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INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

INFINITIES OF NON-LINEAR AND VECTOR MESON LAGRANGIAN THEORIES

R. DELBOURGO

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J. STRATHDEE

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1. It is interesting to remark that if the chiral SU(2) x SU(2) theory set out in Sec. 5 is regarded as a theory of the pion triplet then one finds upon introducing the electromagnetic interaction by the conventional recipe $\partial_\mu \to \partial_\mu - ie A_\mu$, that the resulting interaction Lagrangian is of order M^0, for example, for the Weinberg parametrization. This would mean that the $\pi^+ - \pi^0$ mass difference is finite. A calculation is under way to compute this number.

2. Subsequent to the writing of the paper a number of related papers have come to our notice. These include

(A)	G.V. Efimov	I	Kiev preprint	ITF 68-52
		II	Kiev preprint	ITF 68-54
		III	Kiev preprint	ITF 68-55.

(Some of the material of I and II has been published in English:
G.V. Efimov, Commun. Math. Phys. 7, 138 (1968),
G.V. Efimov, Commun. Math. Phys. 5, 42 (1967) - the rest is in Russian.)

These papers deal with, among other matters, the problems of causality and unitarity that arise with non-polynomial interactions. Efimov discusses the connection of such interactions with quasi-local Lagrangians for which he has found a set of rules for evaluating the perturbation series in a manner compatible with the usual unitarity, relativistic invariance and analyticity requirements. If this programme has been validly carried through "this is a remarkable result, deserving of careful study to be sure that no hidden troubles have been overlooked".*)

*) A.S. Wightman, Proceedings of 14th International Conference on High-Energy Physics, Vienna, 1968.

(B) H.M. Fried, Difficulties with nonlinear (e.g. chiral) dynamics, Nuovo Cimento 52A, 1333 (1967);

H.M. Fried, Correlation between transcendental and polynomial Lagrangians, Phys. Rev. 174, 1725 (1968).

We should like to acknowledge that Fried was the first to consider using E-F methods for the chiral Lagrangians although he did this in a qualitative way only by ignoring the problems of treating derivatives in the interaction.

The difficulty in the E-F method considered by Fried concerns, in the simplest instance, the possible existence of a singularity of the vacuum matrix element $\langle T(L_{int}(x) L_{int}(y))\rangle$ for spacelike (x-y). Such a singularity would upset the unitarity and causality of the theory. However, it seems to us that this problem has been dealt with in a convincing way by Efimov (JETP 17, 1417 (1963)) where he shows that this matrix element considered as a function of the causal propagator, $\Delta(x-y)$, must have an essential singularity at $\Delta = 0$. Such functions do not suffer from the Fried objection. In fact, the essential singularity at $\Delta = 0$ is a basic feature of the amplitudes calculated by the E-F procedure and distinguishes them from amplitudes obtained by conventional perturbation methods.

3. We have found that in the treatment of the massive Yang-Mills theory, integrations over the auxiliary variables u_λ and v_λ do not give the behaviour $\varphi \sim M^2$ uniformly for all matrix elements; the behaviour of φ appears to vary in a complicated manner with the number of external A lines. To make the normality of the theory explicit a further change of variables appears to be necessary. This we shall discuss in detail in a later note dealing also with weak interactions.

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ABSTRACT

The unitarity and causality preserving technique for summing perturbation contributions introduced by Efimov and Fradkin is extended and applied to non-linear (chiral) type Lagrangian theories and also to theories of massive Yang-Mills and neutral pseudovector mesons. It is shown that the only likely infinities in these theories are those associated with self-mass and self-charge. The same conclusions appear to hold for weak interactions mediated by intermediate bosons. Crucial to the treatment of vector meson theories is the use of (purely self-interacting) unitarity-preserving Stückelberg-like auxiliary fields.

INFINITIES OF NON-LINEAR AND VECTOR MESON LAGRANGIAN THEORIES

1. INTRODUCTION

(A) One of the significant recent advances in particle theory has been the formulation of chirally invariant Lagrangian theories[1). These theories have so far been used with reasonable success for predicting low-energy (soft meson) amplitudes in the following way: the interaction Lagrangian - an exponential or rational function of the spin zero meson fields ϕ^i - is expanded as an infinite power series in ϕ^i and then used to evaluate tree diagram [2)] contributions to the amplitudes. Clearly at the next level of sophistication one is interested in the closed loop contributions at which stage two related problems arise.

(i) Since the Lagrangian itself is expressed as an infinite power series, $\mathcal{L}_{int} = \sum_n a_n g^n \phi^n (\partial\phi)^2$, the number of perturbation diagrams in each order n increases (typically) as fast or faster than n! . On any reasonable estimate the perturbation expansion must be a divergent series. For respectable theories like quantum electrodynamics, with Lagrangians which are polynomial in field variables, one has always suspected [3)] that the perturbation expansion provides an asymptotic series in $e^2/\hbar c$; here, with Lagrangians which are themselves infinite series, this behaviour appears a virtual certainty.

(ii) Each of the terms in the expansion of the Lagrangian (terms like $\phi^{n'} (\partial\phi)^2$; $n \geqslant 1$) represents a non-renormalizable interaction in the conventional sense. The ultraviolet infinities of the perturbation expansion therefore get progressively more virulent. On the face of it this is rather surprising, since it is well known that every non-linear theory can be reformulated as a theory of linear group representations [4)] with polynomial Lagrangians together with a certain number of constraints on the fields ϕ^i. Before the imposition of the constraint the theories are renormalizable; if any non-renormalizability occurs, it must arise through the imposition of the constraint.

In this paper we argue that both difficulties (i) and (ii) stem from the same circumstance, namely the expansion of the Lagrangian in a power series of field variables, and that a summation [5], or even a partial summation, of the divergent perturbation series is likely at the same time to reduce the problem of ultraviolet infinities [6].

(B) A basic and much neglected advance was made towards the (partial) summation of perturbation series arising from rational and exponential Lagrangians in a series of papers by Efimov and Fradkin [7] during 1963. Like all summation methods for divergent series the problem of uniqueness of the sum remains unresolved in their technique too. Efimov however has shown that besides satisfying the usual analyticity requirements, the Efimov-Fradkin (E-F) summation method meets the demand of consistency with Landau-Cutkosky unitarity at least for the self-energy and vertex functions. In this paper we wish to apply the E-F method for summing the perturbation series of non-linear Lagrangians of the chiral variety and for the related problem of theories with gauge vector mesons. We wish to show that the infinities in such theories are no worse after summation than those encountered in conventionally renormalizable theories. Central to our discussion is the result which states that the degree of ultraviolet infinity of E-F sums depends on the growth of $\mathcal{L}_{int}(\phi)$ as $\phi \to \infty$ for non-linear theories just as for usual linear theories. To be more specific, the result (extended below to include derivative couplings so essential in non-linear chiral Lagrangians) can be stated as follows:

(i) Assign to each scalar field $\phi(x)$ (with the propagator $< T\{\phi(x)\ \phi(0)\} > = \Delta(x) \approx x^{-2}$ as $x^2 \to 0$) the "singularity" behaviour $\phi(x) \approx 1/\sqrt{x^2} \approx 1/x$ as $x \to 0$ or equivalently $\phi \sim M$ with $M \to \infty$.

(ii) Likewise assign the behaviours

$$\partial_\mu \phi(x) \underset{x \to 0}{\approx} 1/x^2 \quad \text{or} \quad \partial_\mu \phi \underset{M \to \infty}{\sim} M^2$$

$$\psi(x) \underset{x \to 0}{\approx} 1/x^{3/2} \quad \text{or} \quad \psi \underset{M \to \infty}{\sim} M^{3/2} \; ; \; \psi = \text{spin } \tfrac{1}{2} \text{ field}$$

$$U_\mu(x) \underset{x \to 0}{\approx} 1/x^2 \quad \text{or} \quad U_\mu \underset{m \to \infty}{\sim} M^2 \; ; \; U = \text{spin 1 field} \quad .$$

A theory is expected to be renormalizable, with only a few types of integrals that are ultraviolet infinite, if $\mathcal{L}_{int} \underset{M \to \infty}{\sim} M^4$.

This criterion applies equally to integrals in conventional polynomial Lagrangians like $\mathcal{L}_{int} = g\,\phi^4$ or $g\,\bar{\psi}\,\psi\,\phi$, as well as to E-F sums in theories with Lagrangians like $g\,\phi^2(\partial\phi)^2/(1+\phi^2)$. We shall call such theories normal. Theories like $\mathcal{L}_{int} = g\,\phi^3$ or $g(\partial\phi)^2/(1+\phi^2)$ which behave like M^3 or M^2 or lower ($\mathcal{L}_{int} \sim M^n$; $n < 4$) will be called supernormal. All theories which behave worse than ϕ^4 , i.e., for which $\mathcal{L}_{int} \sim M^n$, $n > 4$, will be called abnormal. For supernormal theories there is the attractive possibility that when $n < 2$ all integrals including those for self-mass and self-charge are finite.

(C) In this paper we shall consider in detail the problem of infinities in two types of Lagrangian theories.

(i) Non-linear Lagrangians of the chiral type which we show are normal but not supernormal.

(ii) Vector meson Lagrangians, particularly the massive Yang-Mills theory. It is well known that the interaction of the spin zero component of the vector field can be shown to be equivalent [8)] to an exponential type of interaction in a Stückelberg formulation of the theory. By adopting a variant of the Stückelberg form studied recently by Veltman and Ghose [9)] and by using equivalence theorems [8)] we prove that theories of neutral pseudovector mesons interacting with nucleons and of massive Yang-Mills fields are indeed <u>normal</u> in the E-F sense with only a few types of integrals in the theories which are possibly infinite [10].

(D) The plan of the paper is as follows: In Sec. 2 we give an outline of the E-F method which has two ingredients, (i) Hori's exponential representation [11] of Wick's normal ordering theorem and (ii) the E-F integral representation [7] of Hori's exponential operator. The power counting rules for estimating over-all ultraviolet infinities of E-F sums is given in Sec. 3. We consider derivative couplings in Sec. 4 and formulate the rules for writing E-F sums in such a manner that the ultraviolet power counting estimate can also be stated here. Sec. 5 contains the application of these results to the non-linear (chiral type) Lagrangians in an SU(2) ⊗ SU(2) symmetric theory. Since equivalence theorems, which state that on-mass-shell S-matrix elements are unaltered by contact transformations in field space, play such a critical role [12], we devote Sec. 6 to a non-rigorous discussion of the circumstances in which such transformations are permissible. Finally in Secs. 7 and 8 we study vector meson interactions and their ultraviolet infinities, both for a neutral pseudovector theory and for the Yang-Mills case.

The companion paper by Koller (in preparation) is concerned with explicit computations of E-F sums for derivative coupling Lagrangians and the techniques needed for their evaluation. In a subsequent paper we shall consider weak interaction Lagrangians to show that the methods of this paper carry over to achieve in a straightforward manner a weak interaction Lagrangian theory with just the <u>normal</u> infinities.

2. THEORIES WITH NON-DERIVATIVE COUPLINGS

We summarize below the steps needed to arrive at the Efimov-Fradkin (E-F) representation [7] of the S-matrix, assuming that the interaction Lagrangian contains no field derivatives. (In Sec. 4 we shall extend the techniques to cover situations where derivatives are encountered.) An illustrative example is presented to demonstrate the power of the E-F method.

Step 1. Begin with the standard perturbation expansion of the S-matrix $S = \sum_N \frac{i^N}{N!} S^{(N)}$, where

$$S^{(N)} = g^N \int d^4z, \dots d^4z_N \; T[L\{\varphi(z_1)\} \dots L\{\varphi(z_N)\}] \qquad (1)$$

and we are supposing in this section that

$$\mathcal{L}_{int} = g\, L\{\varphi(x)\} \qquad (2)$$

where φ denotes a real scalar field.

