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ABSTRACT

Any Lorentz-invariant Lagrangian can be made conformally invariant provided a gauge field ϕ_μ is introduced as part of a covariant derivative. The field ϕ_μ, whose transformations are thus symptomatic of spontaneous symmetry breakdown, is decomposed into transverse and longitudinal pieces whose properties and possible manifestations are discussed. Among them is a natural place for C-violation.

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The purpose of this note is to clarify some aspects of the theory of spontaneously broken conformal symmetry as presented recently in an article by two of the authors [1]. The discussion of Ref.1 was framed within the context of non-linear realizations and culminated in a spontaneous symmetry breaking associated with a set of Goldstone particles in accordance with general theory [2]. With this approach it becomes possible to construct conformal invariant Lagrangians for massive fields. However, there were some peculiar aspects of this application of the general method: firstly, the realizations turned out to be linear (although inhomogeneous in the case of the Goldstone fields) and, secondly, the Goldstone particles were able to acquire mass. In effect, the massless vector field which, at first sight, would seem to be associated with the spontaneous breaking of the conformal symmetries has conspired - according to a well-known mechanism [3] - to obliterate the "classical" Goldstone effect. This means that the full apparatus of non-linear realizations can be dispensed with and the results of Ref.1 derived more directly.

Let us begin by reviewing the definition of the conformal group on space-time. It is a 15-parameter group which includes the following transformations:

a) Inhomogeneous Lorentz transformations

$$x'_\mu = \Lambda_{\mu\nu} x_\nu + a_\mu \quad , \quad \Lambda \in O(3,1) \quad , \qquad (1)$$

b) special conformal transformations

$$x'_\mu = \frac{x_\mu + x^2\beta_\mu}{1 + 2x\beta + x^2 \beta^2} \quad , \qquad (2)$$

c) dilations

$$x'_\mu = e^\lambda x_\mu \qquad (3)$$

where $\Lambda_{\mu\nu}$, a_μ , β_μ and λ are all real quantities independent of x. We are using the Minkowskian metric

$$\eta_{\mu\nu} = \mathrm{diag}\ (+ - - -)$$

and the tacit summation conventions, $\beta x = \beta_\mu x_\mu = \eta^{\mu\nu} \beta_\mu x_\nu$, etc. This will not cause confusion since we shall always view the x_μ as rectangular co-ordinates in a flat space-time.

The defining characteristic of conformal transformations is the preservation of <u>angles</u> but not lengths. In fact, the various transformations (1), (2) and (3) can be summed as the totality of mappings $x_\mu \to x'_\mu$ which satisfy the identity

$$\frac{\partial x'_\mu}{\partial x_\kappa} \frac{\partial x'_\nu}{\partial x_\lambda} \eta^{\mu\nu} = \left| \det\left(\frac{\partial x'}{\partial x}\right)\right|^{\frac{1}{2}} \eta^{\kappa\lambda} . \tag{4}$$

This basic formula can be used to set up the linear representations of the conformal group supposing one is already given the representations of the inhomogeneous Lorentz group. To this end, consider the matrix

$$\Lambda_{\mu\nu}(x) = \left| \det\left(\frac{\partial x'}{\partial x}\right)\right|^{-\frac{1}{4}} \frac{\partial x'_\mu}{\partial x_\nu} . \tag{5}$$

According to (4) it is a Lorentz matrix, $\Lambda(x) \in O(3,1)$. Moreover, it constitutes a linear representation of the conformal group (since both $|\det(\partial x'/\partial x)|$ and $\partial x'_\mu/\partial x_\nu$ are themselves linear representations). Therefore, given a set of fields $\psi_\alpha(x)$ which belong to a linear representation of the inhomogeneous Lorentz group

$$\psi'(x') = D(\Lambda)\, \psi(x)$$

where the representation $\Lambda \to D(\Lambda)$ is defined for all $\Lambda_{\mu\nu}$ which belong to $O(3,1)$, it is clearly possible to represent the conformal transformations (1), (2) and (3) by

$$\psi'(x') = \left| \det\left(\frac{\partial x'}{\partial x}\right)\right|^{\ell/4} D(\Lambda(x))\, \psi(x) \tag{6}$$

where $\Lambda(x)$ is given by (5), and the factor $|\det \partial x'/\partial x|^{\ell/4}$ has been included for greater generality. Consistency requires that ℓ be a Lorentz scalar: a pure number if $D(\Lambda)$ is irreducible, otherwise a matrix which commutes with D. The infinitesimal form of the global transformation (6) coincides with that derived in Ref.1 by non-linear realization techniques and in Ref.4 by the method of induced representations.

