IC/68/107

INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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1968

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3007

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ABSTRACT

Use is made of the formalism of non-linear realizations to treat the spontaneous violation of conformal symmetry. The associated "Goldstone" particles, which are characterized by a 4-vector and a scalar field, are unusual in that they possess mass.

MIRAMARE - TRIESTE

December 1968

* To be submitted for publication.

** On leave of absence from Imperial College, London, England.

NON-LINEAR REALIZATIONS - II: CONFORMAL SYMMETRY

1. INTRODUCTION

In the accompanying paper [1] some formal developments of the technique of non-linear realizations were discussed. The discussions of paper I were confined to internal symmetry groups. In the present paper the non-linear method is extended to include space-time symmetry as well. Thus, it will be shown that, with appropriate modifications, those field equations which manifestly exhibit Poincaré invariance can be rendered in a form possessing conformal invariance of the non-linear type.

The group of conformal transformations [2] in space-time includes, in addition to translations and homogeneous Lorentz transformations, the characteristic special conformal transformations,

$$\frac{x_\mu}{x^2} \rightarrow \frac{x_\mu}{x^2} + \beta_\mu \tag{1.1}$$

and the dilatations,

$$x_\mu \longrightarrow x_\mu e^\lambda \tag{1.2}$$

where $x^2 = x_\mu x_\mu$ denotes the usual Lorentz invariant squared length, and the parameters β_μ and λ are real. There is no reason to expect these transformations to represent symmetries of nature - rather the opposite in fact. However, it is conceivable that they could represent "broken" symmetries and so be useful in physics.

Now in the accompanying paper it was suggested that the non-linear formalism is a natural vehicle for discussions of spontaneously broken internal symmetries. In particular, the massless bosons whose fields appear non-linearly in the effective Lagrangian are to be looked upon as Goldstone bosons. The lack of manifest (i.e., linear) covariance in such Lagrangians is simply a reflection of the truly non-covariant nature of the underlying physics. If the same situation holds in the case of conformal symmetry, we may expect that non-linear realizations of the conformal group (linear relative to the Poincaré group) may be the natural vehicle for expressing this. The following discussion is thus devoted to the proposition that conformal symmetry is broken spontaneously. The Lagrangian but not the ground state is assumed to be invariant under the transformations (1.1) and (1.2). Of course it will be assumed that both the Lagrangian and the ground state are invariant under the usual inhomogeneous Lorentz transformations. In line with the attitude set out in I, we shall treat the broken symmetry by means of non-linear realizations which become linear with respect to the inhomogeneous Lorentz subgroup. There are, however, some important departures from the general features of intrinsically broken internal symmetries discussed in I which we summarize here.

The five preferred fields (or Goldstone bosons) whose existence is necessitated by the non-covariance of the vacuum with respect to the special conformal transformations (1.1) and dilatations (1.2) comprise a 4-vector, ϕ_μ , and a scalar, σ . Unlike the case of internal symmetries, some of the Goldstone bosons have <u>spin</u>. More surprisingly, they all have <u>mass</u>. In Sec. 3, where effective Lagrangians are discussed, it will be shown that it is quite within the rules to include mass-like terms for the preferred fields in the Lagrangian. [3)]

The non-linear realizations of the conformal group are presented in Sec. 2 together with formulae for covariant derivatives. These are used in Sec. 3 to construct effective Lagrangians. These Lagrangians are manifestly invariant under the inhomogeneous Lorentz group only, and can therefore include, for example, mass terms, which are forbidden in a truly conformal invariant theory. An example is discussed in Sec. 3. Finally, the detailed transformation properties of the preferred and other fields are derived in the Appendix.

2. NON-LINEAR REALIZATIONS OF THE CONFORMAL GROUP

Discussion of the conformal transformations of space-time is facilitated by exploiting their equivalence to the orthogonal transformations in six dimensions.[4)] Let us begin, therefore, with linear representations of the group O(4, 2),

$$\Psi(\eta) \rightarrow \Psi'(\eta) = D(\Lambda)\, \Psi(\Lambda^{-1}\eta) \tag{2.1}$$

where η denotes the 6-dimensional co-ordinate vector η_A and Λ a pseudo-orthogonal transformation on these co-ordinates,

$$\eta_A \rightarrow \eta'_A = \Lambda_{AB}\, \eta_B \quad . \tag{2.2}$$

Throughout this section the following summation convention for upper case latin indices is tacitly assumed:

$$\begin{aligned} \xi_A \eta_A &= \xi_\mu \eta_\mu - \xi_5 \eta_5 + \xi_6 \eta_6 \\ &= \xi_0 \eta_0 - \xi_1 \eta_1 - \xi_2 \eta_2 - \xi_3 \eta_3 - \xi_5 \eta_5 + \xi_6 \eta_6 \quad . \end{aligned} \tag{2.3}$$

The matrices $D(\Lambda)$ comprise a finite-dimensional representation of O(4, 2). Corresponding to the infinitesimal transformation $\Lambda = 1 + \epsilon$ they take the form

$$D(\Lambda) = 1 - \frac{i}{2}\, \epsilon_{AB}\, J_{AB} \quad , \tag{2.4}$$

where the generators of infinitesimal transformations J_{AB} must satisfy the usual commutation rules,

$$\frac{1}{i}\, [J_{AB}, J_{CD}] = g_{BC} J_{AD} - g_{BD} J_{AC} + g_{AD} J_{BC} - g_{AC} J_{BD} \quad , \tag{2.5}$$

where g_{AB} is the metric tensor defined by the form (2.3). The generators of the 6-dimensional self-representation are given by

$$(J_{AB})_{CD} = i(g_{AC}g_{BD} - g_{AD}g_{BC}) \ . \tag{2.6}$$

The fifteen generators of conformal transformations are defined in terms of the set J_{AB} as follows:

$$\begin{aligned} &\text{homogeneous transformations,} && J_{\mu\nu} \\ &\text{translations,} && P_\mu = J_{5\mu} + J_{6\mu} \\ &\text{special conformal transformations,} && K_\mu = J_{5\mu} - J_{6\mu} \\ &\text{dilatations,} && D = J_{56} \ . \end{aligned} \tag{2.7}$$

