IC/68/105

INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

NON-LINEAR REALIZATIONS - I: THE ROLE OF GOLDSTONE BOSONS

ABDUS SALAM

and

J. STRATHDEE

1968

MIRAMARE - TRIESTE

IC/68/105

INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

NON-LINEAR REALIZATIONS - I: THE ROLE OF GOLDSTONE BOSONS*

ABDUS SALAM**

and

J. STRATHDEE

ABSTRACT

A method, based upon analogies with the Wigner boost technique, is presented for setting up the non-linear realizations of any continuous symmetry group. It is argued that such realizations are relevant to and useful in the treatment of cases of spontaneous symmetry breakdown.

MIRAMARE - TRIESTE

December 1968

* To be submitted for publication.

** On leave of absence from Imperial College, London, England.

NON-LINEAR REALIZATIONS - I:

THE ROLE OF GOLDSTONE BOSONS

1. INTRODUCTION

The equations of physics are usually expressed in a form showing manifest covariance under the transformations of the space-time and internal symmetry groups. Recently, however, some attention has been paid to the possibility of expressing internal symmetries of the chiral type in a fashion which is not manifestly covariant, some of the transformations of these groups being realized non-linearly [1]. In the present paper the non-linear method is shown to provide an economical framework within which to treat the problems of spontaneously broken symmetries [2], i.e., symmetries which are present in the Lagrangian but not in the ground state. Beyond this, the view will be presented that the non-linear method, as a method embodying dynamics rather than pure group theory, is applicable only to situations in which the symmetry is broken spontaneously. Much of the following discussion will be in the nature of a review. However, one purpose of this is to present the notational developments that emerge through a consistent use of the language of Wigner boosts which, it seems to us, greatly clarifies the subject of non-linear realizations. We shall need this development in particular in the second paper where we consider non-linear realizations of the conformal group in space-time.

Central to the method of non-linear realizations is the notion of a preferred field which provides a bridge to representations which are linear but constrained. There is in fact a complete duality between sets of fields, on the one hand, which transform linearly while being subject to certain non-linear constraints and, on the other hand, equivalent sets of unconstrained fields which transform according to non-linear realizations. The preferred field is used in the formation of covariant equations of constraint upon the set of fields which transforms linearly. If these constraints are then used to eliminate all dependent components there results what is commonly called a non-linear realization. More specifically, the realization will be linear with respect to all but the preferred field itself which generally enters non-linearly. The structure of the realizations arrived at in this way must of course depend critically upon the particular set of fields which are chosen to play the preferred role. This choice depends in turn upon the nature of the vacuum symmetry since, as will be seen in the following, the preferred fields are neither more nor less than the field of the Goldstone bosons [3].

A rather trivial and very familiar example of the procedure just outlined is provided in the case of chiral [3] SU(2) x SU(2). Let the chiral 4-vector π_α play the role of preferred field and let it be employed in the formulation of a set of algebraic constraints upon the fields ϕ_α and $F_{\alpha\beta}$, a chiral 4-vector and 6-tensor, respectively. For the equations of constraint one might take, for example,

$$\pi_\alpha \phi_\beta - \pi_\beta \phi_\alpha = f F_{\alpha\beta}$$

$$\pi_\alpha F_{\alpha\beta} - f \phi_\beta = 0 \qquad (1.1)$$

where f denotes a numerical constant. The question of how such constraints could arise within the context of a dynamical model is not, for the present, at issue. It follows from (1.1) that, in particular,

$$\pi_\alpha \pi_\alpha = f^2$$

$$\pi_\alpha \phi_\alpha = 0 \qquad (1.2)$$

so that the sixteen various components can be expressed in terms of six independent ones, say $\underline{\pi}$ and $\underline{\phi}$. The linear transformation laws appropriate to the chiral 4-vectors π_α and ϕ_α are exemplified by

$$\pi_{1,2} \rightarrow \pi_{1,2}$$

$$\pi_3 \rightarrow \pi_3 \cos\omega - \pi_4 \sin\omega$$

$$\pi_4 \rightarrow \pi_3 \sin\omega + \pi_4 \cos\omega \qquad (1.3)$$

corresponding to a purely chiral transformation. It remains only to eliminate the dependent components π_4 and ϕ_4 by means of the constraints (1.2). The independent components $\underline{\pi}$ and $\underline{\phi}$ then transform according to non-linear realizations which are exemplified by

$$\pi_{1,2} \rightarrow \pi_{1,2}$$
$$\pi_3 \rightarrow \pi_3 \cos\omega - \sqrt{f^2 - \underline{\pi}^2}\,\sin\omega \qquad (1.4)$$

