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INFINITIES IN EINSTEIN'S GRAVITATIONAL THEORY

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ABSTRACT

By means of a recently developed technique for computing with non-polynomial Lagrangians one can demonstrate that the ultraviolet divergences of the gravitational interaction are limited to self-mass, self-charge and graviton-graviton scattering.

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The Einstein Lagrangian

$$L = \frac{1}{\kappa^2} \sqrt{-g}\, g^{\mu\nu} (\Gamma^{\lambda}_{\mu\rho} \Gamma^{\rho}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu} \Gamma^{\rho}_{\lambda\rho}) \tag{1}$$

is a non-polynomial function when expressed in terms of the contravariant components, $g^{\mu\nu}$, of the metric tensor. The covariant components, $g_{\mu\nu}$, which enter the expression for $g = \det g_{\alpha\beta}$ and the Christoffel symbol

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu}) \tag{2}$$

can be given as a ratio of two polynomials in $g^{\mu\nu}$.

Conventionally[1], the Lagrangian (1) is linearized by defining a new field variable, $h_{\mu\nu}$, by

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu} \tag{3}$$

where κ denotes the gravitational constant and $\eta^{\mu\nu}$ the Minkowski metric. Expanding in powers of κ one finds

$$L = L_0 + L_{int} (\kappa, h) \tag{4}$$

where[2]

$$L_0 = -\frac{1}{4} (\partial_{\mu} h^{\lambda\rho} \partial_{\mu} h^{\lambda\rho} - 2\partial_{\rho} h^{\lambda\mu} \partial_{\lambda} h^{\rho\mu} - \partial_{\lambda} h^{\mu\mu} \partial_{\lambda} h^{\rho\rho} + 2\partial_{\mu} h^{\mu\lambda} \partial_{\lambda} h^{\rho\rho}) \tag{5}$$

and

$$L_1 = \kappa h^3 + \kappa^2 h^4 + \ldots \tag{6}$$

Here we have collected all the bilinear terms, which are independent of κ, in the free Lagrangian L_0 while, for L_{int} (which is a function of the h's and their derivatives) an expansion in powers of

κ is implied. Perturbation calculations can be made in the usual way with vertices determined by (6) and a free propagator obtained from (5) together with a gauge condition. In a gauge of the Landau type[3], for example, one might use the propagator

$$G^{\kappa\lambda,\mu\nu}(p) = \frac{1}{p^2+i\epsilon} \left\{ \frac{1}{2}(d_{\kappa\mu}d_{\lambda\nu} + d_{\kappa\nu}d_{\lambda\mu}) - \frac{2}{3}d_{\kappa\lambda}d_{\mu\nu} - d_{\kappa\lambda}e_{\mu\nu} - e_{\kappa\lambda}d_{\mu\nu} - 3e_{\kappa\lambda}e_{\mu\nu} \right\} \qquad (7)$$

where $e_{\mu\nu} = p_\mu p_\nu / p^2 = \eta_{\mu\nu} - d_{\mu\nu}$. In succeeding orders, however, one finds integrals which diverge more and more virulently and calculations become pointless.

Recently we have studied a treatment of non-polynomial Lagrangians with the S-matrix expressed as a power series not in κ but in the non-polynomial L_{int} itself[4]. The method used is the one pioneered by Efimov[5] and Fradkin[6] who explicitly exhibit S-matrix elements as closed expressions to a given order in L_{int} . One can examine these expressions for possible ultraviolet infinities and the results of Efimov and Fradkin can be stated in the following form.

Associate with the field $h^{\mu\nu}$ a factor M and with each derivative an additional power of M reflecting the high frequency behaviour, i.e., $h^{\mu\nu} \sim M$, $\partial_\lambda h^{\mu\nu} \sim M^2$, etc., for $M \to \infty$. Consider the limiting behaviour of L_{int} as $M \to \infty$. If $L_{int} \sim M^4$ then the theory possesses the conventional infinities of self-mass and self-charge and may be called normal. If $L_{int} \sim M^n$, $n > 4$, the infinities proliferate in the manner of conventionally unrenormalizable theories. These we call <u>abnormal</u>. On the other hand, if $L_{int} \sim M^n$ with $n < 4$ then we have a <u>supernormal</u> theory with fewer infinities than in a normal theory. If $n < 2$ there are no ultraviolet infinities.

Let us consider the Lagrangian (1) from this point of view. Clearly if $g^{\mu\nu} \sim M$ then the covariant components behave like $g_{\mu\nu} \sim 1/M$ while $\sqrt{-g} \sim 1/M^2$ and, therefore, $L \sim M$. However, since $L_0 \sim M^4$ it follows that $L_{int} \sim M^4$ also. This shows that in the Efimov-Fradkin treatment Einstein's gravitational theory would be normal.

