ELASTIC ELECTRON SCATTERING
FROM NUCLEAR MAGNETIC MOMENTS IN $^9\text{Be}$
AND PROJECTED HARTREE-FOCK WAVE FUNCTIONS

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ABSTRACT

General formulas are given for calculating magnetic form factors with projected Hartree-Fock functions. As an application, the magnetic form factor of $^9\text{Be}$ is calculated. It is found that the octupole part of the magnetic form factor depends on the deformation of the Hartree-Fock field. The agreement with experiment is satisfactory.

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1. INTRODUCTION

Elastic scattering of high-energy electrons has been developed into a powerful tool for investigating the charge distribution and the magnetic structure of nuclei. Especially in light nuclei, where the Born approximation should be justified, the interpretation of the experimental cross-sections is simple.

The charge form factors which, at low momentum transfers, are essentially equal to the contribution of the monopole part alone, have been found to contain a large contribution from the quadrupole term at higher momentum transfers \( q^2 > 2 \text{ fm}^{-2} \). This latter part is systematically underestimated by shell-model calculations by about a factor of four in the 0p-shell nuclei. In previous papers \(^1\), \(^2\), it has been shown that the experimental charge form factors are accurately reproduced over the whole range of momentum transfers by wave functions obtained from a projected Hartree-Fock (PHF) calculation. The large deformation of the Hartree-Fock field provides the necessary enhancement of the quadrupole term.

The magnetic form factors which at low momentum transfers are essentially equal to the contribution of the dipole part alone, have been found \(^3\) to contain a large contribution from the octupole term at higher momentum transfers \( q^2 > 1.5 \text{ fm}^{-2} \). The experimental data \(^3\) for \(^9\)Be and \(^11\)B have been compared \(^4\) with the predictions of the shell model both in the LS and jj-coupling limits and also with the single-particle model prediction. No satisfactory fit is obtained for any of these three theoretical descriptions. A very good fit can be obtained if the ratio of the static magnetic octupole moment to the magnetic dipole moment is treated as an adjustable parameter. The
calculated form factor is, moreover, adjusted to the experimental form factor at $q^2 = 0$, which should be done only if the static magnetic dipole moment is accurately reproduced. This is not the case in the theoretical models considered.

In this paper, we want to investigate whether the PHF wave function, which gives a satisfactory description of the charge form factor of $^9$Be, can also reproduce its magnetic form factor. It is particularly interesting to study the influence of the deformation of the Hartree-Fock field on the octupole part of the form factor. Since the PHF function reproduces the magnetic dipole moment of the ground state, the calculated form factor will fit the experimental form factor at $q^2 = 0$ and we are allowed to normalize them both to unity at zero momentum transfer.

In Sec. 2, general formulas are given for calculating the magnetic form factors with our PHF functions. These formulas are different and slightly more complicated than the general formula, derived by Gunye and Warke, for calculating the matrix element of a general one-body tensor operator between projected Hartree-Fock functions. The difference stems from the fact that Gunye and Warke consider a J-projection from a Slater determinant whose orbitals are expanded in eigenfunctions of $j^2$, whereas our PHF functions are L-projections from a Slater determinant whose orbitals are expanded in eigenfunctions of $I^2$. For the light nuclei in the 0p-shell, it is to be expected that an L-projection gives a better wave function since it is well known that the best shell-model description is close to LS-coupling.

In Sec. 3, these general formulas are specialized to the case of $^9$Be and in Sec. 4 the results are compared with the experimental data and with the results of a shell-model calculation.
2. CALCULATION OF MAGNETIC FORM FACTORS WITH PHF FUNCTIONS

The magnetic form factor $F_M(q^2)$ as well as partial magnetic form factors for each (odd) multipole $F_M^{\lambda}(q^2)$ have been defined by Griffy and Yu. All these form factors are normalized to unity at $q^2 = 0$. In the most useful case where the ground state spin $J$ is not larger than two, only a dipole and an octupole term are present:

$$\left| F_M(q^2) \right|^2 = \left| F_M^1(q^2) \right|^2 + \frac{2q^4}{75} \left( \frac{\mu_3}{\mu_1} \right)^2 \left| F_M^3(q^2) \right|^2 .$$

(1)

The form factor of rank $\lambda$ can be defined as

$$\left| F_M^\lambda(q^2) \right|^2 = \left| \frac{\langle M^\lambda(q) \mid J \rangle}{\langle M^\lambda(0) \mid J \rangle} \right|^2 .$$

(2)

Here $M^\lambda_M(q)$ is the operator given by Willey.

