THE PION ELECTROMAGNETIC
FORM FACTOR
AND THE VENEZIANO MODEL

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ABSTRACT

Elastic unitarity is used to connect the pion electromagnetic form factor to the $\pi\pi$ Veneziano amplitude. A model for the form factor exhibiting this connection is obtained. It is in fair agreement with experiment.

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I. INTRODUCTION

The successful application of the Veneziano amplitude \(^1\) to describe low-energy meson phenomenology \(^2\) has encouraged some authors \(^3\)-\(^6\) to consider the corresponding formulation for the three-point function. This is of considerable interest since it extends the theory into the realm of weak and electromagnetic interactions. However, since the essential ingredients of crossing and duality are missing in the case of the three-point function, direct generalization of the original Veneziano amplitude is not feasible. In Ref.\(^3\) use is made of PCAC, field-current identity and current algebra to connect the vertex function to the Veneziano four-point amplitude extended off the mass shell. The resulting expression for the pion electromagnetic form factor \(F_\pi(s)\) is in good agreement with experiment for \(s < 0\). The positive energy region is left out of consideration due to lack of any knowledge of the imaginary part of the \(\rho\)-trajectory function \(\pi(s)\).

In Refs. \(^5\) and \(^6\) a spurion method is proposed for constructing the three-point function given the off-shell four-point Veneziano formula. In Ref.\(^4\) a Veneziano-like representation of the three-point function is directly proposed. This again contains an infinite number of zero-width resonances and gives good agreement with experiment in \(s < 0\) for the isovector nucleon form factor.

In this paper we adopt a different approach and attempt to connect the three-point function direct to the Veneziano amplitude by imposing elastic unitarity. For consistency the Veneziano amplitude must itself be subjected to elastic unitarity. Since for the form factor only one channel exists, the single-channel unitarized Veneziano form \(^7\),\(^8\) is adequate for this purpose. The resulting expression for the form factor is then expected to give better agreement with experiment for \(s > 0\).

Now the elastic unitarity condition on a meson form factor \(F(s)\) is of the general form

\[
\text{Im } F(s) = \delta(s-s_0) F(s) \int g(s,t) T^*(s,t) d\Omega(t) \quad (1.1)
\]

where \(s_0\) is the elastic energy threshold, \(T(s,t)\) the elastic
scattering amplitude for the mesons involved, $g(s,t)$ a kinematical factor and the integration is over the angular variation of the two-particle intermediate state in the centre-of-mass system. The unitarity condition (1.1) has been made use of to connect form factors and strong interaction amplitudes in several approaches such as dispersion relations, $N/D$ method and effective range formulae. In this paper we shall follow the approach of Ref.10 in which it is proposed that the unitarity eq. (1.1) be satisfied by the equation

$$F(s) = C(s) + i\phi(s) F(s) \int g(s,t) K(s,t) \, d\Omega(t)$$

(1.2)

where $\phi(s)$ is a phase space factor, $K(s,t)$ is the strong interaction $K$-matrix and $C(s)$ is a function that lacks the elastic cut. One may indeed think of (1.2) as a definition of this reduced form factor function $C(s)$ and use eq. (1.1) to show that it is real below the first inelastic threshold.

In the single-channel unitarization of the Veneziano amplitude, the $K$-matrix is taken to be a constant times the original Veneziano formula. Eq. (1.2) is thus particularly suited for the connection of $F(s)$ to the unitarized Veneziano amplitude. In this paper we apply the method to the pion electromagnetic form factor. The details of the model are given in Sec.II. Besides the restriction to elastic unitarity and the identification of the $K$-matrix with the Veneziano amplitude, the model consists of the choice of a definite real function $C(s)$ which is, of course, not entirely arbitrary. The final formula that we obtain for the pion electromagnetic form factor $F_\pi(s)$ contains a single arbitrary parameter which we fix using one experimental value of $|F_\pi(s)|^2$. The resulting agreement with experiment is good for $s > 0$ and fair for $s < 0$.

II. THE MODEL

As we mentioned in the introduction, the general formalism we use is that of Ref. 10. We now describe this formalism in some detail. Consider a scalar current $J$ and denote by $J_2(s)$ its matrix element $\langle 0|J|i\rangle$ where $|i\rangle = \{|a,b\rangle\}$ is a two-particle
state and \( s = (p_a + p_b)^2 \). One may then define a reduced current \( C \), with \( C_i(s) = \langle 0 | C | i \rangle \), carrying the same set of internal quantum numbers as \( J \), by

\[
J_i(s) = C_i(s) + i \sum_r \phi_r(s) J_r(s) G_{ri}(s)
\]

(2.1)

\[
G_{ri}(s) = \int_{-1}^1 K_{ri}(s, \cos \theta) d \cos \theta
\]

(2.2)

where, in eq. (2.1), the sum is over all the possible two-particle states \( r \) and \( \phi_r(s) \) is a phase space factor so chosen that \( C_i(s) \) is free from the two-particle cut in \( J_i(s) \). In eq. (2.2) \( K_{ij} \) is the strong interaction \( K \)-matrix. Eq. (2.1) may be considered a "solution" to the two-particle unitarity equation satisfied by \( J_i(s) \).