and

$$\phi_{1,2} \rightarrow \phi_{1,2}$$
$$\phi_3 \rightarrow \phi_3 \cos\omega + \frac{\underline{\phi}\cdot\underline{\pi}}{\sqrt{f^2 - \underline{\pi}^2}}\,\sin\omega \qquad (1.5)$$

and one sees that only the preferred field $\underline{\pi}$ enters non-linearly. As is well known, the form of this result depends very much on the manner in which π_α is parametrized. An alternative scheme would present π_α in the form

$$\pi_i = \frac{\varphi_i}{\sqrt{1 + \underline{\varphi}^2/f^2}} \quad , \qquad i = 1,\ 2,\ 3$$

$$\pi_4 = \frac{f}{\sqrt{1 + \underline{\varphi}^2/f^2}} \qquad (1.6)$$

where the 3-vector φ_i is independent. Corresponding to the purely chiral transformations (1.3) one finds for φ_i the transformation law

$$\varphi_{1,2} \rightarrow \frac{\varphi_{1,2}}{\cos\omega + (\varphi_3/f)\sin\omega}$$

$$\varphi_3 \rightarrow \frac{\varphi_3 \cos\omega - f\sin\omega}{\cos\omega + (\varphi_3/f)\sin\omega} \qquad (1.7)$$

and, for ϕ_i ,

$$\phi_{1,2} \rightarrow \phi_{1,2}$$
$$\phi_3 \rightarrow \phi_3 \cos\omega + \frac{1}{f}\,\underline{\phi}\cdot\underline{\varphi}\,\sin\omega \qquad (1.8)$$

which replace (1.4) and (1.5) respectively. The realization (1.6) was adopted by Schwinger [4]. We have reproduced it here because it lends itself readily to "proving" the non-invariance of the vacuum. Thus, for infinitesimal ω the transformation (1.7) gives, in particular,

$$-\delta\varphi_3 = \omega f\left(1 + \frac{\varphi_3^2}{f^2}\right) \qquad (1.9)$$

the vacuum expectation value of which cannot possibly vanish [5] since the right-hand side is positive-definite. The realization (1.6) is possible only if the vacuum breaks chiral symmetry. Notice that this argument does not depend on the existence or non-existence of a Lagrangian.

For alternative parametrizations, such as (1.4), it is not possible to make such a categorical statement. However, for practical purposes where the non-linearities are always interpreted by power series expansions in $\underline{\pi}/f$, the implication is the same. Non-linear constraints can be dealt with by power series methods only in theories with intrinsic symmetry breaking. The particles associated with the preferred fields are in fact the Goldstone bosons. One way to see this is to remark that nowhere in a chiral invariant Lagrangian does the field $\underline{\pi}$ appear without being accompanied by $\partial_\mu\underline{\pi}$ as well. It follows that the fields $\underline{\pi}$ must describe massless (spin zero) particles. Thus a vacuum state is indistinguishable from a state with two zero-frequency pions, four zero-frequency pions, etc., provided the pions together form an $I = 0$ multiplet of the subgroup SU(2). Since the pions form an incomplete multiplet of SU(2) x SU(2) it is clear that these physically indistinguishable states of lowest energy are not, in general, chiral invariant.

So far we have said that if non-linear realizations are introduced by considering linear realizations of the preferred fields together with a constraint, the constraint implies that the vacuum state in the theory must be a non-invariant state, and the symmetry a spontaneously broken one, with the independent ones among the preferred fields playing the role of Goldstone bosons. Consider now the converse problem: given a theory with a chiral invariant Lagrangian it may happen that the ground state is not chiral invariant. In this case there must appear Goldstone bosons corresponding to the components of the symmetry which are absent from the vacuum. One may introduce into the theory a set of spin-zero fields describing these bosons. The problem one is presented with is how to couple such mesons with other particles so that the consequences of the vacuum asymmetry are made explicit. These effects must include, for example, the guarantee that these particles remain massless even after interacting with other particles. The formalism must also give a correct account of the perturbations of masses and coupling constants on account of symmetry breaking. Our solution to the problem is to suggest that the appropriate formalism is the one where a non-linear realization is employed with these spin-zero fields playing the role of the preferred fields.