This assertion can be justified by computing the effect of finite transformations generated by P_μ, K_μ and D, respectively, on the 4-vector x_μ defined by

$$x_\mu = \frac{\eta_\mu}{\eta_5 + \eta_6} \ . \tag{2.8}$$

Clearly this quantity is indeed a 4-vector under the transformations generated by $J_{\mu\nu}$. The action on η_A of the other transformations can be got by integrating (2.4) with the explicit generators (2.6). The results are,

$$e^{i\alpha\cdot P}\eta = \begin{bmatrix} \eta_\mu + \alpha_\mu(\eta_5 + \eta_6) \\ \eta_5 + \alpha\cdot\eta + \frac{\alpha^2}{2}(\eta_5 + \eta_6) \\ \eta_6 - \alpha\cdot\eta - \frac{\alpha^2}{2}(\eta_5 + \eta_6) \end{bmatrix}$$

$$e^{i\beta\cdot K}\eta = \begin{bmatrix} \eta_\mu + \beta_\mu(\eta_5 - \eta_6) \\ \eta_5 + \beta\cdot\eta + \frac{\beta^2}{2}(\eta_5 - \eta_6) \\ \eta_6 + \beta\cdot\eta + \frac{\beta^2}{2}(\eta_5 - \eta_6) \end{bmatrix}$$

$$e^{i\sigma D}\eta = \begin{bmatrix} \eta_\mu \\ \eta_5 \,\mathrm{ch}\sigma + \eta_6 \,\mathrm{sh}\sigma \\ \eta_5 \,\mathrm{sh}\sigma + \eta_6 \,\mathrm{ch}\sigma \end{bmatrix} \tag{2.9}$$

The corresponding transformations of the quantity x_μ defined in (2.8) are given by, respectively,

$$\begin{aligned} x_\mu &\to x_\mu + \alpha_\mu \\ \frac{x_\mu}{x^2} &\to \frac{x_\mu}{x^2} + \beta_\mu \\ x_\mu &\to x_\mu e^{-\sigma} \end{aligned} \tag{2.10}$$

which are indeed a translation, a special conformal transformation and a dilatation. This completes the interpretation of the elements of the group SO(4, 2) as conformal transformations on space-time. What remains is to interpret the functions $\Psi(\eta)$ in terms of space-time fields.

To this end consider the "orbital" contribution to the generators J_{AB}. In particular, the infinitesimal change in a scalar field $\Phi(\eta)$ takes the form

$$\begin{aligned} \delta\Phi(\eta) &= \frac{1}{2}\epsilon_{AB}\,\partial_{AB}\,\Phi(\eta) \\ &= \frac{1}{2}\epsilon_{AB}\Big(\eta_A \frac{\partial}{\partial\eta^B} - \eta_B \frac{\partial}{\partial\eta^A}\Big)\,\Phi(\eta) \quad . \end{aligned} \tag{2.11}$$

In terms of the new set of independent variables,

$$x_\mu = \frac{\eta_\mu}{\eta_5 + \eta_6} \quad , \quad \kappa = \eta_5 + \eta_6 \;, \quad \lambda = \eta_5 - \eta_6 \;, \tag{2.12}$$

the components of the operator ∂_{AB} are given by

$$\partial_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

$$\partial_{5\mu} + \partial_{6\mu} = \partial_\mu + 2x_\mu \kappa \frac{\partial}{\partial\lambda}$$

$$\partial_{5\mu} - \partial_{6\mu} = \frac{\lambda}{\kappa} \partial_\mu - 2x_\mu (x_\nu \partial_\nu - \kappa \frac{\partial}{\partial\kappa}) \tag{2.13}$$

$$\partial_{56} = -x_\nu \partial_\nu + \kappa \frac{\partial}{\partial\kappa} - \lambda \frac{\partial}{\partial\lambda} \quad .$$

These forms can be simplified somewhat when it is recognized that the surface η^2 = constant is invariant under the group. If physical space-time is mapped into one such surface, $\eta^2 = 0$, then it is possible to eliminate the derivative, $\partial / \partial\lambda$, by suitably adjusting the behaviour of $\Phi(\eta)$ in the neighbourhood of the preferred surface. Thus, a replacement like

$$\Phi(\eta) \rightarrow \Phi(\eta) - (\lambda - \kappa x^2) \frac{\partial\Phi(\eta)}{\partial\lambda} \quad ,$$

while leaving $\Phi(\eta)$ unchanged on the surface $\eta^2 = 0$, does make $\partial\Phi/\partial\lambda$ vanish there. Henceforward, therefore, it will be assumed that $\lambda = \kappa x^2$ and that the necessary adjustment has been made, i.e.,

$$\partial_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

$$\partial_{5\mu} + \partial_{6\mu} = \partial_\mu$$

$$\partial_{5\mu} - \partial_{6\mu} = x^2 \partial_\mu - 2x_\mu (x_\nu \partial_\nu - \kappa \frac{\partial}{\partial\kappa}) \tag{2.14}$$

$$\partial_{56} = -x_\nu \partial_\nu + \kappa \frac{\partial}{\partial\kappa} \quad .$$

The only remaining obstacle to a space-time interpretation is the presence of the variable κ . The standard way to deal with this is to suppose that $\Phi(\eta)$ is a homogeneous function of degree ℓ , in which case the dependence on κ factors out as κ^ℓ . This factorization occurs only on the hypercone $\eta^2 = 0$, which is the reason for preferring it.

The decomposition of irreducible representations of O(4, 2) under the subgroup consisting of just the inhomogeneous Lorentz transformations is generally rather complicated. For example, the 6-vector

Φ_A contains two scalars and a 4-vector,

$$\eta_A \Phi_A \quad , \quad \Phi_5 + \Phi_6 \quad \text{and} \quad \Phi_\mu - x_\mu(\Phi_5 + \Phi_6) \quad , \tag{2.15}$$

as can be verified with the help of (2.9). This particular decomposition is the only one we shall need below. In general we shall be concerned in the following with an entirely different method of extracting space-time tensors from the linear representations of O(4, 2).

Turning now to the problem of constructing non-linear realizations of the conformal group, we must invent a set of constrained fields, or reducing matrix,[1)] L_ϕ , with the anomalous transformation behaviour

$$L_\phi(\eta) \rightarrow \Lambda L_\phi(\Lambda^{-1}\eta)\, h^{-1}(\Lambda^{-1}\eta, \Lambda) \quad , \tag{2.16}$$

where h is a matrix of some subgroup of O(4, 2). The non-linear fields ψ are then to be defined by

$$\psi(\eta) = D(L_\phi^{-1})\, \Psi(\eta) \quad , \tag{2.17}$$

where $\Psi(\eta)$ is the linear field (2.1). In order that the $\psi(\eta)$ so defined should be interpretable as a space-time distribution we must require that it be a homogeneous function of η . Moreover, since the preferred fields $L_\phi(\eta)$ satisfy a number of quadratic and inhomogeneous constraints, we must have them homogeneous of degree zero,

$$L_\phi(\lambda\eta) = L_\phi(\eta) \quad . \tag{2.18}$$

It then follows that $\psi(\eta)$ is homogeneous to the same degree as the linear field $\Psi(\eta)$ in which it is embedded.