The convergent behaviour of $g_{\mu\nu}$ and of $\sqrt{-g}$ gives rise to beneficial effects in the interaction of gravitation with matter. The coupling of gravitation to a real scalar field, ϕ, for example, is represented by the Lagrangian

$$\frac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$$

the kinetic part of which goes like M^3 rather than M^4 as it would in the absence of gravitation. It can even happen that interactions which are unrenormalizable without gravitation become normal in its presence [7].

There is a special complication in forming perturbation series with the massless graviton propagator (7). It was first shown by Feynman [8] and later generalized by De Witt [9], Faddeev and Popov [10] and by Mandelstam [11] that in order to preserve S-matrix unitarity one must supplement the purely graviton graphs with closed loops of a (fictitious) zero-mass vector fermion. These can be accounted for by adding to the Lagrangian (1) an effective term

$$\frac{1}{2} \sqrt{-g}\, g^{\mu\nu} \partial_\mu A^*_\alpha \partial_\nu A_\alpha \tag{8}$$

with the stipulation that no external A-lines are admitted. From (8) we can separate a free Lagrangian for the A-field from which is obtained the propagator

$$G_{\mu\nu} = \frac{\eta_{\mu\nu}}{p^2 + i\epsilon} \quad . \tag{9}$$

Accordingly we can associate the asymptotic factor M with A_μ and observe that the effective Lagrangian (8) goes like M^3 and so does not disturb the normality of the gravitational theory.

In this note we have not given the detailed rules for writing down S-matrix elements. Their form follows the pattern set out in Ref. 4. The main purpose of this note is to draw the important distinction between the contravariant field $g^{\mu\nu}$, which for ultraviolet infinities behaves like M when $M \to \infty$, and its covariant counterpart $g_{\mu\nu}$ which behaves like $1/M$ in our non-polynomial calculational technique.

REFERENCES AND FOOTNOTES

1) There are at least two conventions. Gupta, for example, in Phys. Rev. 172, 1303 (1968), uses the variable $\gamma^{\mu\nu}$ defined by

$$\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} + \kappa\gamma^{\mu\nu} \quad .$$

Mandelstam, on the other hand, uses $g^{\mu\nu} = \eta^{\mu\nu} + \kappa\phi^{\mu\nu}$.

2) For brevity we shall implicitly contract indices with the Minkowskian metric $\eta_{\mu\nu}$. Thus, for example,

$$h^{\mu\mu} = \eta_{\mu\nu} h^{\mu\nu} \text{ and } \partial_\mu \partial_\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu \quad ,$$

where $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+---)$.

3) This gauge is specified by the subsidiary condition $\partial_\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h_{\mu\mu} = 0$ which is the infinitesimal form of the Fock-de Donder condition for harmonic co-ordinates $\partial_\mu(\sqrt{-g}\, g^{\mu\nu}) = 0$. Harmonic co-ordinates are well adapted to the Minkowski nature of the metric at infinity.

4) R. Delbourgo, Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/69/17.

5) G. V. Efimov, Soviet Phys.-JETP 17, 1417 (1963); Nuovo Cimento 32, 1046 (1964); Nucl. Phys. 74, 657 (1965).

6) E. S. Fradkin, Nucl. Phys. 49, 624 (1963); 76, 588 (1966).

7) For example, the conventional interaction $\phi^2 \partial_\mu \phi \partial_\mu \phi \sim M^5$ which with the inclusion of gravity reads $\sqrt{-g}\, g^{\mu\nu} \phi^2 \partial_\mu \phi \partial_\nu \phi$ becomes $\sim M^4$. The more interesting cases of this phenomenon involve fermions (e.g., the pseudovector nucleon-meson interaction). For

a proper treatment these necessitate the introduction of a vierbein field. We hope to discuss such a formalism in a future publication. The conjecture that the gravitational interaction may suppress infinities in conventional field theory has been made before by de Witt (see Ref. 9) and Deser (Rev. Mod. Phys. 29, 417 (1957)) in the context of functional integration methods. In our treatment of the theory this is exhibited explicitly in the (non-polynomial) S-matrix elements.

8) R. P. Feynman, Acta Phys. Polon. 24, 697 (1963).

9) B. S. de Witt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967); 171, 1834 E (1968).

10) L. D. Faddeev and V. N. Popov, Kiev preprint, ITF-67-36 and Phys. Letters 25B, 29 (1967).

11) S. Mandelstam, Phys. Rev. 175, 1604 (1968).