$$M^\lambda_M(q) = \frac{|e|}{\lambda + 1} \frac{(2\lambda + 1)\mu}{q} \left( \frac{4\pi q}{\mu \cdot c^2} \right) \sum_{i=1}^A \left[ \varepsilon_i \left( \frac{\lambda + 1}{2\lambda + 1} \right)^{1/2} j_{\lambda+1}(q\mathbf{r}_i) \right]$$

$$\times \left[ \frac{\lambda}{2} \left( \frac{\lambda + 1}{2\lambda + 1} \right)^{1/2} \right]$$

$$j_{\lambda+1}(q\mathbf{r}_i) \frac{\mathbf{Y}_{\lambda+1}(\mathbf{r}_i \cdot \mathbf{J}_i) \cdot \mathbf{r}_i}{\lambda + 1} + \frac{\lambda + 1}{2} \left( \frac{\lambda}{2\lambda + 1} \right)^{1/2} j_{\lambda-1}(q\mathbf{r}_i) \frac{\mathbf{Y}_{\lambda-1}(\mathbf{r}_i \cdot \mathbf{J}_i) \cdot \mathbf{r}_i}{\lambda - 1}$$

(3)

where $\varepsilon_i$ and $\gamma_i$ are respectively the charge and the magnetic dipole moment of the $i$th nucleon and $\mathbf{Y}_{\lambda \mu}$ is a vector spherical harmonic. The operator $M^\lambda_M(q)$ is defined in such a way that the static magnetic multipole moment of rank $\lambda$ is given by
\[ \mu_\lambda = \sqrt{\frac{4\pi^3}{2\lambda+1}} \langle JJ | M_{\lambda 0}(q=0) | JJ \rangle \quad \text{(4)} \]

It should be noticed here that for \( \lambda = 3 \) this definition of \( \mu_3 \) coincides with that of Elliott and Lane \(^9\) but differs in sign from that given by Schwartz \(^10\).

The PHF functions obtained in our calculations \(^1\), \(^2\) will be denoted \( |\text{KLSJM}\rangle \). They are \( L \)-projections from a single Slater determinant \( \Phi_K \) which is an eigenfunction of \( L_z \) and of \( S^2 \) with eigenvalues \( K \) and \( S(S+1) \), respectively. The orbital angular momentum \( L \) and the spin angular momentum \( S \) are next coupled to a total \( J \). The following notation is used for \( \Phi_K \):

\[ \Phi_K = \frac{1}{\sqrt{\lambda!}} \det |u_1 \ u_2 \ldots \ u_\lambda| \quad \text{(5)} \]

where the orbitals \( u_k \) \((k = 1, \ldots, \lambda)\) are expanded in a finite basis of oscillator functions \( |n \ell \sigma \rangle \)

\[ \mu_k = \sum_{\ell \sigma} x_{nk}^k |n \ell \ m_k \sigma_k \rangle \quad \text{(6)} \]

The orbitals \( u_k \) are eigenfunctions of \( L_z, \sigma_z \) and \( \tau_3 \).

In order to calculate \( F_{H\lambda}(q^2) \), we make use of the following general formulas, which can also be used in the case of inelastic scattering (the reduced matrix element being that defined by Edmonds \(^8\)):

\[ \langle K'L'SJ' | \sum_{i=1}^A \epsilon_i j_{\lambda i}(qr_i) \mathcal{F}_{\lambda\lambda_i}(\hat{r}_i) \mathcal{F}_{i} | KLSJ \rangle = (-)^{J+S+K+K'+\lambda} \]

-4-
where \( \det O(k'|k) \) is the determinant of the matrix obtained by erasing the row \( k' \) and the column \( \ell \) in the overlap matrix of the orbitals \( \Phi_{k'} \) and \( \Phi_{\ell} \). The \( \sum_{n^L} \) runs over the \( A \) orbitals in \( \Phi_{k} \) and the \( \sum_{k'=1}^{A} \) runs over all the basis states in the expansion of the orbital \( u_{k} \). Similarly, one has

\[
\langle k'| L' \ S \ J' || \sum_{i=1}^{A} Y_{i}^{\dagger} [\gamma_{i}(q_{i})] \Delta q_{i}^{\dagger} \gamma_{i}^{\dagger} \rangle \mid \ell \ L \ S \ J \rangle = \]

\[
(-)^{\lambda' - L' - K - K' \atop (J' + 1)(2J' + 1) \over 2} \left( \frac{(2J+1)(2J' + 1)(2\lambda + 1)(S+1)(2S' + 1)}{S' \langle \Phi_{k}^{*} P_{L}^{*} \Phi_{k} \rangle \Phi_{k'}^{*} P_{L}^{*} \Phi_{k'} \rangle} \right)^{1/2} \]
The matrix elements \( \langle n'\ell'0 | j_\lambda, Y_{\lambda'0} | n\ell0 \rangle \) can be taken over from the calculation of the charge form factor \( \gamma \).