The structure of the subgroup H to which the matrix $h(\eta, \Lambda)$ belongs is determined by the system of constraints to be imposed on the reducing matrix. There are, presumably, several distinct models from among which one can choose. For the present paper we choose the 5-parameter boost given by

$$L_\phi(\eta) = e^{ix\cdot P}\, e^{i\phi\cdot K}\, e^{-i\sigma D} \tag{2.19}$$

where P_μ, K_μ and D are the 6 x 6 matrices defined in (2.9). The parameters ϕ_μ and σ are to be the preferred fields. The factor exp(ixP) does not play an essential role. It has been included mainly to simplify the formulae for covariant derivatives to be derived below.

The laws according to which the preferred fields ϕ_μ and σ transform are determined by requiring that the two columns $(L_\phi)_{A5}$ and $(L_\phi)_{A6}$ transform as true 6-vectors. In other words, the matrices $h(\eta, \Lambda)$ are to act only in the subspace 0, 1, 2, 3. The subgroup H is to coincide with the ordinary homogeneous Lorentz group. That such conditions can be imposed with only five parameters at our disposal is by no means evident: the general programme discussed in I suggests that nine fields are needed. However, we shall demonstrate by an explicit calculation that five parameters suffice.

The two columns can be obtained with the help of the matrices given in (2.9). They are

$$\Phi_A = \frac{1}{2}\left[(L_\phi)_A{}^5 - (L_\phi)_A{}^6\right] = \begin{bmatrix} (\phi_\mu + x_\mu \phi^2)\, e^\sigma \\ \frac{1}{2}(1 + 2x\phi + x^2\phi^2 + \phi^2)\, e^\sigma \\ -\frac{1}{2}(1 + 2x\phi + x^2\phi^2 - \phi^2)\, e^\sigma \end{bmatrix} \quad (2.20)$$

$$\Psi_A = \frac{1}{2}\left[(L_\phi)_A{}^5 + (L_\phi)_A{}^6\right] = \begin{bmatrix} x_\mu\, e^{-\sigma} \\ \frac{1}{2}(1 + x^2)\, e^{-\sigma} \\ \frac{1}{2}(1 - x^2)\, e^{-\sigma} \end{bmatrix} \quad (2.21)$$

The parameter fields ϕ_μ and σ are clearly expressible in terms of the components Φ_A alone and their transformation properties are fixed by the requirement that Φ_A transform as a true 6-vector,

$$\Phi_A(\eta) \longrightarrow \Lambda_{AB}\, \Phi_B(\Lambda^{-1}\eta) \quad .$$

From the transformation behaviour of σ so deduced one can derive the law for Ψ_A by substitution into (2, 21). It transpires that Ψ_A is a 6-vector also.

Let us consider the various transformations in turn.

(a) Inhomogeneous Lorentz transformations: according to (2.15) there is in Φ_A a 4-vector and a scalar, viz.

$$\Phi_\mu - x_\mu(\Phi_5 + \Phi_6) = \phi_\mu e^\sigma$$

$$(\Phi_5 + \Phi_6) = \phi^2 e^\sigma \quad , \qquad (2.22)$$

respectively. This determines that ϕ_μ and σ are indeed a true 4-vector and a true scalar under the group of translations and homogeneous Lorentz transformations: the Poincaré group.

The corresponding quantities contained in Ψ_A are given by

$$\Psi_\mu - x_\mu(\Psi_5 + \Psi_6) = 0$$

$$(\Psi_5 + \Psi_6) = e^{-\sigma} \qquad (2.23)$$

which are a 4-vector (trivially) and a scalar consistent with the proposition that Ψ_A is a 6-vector.

(b) Special conformal transformations: this group taken together with the homogeneous Lorentz transformations comprises a Poincaré-like group which can be dealt with in close analogy to the above. This group is obtained from the space-time Poincaré group by everywhere making the replacement $X_5 + X_6 \rightarrow X_5 - X_6$ so that, for example,

$$x_\mu = \frac{\eta_\mu}{\eta_5 + \eta_6} \rightarrow \frac{\eta_\mu}{\eta_5 - \eta_6} = \frac{x_\mu}{x^2} \quad .$$

Making the appropriate adaptation of (2.15) one can extract from Φ_A the 4-vector and scalar,

$$\Phi_\mu - \frac{x_\mu}{x^2}(\Phi_5 - \Phi_6) = \phi_\mu e^\sigma - \frac{x_\mu}{x^2}(1 + 2x \cdot \phi)\, e^\sigma$$

$$(\Phi_5 - \Phi_6) = (1 + 2x \cdot \phi + x^2 \phi^2)\, e^\sigma \quad . \qquad (2.24)$$

All five of these quantities are scalars under the special conformal transformations,

$$\frac{x_\mu}{x^2} \rightarrow \frac{x_\mu}{x^2} + \beta_\mu \quad .$$

The behaviour of ϕ_μ and σ under these transformations is complicated and not of particular interest apart from the fact, which can be extracted from (2.24), that the quantity $x^2 e^{-\sigma}$ is a scalar. This is enough to assure the correct transformation law for Ψ_A, <u>viz</u>. the invariance of the quantities

$$\Psi_\mu - \frac{x_\mu}{x^2}(\Psi_5 - \Psi_6) = 0$$

$$(\Psi_5 - \Psi_6) = x^2 e^{-\sigma} \quad . \qquad (2.25)$$

(c) Dilatations: these are produced by the hyperbolic rotations in the $\eta_5 \eta_6$ subspace

$$\Lambda^{-1}\eta = \begin{pmatrix} \eta_\mu \\ \eta_5 \operatorname{ch}\lambda + \eta_6 \operatorname{sh}\lambda \\ \eta_5 \operatorname{sh}\lambda + \eta_6 \operatorname{ch}\lambda \end{pmatrix}$$

which implies

$$(\Lambda^{-1} x)_\mu = x_\mu e^{-\lambda} \quad . \qquad (2.26)$$

Under this transformation Φ_μ remains invariant while

$$\Phi_5 + \Phi_6 \rightarrow e^{-\lambda}(\Phi_5 + \Phi_6) \quad , \qquad (2.27)$$

comparison of which with (2.17) yields the law

$$\phi_\mu \rightarrow \phi_\mu e^{-\lambda}$$

$$e^\sigma \rightarrow e^{\sigma + \lambda} \quad . \qquad (2.28)$$

Substitution of the second of these into the expressions (2.18) shows that Ψ_μ is invariant while $\Psi_5 + \Psi_6$ transforms like (2.27).