In the elastic approximation one obtains an algebraic equation for \( J_i(s) \) giving

\[
J_i(s) = \frac{C_i(s)}{1 - i\phi_i G_{ii}(s)}
\]

(2.3)

Now \( G_{ii}(s) \) is, essentially, the s-wave component of \( K_{ii}(s, \cos \theta) \) and is therefore related to \( T_{ii}^{(0)}(s) \), the s partial wave of the strong interaction amplitude \( T_{ii}(s, \cos \theta) \), by

\[
T_{ii}^{(0)}(s) = \frac{G_{ii}(s)}{1 - i\phi_i G_{ii}(s)}
\]

(2.4)

Eqs. (2.3) and (2.4) clearly display the well-known result that \( J_i(s) \) and \( T_{ii}^{(0)}(s) \) possess the same poles and right-hand cut. The resonances in \( T_{ii}^{(0)}(s) \) are obtained by inserting poles in \( G_{ii}(s) \). The corresponding resonances in \( J_i(s) \) are similarly derived from poles in \( C_i(s) \). For the case of a vector current \( J^\mu \), the equation similar to (2.1) is

\[
J^\mu_i(p_a, p_b) = C_i^\mu(p_a, p_b) + i \sum_r \lambda_r(s) J^\mu_r(p_{1r}, p_{2r}) K_{ri}(p_{1r}, p_{2r}, p_a, p_b)
\]

(2.5)

where \( \sum_r \) denotes the summation over the two-particle intermediate states as well as the unitarity phase space integration.
The factor $\lambda_i(s) = \text{const.} \times \rho_r(s) \rho_r^{-1}(s)$ where $\rho_r(s)$ is the two-particle unitarity phase space factor. In the elastic approximation, (2.5) reads

$$J_i^{\mu}(p_a, p_b) = C_i^{\mu}(p_a, p_b) + i \lambda_i(s) \int_U J_i^{\mu}(p_1, p_2) K_{ii}(p_1, p_2, p_a, p_b)$$

(2.6)

Writing

$$J_i^{\mu}(p_a, p_b) = F_i(s) (p_a - p_b) + G_i(s) (p_a + p_b) \mu$$

$$C_i^{\mu}(p_a, p_b) = C_i(s) (p_a - p_b) \mu + D_i(s) (p_a + p_b) \mu$$

(2.7)

eq (2.6) gives

$$F_i(s) = C_i(s) + i \frac{\lambda_i(s) F_i(s)}{s \left\{ \frac{2(m^2 + m_b^2) - s - (m_a^2 - m_b^2)^2}{2(m_a^2 + m_b^2) - s - (m_a^2 - m_b^2)^2} \right\} K_{ii}$$

(2.8)

$$G_i(s) = D_i(s) + i \frac{\lambda_i(s) F_i(s) \cdot (m_a^2 - m_b^2)}{s \left\{ \frac{2(m^2 + m_b^2) - s - (m_a^2 - m_b^2)^2}{2(m_a^2 + m_b^2) - s - (m_a^2 - m_b^2)^2} \right\} K_{ii}$$

$$+ i \lambda_i(s) G_i(s) \int_U K_{ii}$$

(2.9)

With $K_{ii}$ given, (2.8) and (2.9) are algebraic equations, easily solvable for $F_i$ and $G_i$.

We now specialize to the case of the pion electromagnetic form factor. Taking $i = \gamma^\mu \tau^\nu \eta_\nu$ and $J^{\mu}$ the electromagnetic current, we set $G_i = 0$, $F_i(s) = F_\pi(s)$, $m_a = m_b = m_\pi^2$. Eq. (2.8) then
becomes

\[ F_\pi(s) = C_\pi(s) + \frac{i \lambda_\pi(s) F_\pi(s)}{4m_\pi^2 - s} \int \frac{d^4 p_1 \cdot p_2}{4\pi^2} (p_1 - p_2) \cdot (p_a - p_b) K_{\pi\pi}(p_1, p_2, p_a, p_b). \]  

(2.10)