The further expansion of the S-matrix into normal Wick products can be compactly expressed through Hori's functional operator [11] as follows,

$$S^{(N)} = g^N \int d^4z_1 \dots d^4z_N \exp\left(\frac{1}{2}\int d^4x_1 d^4x_2 \Delta(x_1-x_2) \frac{\delta^2}{\delta\varphi(x_1)\delta\varphi(x_2)}\right)$$
$$[L\{\varphi(z_1)\} \dots L\{\varphi(z_N)\}] \qquad (3)$$

where $\Delta(x_1-x_2)$ denotes the bare causal propagator for the scalar field φ. This formula can be simplified to read

$$S^{(N)} = g^N \int d^4z_1 \dots d^4z_N \exp\left(\sum_{i,j=1}^{N} \Delta_{ij} \frac{\partial^2}{\partial\varphi_i \partial\varphi_j}\right)\left[L\{\varphi_1\} \dots L\{\varphi_N\}\right]_{\substack{\varphi_k = \varphi^{ext}(z_k) \\ \Delta_{k\ell} = \Delta(z_k - z_\ell)}} \qquad (4)$$

Here $\varphi_k = \varphi^{ext}(z_k)$ is the wave function of any external particle which may be acting at the point z_k . One may rewrite (4) in a form where these external wave functions are exhibited separately by writing

$$S^{(N)} = \int d^4x_1 \ldots d^4x_n : \varphi(x_1) \ldots \varphi(x_n) : S^{(N)}(x_1, \ldots, x_n) \qquad (5)$$

where the n-point function in the N-th order equals

$$S^{(N)}(x_1, \ldots, x_n) = \langle 0 | \frac{\delta^n}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)} S^{(N)} | 0 \rangle$$

$$= g^N \int d^4z_1 \ldots d^4z_N \sum_{\substack{m_i \\ \Sigma m_i = n}} \delta^{m_1}(x-z_1) \ldots \delta^{(m_N)}(x-z_N) \, S_{m_1 \ldots m_N}(\Delta) \qquad (6)$$

with

$$\delta^{m_1}(x-z_1) \equiv \delta(x_{i_1}-z_1)\, \delta(x_{1_2}-z_1) \ldots \delta(x_{i_{m_1}}-z_1), \text{ etc.,} \qquad (7)$$

and

$$S_{m_1 \ldots m_N}(\Delta) \equiv \exp\left[\frac{1}{2}\sum_{ij} \Delta_{ij} \frac{\partial^2}{\partial\varphi_i \partial\varphi_j}\right] \left(\frac{\partial}{\partial\varphi_1}\right)^{m_1} \ldots \left(\frac{\partial}{\partial\varphi_N}\right)^{m_N} \left[L(\varphi_1) \ldots L(\varphi_N)\right]_{\varphi=0} . \qquad (8)$$

The vacuum graphs are given in their entirety by

$$S = \sum_N \frac{(ig)^N}{N!} \int d^4z_1 \ldots d^4z_N \, S_{00 \ldots 0}(\Delta) \; , \qquad (9)$$

$$S_{00 \ldots 0}(\Delta) = \exp\left(\frac{1}{2}\sum_{ij}^{N} \Delta_{ij} \frac{\partial^2}{\partial\varphi_i \partial\varphi_j}\right) L(\varphi_1) \ldots L(\varphi_N) \; , \qquad (10)$$

while the two-point (self-energy) graphs are completely described by

$$S(x_1, x_2) = \sum_{N \geq 2} \frac{(ig)^N}{N!} \int d^4z_1 \ldots d^4z_N \left[\begin{array}{l} \{\delta(x_1-z_1)\delta(x_2-z_1) S_{200 \ldots}(\Delta) + \\ + \delta(x_1-z_2)\delta(x_2-z_2) S_{020 \ldots}(\Delta) + \ldots \} + \\ \{(\delta(x_1-z_2)\delta(x_2-z_2) + \delta(x_1-z_2)\delta(x_2-z_1)) \cdot \\ \cdot S_{110 \ldots}(\Delta) + \ldots \} \end{array} \right] \qquad (11)$$

with


Fig. 1(a)

$$S_{200}\ldots(\Delta) = \exp\left[\frac{1}{2}\sum \Delta \frac{\partial^2}{\partial\varphi\partial\varphi}\right]\frac{\partial^2}{\partial\varphi_1^2} L(\varphi_1)\ldots L(\varphi_N)\Big|_{\varphi=0}$$


Fig. 1(b)

$$S_{110}\ldots(\Delta) = \exp\left[\frac{1}{2}\sum \Delta \frac{\partial^2}{\partial\varphi\partial\varphi}\right]\frac{\partial^2}{\partial\varphi_1\partial\varphi_2} L(\varphi_1)\ldots L(\varphi_N)\Big|_{\varphi=0}, \text{ etc.} \qquad (12)$$

Step 2. Give a simple integral representation of Hori's exponential operator by making use of the E-F lemma[7]:

$$\exp\left[\Delta\frac{\partial^2}{\partial\varphi\partial\varphi'}\right] F(\varphi\ \varphi')$$
$$= \frac{1}{\pi}\int d^2u \exp\left[-|u|^2 + uc\frac{\partial}{\partial\varphi} + u^*c'\frac{\partial}{\partial\varphi}\right] F(\varphi,\varphi')$$
$$= \frac{1}{\pi}\int d^2u \exp\left[-|u|^2\right] F(\varphi + uc\ ,\ \varphi' + u^*c') \qquad (13)$$

with the parameters c and c' constrained to satisfy $cc = \Delta$, but otherwise arbitrary. [They can be chosen to suit one's purpose. Thus $c = c' = \sqrt{\Delta}$ would correspond to the most symmetric choice, one we often make; $c = \Delta$, $c' = 1$ to the most asymmetric choice. In any event the final result cannot explicitly involve any square roots of Δ and must only depend on the product $cc' = \Delta$.] Since the final expression on the right of (13) involves as integrand the function F shifted from its value at φ, φ' to $\varphi + uc$, $\varphi' + u^*c'$ we shall call this the "exponential shift" lemma.

Applying the lemma to the N-th order S-matrix by introducing complex variables u_{ij}, c_{ij} between every two pairs of points ij, one has the representation

$$\exp\left(\frac{1}{2}\sum_{ij}\Delta_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}\right)[L\{\varphi_1\}\ldots L\{\varphi_N\}]$$
$$= \prod_{i\geqslant j}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\exp\left(-\frac{1}{2}\sum_{ij}|u_{ij}|^2\right)\left[L\{\varphi_1 + \sum_k c_{1k}u_{1k}\}\ldots L\{\varphi_N + \sum_k c_{Nk}u_{Nk}\}\right] \qquad (14)$$

with

$$c_{ij}\, c_{ji} = \Delta_{ij} \qquad \text{(no summation over ij)}$$

and

$$u_{ij} = u^*_{ji} \tag{15}$$

(If we make the restriction $i \neq j$ above, we exclude all graphs in which $\Delta(0)$ appears through a contraction within <u>each</u> $L\{\varphi\}$. This amounts to assuming that $L\{\varphi\}$ is already normally ordered.)

As an application of the lemma consider all vacuum graphs of order g^N. These are given by:

$$S_{00\ldots0}(\Delta) = \prod_{ij}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\exp\left(-\sum|u_{ij}|^2\right)L\left(\sum_k c_{1k}u_{1k}\right)\ldots L\left(\sum_k c_{Nk}u_{Nk}\right) \tag{16}$$

Likewise the self-energy graphs of order g^N are given in terms of

$$S_{20\ldots0}(\Delta) = \prod_{ij}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\exp\left(-\sum|u_{ij}|^2\right)L''\left(\sum_k c_{1k}u_{1k}\right)\ldots L\left(\sum_k c_{Nk}u_{Nk}\right)$$

$$S_{110\ldots}(\Delta) = \prod_{ij}\left(\frac{1}{\pi}\int d^2u_{ij}\right)\exp\left(-\sum|u_{ij}|^2\right)L'\left(\sum_k c_{1k}u_{1k}\right)L'\left(\sum_k c_{2k}u_{2k}\right)\ldots \tag{17}$$

and so on. Here $L' \equiv \frac{\partial L}{\partial\varphi}$, $L'' = \frac{\partial^2 L}{\partial\varphi^2}$, etc.

To see how this works in practice take the model for which $gL(\varphi) = g\varphi^4/(1+\lambda^2\varphi^2)$. The power of the technique, which explicitly displays sums of perturbation series to each order in g, is already apparent since <u>all orders</u> in λ^2 are automatically taken into account by the E-F expressions. Thus to second order in g and all orders in λ^2 the vacuum contribution equals

$$g^2 S_{00}(x_1,x_2) = g^2\int\frac{d^2u}{\pi}\, e^{-|u|^2}\,\frac{c^4u^4}{1+\lambda^2c^2u^2}\;\;\frac{c'^4u^{*4}}{1+\lambda^2c'^2u^{2*}}.$$

where $cc' = \Delta(x_1 - x_2)$. Likewise the two relevant self-energy terms to second order in g but all orders in λ^2 are:

$$g^2 S_{20} = g^2 \int \frac{d^2u}{\pi} e^{-|u|^2} \frac{d^2}{c^2 du^2} \left(\frac{c^4 u^4}{1 + \lambda^2 c^2 u^2} \right) \frac{c'^4 u^{*4}}{1 + \lambda^2 c'^2 u^{2*}}$$

and

$$g^2 S_{11} = g^2 \int \frac{d^2u}{\pi} e^{-|u|^2} \frac{d}{c du} \left(\frac{c^4 u^4}{1 + \lambda^2 c^2 u^2} \right) \frac{d}{c' du^*} \left(\frac{c'^4 u^{*4}}{1 + \lambda^2 c'^2 u^{*2}} \right) .$$

The simplification of these integrals rests on the pair of relations,

$$\frac{1}{\pi} \int d^2u \, u^{*m} u^n f(|u|^2) = \delta_{nm} \int_0^\infty d\xi \, \xi^n f(\xi)$$

$$\frac{1}{\pi} d^2u \frac{f(|u|^2)}{(1 + \alpha n^2)(1 + \beta u^{*2})} = \int_0^\infty d\xi \frac{f(\xi)}{1 - \alpha\beta\xi^2}$$

and derivatives thereof. Thus we find, as expected, that the integrals only involve the product $cc' = \Delta$ and not the parameters c and c' separately. Explicitly,


Fig. 2

$$g^2 S_{00} = g^2 \int_0^\infty d\xi \frac{\Delta^4 \xi^4 e^{-\xi}}{1 - \lambda^4 \Delta^2 \xi^2} \qquad (18)$$

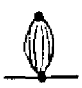
Fig. 3(a)

$$g^2 S_{20} = -g^2 \int_0^\infty d\xi \, \lambda^2 \xi^2 \Delta^4 e^{-\xi} \left[\frac{12}{1 - \lambda^4 \Delta^2 \xi^2} + \frac{10}{(1 - \lambda^4 \Delta^2 \xi^2)^2} + \frac{8}{(1 - \lambda^4 \Delta^2 \xi^2)^3} \right]$$


Fig. 3(b)

$$g^2 S_{11} = g^2 \int_0^\infty d\xi \, \Delta^3 \xi^3 e^{-\xi} \left[\frac{8}{1 - \lambda^4 \Delta^2 \xi^2} + \frac{4}{(1 - \lambda^4 \Delta^2 \xi^2)^2} + \frac{4}{(1 - \lambda^4 \Delta^2 \xi^2)^3} \right]$$

(19)

In particular, when we set $\lambda = 0$ we recover the φ^4 perturbation theory results, viz:

$$S_{00} = 4! \, \Delta^4 \quad , \quad S_{20} = S_{02} = 0 \quad , \quad S_{11} = 4(4!) \Delta^3 .$$

We shall return to the ultraviolet properties of these integrals after we have discussed the question of infinities.