Thus it is proved that the transformations of the parameter fields ϕ_μ and σ can be such as to make true 6-vectors of both Φ_A and Ψ_A, i.e., of the last two columns of the boost matrix $(L_\phi)_{AB}$. The non-linear realizations $\psi(\eta)$ extracted from the linear $\Psi(\eta)$ in the manner of (2.17) with the boost (2.19) transform according to the prescription

$$\psi(\eta) \rightarrow D(h)\, \psi(\Lambda^{-1}\eta) \qquad (2.29)$$

where $h = h(\eta, \Lambda)$ is an ordinary homogeneous Lorentz transformation. The fields $\psi(\eta)$ transform irreducibly under the Lorentz group.

It remains only to set up the formulae for the covariant derivatives. This problem is somewhat obscured in conformal space - even within the context of linear representations - by the fact that the derivative $\partial/\partial x_\mu$ is only part of an operator ∂_{AB} which belongs to the fifteen-dimensional (adjoint) representation. Of course, the other components of this operator can be expressed in terms of $\partial/\partial x_\mu$ and x_μ itself as in formula (2.14) (with $\kappa\partial/\partial\kappa = \ell$, the degree of homogeneity). Let us consider, therefore, the operator Δ_{AB} defined by

$$\Delta_{AB}\,\psi = D(L_\phi^{-1})\, \partial_{AB}\,\Psi\,, \qquad (2.30)$$

where Ψ is given by (2.17) and ∂_{AB} by (2.14). This operator transforms in a hybrid fashion,

$$\Delta_{AB}\,\psi \rightarrow \Lambda_{AA'}\,\Lambda_{BB'}\,D(h)\,\Delta_{A'B'}\,\psi \quad , \qquad (2.31)$$

and in order to avoid the appearance of Λ in the transformation laws it is useful to define the new operator

$$\Delta_{(AB)}\psi = (L_\phi^{-1})_{AA'}\,(L_\phi^{-1})_{BB'}\,\Delta_{A'B'}\,\psi\,, \qquad (2.32)$$

for which the rule (2.31) is modified to

$$\Delta_{(AB)}\,\psi \rightarrow h_{AA'}\,h_{BB'}\,D(h)\,\Delta_{(A'B')}\,\psi \quad . \qquad (2.33)$$

The fifteen components of $\Delta_{(AB)}$ decompose into Lorentz multiplets among which is to be found a 4-vector candidate for the role of covariant derivative.

The computation of $\Delta_{(AB)}$ is very much simplified by remarking that it is possible to represent ∂_{AB} in the form

$$\partial_{AB} = \frac{i}{2} (e^{ixP})_{AA'} (e^{ixP})_{BB'} (K_\mu \partial_\mu + 2\ell D)_{A'B'} \quad . \qquad (2.34)$$

It was for this reason that the factor exp(ixP) was included in the definition of L_ϕ . From (2.30), (2.32) and (2.34) one can derive the formula

$$\Delta_{(AB)}\psi = \frac{i}{2} D(L_\phi^{-1}) \left[(K_\mu)_{AB} \, e^\sigma (\partial_\mu + 2\ell\phi_\mu) + 2\ell D_{AB} \right] \Psi \quad , \qquad (2.35)$$

or, on separating out the irreducible parts,

$$\Delta_{(\mu\nu)}\psi = 0$$

$$(\Delta_{(5\mu)} + \Delta_{(6\mu)})\psi = D(L_\phi^{-1}) \, e^\sigma (\partial_\mu + 2\ell\phi_\mu)\Psi$$

$$(\Delta_{(5\mu)} - \Delta_{(6\mu)})\psi = 0$$

$$\Delta_{(56)}\psi = \ell\psi \qquad (2.36)$$

It is therefore clear that the covariant derivative must be contained in the operator Δ_μ defined by

$$\Delta_\mu \psi = D(L_\phi^{-1}) \, e^\sigma (\partial_\mu + 2\ell\phi_\mu) \, D(L_\phi)\psi \quad , \qquad (2.37)$$

which transforms according to the non-linear prescription,

$$\Delta_\mu \psi \rightarrow h_{\mu\nu} \, D(h) \, \Delta_\nu \psi (\Lambda^{-1}\eta) \quad . \qquad (2.38)$$

To extract the covariant derivative from (2.37) it is necessary to follow the procedure of I in separating off from $\Delta_\mu \psi$ the parts which are proportional to the covariant derivatives of the preferred fields ϕ_μ and σ . Thus one can write

$$\Delta_\mu \psi = e^\sigma (\partial_\mu + 2\ell\phi_\mu)\psi + D(L_\phi^{-1})\, e^\sigma \partial_\mu D(L_\phi)\psi$$

$$= e^\sigma (\partial_\mu + 2\ell\phi_\mu)\psi - \frac{i}{2}\,(L_\phi^{-1}\, e^\sigma \partial_\mu L_\phi)_{AB}\, S_{AB}\,\psi \qquad (2.39)$$

where the S_{AB} generate infinitesimal SO(4, 2) transformations on $\Psi(\eta)$. The covariant parts of the factor $L_\phi^{-1}\, e^\sigma \partial_\mu L_\phi$ are given by

$$(L_\phi^{-1} e^\sigma \partial_\mu \Phi)_A = \frac{1}{2}\left[(L_\phi^{-1} e^\sigma \partial_\mu L_\phi)^5_A - (L_\phi^{-1} e^\sigma \partial_\mu L_\phi)^6_A\right] =$$

$$= \begin{bmatrix} e^{2\sigma}(\partial_\mu \phi_\nu + \phi^2 g_{\mu\nu} - 2\phi_\mu \phi_\nu) \\ e^\sigma(\partial_\mu \sigma + 2\phi_\mu) \\ e^\sigma(\partial_\mu \sigma + 2\phi_\mu) \end{bmatrix}$$

$$(L_\phi^{-1} e^\sigma \partial_\mu \Psi)_A = \frac{1}{2}\left[(L_\phi^{-1} e^\sigma \partial_\mu L_\phi)^5_A + (L_\phi^{-1} e^\sigma \partial_\mu L_\phi)^6_A\right] =$$

$$= \begin{bmatrix} g_{\mu\nu} \\ - e^\sigma(\partial_\mu \sigma + 2\phi_\mu) \\ e^\sigma(\partial_\mu \sigma + 2\phi_\mu) \end{bmatrix} \qquad (2.40)$$