The problem of making explicit the Goldstone bosons can be solved in various ways [6]. Suppose, for simplicity, that the Lagrangian contains a zero-spin chiral 4-vector, Φ_α: then it is possible to eliminate Φ_4 in favour of $\chi = \sqrt{\Phi_\alpha \Phi_\alpha}$. The kinetic energy then takes the form

$$\frac{1}{2}(\partial_\mu \Phi_\alpha \partial_\mu \Phi_\alpha - m^2 \Phi_\alpha \Phi_\alpha) = \frac{1}{2}(\partial_\mu \chi \partial_\mu \chi - m^2 \chi^2) + \frac{1}{2} D_\mu \underline{\Phi} \cdot D_\mu \underline{\Phi} \tag{1.10}$$

where $D_\mu \underline{\Phi}$ denotes the so-called covariant derivative of $\underline{\Phi}$. It is given by

$$\frac{1}{\chi} D_\mu \underline{\Phi} = \partial_\mu \left(\frac{\underline{\Phi}}{\chi}\right) - \frac{\underline{\Phi}}{\chi} \times \partial_\mu \left(\frac{\underline{\Phi}}{\chi}\right) \tag{1.11}$$

and belongs to the non-linear realization (1.5). The second term in (1.10) contains the term $(1/2)(\partial_\mu \underline{\Phi})^2$ together with an infinite number of interaction terms which arise from the expansion of $1/\chi$ in powers, i.e.,

$$\frac{1}{\chi} = \frac{1}{\langle\chi\rangle + \chi'} = \frac{1}{\langle\chi\rangle} - \frac{\chi'}{\langle\chi\rangle^2} + \frac{\chi'^2}{\langle\chi\rangle^3} - \cdots$$

which is meaningful provided $\langle\chi\rangle \neq 0$. The non-vanishing of $\langle\chi\rangle$ must be thought of as a consequence of the supposed vacuum asymmetry [7].

The particle which is characterized by the isoscalar field χ' is of no particular importance in the theory. It arose as a byproduct of the effort to set up an effective Lagrangian with Goldstone bosons. Having got the effective Lagrangian one is at liberty to set $\chi' = 0$. It is only the numerical part $\langle\chi\rangle$ which must be kept.

An important exception to the theorem which requires the presence of massless bosons in situations where a symmetry is broken intrinsically occurs when long-range vector fields are also present.[8)] If the currents of the spontaneously broken symmetries are coupled to a gauge field of the Yang-Mills type then the symmetry breaking manifests itself not through the appearance of Goldstone bosons but rather in the acquiring of mass by some of the components of the gauge field. This phenomenon,which was discovered by Anderson and developed by Higgs and Kibble,will be presented in the non-linear notation in Sec. 2 where it will be shown that the preferred field disappears from the Lagrangian if the symmetry group is gauged. In addition - and this is where we improve on Higgs and Kibble - the residual quantized objects like χ' , whose existence is required in their models, can be set equal to zero without doing any violence to the elegant formulation afforded by the non-linear formalism.

Turning back to the formulation of the non-linear method, it will be remarked that, in eqs. (1.1), the field π_α plays a role which is formally similar to that taken by the 4-momentum p_α in the formulation of relativistically covariant free field equations. The analogy can be extended to interaction terms as well. The involvement of orbital angular momentum in the relativistic coupling of particles with spin is accounted for by the presence of terms like $\gamma_\alpha \partial/\partial x_\alpha$ in the Lagrangian. Likewise, the involvement of soft pions in the chiral invariant coupling of particles with isospin is brought about through terms like $\Gamma_\alpha \pi_\alpha = \pi_4 + \gamma_5 \underline{\tau} \cdot \underline{\pi}$.

We are not advocating the exploitation of this analogy as a practical way to make chiral invariant Lagrangians. The existing method which uses non-linear realizations directly is a simpler one to apply. However, there is another aspect of the analogy between π_α and p_α which leads to a formal development of some power. This lies in the notion of the <u>boost</u>. It was Wigner's discovery [9] that the momentum 4-vector p_α could with great advantage be represented in the form

$$p_\alpha = (L_p)_{\alpha 4}\, m \qquad (1.12)$$

where m denotes the rest mass, $\sqrt{p^2}$ and $(L_p)_{\alpha\beta}$ a 4 x 4 matrix belonging to the Lorentz group. The mass-shell constraint $p_\alpha p_\alpha = m^2$ is accounted for automatically in the representation (1.12) by the (pseudo) orthogonality of the matrix L_p. The power of this representation lies in that it leads to the realization of the Lorentz group in terms of 3 x 3 orthogonal matrices (or 2 x 2 unitary and unimodular ones). Thus, corresponding to the Lorentz transformation

$$p_\alpha \to p'_\alpha = \Lambda_{\alpha\beta}\, p_\beta \ , \qquad (1.13)$$

one has the realization

$$\Lambda \to R(p,\Lambda) = L^{-1}_{\Lambda p}\, \Lambda\, L_p \qquad (1.14)$$

where $R(p,\Lambda)$ is, in effect, an ordinary space-rotation. This realization is of course essentially the same as the non-linear one (1.5) obtained in this instance by setting $p_4 = \sqrt{\underline{p}^2 + m^2}$. It is perhaps worth noticing that, insofar as finite-dimensional realizations are involved, the distinction between the compact group SU(2) x SU(2) and its non-compact relative SL(2, C) is a minor one.