3. ULTRAVIOLET INFINITIES OF THE E-F SUMS

Physically we are only concerned with S-matrix elements in momentum space[13)] i.e., the Fourier transforms

$$\tilde{S}(p) = \prod \left(\int d^4x \, e^{ipx} \right) S(\Delta(x_{ij})) \quad . \tag{20}$$

On account of the causal character of the propagators $\Delta(x)$ the task of defining the x-space contours of integration in integrals like (20) is not trivial. As is well known[14)], the light cone singularity of $\Delta(x)$ is given by the following expression:

$$4\pi i \Delta(x;\mu) = \delta(x^2) - \frac{\mu\theta(x^2)}{2\sqrt{x^2}} J_1(\mu\sqrt{x^2})$$

$$+ i\mu \left[\frac{\theta(x^2)}{2\sqrt{x^2}} N_1(\mu\sqrt{x^2}) + \frac{\theta(-x^2)}{\pi\sqrt{-x^2}} K_1(\mu\sqrt{-x^2}) \right.$$

$$= \delta(x^2) - \frac{i}{\pi x^2} - \frac{\mu^2}{4} \left[\theta(x^2) - \frac{2i}{\pi} \log\left(\frac{1}{2}\mu\sqrt{|x^2|}\right) \right]$$

$$+ O\left(\sqrt{|x^2|} \log |x^2|\right) \quad . \tag{21}$$

The crucial part of Efimov's work is a method of carrying out the x-space integrals, with the demonstration that one may define them so as to preserve the unitarity of the S-matrix in the perturbation sense, i.e., in the expansion of $S(\Delta)$ in powers of Δ . Efimov's procedure consists in concentrating firstly on the Euclidean or Symanzik region of the external momenta[15)]. For this region of p-space it may be shown[7)] that x-space contours of integration can be rotated from the Lorentz into the Euclidean region of x . For other regions of p-space Efimov makes suitably defined continuations from the Symanzik region. In this paper we are only concerned with the ultraviolet infinities associated with integrals (20) so for our purpose it is sufficient to remain in the Symanzik region - or, to make matters simpler, on its edge where all external momenta p_μ are zero. Thus we examine the infinities associated with the Euclidean x-space integrals $(x_{ij}^2 < 0)$

$$\tilde{S}(0) = \int \pi \, d^4x \, S(\Delta) .$$

where the Δ assume real values. A naive power count of the over-all[16] infinities can be made by considering the appropriate proper diagrams and retaining the most singular parts of all the propagators Δ. Setting the lower cut-off $x^2 = M^{-2}$ $(M^2 \to \infty)$ to all x-space integrations it is evident that we can associate a factor M to each $\sqrt{\Delta}$ that occurs; and since what in fact determines the infinities is the powers of $L(\varphi \approx \sqrt{\Delta})$, we may easily estimate the over-all infinity to be expected by setting $\varphi = M$ in $L(\varphi)$ and letting $M \to \infty$.

Consider therefore an n-point function and follow the Dyson power counting procedure[17]. Suppose that $L(\varphi = M)$ behaves as M^{ν} for large M. The integrand of $S_{m_1 \ldots m_N}(\Delta)$ in (14) contains the term (putting $c_{ij} = c_{ji} = \sqrt{\Delta_{ij}}$ for simplicity)

$$\Delta^{-\frac{1}{2}\sum m_i} \left[L(\sqrt{\Delta}\, u) \right]^N \sim M^{-n+N\nu}$$

where n denotes the number of external lines and N the order of the graph (number of "vertices"). The singularity produced at $x^2 = 0$ $(M \to \infty)$ is compensated by $4(N-1)$ integrations, 4 integrations being omitted because the integrand is independent of the over-all centre of mass co-ordinates. Therefore

$$\int (d^4x)^{N-1} \, S(\Delta) \sim M^{-4(N-1)} \, M^{-n+N\nu} . \qquad (22)$$

If the integral is to be regular in the limit $M \to \infty$

$$N(4-\nu) + n > 4 .$$

The theory then will be renormalizable in the conventional sense (normal or supernormal) if $\nu \leqslant 4$. In particular (up to possible logarithmic infinities) if $\nu \leqslant 2$ even self-mass and self-charge are superficially finite.

We return to the example above to see that this naive infinity count is sensible. The self-energy contributions (19) to second order in g (but all orders in λ) read in momentum space,

$$\tilde{S}(p) = 2g^2 \int S_{20}(\Delta(x))\, d^4x + 2g^2 \int S_{11}(\Delta(x))\, e^{ipx}\, d^4x \quad . \tag{23}$$

Taking $p^2 \leqslant 0$ (Euclidean region) the integrals reduce to

$$i\tilde{S}(p) = 4\pi^2 g^2 \int_{1/M}^{\infty} dr \cdot r^3\, S_{20}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right) + \frac{8\pi^2 g^2}{\sqrt{-p^2}} \int_{1/M}^{\infty} dr \cdot r^2\, J_1(r\sqrt{-p^2})\, S_{11}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right) \tag{24}$$

where we cut off the integrations at $r = M^{-1}$ in order to estimate the infinity as $x^2 \to 0$. Since the ultraviolet behaviour of the integral is independent of the value of p^2 we set this equal to 0,

$$i\tilde{S}(0) = 4\pi^2 g^2 \int_{1/M}^{\infty} dr \cdot r^3 \left[S_{20}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right) + S_{11}\left(\frac{\mu K_1(\mu r)}{4\pi^2 r}\right)\right] \quad .$$

As $r \to 0$, $\frac{\mu K_1(\mu r)}{4\pi^2 r} \to \frac{\mu^2}{8\pi^2} \log(\frac{1}{2}\mu r) + \frac{1}{4\pi^2 r^2}$

so the lethal infinities at the lower limit are obtained, using (19), as

$$\lim_{M\to\infty} 4\pi^2 g^2 \int_{1/M} dr \cdot r^3 \left[S_{20}\left(\frac{1}{4\pi^2 r^2}\right) + S_{11}\left(\frac{1}{4\pi^2 r^2}\right)\right]$$

$$= \lim_{M\to\infty} 4\pi^2 g^2 \int_{1/M} dr \,.\, r^3 \left[\frac{-12}{\lambda^2\, 16\pi^4 r^4} - \frac{8}{\lambda^4}\right] \sim \frac{3g^2}{\pi^2 \lambda^2} \log M \quad , \tag{25}$$

i.e., we meet a logarithmic infinity at most. A naive power count (up to these logarithms) would have agreed with this result since when we set $\Delta = M^2 \to \infty$

$$\int d^4x\, [S_{20}(\Delta) + S_{11}(\Delta)] \sim \frac{1}{M^4}\, [S_{20}(M^2) + S_{11}(M^2)] \to -\frac{12}{\lambda^2} \quad . \tag{26}$$

An interesting feature of the result is the pole $1/\lambda^2$ of the "leading infinity" in the λ^2-plane. This is not entirely surprising in view of the fact that for $\lambda = 0$ we must necessarily recover the conventional quadratic perturbation infinity.

4. THEORIES WITH DERIVATIVE COUPLINGS

In this section we extend the summation technique to cases where $\mathcal{L}_{int}$ contains derivatives of the φ field,

$$\mathcal{L}_{int} = g\, L(\varphi, \partial_\mu \varphi) \quad . \tag{27}$$

It is common knowledge that for such situations the Hamiltonian contains surface dependent terms and formula (1) for the S-matrix holds only if suitable modifications are made to the definition of time-ordering products of $\partial_\mu\varphi$. More specifically, using a theorem[18] first proved by Matthews for Lagrangians involving one time derivative and later extended by Dyson to Lagrangians with two time derivatives, the S-matrix is covariantly defined if we invert the order of differentiation and the time-ordering operation in vacuum expectation values of the following variety:

$$\langle T^*\{\varphi_\mu(x_1)\,\varphi(x_2)\}\rangle = \Delta_{\mu_1}(x_1-x_2) \equiv \partial_{\mu_1}\Delta(x_1-x_2)$$

$$\langle T^*\{\varphi_\mu(x_1)\,\varphi_\nu(x_2)\}\rangle \equiv \Delta_{\mu_1\nu_2}(x_1-x_2) \equiv \partial_{\mu_1}\partial_{\nu_2}\Delta(x_1-x_2)$$

where $\varphi_\mu \equiv \partial_\mu \varphi$. Given then the modified time-ordering operation T^* we have

$$S^{(N)} = g^N \int d^4z_1 \ldots d^4z_N\, T^*[L\{\varphi(z_1),\varphi_\mu(z_1)\}\ldots L\{\varphi(z_N),\varphi_\mu(z_N)\}] \; . \tag{28}$$

The Wick reduction can be carried through by extending Hori's exponential operator to include differentiation with respect to the derived fields φ_μ as follows:

$$S^{(N)} = g^N \int d^4z_1 \ldots d^4z_N \exp\left(\frac{\partial}{\partial \underset{\sim}{\varphi}} \underset{\sim}{\Delta} \frac{\partial}{\partial \underset{\sim}{\varphi}}\right)\left[L\{\varphi_1,\varphi_{\mu_1}\}\ldots L\{\varphi_N,\varphi_{\mu_N}\}\right]_{\substack{\varphi_k = \varphi^{ext}(z_k) \\ \varphi_{\mu k} = \partial_\mu \varphi^{ext}(z_k)}} \tag{29}$$

where

$$\frac{\partial}{\partial \underset{\sim}{\varphi}} \underset{\sim}{\Delta} \frac{\partial}{\partial \underset{\sim}{\varphi}} \equiv \frac{1}{2} \sum_{ij} \frac{\partial}{\partial \underset{\sim}{\varphi}_i} \underset{\sim}{\Delta}_{ij} \frac{\partial}{\partial \underset{\sim}{\varphi}_j}$$

$$\equiv \frac{1}{2} \sum_{ij} \Delta(z_i - z_j) \frac{\partial^2}{\partial \varphi_i \partial \varphi_j} + \Delta_{\mu_i}(z_i - z_j) \frac{\partial^2}{\partial \varphi_{\mu i} \partial \varphi_j}$$

$$+ \Delta_{\nu_j}(z_i - z_j) \frac{\partial^2}{\partial \varphi_i \partial \varphi_{\nu j}} + \Delta_{\mu_i \nu_j}(z_i - z_j) \frac{\partial^2}{\partial \varphi_{\mu i} \partial \varphi_{\nu j}}$$

(30)

In order to give a simple integral representation of this generalized operator we must be prepared to introduce auxiliary vector variables. To see how this is achieved it is enough to consider a pair of points since the extension to the whole series of points is easily performed by the method outlined in Sec. 2. Since

$$\Delta_\mu(x) = 2x_\mu \frac{d}{dx^2} \Delta(x^2) \equiv 2x_\mu \Delta'(x^2)$$

$$\Delta_{\mu\nu}(x) = -4x_\mu x_\nu \Delta''(x^2) - 2g_{\mu\nu} \Delta'(x^2) \tag{31}$$

we need to introduce at most one auxiliary vector and four auxiliary scalar complex integrations. Showing this in detail,

$$\exp \left[\frac{\partial}{\partial \underset{\sim}{\varphi}} \Delta(x) \frac{\partial}{\partial \underset{\sim}{\varphi}'} \right] F(\underset{\sim}{\varphi}; \underset{\sim}{\varphi}')$$

$$= \exp \left[\Delta \frac{\partial^2}{\partial \varphi \partial \varphi'} + 2x_\mu \Delta' \frac{\partial^2}{\partial \varphi_\mu \partial \varphi'} - 2x_\nu \Delta' \frac{\partial^2}{\partial \varphi \partial \varphi'_\nu} \right. \left. - 4x_\mu x_\nu \Delta'' \frac{\partial^2}{\partial \varphi_\mu \partial \varphi_\nu} - 2\Delta' \frac{\partial^2}{\partial \varphi_\mu \partial \varphi_\mu} \right] F(\varphi, \varphi_\lambda; \varphi', \varphi'_\lambda) =$$

$$= \frac{1}{\pi^8} \int d^8 c_{(\mu)} \, d^2 c \, d^2 b_1 d^2 b_2 d^2 a \exp\left[-|a|^2 + \alpha a \frac{\partial}{\partial\varphi} + \alpha' a^* \frac{\partial}{\partial\varphi'}\right]$$

$$\exp\left[-|b_2|^2 + \beta_2 b_2 \, 2x_\mu \frac{\partial}{\partial\varphi_\mu} + \beta_2' b_2^* \frac{\partial}{\partial\varphi'}\right]$$

$$\exp\left[-|b_1|^2 + \beta_1 b_1 \frac{\partial}{\partial\varphi} + \beta_1' b_1^* \, 2x_\nu \frac{\partial}{\partial\varphi'_\nu}\right]$$

$$\exp\left[-|c|^2 + \gamma_1 c \, 2x_\mu \frac{\partial}{\partial\varphi_\mu} + \gamma_1' c^* \, 2x_\nu \frac{\partial}{\partial\varphi'_\nu}\right]$$

$$\exp\left[-|c_\lambda c_\lambda|^2 + \gamma_2 c_\mu \frac{\partial}{\partial\varphi_\mu} + \gamma_2' c_\nu^* \frac{\partial}{\partial\varphi'_\nu}\right] F(\varphi, \varphi_\lambda; \varphi', \varphi'_\lambda)$$

$$= \frac{1}{\pi^8} \int d^8 c_{(\mu)} \, d^2 c \, d^2 b_1 d^2 b_2 a \exp -\left[|a|^2 + |b_1|^2 + |b_2|^2 + |c|^2 + |c_\lambda c_\lambda|\right]$$

$$F\begin{pmatrix} \varphi + \alpha a + \beta_1 b_1 \, , \; \varphi_\lambda + 2x_\lambda \beta_2 b_2 + 2x_\lambda \gamma_1 c + \gamma_2 c_\lambda \; ; \\ \varphi' + \alpha' a^* + \beta_2' b_2^* \, , \; \varphi'_\lambda + 2x_\lambda \beta_1' b_1^* + 2x_\lambda \gamma_1' c^* + \gamma_2' c_\lambda^* \end{pmatrix} \qquad (32)$$

where

$$\alpha\alpha' = \Delta, \quad \gamma_1\gamma_1' = -\Delta''$$
$$\beta_2\beta_2' = -\beta_1\beta_1' = -\frac{1}{2}\gamma_2\gamma_2' = \Delta' \qquad (33)$$