3. APPLICATION TO \(^9\)Be

The PHF wave function \(^2\) obtained for the ground state of \(^9\)Be contains a mixture of two L-values

\[
\Psi (J = \frac{3}{2}) = \sqrt{1 - x^2} |L=1 S=\frac{1}{2} J=\frac{3}{2} \rangle + x |L=2 S=\frac{1}{2} J=\frac{3}{2} \rangle
\]

(9)

where \( |LSJ \rangle \) is the function defined in the previous section. The Slater determinant \( \Phi_K \) has the special form

\[
\Phi_{K=1} = \frac{1}{\sqrt{9!}} \det \left| \psi_0^4 \phi_0^4 \phi_1^{n+} \right|
\]

(10)

with an obvious notation for the spin-isospin structure and is an eigenfunction of the spin with eigenvalue \( \frac{1}{2} \). The spatial functions are expanded into harmonic oscillator functions.
\[ \psi_0 = |0s0\rangle + \beta |0d0\rangle \]
\[ \varphi_0 = |0p0\rangle + \delta |0f0\rangle \]
\[ \varphi_1 = |0p1\rangle + \xi |0f1\rangle \]  \( (11) \)

The values of the parameters \( \beta, \delta \) and \( \xi \) are given in Ref. 2 for \( L = 1 \) and \( L = 2 \). Since \( Y(J = \frac{3}{2}) \) contains two terms with different values of \( L \), the calculation of \( F_M \) requires calculating both diagonal and off-diagonal matrix elements. The evaluation of \( (8) \) and \( (9) \) is straightforward but lengthy. It was made on an IBM 360/40 computer. The usual correction for centre-of-mass motion and finite proton size was applied by taking the same correction factor as in the case of the charge form factor \( (2) \).

4. RESULTS

The form factor \( |F_M(q^2)|^2 \) calculated with the PHF function (10) with \( x = -0.3775 \) is shown in Fig. 1. The experimental points are taken from Vanpraet and Kossanyi-Demay \( (11) \) at very low momentum transfers and from Rand, Frosch and Yearian \( (3) \) at higher momentum transfers. Although the agreement is less good than in the case of the charge form factor, we believe it still to be satisfactory since no parameter was adjusted.

The dependence on the deformation is shown in Fig. 2 by plotting also the form factor (curve 2) calculated with the shell-model function (i.e., putting \( \bar{q} = \delta = \xi = 0 \) in (10)). A considerably larger octupole part is obtained than with the PHF function (curve 1). Fig. 2 also illustrates the dependence on the strength of the spin-orbit force: curves 3 and 4 show the form factors obtained in the LS-coupling limit \( (x = 0) \), respectively with the PHF and the shell-model functions. These last two form factors have also been normalized to unity at \( q^2 = 0 \) as was done by Griffy and Yu \( (4) \), although the LS-coupling limit does not
reproduce the experimental magnetic dipole moment. Fig. 2 shows that the octupole part of the form factor increases when going from LS to intermediate coupling.

In Table 1, we have collected the values of the static magnetic dipole and octupole moments, calculated with the PHF and shell model functions, both in LS and intermediate coupling. One sees that increasing the deformation leaves $\mu_1$ unchanged but decreases $\mu_3$, while increasing the spin orbit strength results in a decrease in absolute value of $\mu_1$ and does not affect $\mu_3$ very much.

5. CONCLUSIONS

In this paper we have obtained general formulas for calculating the magnetic form factor with the wave function obtained from a PHF calculation. These formulas have been used to calculate the form factor of $^9$Be. A satisfactory fit is found with the experimental data for the wave function which reproduced the charge form factor. Although, as is seen from Fig. 2, a similarly good fit is obtained with the shell-model function in LS-coupling, this latter fit is not satisfactory since the LS-coupling function gives a quite wrong magnetic dipole moment so that the normalization at $q^2 = 0$ cannot be justified.

The deformation of the HF field has been found to have a clear influence on the octupole term in the magnetic form factor. Although this influence is considerably less than in the case of the quadrupole term in the charge form factor, it results in a considerable improvement of the form factor calculated in the shell-model approximation.

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REFERENCES


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Magnetic dipole on octupole moment for the ground state of $^9$Be calculated with different wave functions.
FIGURE CAPTIONS

Fig. 1  Magnetic form factor for $^9$Be calculated with the PHF function. The dashed lines show the dipole and octupole contributions separately.

Fig. 2  Magnetic form factor for $^9$Be calculated with different functions.
curve 1: PHF function in intermediate coupling
curve 2: shell-model function in intermediate coupling
curve 3: PHF function in LS-coupling
curve 4: shell-model function in LS-coupling.