The remaining part of this antisymmetric matrix,

$$(L_\phi^{-1}\, e^\sigma \partial_\mu L_\phi)_{\lambda\rho} = 2e^\sigma (g_{\mu\lambda}\, \phi_\rho - g_{\mu\rho}\, \phi_\lambda) \;, \qquad (2.41)$$

is not covariant on its own but must be combined with the other terms in (2.39) to make the covariant derivative of ψ , which is therefore given by the expression

$$D_\mu \psi = e^\sigma \left[\partial_\mu + 2\ell\phi_\mu - 2i\, S_{\mu\nu}\, \phi_\nu\right]\psi \;. \qquad (2.42)$$

Finally, the independent parts of (2.40), whose covariance is dependent upon the requirement that $\Phi_A(\eta)$ and $\Psi_A(\eta)$ are homogeneous of degree zero, are adopted as the definitions of the covariant derivatives of ϕ_μ and σ, respectively,

$$D_\mu \phi_\nu = e^{2\sigma}\left[\partial_\mu\phi_\nu + \phi^2 g_{\mu\nu} - 2\phi_\mu\phi_\nu\right] \qquad (2.43$$

$$D_\mu \sigma = e^{\sigma}\left[\partial_\mu\sigma + 2\phi_\mu\right]. \qquad (2.44)$$

Since $\psi(\eta)$ is homogeneous of degree ℓ it contains, in addition to its dependence on x, the factor κ^ℓ which must of course cancel from any field equations which are to have meaning in space-time. The fields ϕ_μ and σ, being of degree zero, have no such factor. Hence, the rules for constructing conformal invariant and meaningful field equations out of the covariant quantities ψ, $D_\mu\psi$, $D_\mu\phi_\nu$ and $D_\mu\sigma$ must be:

a) covariance under the Lorentz group in the linear sense;

b) homogeneity in η-space.

Although the method we have followed in deriving these realizations is exactly the one set out in I for the construction of non-linear realizations, our results are a little unexpected. The realizations of this section are not, strictly speaking, non-linear. This may be due to the operation of an Anderson-Higgs-Kibble mechanism [3].
The details of the transformation laws are derived in the Appendix. Corresponding to the infinitesimal special conformal transformation

$$\delta x_\mu = (x^2 g_{\mu\nu} - 2x_\mu x_\nu)\beta_\nu \quad , \qquad (2.45)$$

the various fields transform according to

$$\begin{aligned}\delta\psi &= -\delta x_\mu\partial_\mu\psi - 2\ell\beta_\mu x_\mu\psi - 2i\beta_\mu x_\nu S_{\mu\nu}\psi \\ \delta\phi_\lambda &= -\delta x_\mu\partial_\mu\phi_\lambda + 2\beta_\mu x_\mu\phi_\lambda + (\beta_\lambda - 2x_\lambda\beta_\mu\phi_\mu) \\ \delta\sigma &= -\delta x_\mu\partial_\mu\sigma - 2\beta_\mu x_\mu \quad .\end{aligned} \qquad (2.46)$$

For the infinitesimal dilatation

$$\delta x_\mu = \lambda x_\mu \qquad (2.47)$$

the fields transform according to

$$\delta\psi = -\delta x_\mu \partial_\mu \psi + \lambda \ell \psi$$

$$\delta\phi_\nu = -\delta x_\mu \partial_\mu \phi_\nu - \lambda \phi_\nu$$

$$\delta\sigma = -\delta x_\mu \partial_\mu \sigma + \lambda \ . \tag{2.48}$$

These transformation laws are all linear. Those for the preferred fields, however, contain inhomogeneous terms. It is these inhomogeneous terms which symptomize the underlying vacuum asymmetry. Thus, an infinitesimal conformal transformation of the vector field at $x_\mu = 0$ takes the form

$$\delta\phi_\mu(0) = \beta_\mu \tag{2.49}$$

the vacuum expectation value of which cannot vanish. Similarly an infinitesimal dilatation of the scalar field at $x_\mu = 0$ takes the form

$$\delta\sigma(0) = \lambda \tag{2.50}$$

which again implies a vacuum asymmetry.

It is necessary to remark that, in order to avoid clutter, the preferred fields have not been normalized throughout this section. This can be corrected by making a trivial adjustment: replace ϕ_μ and σ everywhere by $f\phi_\mu$ and $g\sigma$. Then, for example, the covariant derivatives (2.43) and (2.44) become

$$D_\mu\phi_\nu = e^{2g\sigma}(\partial_\mu \phi_\nu - 2f\,\phi_\mu\,\phi_\nu + f\phi^2\,g_{\mu\nu})$$

$$D_\mu\,\sigma = e^{g\sigma}(\partial_\mu \sigma + \frac{2f}{g}\,\phi_\mu) \quad ,$$

while the infinitesimal variations (2.49) and (2.50) become, respectively,

$$\delta\phi_\mu(0) = \frac{\beta_\mu}{f}$$

$$\delta\sigma(0) = \frac{\lambda}{g}$$

3. CONFORMAL INVARIANT EFFECTIVE LAGRANGIANS

The rules, formulated above, for the construction of conformal invariant field equations can of course be expressed in the language of Lagrangians. However, there is one further requirement to be met. This concerns the degree of homogeneity of the Lagrangian which must equal -4.

In order that the field equations derived by minimizing the action should be conformal invariant it is clearly necessary that the Lagrangian should transform as a scalar <u>density</u>. This means that it is necessary to keep track of the Jacobian $|\partial x'/\partial x|$ which corresponds to the transformation $x \to x'$,

$$|\partial x'/\partial x| \; L(\psi'(x') , \psi'_{\mu}(x') , \ldots) = L(\psi(x) , \psi_{\mu}(x) , \ldots) \tag{3.1}$$

where $\psi_{\mu} = D_{\mu}\psi$. The variables $D_{\mu}\phi_{\nu}$ and $D_{\mu}\sigma$, which must be present, are indicated by dots. The Jacobian is easily computed for the various types of transformation. The results are, respectively,

a) inhomogeneous Lorentz transformations, $|\partial x'/\partial x| = 1$

b) special conformal transformations, $|\partial x'/\partial x| = (x'^2/x^2)^4$

c) dilatations, $x \to x' = \lambda x$, $|\partial x'/\partial x| = \lambda^4$.