The integral in this equation may be evaluated in the centre-of-mass system where

\[ p_1 = (\frac{s}{2}, q), \quad p_2 = (\frac{s}{2}, -q), \]

\[ p_a = (\frac{s}{2}, p), \quad p_b = (\frac{s}{2}, -p), \]

\[ p^2 = q^2 = \frac{s}{4} - m_\pi^2, \quad p \cdot q = \frac{s}{4} - m_\pi^2 \cos \theta. \]  

(2.11)

Ignoring a multiplicative factor (to be absorbed in the function chosen for \( K_{\pi\pi} \)), eq. (2.10) now takes the form

\[ F_\pi(s) = C_\pi(s) + i \phi_\pi(s) F_\pi(s) \int \cos \theta K_{\pi\pi}(s, \cos \theta) d \cos \theta. \]  

(2.12)

For the \( K \)-matrix \( K_{\pi\pi} \), we now take the Veneziano amplitude for \( \pi^+ \pi^- \) scattering involving the \( \rho \) trajectory

\[ K_{\pi\pi} = -\beta \frac{\Gamma(1 - \alpha_\rho(t)) \Gamma(1 - \alpha_\rho(s))}{\Gamma(1 - \alpha_\rho(t) - \alpha_\rho(s))} \]  

(2.13)

with \( \beta \) constant and

\[ \alpha_\rho(s) = \frac{s}{2} + \frac{1}{2m_\rho^2} s. \]  

(2.14)

In the centre-of-mass system \( t = 2(m_\pi^2 - \frac{s}{4})(1 - \cos \theta) \) and (2.13) becomes

\[ K_{\pi\pi}(s, \cos \theta) = -\beta \frac{\Gamma(1 - \alpha) \Gamma\left\{ \frac{1}{2} - \left(\frac{m_\pi^2}{m_\rho^2} + \frac{1}{2} - \alpha/2\right)(1 - \cos \theta)\right\}}{\Gamma\left\{ \frac{1}{2} - \alpha - \left(\frac{m_\pi^2}{m_\rho^2} + \frac{1}{2} - \alpha/2\right)(1 - \cos \theta)\right\}}. \]  

(2.15)

where \( \alpha = \alpha_\rho(s) \).

As we have remarked in connection with the scalar case, the poles in \( s \) of the integral \( G \) over \( K \) are accompanied by similar poles in \( G \), so that the resonances in \( T \) (note from eq. (2.12) it is
now the p-wave component which is picked) and J correspond to each other. From eq. (2.15) these poles are clearly those of \(\rho = (1 - \chi)\) occurring at all the positive integers in \(\chi\). Numerical calculation shows that the integral over the remaining factor on the right-hand side of (2.15) does not vanish at any of these poles lying in the energy interval considered here. Since we are dealing here with a low-energy model we may take for \(C_\pi\) the simplest form consistent with these considerations:

\[
C_\pi(s) = A \Gamma(1 - \alpha_p(s))
\]  

(2.16)

where \(A\) is a constant. In writing this form for \(C_\pi\) we are disregarding the asymptotic behaviour of the form factor which we hope to investigate in some later work. The form (2.16) is, of course, free of the two-particle cut as is required by the original equation (2.6). The form factor \(F^\pi(s)\) obtained from (2.12) on using (2.15) and (2.16) must therefore satisfy elastic unitarity.

It now remains to fix the phase space factor \(\phi(s)\) in eq. (2.12). Writing

\[
\mathcal{H}(\chi) = \int_{-1}^{+1} \frac{\Gamma\left\{\frac{1}{2} - \left(\frac{m_\pi^2}{m_p^2} + \frac{1}{4} - \frac{\alpha}{2}\right)(1 - \chi)\right\}}{\Gamma\left\{\frac{1}{2} - \alpha - \left(\frac{m_\pi^2}{m_p^2} + \frac{1}{4} - \frac{\alpha}{2}\right)(1 - \chi)\right\}} \chi \, dx
\]

(2.17)

eq (2.12) gives

\[
F^\pi(s) = \frac{A}{\Gamma^{-1}(1 - \alpha(s)) + i \beta \phi_\pi(s) \mathcal{H}(\chi(s))}.
\]

(2.18)

The condition for elastic unitarity is satisfied by taking for \(\phi_\pi\) the phase space factors \(\rho_\pi(s) = \frac{4m_\pi^2}{s}\). In order to remove the spurious singularity at \(s = 0\) an additional factor which does not destroy elastic unitarity is required. Since such an additional factor will change the real part of the denominator in eq. (2.18), it must vanish at the resonance positions if these are to remain at the positive integer values of \(\chi\) with the correct physical masses. However, since in the energy range for which this model is proposed the only resonance occurring is the \(\rho\)-meson, it is enough to impose this last requirement only at the \(\rho\)-mass. We shall take for \(\phi_\pi(s)\) the following function:

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\[
\phi_\pi(s) = \rho_\pi(s) - i \psi(s) + i \Theta(s-4m_\pi^2) \left\{ 1 + \gamma(s-m_\rho^2) \right\} \frac{s-4m_\pi^2}{m_\rho^2-4m_\pi^2} \psi(m_\rho^2)
\]

where

\[
\psi(s) = \frac{2}{\pi} \rho_\pi(s) \log \frac{\sqrt{s} + \sqrt{s-4m_\pi^2}}{2m_\pi}
\]

and \( \gamma \) is a constant related to the width \( \Gamma_\rho \) by

\[
\gamma = \frac{2\beta m_\rho \rho_\pi(m_\pi^2) H(1) + \Gamma_\rho}{2\beta m_\rho^2 \psi(m_\pi^2) H(1) \Gamma_\rho} + \frac{1}{\psi(m_\pi^2) \frac{d\psi}{ds}}_{s=m_\rho^2} - \frac{1}{m_\rho^2-4m_\pi^2}
\]

This choice of \( \phi_\pi(s) \) satisfies the elastic unitarity condition, removes the spurious singularity (of \( \rho_\pi(s) \)) at \( s = 0 \) and ensures that the \( \rho \)-resonance has the correct mass and spin. Eq. (2.18) for \( F_\pi(s) \) then becomes

\[
F_\pi(s) = \frac{A}{\Gamma^{-1}(1-\alpha) + \beta H(\alpha) \left[ \psi(\alpha) - \Theta(s-4m_\pi^2) \right] \left( s-4m_\pi^2 \right) \gamma(m_\rho^2) + \gamma \beta_\pi(s) H(\alpha)} \]

One may immediately verify from this equation that \( F_\pi(s) \) is real for \( s < 4m_\pi^2 \) and has the required discontinuity across the cut on \( s > 4m_\pi^2 \) to satisfy elastic unitarity. The normalization condition \( F_\pi(0) = 1 \) gives

\[
A = \Gamma^{-1}(\alpha) + \frac{2}{\pi} H(\alpha) \beta
\]

Eq. (2.22) therefore determines the form factor \( F_\pi(s) \) in terms of a single parameter \( \beta \) when the physical values of \( m_\pi, m_\rho \) and \( \Gamma_\rho \) known. We shall assume that these are known from strong interactions, consistent with our insertion of the Veneziano amplitude as a strong interaction K-matrix.
III. RESULTS

The following values for the physical parameters were used:

\[
\begin{align*}
  m_K &= 0.139 \text{ GeV} \\
  m_p &= 0.765 \text{ GeV} \\
  \Gamma_p &= 0.110 \text{ GeV}
\end{align*}
\]  

(3.1)

A curve for \( H(\omega) \) in the range used in the calculation was plotted. This is shown in Fig.1. The change of variable to the centre-of-mass energy is made by (2.14).

The value of the parameter \( \beta \) was determined by taking as input a single experimental point \(^*)^{12} \)

\[
|F_\pi(s)|^2_{s=m_p^2} = 49
\]  

(3.2)

Eq. (3.2) and the normalization condition (2.23) then give for \( A \) and \( \beta \) the values

\[
\begin{align*}
  A &= 0.58 \\
  \beta &= 0.61
\end{align*}
\]  

(3.3)

Using eqs (3.1) and (3.3) in eq. (2.21) we get \( \gamma = -3.16 \text{ GeV}^{-2} \). The functions \( |F_\pi(s)|^2, F_\pi(s) \) were then plotted for \( s > 0 \) and \( s < 0 \), respectively. Agreement with experiment is very good. For \( s > 0 \) the \( \rho \)-resonance is well reproduced. We could, of course, have satisfied the peak value of Augustin et al. \(^{12}\) by using their measurement in (3.2) instead of the average figure of 49. For \( s < 0 \) a marked improvement over the simple \( \rho \)-pole dominance is obtained. For \( s \approx -0.4 \text{ (GeV)}^2 \), however, this model seems to have a minimum beyond which it should not be expected to give reliable results.

Our conclusion is therefore that elastic unitarity effects on the pion electromagnetic form factor are well described by a Veneziano formula for the \( \pi\pi \) strong interaction \( K \)-matrix.

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* The value quoted in eq. (3.2) is some average of the experimental values at \( s = m_p^2 \).
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12) The experimental results are those of:
Fig. 1

$H(\alpha)$ plotted against $\alpha$.
Fig. 2
\[ F_\pi(s) \text{ plotted against } s \text{ for } s < 0 \text{ from eq. (2.22).} \]