The result cannot depend on the individual α, β, ... but only on the products $\alpha\alpha' = \Delta$, etc. For the remainder of this discussion we choose to make the quasisymmetric split

$$\alpha = \alpha' = \beta_1 = \beta_2' = \sqrt{\Delta}, \quad \gamma_1 = \gamma_1' = \sqrt{-\Delta''}$$
$$\gamma_2 = \gamma_2' = \sqrt{-2\Delta'} \quad \text{and} \quad \beta_2 = -\beta_1' = \Delta'/\sqrt{\Delta}\,. \qquad (34)$$

Now because in the limit $x^2 \to 0$,

$$\Delta'(x^2) \sim 1/x^4 \quad \text{and} \quad \Delta''(x^2) \sim 1/x^6 \qquad (35)$$

one can see that, consistently for all integrations over the shifted functional, we can ascribe the "singularity factors"

$$\varphi \sim M \quad \text{and} \quad \varphi_\mu \sim M^2 \qquad (36)$$

owing to the terms Δ and $(x\Delta'/\sqrt{\Delta} + x\sqrt{\Delta''})$ occurring respectively in the shifted arguments. Perhaps the clearest way to appreciate this conclusion is to realize that most of the auxiliary integrations are redundant and that for the simple case treated above only one auxiliary vector and one auxiliary scalar variable suffice to make the "exponential shift" defined in Sec. 2. Thus, write

$$\exp\left[\frac{\partial}{\partial \underset{\sim}{\varphi}} \underset{\sim}{\Delta} \frac{\partial}{\partial \underset{\sim}{\varphi}'}\right] = \exp\left[\left(c_{\mu\lambda}\frac{\partial}{\partial\varphi_\mu} + c_\lambda \frac{\partial}{\partial\varphi}\right)\left(c'_{\lambda\nu}\frac{\partial}{\partial\varphi'_\nu} + c'_\lambda \frac{\partial}{\partial\varphi'}\right) + cc'\frac{\partial^2}{\partial\varphi\partial\varphi'}\right] \tag{37}$$

with

$$cc' + c_\lambda c'_\lambda = \Delta\ , \quad c_{\mu\lambda}c_\lambda = \Delta_\mu\ , \quad c_{\mu\lambda}c'_{\lambda\nu} = \Delta_{\mu\nu}\ . \tag{38}$$

$$\exp\left[\frac{\partial}{\partial \underset{\sim}{\varphi}} \underset{\sim}{\Delta} \frac{\partial}{\partial \underset{\sim}{\varphi}'}\right] F(\underset{\sim}{\varphi};\underset{\sim}{\varphi}')$$

$$= \frac{1}{\pi^5}\int d^8u_{(\mu)} d^2u \exp -\left[|u|^2 + |u_\lambda u_\lambda|\right]$$

$$F(\varphi + cu + c_\mu u_\mu, \varphi_\lambda + c_{\lambda\mu}u_\mu; \varphi' + c'u^* + c'_\mu u^*_\mu, \varphi'_\lambda + c_{\lambda\nu}u^*_\nu)\ . \tag{39}$$

Again the result can only depend on the products $cc' = \Delta$; if we make the symmetrical choice $c = c'$ for simplicity then we show in the appendix that in the (Euclidean) limit $x \to 0$,

$$c \sim 1/x\ , \quad c_\mu \sim 1/x \quad \text{and} \quad c_{\mu\nu} \sim 1/x^2\ .$$

The association (36) of the ultraviolet factors $\varphi \sim M$ and $\varphi_\mu \sim M^2$ then becomes more obvious.

The accompanying paper by Koller (in preparation) describes how, in a particular model, vector integrations can be performed, based on identities like

$$\frac{1}{\pi^4}\int d^8u_{(\mu)}\, u_\kappa u^*_\lambda\, e^{-u_\mu u^*_\mu} = \frac{1}{4} g_{\kappa\lambda}$$

$$\frac{1}{\pi^4}\int d^8u_{(\mu)}\, u_\kappa u_\lambda\, f(u_\nu u^*_\nu) = 0, \text{ etc.}$$

Here we shall only be concerned with a superficial count of the over-all infinities to be expected in a given S-matrix element.

The procedure is the same as before. Consider a graph with n_0 external scalar and n_1 external vector (derived scalar) lines which contributes to the n-point function $n_0 + n_1 = n$. Thus

$$S^{total}(x_1, \ldots, x_n) = \sum_{n_1=0}^{n} \partial_{\mu_1} \ldots \partial_{\mu_{n_1}} S_{\mu_1 \ldots \mu_{n_1}}(x_1, \ldots, x_n). \quad (40)$$

The degree of the over-all infinity (up to logarithms) derives from the integral

$$\int (d^4x)^{N-1} \left(\frac{\partial}{\partial\varphi}\right)^{n_0} \left(\frac{\partial}{\partial\varphi_\mu}\right)^{n_1} [L(\sqrt{\Delta})]^N \sim M^{-n_0} M^{-2n_1} M^{N\nu - 4(N-1)} \quad (41)$$

where we ascribe $\varphi \sim M$, $\varphi_\mu \sim M^2$ and assume $L \sim M^\nu$. Such a graph therefore behaves in the limit $M \to \infty$ as

$$M^{(\nu-4)N+4-n-n_1} \; ; \; n_1 = 0, 1, \ldots, n.$$

Thus a theory is normal if $4 - \nu \geqslant 0$ (supernormal if $4 > \nu$).

5. NON-LINEAR REALIZATIONS OF SU(2) ⊗ SU(2)

The simplest practical applications of our conclusions about derivative couplings are to be found in the non-linear realizations of chiral groups. We shall study the case of SU(2) ⊗ SU(2) symmetry for definiteness as the features which emerge will apply to more complicated cases as well.

Describe the mesons of the $(\frac{1}{2}, \frac{1}{2})$ representation by the field matrix,

$$\mathcal{S} = \sigma + i \underset{\sim}{\tau} \cdot \underset{\sim}{\varphi} \, \Lambda(\varphi^2) \quad (42)$$

where the non-linearity is introduced by imposing the constraint

$$\mathcal{S}\mathcal{S}^\dagger = 1 \quad \text{or} \quad \sigma^2 + \underset{\sim}{\varphi}^2 \Lambda^2(\varphi^2) = 1 \quad . \tag{43}$$

The choice of function $\Lambda(\varphi^2)$ corresponds to different parametrizations of the non-linear "co-ordinates" (σ and $\underset{\sim}{\varphi}$ are "co-ordinates" of the differential manifold (43)) and with each such choice of Λ the corresponding interpolating field φ is different[19]. (However, we shall use the same symbol in every case.)

The unique[20] SU(2) ⊗ SU(2) invariant Lagrangian which contains only two derivatives of the fields is

$$\mathcal{L} = \frac{1}{4}\Lambda^{-2}(0)\ \mathrm{Tr}[(\partial_\mu \mathcal{S})(\partial_\mu \mathcal{S}^+)] = \frac{1}{4}\Lambda^{-2}(0)\ \mathrm{Tr}[\mathcal{J}_\mu \mathcal{J}_\mu] \tag{44}$$

where we write

$$\mathcal{J}_\mu \equiv -i\mathcal{S}^+ \partial_\mu \mathcal{S} = \mathcal{J}_\mu^+ \quad . \tag{45}$$

If we substitute for $\mathcal{S}$ the expression (42) and eliminate σ by means of the constraint equation (43) then we find

$$\begin{aligned}\mathcal{J}_\mu &= \underset{\sim}{\tau}\cdot[\Lambda(\sigma\partial_\mu \underset{\sim}{\varphi} - \underset{\sim}{\varphi}\,\partial_\mu \sigma + \Lambda \underset{\sim}{\varphi}\times\partial_\mu \underset{\sim}{\varphi}) + 2\Lambda'\sigma\underset{\sim}{\varphi}\,(\underset{\sim}{\varphi}\cdot\partial_\mu \underset{\sim}{\varphi})] \\ &= \Lambda\underset{\sim}{\tau}\cdot\left[\sqrt{1-\Lambda^2\varphi^2}\,\partial_\mu \underset{\sim}{\varphi} + \frac{\underset{\sim}{\varphi}(\underset{\sim}{\varphi}\,\partial_\mu \underset{\sim}{\varphi})}{\sqrt{1-\Lambda^2\varphi^2}}(\Lambda^2 + 2\Lambda^{-1}\Lambda') + \Lambda\underset{\sim}{\varphi}\times\partial_\mu \underset{\sim}{\varphi}\right]\end{aligned} \tag{46}$$

and

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \frac{1}{4}\Lambda^{-2}(0)\ \mathrm{Tr}[\mathcal{J}_\mu \mathcal{J}_\mu] - \frac{1}{2}(\partial_\mu \underset{\sim}{\varphi})\cdot(\partial_\mu \underset{\sim}{\varphi}) \\ &= \frac{1}{2}[\Lambda^{-2}(0)\,\Lambda^2 - 1]\,(\partial_\mu \underset{\sim}{\varphi})\cdot(\partial_\mu \underset{\sim}{\varphi}) + \\ &\quad + \frac{(\underset{\sim}{\varphi}\cdot\partial_\mu \underset{\sim}{\varphi})(\underset{\sim}{\varphi}\cdot\partial_\mu \underset{\sim}{\varphi})}{2\Lambda^2(0)(1-\Lambda^2\varphi^2)}\,[\Lambda^4 + 4\Lambda\Lambda' + 4\Lambda'^2\varphi^2]\end{aligned} \tag{47}$$

where $\Lambda' = d\Lambda/d\varphi^2$. The ensuing equations of motion can be conveniently remembered in the Sugawara form[21],

$$\partial_\mu \mathcal{J}_\mu = 0 \ , \quad \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu + i[\mathcal{J}_\mu, \mathcal{J}_\nu] = 0 \ . \tag{48}$$

We may now inquire about the "ultraviolet behaviour" of the interaction Lagrangian with a view to possible renormalizability[22). Begin by supposing that for large $\varphi \sim M$

$$\Lambda(\varphi^2) \to \varphi^k \sim M^k \ . \qquad \sigma \sim [1 - M^{2+2k}]^{\frac{1}{2}},$$

$$\mathcal{J}_\mu \sim M^{k+2} \left[(1 - M^{2+2k})^{\frac{1}{2}} + \frac{M^2(M^{2k} + M^{-2})}{(1 - M^{2+2k})^{1/2}} + M^{k+1} \right] \tag{49}$$

and

$$\mathcal{L}_{int} \sim (M^{2k} - 1)\, M^4 + \frac{M^6}{1 - M^{2+2k}} \,(M^{4k} + M^{2k-2}) \ . \tag{50}$$

Hence for $k > 0$, $\mathcal{J}_\mu \sim M^{2k+3}$ and $\mathcal{L}_{int} \sim M^{2k+4}$,

for $-1 < k < 0$, $\mathcal{J}_\mu \sim M^{2k+3}$ and $\mathcal{L}_{int} \sim M^4$

and for $k < -1$, $\mathcal{J}_\mu \sim M^{k+2}$ and $\mathcal{L}_{int} \sim M^4$.