Clearly, since the $\Lambda_{\mu\nu}$ in (6) depends upon x_μ, the ordinary partial derivative $\psi_{,\mu} = \partial_\mu \psi$ will not transform in a simple fashion. Rather, one finds

$$\psi'_{,\mu}(x') = \frac{\partial x_\nu}{\partial x'_\mu} \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|^{\ell/4} D(\Lambda(x)) \left\{\psi_{,\nu}(x) + \frac{1}{4}(\ell\eta_{\nu\rho} - iS_{\nu\rho})\partial_\rho \ln\left|\det\frac{\partial x'}{\partial x}\right|\psi(x)\right\} \tag{7}$$

where $S_{\mu\nu} = -S_{\nu\mu}$ denotes the usual spin matrices which generate the infinitesimal form of $D(\Lambda)$. Following the standard procedure we can invent a covariant derivative $\psi_{;\mu}$ by introducing a gauge field (or connection) ϕ_μ as follows:

$$\psi_{;\mu} = \psi_{,\mu} + 2(\ell\eta_{\mu\nu} - iS_{\mu\nu})\,\phi_\nu\,\psi \tag{8}$$

where ϕ_μ transforms according to the rule

$$\phi'_\mu(x') = \frac{\partial x_\nu}{\partial x'_\mu}\left(\phi_\nu(x) - \frac{1}{8}\partial_\nu \ln\left|\det\frac{\partial x'}{\partial x}\right|\right). \tag{9}$$

With this construction, the covariant derivative (8) transforms according to the rule

$$\psi'_{;\mu}(x') = \left|\det\frac{\partial x'}{\partial x}\right|^{\frac{\ell-1}{4}} \Lambda_{\mu\nu}\, D(\Lambda)\,\psi_{;\nu}(x)$$

and is seen to be a covariant object of "weight" $\ell - 1$.

Having described the conformal transformations of fields and defined their covariant derivatives, let us now consider the problem of constructing conformal invariant Lagrangians. Actually, since the Jacobian determinant is not trivial we must take care to make the Lagrangian density $L(\psi)$ a scalar density of weight $\ell = -4$, i.e.

$$L'(\psi') = \left| \det \frac{\partial x'}{\partial x} \right|^{-1} L(\psi) \quad .$$

In order to meet this condition it was assumed in Ref.1. that there exists a Lorentz scalar field $\sigma(x)$ with the anomalous transformation behaviour

$$\sigma'(x') = \sigma(x) + \frac{1}{4} \ln \left| \det \frac{\partial x'}{\partial x} \right| \quad . \tag{10}$$

With such a field at our disposal we can construct the Lorentz scalar of weight ℓ, given by $\exp(\ell\sigma)$ and multiply this factor - with ℓ chosen appropriately - into any given Lorentz scalar term in the Lagrangian thereby making it a scalar density of weight -4.

Thus, with the fields ϕ_μ and σ we can turn any Lorentz-invariant Lagrangian into a conformally invariant one. However, these fields evidently transform in a manner which is not consistent with the invariance of the vacuum state (except under the subgroup (1) for which $\det \left| \partial x'/\partial x \right| = 1$) and it appears that we are dealing with a theory in which the conformal symmetry is spontaneously broken. This was the point of view adopted in Ref.1.

However, there remains some ambiguity in this programme. It appears on comparing (9) and (10) that ϕ_μ and $-\frac{1}{2}\partial_\mu\sigma$ transform identically. Therefore, we can everywhere replace ϕ_μ by $C_\mu - \frac{1}{2}\partial_\mu\sigma$ where C_μ transforms like an ordinary field with $\ell = -1$. It follows that the couplings of C_μ are not controlled by the conformal requirements. Any Lorentz-invariant coupling is permissible for C_μ.

