\mu}{}^{\lambda}$ and that the covariant derivative of the metric tensor should vanish, we obtain the unique form for Γ:

$$\Gamma_{\mu\nu}{}^{\lambda} = \left\{ {\lambda \atop \mu\,\nu} \right\} = \frac{1}{2} g^{\lambda\alpha} \left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \right). \tag{3.22}$$

Under these conditions the acts of raising and lowering indices and taking covariant derivatives commute.

The matrix $L_{\mu}{}^{a}$ may be used to interchange latin and greek indices. Thus, given any linearly transforming vector ψ_{μ} we may associate with it a canonical non-linearly transforming field ψ_a defined by

$$\psi_a = L^{\mu}{}_a \psi_{\mu} . \tag{3.23}$$

Conversely,

$$\psi_\mu = L_\mu^{\ a} \psi_a \tag{3.24}$$

is a linear field if ψ_a transforms non-linearly on the index a. These equations are the concrete form, for Lorentz vectors, of the embedding construction mentioned earlier. We would like this interchanging of indices to commute with covariant derivation, that is

$$(L^\mu_{\ a} \psi_\mu)_{;\nu} \equiv \psi_{a;\nu} = L^\mu_{\ a} \psi_{\mu;\nu}$$

which requires the vanishing of the covariant derivative of $L_{\mu a}$,

$$L_{\mu a;\nu} = L_{\mu a,\nu} - \Gamma_{\nu\mu}^{\ \ \lambda} L_{\lambda a} + B_{\nu ab} L_\mu^{\ b} \quad . \tag{3.25}$$

Equating this expression to zero we may solve for $B_{\nu ab}$ in terms of the preferred fields

$$B_{\nu ab} = L_{\lambda a} \Gamma_{\nu\mu}^{\ \ \lambda} L^\mu_{\ b} - L_{\mu a,\nu} L^\mu_{\ b} \tag{3.26}$$

$$= \frac{1}{2} (L^\mu_{\ a} L_{\mu b,\nu} - L^\mu_{\ b} L_{\mu a,\nu}) - \frac{1}{2} L_{\nu c} L^\lambda_{\ a} L^\mu_{\ b} (L_{\mu,\lambda}^{\ \ c} - L_{\lambda,\mu}^{\ \ c})$$

$$- \frac{1}{2} (L^\mu_{\ a} L_{\nu b,\mu} - L^\mu_{\ b} L_{\nu a,\mu}) \tag{3.27}$$

which clearly transforms as in eq. (3.21) and is, as claimed earlier, the usual spinor connection of the Vierbein formalism thereby demonstrating the equivalence of the two approaches.

The covariant derivative of a hybrid field will transform as

$$\psi_{;\mu}(x) \to \psi'_{;\mu}(\bar{x}) = \frac{\partial x^\nu}{\partial \bar{x}^\mu} D^{(1)}\left(\frac{\partial x}{\partial \bar{x}}\right) D^{(2)}(h) \psi_{;\nu}(x) \ . \tag{3.28}$$

We can also define a "non-linear" covariant derivative (cf. eq. (3.23) and (3.24)) as

$$\psi_{;a} = L^\mu_{\ a} \psi_{;\mu}$$

which transforms as

$$\psi_{;a}(x) \to \psi'_{;a}(\bar{x}) = h_a^{\ b} \, D^{(1)}\left(\frac{\partial x}{\partial \bar{x}}\right) D^{(2)}(h) \, \psi_{;b} \tag{3.29}$$

and which may be useful on occasion .

A bilinear expression in Dirac spinors has the transformation property

$$\bar{\psi}\gamma_a\psi \to \bar{\psi}\Lambda^{-1}(h)\,\gamma_a\,\Lambda(h)\psi = h_a^{\ b}\,\bar{\psi}\gamma_b\psi \tag{3.30}$$

where Λ are the representation matrices of the spinor ψ, and γ_a are the usual constant Dirac matrices satisfying the anticommutation law

$$\left\{\gamma_a, \gamma_b\right\} = 2\eta_{ab} \quad .$$

Such a bilinear, being a Lorentz vector, may be converted to a linearly transforming form using eq. (3. 24), that is

$$L_\mu^{\ a}\,\bar{\psi}\gamma_a\psi \to \frac{\partial x^\nu}{\partial \bar{x}^\mu}\,(L_\nu^{\ a}\,\bar{\psi}\gamma_a\psi) \quad . \tag{3.31}$$

This conversion may be regarded as a redefinition of the Dirac matrices as

$$\gamma_\mu(x) = L_\mu^{\ a}(x)\,\gamma_a$$

which satisfy the new anticommutation law

$$\left\{\gamma_\mu, \gamma_\nu\right\} = 2g_{\mu\nu}$$

of the generalized Clifford algebra which frequently appears in the standard treatment of spin structure on space-time.

Invariant Lagrangian densities may now be readily constructed by saturating latin indices with η_{ab}, greek indices with $g_{\mu\nu}$ and multiplying by the appropriate powers of $\sqrt{-g}$ to achieve the desired weight of one. There is nothing in principle to stop our adding into the theory the σ and ϕ_a fields of Sec. II after suitably extending their transformation laws in the obvious manner suggested by eqs. (2.17) and (2.18). The σ-field might be introduced as a factorization of the gravitational metric tensor $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\, e^{-2\sigma} \quad .$$

The density $\tilde{g}_{\mu\nu}$ describes the pseudo-tensorial component of the gravitational force and in the absence of this interaction reduces to the Minkowskian metric $\eta_{\mu\nu}$ with a resulting restriction of the invariance group to the conformal subgroup. Alternatively, we could form the affine sum of the natural connection defined by the σ-field, with the Christoffel symbol associated with the space-time metric $g_{\mu\nu}$. Another possibility is simply to introduce $e^{-4\sigma}$ as a normal field in its own right, since it is a scalar density of weight one. In this case it has no intrinsic geometric significance.

The ϕ field may best be incorporated as an independent massless vector by forming the affine sum of the appropriate connections. The ensuing theory with its non-Riemannian geometry is very similar to Weyl's[15] formulation of electromagnetism.

We have throughout regarded the conformal group as a subgroup of the group of general co-ordinate transformations. Indeed in our formulation of conformal symmetry we deliberately constructed the metric tensor $g_{ab} = \eta_{ab}\, e^{-2\sigma}$ so that the line element $ds^2 = g_{ab}\, dx^a\, dx^b$ was invariant. Invariance under the latter group then automatically implies invariance under the former and from this point of view the σ field is not necessary if we are provided with the gravitational field $g_{\mu\nu}$. It is possible to regard conformal actions as being something quite distinct from general co-ordinate transformations, which act directly on the gravitational metric tensor, resulting in a non-invariance of the line element on the space-time manifold. We, however, prefer the interpretation above with its implied hierarchy of space-time groups: Poincaré, conformal and general co-ordinate. Invariance under the latter two then enforces the successive, but not mutually necessary, introduction of the σ and $g_{\mu\nu}$ fields with their respective coupling strengths.

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