This shows that non-linear realizations of chiral groups, for the preferred meson fields, yield normal $(k < 0)$ or seemingly abnormal $(k > 0)$ Lagrangians, but not supernormal ones. The reason for this is not far to seek. For $k < 0$, $\mathcal{L} \sim M^{2k+4}$ so that subtracting off $\mathcal{L}_f = (\partial_\mu \varphi)^2 \sim M^4$ we meet a normal situation.

The question now poses itself: since we can pass from one set of co-ordinates φ to another $\bar{\varphi}$ by a point transformation

$$\mathcal{S} = \sigma + i\underset{\sim}{\tau} \cdot \underset{\sim}{\varphi}\, \Lambda(\varphi^2) = \bar{\sigma} + i\underset{\sim}{\tau} \cdot \underset{\sim}{\bar{\varphi}}\, \Lambda(\bar{\varphi}^2) \tag{51}$$

what is the significance of the abnormal parametrizations $(k > 0)$? In the next section we argue that the invariance of the total Lagrangian $(\mathcal{J}_\mu \mathcal{J}_\mu)$ should imply that the S-matrix elements on the mass shell do not differ from one parametrization to the next, so that the theory is normal irrespective of the possibility $k > 0$. We list below some special choices of parametrization.

(i) Gasiorowicz-Geffen co-ordinates

$$\Lambda(\varphi^2) = \lambda \quad , \quad \text{a constant} \quad (\text{i.e., } k = 0)$$

$$\mathcal{J}_\mu = \lambda \underset{\sim}{\tau} \cdot [\sigma \partial_\mu \underset{\sim}{\varphi} - \underset{\sim}{\varphi} \partial_\mu \sigma + \lambda \underset{\sim}{\varphi} \times \partial_\mu \underset{\sim}{\varphi}]$$

with $\sigma = (1 - \lambda^2 \varphi^2)$. Also,

$$2\mathcal{L}_{int} = \frac{\lambda^2 (\underset{\sim}{\varphi} \cdot \partial_\mu \underset{\sim}{\varphi})(\underset{\sim}{\varphi} \cdot \partial_\mu \varphi)}{(1 - \lambda^2 \varphi^2)} \sim M^4 \ . \tag{52}$$

(ii) Schwinger co-ordinates

$$\Lambda(\varphi^2) = \lambda(1 + \lambda^2 \varphi^2)^{-\frac{1}{2}} \ ; \quad \lambda \ \text{constant} \quad (\text{i.e., } k = -1) \ .$$

Thus $\sigma = (i + \lambda^2 \varphi^2)^{-\frac{1}{2}}$,

$$\mathcal{J}_\mu = \frac{\lambda \underset{\sim}{\tau}}{1 + \lambda^2 \varphi^2} [\partial_\mu \underset{\sim}{\varphi} + \lambda \underset{\sim}{\varphi} \times \partial_\mu \underset{\sim}{\varphi}]$$

and

$$2\mathcal{L}_{int} = -\frac{\lambda^2}{1 + \lambda^2 \varphi^2} \left[\varphi^2 (\partial_\mu \underset{\sim}{\varphi}) \cdot (\partial_\mu \underset{\sim}{\varphi}) + \frac{(\underset{\sim}{\varphi} \cdot \partial_\mu \underset{\sim}{\varphi})(\underset{\sim}{\varphi} \cdot \partial_\mu \underset{\sim}{\varphi})}{1 + \lambda^2 \varphi^2} \right] \sim M^4 \ . \tag{53}$$

(iii) Weinberg co-ordinates

$$\Lambda(\varphi^2) = 2\lambda(1 + \lambda^2 \varphi^2)^{-1} \ ; \quad \lambda \ \text{constant} \quad (\text{i.e., } k = -2)$$

$\sigma = (1 - \lambda^2 \varphi^2)(1 + \lambda^2 \varphi^2)^{-1}$ giving

$$\mathcal{J}_\mu = \frac{2\lambda \underset{\sim}{\tau}}{(1 + \lambda^2 \varphi^2)^2} \left[(1 - \lambda^2 \varphi^2) \partial_\mu \underset{\sim}{\varphi} + 2\lambda \underset{\sim}{\varphi} \times \partial_\mu \underset{\sim}{\varphi} + 2\lambda^2 (1 + \lambda^2 \varphi^2) \underset{\sim}{\varphi} (\underset{\sim}{\varphi} \cdot \partial_\mu \underset{\sim}{\varphi}) \right]$$

and

$$2\mathcal{L}_{int} = (\partial_\mu \underset{\sim}{\varphi}) \cdot (\partial_\mu \underset{\sim}{\varphi}) \left[\frac{1}{(1 + \lambda^2 \varphi^2)^2} - 1 \right] \sim M^4 \ . \tag{54}$$

(iv) <u>Harmonic co-ordinates</u>

A set of co-ordinates which may prove useful in the vector problem is defined by the condition

$$\Lambda \sqrt{1 - \varphi^2\Lambda^2} = \lambda^2$$

where λ is a constant. In these co-ordinates, which we shall call harmonic, the current operator is given by

$$\underset{\sim}{\mathcal{J}}_\mu = \left\{\partial_\mu \underset{\sim}{\varphi} + \frac{2\lambda}{1 + \sqrt{1 - 4\lambda^2 \underset{\sim}{\varphi}^2}} \left(\underset{\sim}{\varphi} \times \partial_\mu \underset{\sim}{\varphi} + \frac{2\lambda \underset{\sim}{\varphi}\underset{\sim}{\varphi} \cdot \partial_\mu \underset{\sim}{\varphi}}{1 - 4\lambda^2 \underset{\sim}{\varphi}^2} \right)\right\} .$$

In this form the linear term $\partial_\mu \varphi$ appears multiplied by a constant rather than by a function of φ^2 .

6. FIELD TRANSFORMATIONS

In the previous section, as in the next, we are assuming the correctness of the basic equivalence theorem which states that if a local point transformation of fields is made such that the physical spectrum associated with these fields is unaltered - and therefore also the Hilbert spaces of in-and out states remains the same - then the on-mass-shell (physical) S-matrix elements, computed using either the original or the transformed Lagrangians, are identical. This theorem[23)] first stated by Chisholm, Kamefuchi, O'Raifeartaigh and Salam, has been proved to varying degrees of restrictiveness on field transformations and rigour by the above-mentioned authors and by Borchers. It has latterly been extended[24)] by Coleman, Wess and Zumino who claim to sharpen the result to apply even to diagrams with equal numbers of closed loops. The weak point when one comes to applying the theorem in practical cases is the lack of criteria whereby one may judge what transformations leave unchanged the in and out limits of the interpolating fields. For practical purposes the only procedure known to us is the adiabatic switching on and off of charges; this implies that a point transformation is allowed if

(i) In the limit $g \to 0$ for a transformation like

$$\varphi(x) \to a_1 \; \bar{\varphi}(x) + a_2 \bar{\varphi}^2(x) + \ldots ,$$

the $a_i \to 0$, $i > 1$, $a_1 \longrightarrow \text{constant} \neq 0$.

($a_1 \neq 1$ implies a wave-function renormalization).

(ii) No derivatives are involved in the transformation, otherwise the particle spectra associated with the two sets of fields are likely to differ. For example if we let

$$\phi = \bar{\phi} + \mu^{-2} \, \partial^2 \bar{\phi} \qquad , \qquad \text{with } \partial^2 \phi = 0 ,$$

the transformed equation $\partial^2(\partial^2 + \mu^2)\,\bar{\phi} = 0$ leads to a different spectrum for $\bar{\phi}$. (In certain circumstances first-order derivatives may be allowed as we shall see for the case of transformations involving vector mesons.)

(iii) The only known procedure for computing S-matrix elements for given Lagrangians is essentially the Dyson perturbation procedure which relies on identifying that part of the Lagrangian which depends bilinearly on field variables as $\mathcal{L}_f$. In the sequel, when making point transformations we shall separate out all bilinear terms; thus a term like $\mathcal{L} = \frac{(\partial_\mu \phi)^2}{1+\phi^2}$ will contribute $(\partial_\mu \phi)^2$ to $\mathcal{L}_f$ and

$\frac{\phi^2}{1+\phi^2} (\partial_\mu \phi)^2$ to $\mathcal{L}_{int}$.

(iv) A consequence of the split mentioned in (iii) is that in our power-counting theorem $\mathcal{L} = \frac{(\partial_\mu \phi)^2}{1+\phi^2}$ does not behave supernormally like M^2 (assuming $\phi \sim M$, $\partial\phi \sim M^2$) but normally like $\frac{\phi^2}{1+\phi^2} (\partial\phi)^2 \sim M^4$. This may mean that our estimates of singularity behaviour are likely to be over-estimates and that a future formulation of a new computational procedure may depress our estimates of likely infinities.

(v) Regarding our discussion of non-linear realizations of chiral groups in Sec. 5 it is important to realize that the interpolating fields for two different choices of co-ordinates can be related to each other; thus writing

$$\mathcal{S} = \sigma(\varphi^2) + i\underset{\sim}{\tau} \cdot \underset{\sim}{\varphi}\, \Lambda(\varphi^2) = \bar{\sigma}(\bar{\varphi}^2) + i\underset{\sim}{\tau} \cdot \underset{\sim}{\bar{\varphi}}\, \Lambda(\bar{\varphi}^2) \qquad (51')$$

one can express $\underset{\sim}{\varphi}$ fields in terms of $\underset{\sim}{\bar{\varphi}}$ fields by comparing terms of the power series in the φ . We have assumed that the adiabatic limits of both φ and $\bar{\varphi}$ are the same so that the on-mass-shell S-matrices are equal and so is the singularity behaviour of S-matrix elements. It is well known that this result does not apply to the n-point Green's functions.

7. MASSIVE VECTOR MESON COUPLINGS

It is a well-known fact [25)] that the infinities in vector meson theories stem from the longitudinal components of the vector field propagator, viz. the spin zero projection ($e_{\mu\nu}$) terms of $\Delta_{\mu\nu}$;

$$\left.\begin{aligned} \Delta_{\mu\nu} &= \langle T\{U_\mu U_\nu\}\rangle = \frac{d_{\mu\nu}(1)}{p^2-\mu^2} - \frac{e_{\mu\nu}(p)}{\mu^2} \\ &\text{where} \\ d_{\mu\nu}(p) &\equiv g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2+i\epsilon} \quad , \quad e_{\mu\nu}(p) \equiv \frac{p_\mu p_\nu}{p^2+i\epsilon} \end{aligned}\right\} \qquad (55)$$

The vector (d) and scalar (e) parts of Δ may be associated with the following non-local transverse [26)] and longitudinal split of the field U ,

$$U^1_\mu(x) = d_{\mu\nu}(\partial)\, U_\nu(x) \quad , \quad U^0_\mu(x) = e_{\mu\nu}(\partial)\, U_\nu(x) \quad .$$

The transverse field U^1 is <u>normal</u> since its propagator is singular like $1/x^2$ as $x \to 0$, while the longitudinal field U^0 is <u>abnormal</u> with its more singular propagator like $1/x^4$. It is the coupling of U^0 (being essentially a derivative coupling of a spin zero object χ defined by the relation

$$U^0_\mu \equiv \partial_\mu \chi \quad \text{with} \quad \chi \equiv \partial^{-2}\, \partial_\nu U_\nu \qquad)$$

which gives rise to non-renormalizable infinities. To exhibit these infinities more transparently one can convert this derivative coupling of χ into a non-derivative coupling by a set of well-known and standard contact transformations obtaining an equivalence theorem[12] for the two theories. The transformed (non-derivative) Lagrangians contain terms which as a rule are exponential or rational functions of χ and which, although they produce intractable infinities in a straightforward perturbation approach, are of just the right form for the summation techniques of Efimov and Fradkin.