(3.2)

Suppose now that from the homogeneous function of degree -4, defined over η-space,

$$\mathcal{L}(\psi(\eta) , \psi_{\mu}(\eta) , \ldots) = \lambda^4 \mathcal{L}(\psi(\lambda\eta) , \psi_{\mu}(\lambda\eta) , \ldots) , \tag{3.3}$$

is obtained the Lagrangian of (3.1) by the prescription

$$\mathcal{L}(\psi(\eta) , \psi_{\mu}(\eta) , \ldots) = \kappa^{-4} L(\psi(x) , \psi_{\mu}(x) , \ldots) \tag{3.4}$$

where the arguments are related by

$$\psi(\eta) = \kappa^{\ell}\psi(x)$$

$$\psi_{\mu}(\eta) = \kappa^{\ell}\psi_{\mu}(x) . \tag{3.5}$$

That the fields $\psi(\eta)$ and $\psi_\mu(\eta) = D_\mu \psi(\eta)$ have the same degree of homogeneity, ℓ, is a consequence of the definition of the covariant derivative operator which was adopted in Sec. 2. If $\mathcal{L}$ is a true scalar in η-space

$$\mathcal{L}(\psi'(\eta'), \psi'_\mu(\eta'), \ldots) = \mathcal{L}(\psi(\eta), \psi_\mu(\eta), \ldots) \tag{3.6}$$

for the transformations of O(4, 2), then it follows that the x-space Lagrangian defined by (3.4) satisfies the condition

$$(\kappa/\kappa')^4 \, L(\psi'(x'), \psi'_\mu(x'), \ldots) = L(\psi(x), \psi_\mu(x), \ldots) \tag{3.7}$$

and it is a simple matter to show that, in each case, the factor $(\kappa/\kappa')^4$ coincides precisely with the Jacobian determinant of (3.1)

$$(\kappa/\kappa')^4 = \left| \partial x'/\partial x \right| . \tag{3.8}$$

Thus, we can conclude that the Lagrangian obtained by the prescription (3.4) from the homogeneous η-space scalar of degree -4 is indeed a scalar density under the conformal transformations of space-time.

To summarize: Conformal invariant effective Lagrangians can be built out of the homogeneous field variables $\psi(\eta)$ and the covariant derivatives $D_\mu\psi$, $D_\mu\phi_\nu$ and $D_\mu\sigma$ which belong to non-linear realizations, by making combinations which are manifestly Lorentz invariant and which are homogeneous of degree -4. The degree of homogeneity of any given Lorentz scalar combination can be brought to the required value through multiplication with the appropriate power of the conformal scalar $\eta_A\Phi_A$ which has degree 1. Finally, extract the common factor κ^{-4}.

In the parametrization of Sec. 2 the scalar field is given by

$$-2 \, \eta_A\Phi_A = \kappa \, e^\sigma . \tag{3.9}$$

This means that only the power of e^σ is affected by such multiplications as are needed to bring the degree of any term to the value -4 . Since, in the application of effective Lagrangians to the computation of tree graphs, it is customary to expand all functions of the preferred fields in power series, this replacement of σ by some multiple of itself is a very mild adjustment.

It is of course necessary that the Lagrangian be real (up to a 4-divergence). For this reason the "spin coupling" of the field ϕ_μ which is suggested by the form of the covariant derivative

$$D_\mu \psi = e^{g\sigma} (\partial_\mu + 2\ell f \phi_\mu - 2ifS_{\mu\nu}\phi_\nu) \psi$$

is not present generally. If ψ is a Dirac spinor the term $-2ifS_{\mu\nu}\phi_\nu\psi$ gives a purely imaginary contribution. In fact, for this case the covariant derivative appears in the combination

$$\gamma_\mu D_\mu \psi = \gamma_\mu e^{g\sigma} (\partial_\mu + f(2\ell - 3)\phi_\mu)\psi \quad . \tag{3.10}$$

A possible Lagrangian for the system comprising a Dirac field ψ of degree ℓ in interaction with the preferred fields ϕ_μ and σ might be given by

$$L = L_\psi + L_{\phi\sigma} \tag{3.11}$$

where

$$L_\psi = e^{-(\ell + \bar{\ell} + 4)g\sigma} \left\{ \frac{i}{2} (\bar{\psi}\gamma_\mu D_\mu \psi - D_\mu \bar{\psi}\gamma_\mu \psi) - m\bar{\psi}\psi \right\} \tag{3.12}$$

and

$$L_{\phi\sigma} = e^{-4g\sigma} \left\{ \frac{1}{4} (D_\mu \phi_\nu - D_\nu \phi_\mu)^2 + \frac{1}{2} (D_\mu \sigma)^2 - \frac{1}{2} \left(\frac{\kappa}{4g}\right)^2 \right\} \tag{3.13}$$

where $\bar{\ell}$ denotes the complex conjugate of ℓ and κ is a constant. The exponential factors in L_ψ and $L_{\phi\sigma}$ are needed to make the Lagrangian transform like a scalar density. With the explicit forms for the covariant derivatives inserted, the forms (3.12) and (3.13) become

$$L_\psi = e^{-2g(\mathrm{Re}\,\ell + \frac{3}{2})\sigma} \left\{ \frac{i}{2} (\bar{\psi}\gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma_\mu \psi) - 2f(\mathrm{Im}\,\ell)\, \bar{\psi}\gamma_\mu \psi \phi_\mu + m\, e^{-g\sigma} \bar{\psi}\psi \right\} \tag{3.14}$$

$$L_{\phi\sigma} = \frac{1}{4} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)^2 + \frac{1}{2} e^{-2g\sigma} (\partial_\mu \sigma + \frac{2f}{g} \phi_\mu)^2 - \frac{1}{2} \left(\frac{\kappa}{4g}\right)^2 e^{-4g\sigma} \tag{3.15}$$

The bare masses of the preferred fields are, respectively, $2f/g$ for the vector [5] and κ for the scalar.

Formulae for the conserved currents $K_{\mu\gamma}$, corresponding to special conformal transformations, and D_μ, corresponding to dilatations, are derived in the Appendix. For the Lagrangian (3.11) one finds, at $x = 0$,

$$f K_{\mu\nu}(0) = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \tag{3.16}$$

$$g D_\mu(0) = (\partial_\mu \sigma + \frac{2f}{g}) e^{-2g\sigma} + (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \phi_\nu + $$

$$+ (\mathrm{Im}\,\ell) \bar{\psi} \gamma_\mu \psi e^{-(\ell + \bar{\ell} + 3) g\sigma} \quad . \tag{3.17}$$

APPENDIX

TRANSFORMATION DETAILS

In order to construct the conserved currents which in a Lagrangian theory correspond to the various symmetries, it is necessary to have explicit expressions for the infinitesimal variations in the fields. In particular it is necessary to have explicitly the infinitesimal form of $h(\eta, \Lambda)$. This appendix is concerned with the computation of these expressions.