The interactions of $\sigma(x)$, on the other hand, are governed completely by the requirements of conformal invariance. In fact, the equation of motion for σ, derived from a conformal invariant action, can be put into the form

$$\Box(e^{-2\sigma}) = 2\Theta_{\mu\mu} \tag{11}$$

where $\Theta_{\mu\nu}$ denotes the symmetrical energy-momentum tensor of all the fields (including σ itself).

From the appearance of (11) one could choose to interpret $\sigma(x)$ as a scalar gravitational field[5] and make its couplings correspondingly weak. It is even possible to give this "gravitation" a geometrical meaning by interpreting the expression $g_{\mu\nu} = \eta_{\mu\nu} \exp(-2\sigma)$ as the metric tensor in a Riemannian space-time. This point of view will be presented in some detail in a forthcoming article.

A second possibility is to interpret $\sigma(x)$ as a strongly coupled field with a finite range. The conformal invariance of the action is not disturbed by including the term $\exp(-4\sigma)$ in the Lagrangian density and from this term a mass can be extracted. If this interpretation is adopted then it would be necessary to break the symmetry insofar as leptons are concerned. It would not be acceptable to have $\sigma(x)$ coupled strongly to the leptons. Hence, the right-hand side of (11) must be replaced by $2\Theta_{\mu\mu}^{(\text{hadron})}$. It may be remarked, however , that if the leptons were truly massless then they would not contribute to $\Theta_{\mu\mu}$. We should perhaps look upon the lepton mass terms as spontaneous breakers of conformal symmetry, which means that the self mass *) exactly equals the physical mass, the bare mass being zero.

Can nature have a direct use for the field $C_\mu = \phi_\mu(x) + \frac{1}{2}\partial_\mu \sigma(x)$? So far as conformal invariance is concerned, it could be dispensed with at least so long as we have **) $\sigma(x)$. However its manner of appearance

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*) It was suggested in Ref. 2 that it might prove possible to compute the pion mass - in the context of chiral SU(2) x SU(2) - by treating it as a counter term: something to be added to and subtracted from the Lagrangian and computed self-consistently. This programme is being pursued at the present time by workers at Imperial College.

**) It is possible to set up conformal invariant theories in which σ does not appear. They are characterized by the absence of mass terms. In addition, the ℓ-values of the fields are highly constrained. Thus $\mathrm{Re}\,\ell = -1$ for scalar and vector fields and $\mathrm{Re}\,\ell = -3/2$ for the Dirac spinor. (These assignments are discussed in a recent article by G. Mack and Abdus Salam (Ref. 4). They can be referred to as the <u>canonical</u> weights.) The imaginary parts of ℓ are unrestricted.

in the covariant derivative and its characteristic couplings are extremely suggestive in themselves - particularly when we remark that whenever $\mathrm{Re}\,\ell = -1$ (bosons), $-\frac{3}{2}$ (fermions) and $\mathrm{Im}\,\ell \neq 0$, C_μ couples to a conserved neutral current. We wish to list some attractive possibilities of the role this field might play in particle physics.

a) <u>As the electromagnetic field</u>: If $\mathrm{Im}\,\ell$ is identified with the electric charge e and C_μ is massless, the field could be identified with the electromagnetic field.

b) <u>As the baryon-conserving field</u>: If C_μ is massive and $\mathrm{Im}\,\ell$ is proportional to baryon number, C_μ could be identified with a baryon-conserving field possibly with the $J^{PC} = 1^{--}$ ω-particle.

c) If $\mathrm{Im}\,\ell = 0$ and $\mathrm{Re}\,\ell \neq \ell_{\text{canonical}}$ *) C_μ would be an exotic particle of the second kind with $J^{PC} = 1^{-+}$, assuming that the theory is invariant for antiparticle conjugation.

d) It cannot have escaped the reader's notice that the conflicting charge-conjugation properties postulated in b) and c) immediately suggest a possible mechanism for C-violation. This may be the most important role of the C-particle.

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* See second footnote on p.6.

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