Since the fields U^1_λ and $\chi = \partial^{-2}\partial_\nu U_\nu$ are non-local, the equivalence theorems are stated, in practice, not in terms of the split $U_\lambda = U^1_\lambda + U^0_\lambda = U^1_\lambda + \partial_\lambda \chi$, but in terms of the local Stückelberg split[27]

$$U_\lambda = A_\lambda + \frac{1}{\mu}\,\partial_\lambda\, B$$

where A_λ and B are five independent local fields, the replacement of U by A and B in the Lagrangian being made in such a manner that the Hilbert spaces generated by incoming and outgoing transverse components of the field A^1_μ coincide with the corresponding spaces generated by U^1_μ . There are numerous formulations to achieve this; we use in this section a particularly elegant one originally introduced by Schwinger and latterly used by Veltman and Ghose[9]. To illustrate the essential idea we shall first consider neutral pseudovector meson theory and then, by making a non-trivial extension, consider the theory of self-interacting massive Yang-Mills mesons.

(i) Neutral axial mesons coupling to a matter field

To the conventional Lagrangian $\mathcal{L}(U)$ of the vector meson add a Lagrangian $\mathcal{L}_f(B)$ for a zero mass free particle B of negative metric

$$\mathcal{L} = -\frac{1}{4} U_{\mu\nu}U_{\mu\nu} + \frac{1}{2}\mu^2 U_\nu U_\nu - \frac{1}{2}(\partial_\nu B)(\partial_\nu B) - J_\nu U_\nu + \mathcal{L}_f\,(\text{matter}) \qquad (56)$$

where $U_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu$ and J_μ is a (not necessarily conserved) matter-current which does not involve U_μ. Because U_μ and B are non-interacting, the completeness relation for in and out states reads

$$1 \equiv 1_U \otimes 1_B \quad \text{with} \quad 1_U \equiv \sum |U'\rangle\langle U'|$$

$$1_B \equiv \sum |B\rangle\langle B| \tag{57}$$

and the S-matrix in the transverse U-sector alone, defined by

$$S = \sum |U^1_{out}\rangle\langle U^1_{in}| \tag{58}$$

is necessarily unitary.

Let us now make the transformation

$$U_\lambda = A_\lambda + \frac{1}{\mu} \partial_\lambda B \quad , \qquad B \text{ unchanged.} \tag{59}$$

The transformed Lagrangian equals

$$\mathcal{L} = -\frac{1}{4} A_{\mu\nu} A_{\mu\nu} + \frac{1}{2}\mu^2 A_\nu A_\nu + \mu A_\nu \partial_\nu B - J_\nu (A_\nu + \frac{1}{\mu} \partial_\nu B) + \mathcal{L}_f \text{ (matter)} \tag{60}$$

and no longer contains second derivative terms of the B field. The B field may be looked upon in Schwinger's language as a Lagrange multiplier "extended operator". From the equations of motion

$$\left.\begin{aligned} \partial_\mu A_{\mu\nu} + \mu^2 (A_\nu + \frac{1}{\mu} \partial_\nu B) &= J_\nu \\ \mu^2 \partial_\nu A_\nu &= \partial_\nu J_\nu \end{aligned}\right\} \tag{61}$$

we recover

$$\partial^2 B = 0 \quad , \quad \partial_\mu A_\mu = \partial_\mu U_\mu \quad \text{or} \quad A^1_\mu = U^1_\mu \quad . \tag{62}$$

Thus $|U^1_\mu\rangle = |A^1_\mu\rangle$ and a unitary S-matrix can be written in the equivalent form

$$S = \sum |A^1_{out}\rangle\langle A^1_{in}| \tag{58'}$$

with B states still making no appearance [28].

Consider now the propagators in the new theory. According to prescription (iii) of Sec. 5, we take for our free Lagrangian

$$\mathcal{L}_f = -\frac{1}{4}A_{\mu\nu}A_{\mu\nu} + \frac{1}{2}\mu^2 A_\nu A_\nu + \mu A_\nu \partial_\nu B \quad . \tag{63}$$

Using the well-known ansatz which gives the propagator matrix Δ in terms $(\mathcal{L}_f)^{-1}$, we obtain the momentum-space propagator [29)]

$$\Delta(p) = \begin{pmatrix} \langle A_\mu, A_\nu \rangle & \langle A_\mu, B \rangle \\ \langle B, A_\nu \rangle & \langle B, B \rangle \end{pmatrix}$$

$$= \begin{pmatrix} -d_{\mu\nu}(p)\,\Delta(p,\mu) & ip_\mu\,\Delta(p,0)/\mu \\ -ip_\nu\,\Delta(p,0)/\mu & -\Delta(p,0) \end{pmatrix} \quad . \tag{64}$$

In x-space note that $\langle T(A_\mu, A_\nu)\rangle \approx \langle T(B,B)\rangle \approx 1/x^2$ are <u>normal</u> propagators whereas $\langle T(A_\mu, B)\rangle \approx 1/\mu x^3$ is <u>abnormal</u>.

The next step is to convert the derivative coupling $J_\mu \partial_\mu B$ into a non-derivative coupling in preparation for applying the E-F summation technique by an operator gauge transformation. Taking the concrete case where

$$J_\mu = ig\bar{\psi}\gamma_\mu\gamma_5\psi \quad \text{and} \quad \mathcal{L}_f(\text{matter}) = \bar{\psi}(i\gamma_\mu\partial_\mu - m)\,\psi \tag{65}$$

transform $\psi' = e^{g\gamma_5 B/\mu}\,\psi$. This gives

$$\mathcal{L} = \bar{\psi}'(i\gamma_\mu\partial_\mu + ig\gamma_\mu\gamma_5 A_\mu - m e^{2\gamma_5 gB/\mu})\,\psi'$$
$$-\frac{1}{4}A_{\mu\nu}A_{\mu\nu} + \frac{1}{2}\mu^2 A_\nu A_\nu + \mu A_\nu\partial_\nu B \quad . \tag{66}$$

The exponential interaction can be treated by E-F methods as it stands. One can if desired bring it into a rational form by making a further change of variables (to Weinberg co-ordinates φ):

$$\frac{g\varphi}{\mu} = \tan\frac{gB}{\mu} \quad . \tag{67}$$

The fields φ and B possess the same adiabatic limit $\varphi_{in} = B_{in}$ when the coupling parameter g as a function of time t vanishes for $t \to \pm \infty$. In terms of φ , A and ψ' the Lagrangian reads $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_{int}$

$$\left. \begin{aligned} \mathcal{L}_f &= -\frac{1}{4} A_{\mu\nu} A_{\mu\nu} + \frac{1}{2}\mu^2 A_\nu A_\nu + \mu A_\nu \partial_\nu \varphi + \bar{\psi}' (i\gamma\partial - m)\psi' \\ \mathcal{L}_{int} &= ig\bar{\psi}' A_\mu \gamma_\mu \gamma_5 \psi' + \mu A_\nu \partial_\nu \varphi \left(\frac{g\varphi}{\mu}\right)^2 \left[1 + \left(\frac{g\varphi}{\mu}\right)^2\right]^{-1} \\ &\quad + 2m \frac{g\varphi}{\mu} \left[1 + \left(\frac{g\varphi}{\mu}\right)^2\right]^{-1} \bar{\psi}' \left(\frac{g\varphi}{\mu} - \gamma_5\right)\psi' \end{aligned} \right\} \tag{68}$$

In the next section we prove the result that the singularity limit of relevant fields is given by the estimates

$$A_\mu \sim M \ , \quad \psi' \sim M^{3/2} \ , \quad \varphi \sim M^2 \quad \text{and} \quad \partial_\mu \varphi \sim M^3 \ , \tag{69}$$

i.e., the fields A_μ and ψ' are normal but φ and $\partial\varphi$ are abnormal. The important point however is that, notwithstanding this abnormality of φ , the interaction Lagrangian behaves like M^4 and is perfectly normal, the denominator $1 + (g\varphi/\mu)^2$ in (68) providing the necessary damping of infinities.

(ii) Massive Yang-Mills theory

If $\underset{\sim}{U}_\mu$ are the three Yang-Mills fields of an SU(2) gauge group, the Lagrangian of the massive theory can be summarized in the form

$$\mathcal{L} = \frac{1}{2} \mathrm{Tr} \left[-\frac{1}{4} U_{\mu\nu} U_{\mu\nu} + \mu^2 U_\nu U_\nu \right] \tag{70}$$

where U_ν is the matrix $\underset{\sim}{U}_\nu \cdot \underset{\sim}{\tau}$ and

$$U_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu + ig[U_\mu, U_\nu] \ . \tag{71}$$

As is well known, when $\mu = 0$ the Lagrangian is invariant for the operator gauge transformations

$$U_\mu = \mathcal{S}^\dagger (A_\mu + ig^{-1} \mathcal{S} \partial_\mu \mathcal{S}^\dagger) \tag{72}$$

where $\mathcal{S}(\underset{\sim}{\varphi})$ is any unitary matrix of the type discussed in Sec. 5 (with the coupling parameter $\lambda = \frac{g}{\mu}$) depending on three independent fields $\underset{\sim}{\varphi}$, i.e.,

$$\mathrm{Tr}\,[U_{\mu\nu} U_{\mu\nu}] = \mathrm{Tr}\,[A_{\mu\nu} A_{\mu\nu}] \quad .$$

With this background start by considering a theory of three Yang-Mills fields $\underset{\sim}{U}_\mu$ and three spin zero fields $\underset{\sim}{\varphi}$ such that the U_μ interact among themselves and so do the φ but U_μ and φ do not interact with each other. The Lagrangian is postulated to be

$$2\mathcal{L} = \mathrm{Tr}\,[-\frac{1}{4} U_{\mu\nu} U_{\mu\nu} + \frac{1}{2}\mu^2 U_\nu U_\nu] - \frac{\mu^2}{2g^2} \mathrm{Tr}\,[(\partial_\nu \mathcal{S})(\partial_\nu \mathcal{S}^\dagger)] \quad . \tag{73}$$

The field φ enters with a negative metric as in the neutral theory. Since U_μ and φ do not interact, the completeness relation for in and out states is $1 = 1_U \otimes 1_\phi$ where

$$1_U = \sum |U^1\rangle\langle U^1| \ , \quad 1_\varphi = \sum |\varphi\rangle\langle\varphi| \tag{74}$$

and the S-matrix defined in the U-sector by

$$S = \sum |U^1_{out}\rangle\langle U^1_{in}| \tag{75}$$

is perforce unitary. (Recall that U^1_μ are the transverse components of the field U_μ .)

Let us now make a gauge transformation (72) to recast the Lagrangian in the form

$$2\mathcal{L} = \mathrm{Tr}\,[-\frac{1}{4} A_{\mu\nu} A_{\mu\nu} + \frac{1}{2}\mu^2 A_\nu A_\nu + \frac{\mu^2}{g} A_\nu \mathcal{J}_\nu] \tag{76}$$

$$\mathcal{J}_\nu \equiv i\mathcal{S}\partial_\nu \mathcal{S}^\dagger = -i\mathcal{S}^\dagger \partial_\nu \mathcal{S} . \tag{45}$$

The original equations of motion

$$\partial_\nu U_\nu = 0 \quad , \quad \partial_\nu \mathcal{J}_\nu = 0 \tag{48}$$

allow us to deduce that A satisfies the equation

$$\partial_\mu(\mathcal{S}^\dagger A_\mu \mathcal{S}) = 0 \quad . \tag{77}$$

In the adiabatic limit $g(\pm \infty) \to 0$ when $U_\mu \to U_\mu^{in}$, $A_\mu \to A_\mu^{in}$, $\mathcal{S}(\varphi) \to 1$, $\frac{i}{g} \mathcal{S}^\dagger \partial_\mu \mathcal{S} \to \partial_\mu \varphi^{in}$, we recover the important result that the <u>transverse</u> components $A_\mu^1 = U_\mu^1$ for in and out states. Crucial to this derivation is the requirement that no second-order derivatives or higher be involved in $\partial_\mu(\mathcal{S}^\dagger A_\mu \mathcal{S})$. The U-sector S-matrix can therefore be re-expressed as

$$S = \sum | A^1_{out} \rangle \langle A^1_{in} | \tag{75'}$$

and states that if there are no φ particles initially there will be none finally.