The Lorentz transformation $h(\eta, \Lambda)$ corresponding to the O(4, 2) transformation Λ is defined by the transformation law of the boost components

$$L_\phi(\eta) \rightarrow L_{\phi'}(\eta) = \Lambda \, L_\phi(\Lambda^{-1}\eta) \, h^{-1}(\Lambda^{-1}\eta, \Lambda) \tag{A. 1}$$

where $L_{\phi'}$ is determined indirectly by the requirement

$$h_{\mu 5} + h_{\mu 6} = g_{\mu 5} + g_{\mu 6} \quad , \tag{A. 2}$$

which forces the combination

$$-2\Phi_A = (L_\phi)_{A5} + (L_\phi)_{A6}$$

in terms of which the parameters ϕ_μ and σ and, therefore, the entire matrix L_ϕ, can be expressed to transform like a true 6-vector:

$$\Phi_A(\eta) \rightarrow \Phi'_A(\eta) = \Lambda_{AB} \, \Phi_B \, (\Lambda^{-1}\eta) \quad . \tag{A. 3}$$

The parameters ϕ_μ and σ are given by the expression (2. 20) for Φ_A. Inverting this expression gives

$$\begin{aligned} \phi_\mu \, e^\sigma &= \Phi_\mu - x_\mu \, (\Phi_5 + \Phi_6) \\ \phi^2 \, e^\sigma &= \Phi_5 + \Phi_6 \quad . \end{aligned} \tag{A. 4}$$

From these relations it is straightforward to deduce the transformation laws for ϕ_μ and σ by substituting (A. 3). The results are, for the various types of transformation:

a) <u>Inhomogeneous Lorentz transformations</u>

$$x'_\mu = \Lambda_{\mu\nu}\, x_\nu + \alpha_\mu$$

$$\phi'_\mu(x') = \Lambda_{\mu\nu}\, \phi_\nu(x)$$

$$\sigma'(x') = \sigma(x) \quad . \tag{A.5}$$

b) <u>Special conformal transformations</u>

$$x'_\mu = \frac{x_\mu + x^2\beta_\mu}{1 + 2\beta x + \beta^2 x^2}$$

$$\phi'_\mu(x') = (1 + 2\beta x + \beta^2 x^2)\, \phi_\mu(x) + ((1 + 2x\phi)(1 + 2\beta x) - 2x^2\beta\phi)\, \beta_\mu - (2\beta\phi + \beta^2\, (1 + 2x\phi))\, x_\mu$$

$$\sigma'(x') = \sigma(x) - \ell n(1 + 2\beta x + \beta^2 x^2) \quad . \tag{A.6}$$

c) <u>Dilatations</u>

$$x'_\mu = e^\lambda\, x_\mu$$

$$\phi'_\mu(x') = e^{-\lambda}\, \phi_\mu(x)$$

$$\sigma'(x') = \sigma(x) + \lambda \quad . \tag{A.7}$$

These relations, (A.5), (A.6) and (A.7), define the transformation law for the entire boost matrix which is given in terms of ϕ_μ and σ by

$$L_\phi(\eta) = e^{ix\cdot P}\, e^{i\phi\cdot K}\, e^{-i\sigma D} \tag{A.8}$$

which, since it is homogeneous of degree zero, is in fact a function of x_μ only.

The transformation $h(\eta,\Lambda)$ defined by (A.1) can be expressed in the form

$$h(\eta,\Lambda) = e^{i\sigma' D}\, e^{-i\phi' K}\, e^{-ix' P}\, \Lambda e^{ixP}\, e^{i\phi K}\, e^{-i\sigma D} \tag{A.9}$$

and it remains only to substitute from (A. 5), (A. 6) and (A. 7) to obtain the explicit form of h in terms of ϕ_μ, σ, x_μ and Λ. The results are as follows:

a) Inhomogeneous Lorentz transformations

$$x_\mu \to \Lambda_{\mu\nu} x_\nu + \alpha_\mu$$

$$h_{\mu\nu}(\eta, \Lambda) = \Lambda_{\mu\nu} \quad , \qquad (A.10)$$

which means that for pure translations, $x \to x + \alpha$, we have $h = 1$.

b) Special conformal transformations

For these the computation is a rather lengthy one and so we give the result only for infinitesimal transformations,

$$x'_\mu = x_\mu + (x^2 g_{\mu\nu} - 2x_\mu x_\nu)\, \beta_\nu + \ldots$$

$$h_{\mu\nu}(\eta, \Lambda) = g_{\mu\nu} + 2(\beta_\mu x_\nu - \beta_\nu x_\mu) + \ldots \quad . \qquad (A.11)$$

c) Dilatations

$$x'_\mu = e^\lambda x_\mu$$

$$h_{\mu\nu}(\eta, \Lambda) = g_{\mu\nu} \qquad (A.12)$$

Thus, for translations and dilatations the transformation h coincides with the identity, for ordinary Lorentz transformations h coincides with Λ, while for special conformal transformations h becomes a more complicated Lorentz transformation.

Let us now compute the infinitesimal variations in the field ψ which are induced by transformations of the conformal group. Firstly, in η-space,

$$\psi'(\eta') = (1 - \frac{i}{2}\; \delta h_{\mu\nu}\, S_{\mu\nu})\, \psi(\eta) \quad . \qquad (A.13)$$

Assuming that $\psi(\eta)$ is homogeneous of degree ℓ, then it is possible to define the space-time field $\psi(x)$ by

$$\psi(x) = \kappa^{-\ell}\, \psi(\eta)$$

where η is expressed in terms of x by (2.12). Then we have

$$\psi'(x') = (1 - \frac{\ell\,\delta\kappa}{\kappa} - \frac{i}{2}\,\delta h_{\mu\nu}\,S_{\mu\nu})\psi(x)$$

so that, finally,

$$\delta\psi(x) = \psi'(x) - \psi(x)$$

$$= -(\delta x_\mu\,\partial_\mu + \ell\,\frac{\delta\kappa}{\kappa} + \frac{i}{2}\,\delta h_{\mu\nu}\,S_{\mu\nu})\psi \quad . \qquad (A.14)$$

The infinitesimal quantity $\delta\kappa/\kappa$ appearing here is given by

$$\frac{\delta\kappa}{\kappa} = \begin{cases} 0 & , \text{ inhomogeneous Lorentz transformations} \\ 2\beta\cdot x, & \text{special conformal transformations} \\ -\lambda & , \text{ dilatations} \quad . \end{cases} \qquad (A.15)$$