Separating out the free part from the gauge-transformed Lagrangian we arrive finally at $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_{int}$ with

$$\left.\begin{aligned} 2\mathcal{L}_f &= \mathrm{Tr}\left[-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{2}\mu^2 A_\nu A_\nu + \mu A_\nu \partial_\nu \varphi\right] \\ 2\mathcal{L}_{int} &= \mathrm{Tr}\left[-\tfrac{1}{2} i g [A_\mu, A_\nu](\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4} g^2 [A_\mu, A_\nu][A_\mu, A_\nu] \right. \\ &\qquad \left. + \mu A_\nu (g^{-1} \mu \mathcal{J}_\nu - \partial_\nu \varphi)\right] \end{aligned}\right\} \tag{78}$$

As we show later, $\mathcal{L}_f$ gives the ultraviolet characteristics

$$A_\mu \sim M \quad , \quad \varphi \sim M^2 \quad \text{and} \quad \partial_\mu \varphi \sim M^3 \quad (\text{as } M \to \infty) \tag{79}$$

for the free fields. Clearly, in spite of the abnormal growth of the scalar fields $\underset{\sim}{\varphi}$, $\mathcal{L}_{int}$ in (78) is $\approx M^4$ in the E-F sense with the conventional parametrizations of $\mathcal{S}(\varphi)$ and $\mathcal{J}_\mu$ and the massive Yang-Mills is normal.

8. ULTRAVIOLET INFINITIES OF VECTOR MESON THEORIES

We give in this section the explicit details of the infinities that are encountered in vector meson Lagrangians. As stated before, the new point that emerges is the abnormal behaviour of the (unphysical) scalar field which couples to the longitudinal vector component.

(i) Neutral vector mesons

The free Lagrangian used for estimating the ultraviolet infinities of the E-F sums,

$$\mathcal{L}_f(A,\varphi) = -\frac{1}{4} A_{\mu\nu} A_{\mu\nu} + \frac{1}{2}\mu^2 A_\nu A_\nu + \mu A_\nu \partial_\nu \varphi$$

leads naturally to A-φ mixing as the propagator (64) clearly shows. The time-ordering operation must therefore include A-φ cross terms and, since the interaction Lagrangian involves A_μ, $\partial_\mu \varphi$ and φ, Hori's exponential operator must contain all combinations of A_μ, φ_μ and φ. Accordingly this operator reads, for every pair of points,

$$\exp\left[\frac{\partial}{\partial \underset{\sim}{\varphi}} \underset{\sim}{\Delta} \frac{\partial}{\partial \underset{\sim}{\varphi}'}\right] = \exp\left[\Delta_{\mu\nu}\frac{\partial^2}{\partial A_\mu \partial A'_\nu} + \frac{D_{\mu\nu}}{\mu}\left(\frac{\partial^2}{\partial \varphi_\mu \partial A'_\nu} - \frac{\partial^2}{\partial A_\mu \partial \varphi'_\nu}\right) + \frac{D_\nu}{\mu}\left(\frac{\partial^2}{\partial\varphi \partial A'_\nu} - \frac{\partial^2}{\partial A_\nu \partial \varphi'}\right) + D_{\mu\nu}\frac{\partial^2}{\partial\varphi_\mu \partial\varphi'_\nu} + D_\nu\left(\frac{\partial^2}{\partial\varphi_\nu \partial\varphi'} - \frac{\partial^2}{\partial\varphi\partial\varphi'_\nu}\right) + D\frac{\partial^2}{\partial\varphi\partial\varphi'}\right] \quad (80)$$

where

$$\left.\begin{aligned} \Delta_{\mu\nu} &\equiv \left[- g_{\mu\nu} + \partial^{-2} \partial_\mu \partial_\nu \right] \Delta(x, \mu^2) \\ D &\equiv \Delta(x, 0) \ , \ D_\mu = \partial_\mu D \ , \ D_{\mu\nu} = \partial_\mu \partial_\nu D \end{aligned}\right\} . \tag{81}$$

To apply the "exponential shift lemma" one would, in the most general situation, introduce four vector and nine scalar (complex) auxiliary variables of integration. However, as we showed in Sec. 4, most of these are redundant. In order to demonstrate what ultraviolet infinities we may expect in the theory, we shall eliminate a number of redundant variables and set out the power counting theorem using only[30)] three vector and one scalar variables. Thus write symmetrically,

$$\exp\left[\frac{\partial}{\partial \underset{\sim}{\varphi}} \underset{\sim}{\Delta} \frac{\partial}{\partial \underset{\sim}{\varphi}'}\right] = \exp\Bigg[c^2 \frac{\partial^2}{\partial\varphi \partial\varphi'} + \left(c_{\mu\lambda} \frac{\partial}{\partial\varphi_\mu} + c_\lambda \frac{\partial}{\partial\varphi}\right)\left(c_{\lambda\nu} \frac{\partial}{\partial\varphi'_\nu} + c_\lambda \frac{\partial}{\partial\varphi'}\right)$$

$$+ \left(a_{\mu\nu} \frac{\partial}{\partial A_\mu} + b_{\mu\lambda} \frac{\partial}{\partial\varphi_\mu} + b_\lambda \frac{\partial}{\partial\varphi}\right)\left(a_{\lambda\nu} \frac{\partial}{\partial A'_\nu} + b_{\lambda\nu} \frac{\partial}{\partial\varphi'_\nu} + b_\lambda \frac{\partial}{\partial\varphi'}\right)$$

$$+ a^2 \frac{\partial^2}{\partial A_\lambda \partial A'_\lambda} \Bigg] \tag{82}$$

where

$$\left.\begin{aligned} & a^2 = -\Delta \ , \ a_{\mu\lambda} a_{\lambda\nu} = \partial^{-2} \partial_\mu \partial_\nu \Delta \\ & a_{\nu\lambda} b_\lambda = D_\nu / \mu \ , \ a_{\mu\lambda} b_{\lambda\nu} = D_{\mu\nu} / \mu \\ & b_{\mu\lambda} b_{\lambda\nu} + c_{\mu\lambda} c_{\lambda\nu} = D_{\mu\nu} \ , \ c_{\mu\lambda} c_\lambda + b_{\mu\lambda} b_\lambda = D_\mu \\ & c^2 + c_\lambda c_\lambda + b_\lambda b_\lambda = - D \ . \end{aligned}\right\} \tag{83}$$

As before, we obtain the identity

$$\exp\left[\frac{\partial}{\partial \underset{\sim}{\varphi}} \underset{\sim}{\Delta} \frac{\partial}{\partial \underset{\sim}{\varphi}'}\right] F(A_\lambda, \varphi, \varphi_\lambda; A'_\lambda \varphi', \varphi', \varphi'_\lambda)$$

$$= \frac{1}{\pi^{13}} \int d^8 w_{(\mu)}\, d^8 v_{(\mu)}\, d^8 u_{(\mu)}\, d^2 u \quad \exp - \left[|u|^2 + u_\mu u_\mu^* + v_\mu v_\mu^* + w_\mu w_\mu^*\right]$$

$$\cdot F \begin{pmatrix} A_\lambda + a w_\lambda + a_{\lambda\nu} v_\nu,\ \varphi + cu + c_\lambda u_\lambda + b_\lambda v_\lambda,\ \varphi_\lambda + c_{\lambda\nu} u_\nu + b_{\lambda\nu} v_\nu; \\ A'_\lambda + a w_\lambda^* + a_{\lambda\nu} v_\nu^*,\ \varphi' + cu^* + c_\lambda u_\lambda^* + b_\lambda v_\lambda^*,\ \varphi'_\lambda + c_{\lambda\nu} u_\nu^* + b_{\lambda\nu} v_\nu^* \end{pmatrix} . \quad (84)$$

In the Appendix we prove that in the singular limit $x^2 \to 0$,

$$\left.\begin{array}{l} a \sim 1/x \;,\; a_{\mu\nu} \sim 1/x \;,\; b_\mu \sim 1/\mu x^2 \;,\; b_{\mu\nu} \sim 1/\mu x^3 \\ c \sim 1/x \;,\; c_\mu \sim 1/x^2 \text{ and } c_{\mu\nu} \sim 1/x^3 \end{array}\right\} \quad (85)$$

Clearly our parametrization has ensured that the most singular behaviour associated with A_μ as a result of the shift is $1/x$ (since both a and $a_{\mu\nu} \sim 1/x$). Likewise for $\varphi \sim 1/x^2$ and $\partial_\mu \varphi \sim 1/x^3$. This means that the parametrization adopted has ensured that for the E-F sums carried out in this manner $A_\mu \sim M$ is <u>normal</u> while the spin zero field $\varphi \sim M^2$ behaves <u>abnormally</u> 31) .

(ii) <u>Yang-Mills theory</u>

The pattern here is the same; the isospin complications being quite inessential to the ultraviolet count - to take account of them an additional isospin index is acquired by every auxiliary variable of integration, $c \to c^i$. The basic free Lagrangian is otherwise identical to the neutral case and the power count goes through similarly: viz.

$$\underset{\sim}{A}_\mu \sim M \ \underline{\text{normal}}\ , \text{ but } \underset{\sim}{\varphi} \sim M^2 ,\ \partial_\mu \underset{\sim}{\varphi} \sim M^3 \ \underline{\text{abnormal}} .$$

To investigate the ultraviolet character of the interaction term it is only the $\underset{\sim}{\varphi}$ containing part of the Lagrangian

$$\mathcal{L}_{int}(\varphi) \approx (\mathcal{J}_\nu - \frac{g}{\mu} \partial_\nu \varphi) A_\nu$$

which needs to be studied, since the purely A-dependent part of $\mathcal{L}_{int}$ is obviously normal. Now whenever $\mathcal{S}(\varphi) \sim 1$ ($k \leqslant -1$, for example, in the Schwinger and Weinberg parametrizations), $\mathcal{J}_\mu = i \mathcal{S}^\dagger \partial_\mu \mathcal{S} \sim M$. Hence $\mathcal{J}_\nu A_\nu \sim M^2$ and $\mathcal{L}_{int} \sim M^4$ is normal.

9. CONCLUSIONS

We have shown in this paper that a simple power count of ultraviolet infinite integrals in Efimov-Fradkin sums of perturbation diagrams suggests that non-linear meson theories have only a few types of integrals which are infinite. It is likely that these infinities can be absorbed into a renormalization of a few constants. (This part of the programme not discussed here we hope to attempt in a separate paper.) Likewise we have shown that the same conclusion holds for vector meson interactions (including, as we hope to show elsewhere, weak interaction theories mediated by intermediate bosons) and non-linear gravitational theory of Einstein. An essential ingredient here was the introduction of non-interacting (or self-interacting) spin zero fields which by suitable transformations are made to mix with the redundant longitudinal components of the vector field. The E-F sums of perturbation theory are carried out at this stage of mixing and this procedure apparently leads to the normality of the theory. There appears in principle no reason why this method of introducing non-interacting auxiliary fields could not be used to renormalize also theories of higher spins like $J = 2$.

A number of fundamental problems remain, basic to the whole approach, which are unresolved. There is the difficult problem of uniqueness of the sums, the renormalization programme, the problem of contours in the auxiliary variable planes. There is the problem of treating zero mass Yang-Mills field recently discussed by Feynman, de Witt, Fadeev and Popov and Mandelstam[32]. (Does the massive Yang-Mills theory of Sec. 7 coincide in the limit $\mu \longrightarrow 0$ with the theory discussed by the above authors? It can be easily checked that this limit is finite in our theory.) In this paper we have been conservative with our Lagrangians; there is the whole question of whether by modifying the starting Lagrangians - for example the mass term in Yang-Mills case - one may possibly obtain supernormal theories where self-mass and self-charge are also finite. As Lee [33] has remarked, the most noticeable feature of the infinities is their total absence in nature. The present approach appears to open up fascinating possibilities in these directions.