The infinitesimal variations in the fields ψ, ϕ_μ and σ and the transformations which induce them are listed as follows:

a) Inhomogeneous Lorentz transformations

$$x'_\mu = x_\mu + \epsilon_{\mu\nu}\,x_\nu + \alpha_\mu \quad , \quad \epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$$

$$\delta\psi = -(\epsilon_{\mu\nu}x_\nu + \alpha_\mu)\,\partial_\mu\psi - \frac{i}{2}\,\epsilon_{\mu\nu}\,S_{\mu\nu}\,\psi$$

$$\delta\phi_\lambda = -(\epsilon_{\mu\nu}\,x_\nu + \alpha_\mu)\,\partial_\mu\,\phi_\lambda + \epsilon_{\lambda\mu}\,\phi_\mu$$

$$\delta\sigma = -(\epsilon_{\mu\nu}\,x_\nu + \alpha_\mu)\,\partial_\mu\,\sigma \quad . \qquad (A.16)$$

b) Special conformal transformations

$$x'_\mu = x_\mu + (x^2\,g_{\mu\nu} - 2x_\mu\,x_\nu)\,\beta_\nu$$

$$\delta\psi = -\beta_\mu(x^2\,g_{\mu\nu} - 2x_\mu\,x_\nu)\,\partial_\nu\,\psi - 2\,\ell\,\beta_\mu\,x_\mu\,\psi - 2i\,\beta_\mu\,x_\nu\,S_{\mu\nu}\,\psi$$

$$\delta\phi_\lambda = -\beta_\mu(x^2\,g_{\mu\nu} - 2x_\mu\,x_\nu)\,\partial_\nu\,\phi_\lambda + 2\beta_\mu\,x_\mu\,\phi_\lambda + (\beta_\lambda - 2x_\lambda\,\beta_\mu\,\phi_\mu)$$

$$\delta\sigma = -\beta_\mu(x^2\,g_{\mu\nu} - 2x_\mu\,x_\nu)\,\partial_\nu\,\sigma - 2\beta_\mu\,x_\mu \quad . \qquad (A.17)$$

c) Dilatations

$$x'_\mu = x_\mu + \lambda x_\mu$$

$$\delta\psi = -\lambda(x_\mu \partial_\mu - \ell)\psi$$

$$\delta\phi_\nu = -\lambda(x_\mu \partial_\mu + 1)\phi_\nu$$

$$\delta\sigma = -\lambda(x_\mu \partial_\mu \sigma - 1) \quad . \qquad \text{(A.18)}$$

These infinitesimal forms can be used in the canonical expression for the conserved currents

$$\epsilon\, j_\mu = \frac{\partial L}{\partial \psi_{,\mu}} \delta\psi + \frac{\partial L}{\partial \phi_{\lambda,\mu}} \delta\phi_\lambda + \frac{\partial L}{\partial \sigma_{,\mu}} \delta\sigma + \delta x_\mu L \qquad \text{(A.19)}$$

where $\psi_{,\mu} = \partial_\mu \psi$, etc. Corresponding to the inhomogeneous Lorentz transformations are the well-known tensors $T_{\mu\nu}$ and $M_{\mu\nu\lambda}$. The new currents, which correspond to the special conformal transformations and the dilatations are, respectively,

$$K_{\mu\nu} = \frac{\partial L}{\partial \psi_{,\mu}} \left\{ (x^2 g_{\nu\rho} - 2x_\nu x_\rho)\partial_\rho \psi + 2(\ell x_\nu + i S_{\nu\rho} x_\rho)\psi \right\}$$

$$+ \frac{\partial L}{\partial \phi_{\lambda,\mu}} \left\{ (x^2 g_{\nu\rho} - 2x_\nu x_\rho)\partial_\rho \phi_\lambda - 2x_\nu \phi_\lambda - g_{\nu\lambda} + 2x_\lambda \phi_\nu \right\}$$

$$+ \frac{\partial L}{\partial \sigma_{,\mu}} \left\{ (x^2 g_{\nu\rho} - 2x_\nu x_\rho)\partial_\rho \sigma + 2x_\nu \right\}$$

$$- L(x^2 g_{\nu\mu} - 2x_\nu x_\mu) \qquad \text{(A.20)}$$

$$D_\mu = \frac{\partial L}{\partial \psi_{,\mu}} (x_\nu \partial_\nu - \ell)\psi + \frac{\partial L}{\partial \phi_{\lambda,\mu}} (x_\nu \partial_\nu + 1)\phi_\lambda - \frac{\partial L}{\partial \sigma_{,\mu}} (x_\nu \partial_\nu \sigma - 1) -$$

$$- x_\mu L$$

$$\text{(A.21)}$$

where it is to be understood that the derivatives with respect to ψ include all fields and their adjoints, if necessary, excepting the preferred fields ϕ_μ and σ. The expressions (A.20) and (A.21) simplify at the point $x = 0$,

$$K_{\mu\nu}(0) = - \frac{\partial L}{\partial \phi_{\nu,\mu}} \qquad (A.22)$$

$$D_\mu(0) = - \ell \frac{\partial L}{\partial \psi_{,\mu}} \psi + \frac{\partial L}{\partial \phi_{\lambda,\mu}} \phi_\lambda + \frac{\partial L}{\partial \sigma_{,\mu}} . \qquad (A.23)$$

REFERENCES AND FOOTNOTES

1) Abdus Salam and J. Strathdee, Non-linear realizations - I: "The role of Goldstone bosons," ICTP, Trieste, preprint IC/68/105.

2) A recent discussion of this group with references to earlier work is contained in G. Mack and Abdus Salam, "Finite-component field representations of the conformal group," ICTP, Trieste, preprint IC/68/68.

3) One may perhaps be able to understand this appearance of massive fields as a consequence of the Anderson-Higgs-Kibble mechanism, discussed in I, which seems to operate in this case. The vector field ϕ_μ couples like a gauge field in that it enters the covariant derivatives of other fields in the combination $\partial_\mu - ig\phi_\mu$. If these were the only couplings, one could perhaps surmise that this is a consequence of the operation of the Anderson-Higgs-Kibble mechanism. However, there could be other couplings as well, which make the situation unclear.

4) P.A.M. Dirac, Annals of Mathematics 37, 429 (1936).

5) If the fields ϕ_μ and σ are correctly normalized then f is dimensionless while $1/g$ has the dimensions of mass.