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APPENDIX I

We give here proofs of the singular behaviours of the shifted arguments occurring in (39) for derivative coupling theories. Our only concern is the (Euclidean) limit $x \to 0$, of eqs. (38) where $\Delta(x) \sim 1/x^2$. To solve eqs. (38) let

$$c_{\mu\nu} \equiv d_{\mu\nu}(x)\, c_1 + e_{\mu\nu}(x)\, c_0$$

and

$$c_\mu = x_\mu c_2$$

with

$$d_{\mu\nu}(x) \equiv g_{\mu\nu} - x^{-2} x_\mu x_\nu \equiv g_{\mu\nu} - e_{\mu\nu}(x)$$

and make the symmetrical choice $c = c'$ as in the text. Since

$$\Delta_{\mu\nu} = -2\Delta' \, d_{\mu\nu} - (2\Delta' + 4x^2 \Delta'')\, e_{\mu\nu}$$

and

$$\Delta_\mu = 2x_\mu \Delta'$$

we obtain the equations

$$c_1^2 = -2\Delta' \ , \quad c_0^2 = -2(\Delta' + 2x^2 \Delta'')$$

$$c_0 c_2 = 2\Delta' \ , \quad c^2 + x^2 c_2^2 = \Delta$$

which are solved by

$$c_1 = [-2\Delta']^{\frac{1}{2}} \sim 1/x^2$$

$$c_0 = [-2(\Delta' + 2x^2\Delta'')]^{\frac{1}{2}} \sim i\sqrt{6}/x^2$$

$$c_2 = 2\Delta'/c_0 \sim i\sqrt{2}/\sqrt{3}\,x^2$$

and

$$c = [\Delta - x^2 c_2^2]^{\frac{1}{2}} \sim 1\Big/\sqrt{3x^2} \ .$$

This proves the statement that a correct estimate of the most singular behaviour is given by

$$c_{\mu\nu} \sim 1/x^2 \ , \quad c_\mu \sim 1/x \ \text{ and } \ c \sim 1/x \ .$$

APPENDIX II

We perform the same steps as in Appendix I to discuss the ultraviolet characteristics of vector meson theories. The equations to be solved are (83). Set

$$a_{\mu\nu}(x) = d_{\mu\nu}(x)\, a_1 + e_{\mu\nu}(x)\, a_0 \qquad \text{and, similarly ,}$$

$$b_{\mu\nu} = d_{\mu\nu}\, b_1 + e_{\mu\nu} b_0 \ , \qquad b_\mu = x_\mu b_2$$

$$c_{\mu\nu} = d_{\mu\nu} c_1 + e_{\mu\nu} c_0 \ , \qquad c_\mu = x_\mu c_2 \ .$$

Since, near $x^2 = 0$,

$$\partial^{-2} \partial_\mu \partial_\nu \Delta(x) \sim \partial_\mu \partial_\nu \log \sqrt{x^2} = [d_{\mu\nu}(x) - e_{\mu\nu}(x)] / x^2$$

we have the limiting equations

$$a^2 \sim -1/x^2 \ , \quad a_1^2 = -a_0^2 \sim 1/x^2$$

$$a_0 b_2 \sim -2/\mu x^4 \ , \quad a_1 b_1 \sim 2/\mu x^4 \ , \quad a_0 b_0 \sim -6/\mu x^4$$

$$b_1^2 + c_1^2 \sim 2/x^4 \ , \quad b_0^2 + c_0^2 \sim -6/x^4 \ , \quad c_0 c_2 + b_0 b_2 \sim -2/x^4$$

$$c^2 + x^2 c_2^2 + x^2 b_2^2 \sim -1/x^2$$

which solve as

$$a_1 \sim i a_0 \sim i a \sim 1/x$$

$$b_1 \sim \frac{1}{3} i b_0 \sim i b_2 \sim 2/\mu x^3$$

$$i c_1 \sim \frac{1}{3} c_0 \sim c_2 \sim 2/\mu x^3$$

but $\qquad c \sim i/x$.

For the tensorial quantities therefore we obtain, up to proportionality factors,

$$a_{\mu\nu} \sim 1/x \quad , \qquad a \sim 1/x$$

$$b_{\mu\nu} \sim 2/\mu x^3 \quad , \qquad b_{\mu} \sim 2/\mu x^2$$

$$c_{\mu} \sim 2/\mu x^3 \quad , \qquad c_{\mu} \sim 2/\mu x^2 \quad , \quad c \sim 1/x$$

as stated in the text. The most singular terms of the shifted arguments in (84) thereby demonstrate the ultraviolet characteristics

$$A \sim 1/x \quad , \quad \varphi \sim 1/x^2 \quad , \quad \partial_{\mu}\varphi \sim 1/x^3 \quad .$$

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C. Isham, Imperial College, London, preprint ICTP/68/15 (Nuovo Cimento to be published).

5) We list below some of the papers where summation of perturbation diagrams or infinite parts of such diagrams has been carried out, using widely different techniques.
R. Arnowitt and S. Deser, Phys. Rev. 100, 349 (1955);
G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963);
T.D. Lee and C.N. Yang, Phys. Rev. 128, 885 (1962);
Abdus Salam, Phys. Rev. 130, 1287 (1963);
Abdus Salam and R. Delbourgo, Phys. Rev. 135, B1398 (1964);
M. Baker, K. Johnson and R. Willey, Phys. Rev. 136, B1111 (1964);
T.D. Lee, TH140 (CERN preprint 1968).

6) Throughout this paper we use the terms "divergent" for series and "ultraviolet infinite" for integrals.

7) G.V. Efimov, Soviet Phys.-JETP, 17, 1417 (1963);
E.S. Fradkin, Nucl. Phys. 49, 624 (1963);
See also subsequent work by
G.V. Efimov, Nuovo Cimento 32, 1046 (1964) and Nucl. Phys. 74, 657 (1965).

8) Equivalence theorems for zero spin derivative coupling theories were first given by E.C. Nelson, Phys. Rev. 60, 830 (1941);
K.M. Case, Phys. Rev. 76, 14 (1949);
F.J. Dyson, Phys. Rev. 73, 929 (1948).
For spin one case see
Abdus Salam, Nucl. Phys. 18, 681 (1960);
A. Komar and Abdus Salam, Nucl. Phys. 21, 624 (1960);
S. Kamefuchi and H. Umezawa, ibid 23, 399 (1961);
Abdus Salam, Phys. Rev. 127, 331 (1962).

9) M. Veltman, preprint 68/28, Utrecht;
P. Ghose, preprint (Santineketan University).
This formulation has close connections with Schwinger's theory of extended operators: J. Schwinger, Phys. Rev. 125, 324 (1962).

10) This is contrary to the conclusion of T.D. Lee (Ref. 5) who has recently studied neutral pseudovector mesons interacting with nucleons. Lee finds a number of relations between the infinite renormalization constants but no limit to the types of different infinities in the theory. We believe the difference between our result and that of Lee comes from the use of the Veltman-Ghose mixing which we introduce.

11) S. Hori, Progr. Theoret. Phys. 7, 589 (1952).

12) The type of problem which one meets is that $\mathcal{L}_{int}(\varphi)$ may look abnormal when expressed in terms of one set of fields but normal in another formulation. Consider for example a theory with $\mathcal{L}_{int} = 2\varphi\,(1+\varphi)\,(\partial\varphi)^2 + \varphi^3\,(1+\varphi)^3$ which looks hideously

abnormal but can be transformed into the familiar supernormal $\bar{\varphi}^3$ theory with the substitution $\bar{\varphi} = \varphi\,(1 + \varphi)$.

13) The integrands (18) and (19) exhibit poles in the ξ-plane and pose the problem of defining the correct contour of integration in the ξ-plane, such that a power series expansion of EF integrals coincides with the perturbation expansion. We believe (contrary to the opinion expressed by Efimov and Fradkin) that this problem is bound up with the problem of defining the Fourier integral (20) away from the Symanzik region in p-space. In other words the ξ-integrations should be done after the Fourier transform (20) is taken.

14) See for instance N. N. Bogolubov and D. V. Shirkov, "Introduction to the theory of quantized fields", (Interscience, 1959).

15) The Symanzik region is defined by the condition that the linear combinations $\sum_j \alpha_j p_j$ be spacelike for any choice of real parameters α_j, i.e., the Gram determinants of $-(p_i \cdot p_j)$ are positive for all sets of i and j.

16) There may of course be hidden infinities from sub-integrations which, though they are not discussed here, form an integral part of the renormalization programme. We hope to study these in future work.

17) F. J. Dyson, Phys. Rev. 75, 1736 (1949).

18) P. T. Matthews, Phys. Rev. 75, 1270 (1949),
F. J. Dyson, Phys. Rev. 83, 608 (1951).

19) For criteria when the S-matrices are equivalent see Sec. 6 and Refs. 23 and 24.

20) The co-ordinate independence of the Lagrangian on the differential manifold has been proved by S. Coleman, J. Wess and B. Zumino (Phys. Rev. to be published) and C. Isham (Nuovo Cimento to be published).

21) H. Sugawara and M. Yoshimura, Chicago preprint COO 264-429.

22) In applying the exponential shift lemma for making an ultra-violet count one has to introduce isotopic labels U^i to the auxiliary variables of integration. See the forthcoming paper by Koller.

23) J.S.R. Chisholm, Nucl. Phys. 26, 469 (1961);
H.J. Borchers, Nuovo Cimento 15, 784 (1960);
S. Kamefuchi, L.O'Raifeartaigh and Abdus Salam, Nucl. Phys. 28, 529 (1961).

24) S. Coleman, J. Wess and B. Zumino (Phys. Rev. to be published).

25) Abdus Salam, Phys. Rev. 127, 331 (1962).

26) The transverse and longitudinal parts refer to a four-dimensional split of the 4-vector U_μ.

27) E.C.G. Stückelberg, Helv. Phys. Acta 11, 225 and 299 (1938).

28) The B field does possess a mixing with the A_μ field, but only with the longitudinal piece $\partial_\mu A_\mu$, not with the transverse part A^1_μ.

29) It is worth recording that when we attach a mass κ to the B-field the modified propagator (64) reads

$$\underset{\sim}{\Delta} = \left(\begin{array}{c|c} -d_{\mu\nu}(p)\,\Delta(p,\mu) - \kappa^{-2}\mu^{2}\, e_{\mu\nu}\,\Delta(p,\kappa) & i\,p_\mu\,\Delta(p,\kappa)/\mu \\ \hline -i\,p_\nu\,\Delta(p,\kappa)/\mu & -\Delta(p,\kappa) \end{array}\right)$$

so that in the special case $\kappa = \mu$, the vector sector is just the Stückelberg propagator.

30) We could have eliminated one further scalar variable (a) but at the expense of complicating the factors $a_{\mu\lambda}$, $b_{\mu\lambda}$, etc.

31) In contrast to the case of non-linear Lagrangians treated in Sec. 5, where the scalar fields φ were <u>physical</u> and behaved <u>normally</u>.

32) R.P. Feynman, Acta Phys. Polon. 24, 697 (1963);
B.S. De Witt, Relativity Groups and Topology (Blackie & Son Ltd., 1964) pp. 587-820;
L.D. Faddeev and V.N. Popov, Phys. Letters 25B, 29 (1967);
S. Mandelstam, Phys. Rev. 175, 1580 (1968); Phys. Rev. 175, 1604 (1968).

33) T.D. Lee, CERN Conference on Weak Interactions, 1969.

FIGURE CAPTIONS

Fig. 1(a) Self-energy diagrams $S_{200...}$

Fig. 1(b) Self-energy diagrams $S_{110...}$

Fig. 2 Typical vacuum transition S_{00}.

Fig. 3(a) Self-energy diagram S_{20}.

Fig. 3(b) Self-energy diagram S_{11}.