We propose to adopt the method of Wigner for dealing with non-linear realizations. That is, we shall express the preferred field in the form

$$\pi_\alpha = (L_\pi)_{\alpha 4}\, f \qquad (1.15)$$

where $(L_\pi)_{\alpha\beta}$ denotes a 4 x 4 matrix whose components are dynamical variables. These variables are not all independent. They are subject to the constraints

$$\sum_{\beta} (L_\pi)_{\alpha\beta} (L_\pi)_{\gamma\beta} = \delta_{\alpha\gamma}$$

$$\det (L_\pi) = 1 \qquad (1.16)$$

or, in other words, L_π belongs to SO(4). Included among the constraints (1.16) is of course the principal one, $\pi_\alpha \pi_\alpha = f^2$.

The main advantage to be gained by replacing the preferred field π_α with the matrix $(L_\pi)_{\alpha\beta}$ is the ease with which such a matrix can be used to effect a passage between linear and non-linear realizations. Moreover, it enables one to discover the general features of a class of non-linear realizations without the complication of having to commit oneself to a particular parametrization. It is this formal power which makes the boost approach useful for generalizing beyond the chiral groups. However, to avoid semantic confusion we shall invent a new name, reducing matrix, for the matrix L_π and its generalizations, since the word boost has already a rather precise meaning within the context of the inhomogeneous Lorentz group and its representations.

The realizations of a continuous group G which become linear when restricted to some specified subgroup H are treated by means of the reducing matrix in Sec. 2. These realizations are then gauged in the Yang-Mills manner. Sec. 3 contains some general remarks about symmetry breaking both spontaneous and explicit. The formal techniques are illustrated in Sec. 4 on a model which could have practical interest, the non-linear realizations of SU(3) which become linear with respect to $SU(2)_I \times U(1)_Y$.

2. NON-LINEAR REALIZATIONS

Consider the problem of constructing non-linear realizations of a continuous group G which become both linear and irreducible under some specified subgroup H . One may suppose that the linear irreducible representations of G

$$\Psi \rightarrow D(g)\, \Psi \quad , \quad g \in G \tag{2.1}$$

and their decomposition into linear irreducible representations of H are known. It will prove convenient to assume that the basis has been chosen so as to render the matrices, D(h) where $h \in H$, block diagonal in form.

The first stage in solving the realization problem is the definition of a matrix, $(L_\phi)_{\alpha\beta}$, the elements of which are field variables. This matrix, which we shall call the <u>reducing matrix</u>, will be subject to a number of algebraic constraints and will be endowed with a peculiar transformation behaviour under the operations of the group G . The basic requirements are:

(a) The matrix L_ϕ is constrained to belong to the group G , i.e., to its self-representation. This is in order that, for any finite-dimensional representation $g \rightarrow D(g)$, the functional $D(L_\phi)$ shall be well defined. The number of independent fields, ϕ_a , needed to parametrize L_ϕ is therefore equal to or less than the dimensionality of G .

(b) Under the operations of the group G the fields which make up the reducing matrix transform according to

$$L_\phi \rightarrow g\, L_\phi\, h^{-1}\, (\phi, g) \tag{2.2}$$

where $g \in G$ and $h(\phi, g) \in H$. In other words, the columns of the reducing matrix must be arranged into sets which transform among themselves according to some representation of the subgroup H .

(c) Under the operations of the subgroup H the reducing matrix transforms in the ordinary way,

$$L_\phi \rightarrow h\, L_\phi\, h^{-1} \quad . \tag{2.3}$$

It follows from (2.2) that the functionals $D(L_\phi)$, which are defined for any finite-dimensional representation, transform according to

$$D(L_\phi) \rightarrow D(g\, L_\phi\, h^{-1})$$
$$= D(g)\, D(L_\phi)\, D(h^{-1}) \quad . \tag{2.4}$$

It is this property which enables one to project non-linear realizations out of linear ones like (2.1) by the operation

$$\psi = D(L_\phi^{-1})\, \Psi \quad . \tag{2.5}$$

A comparison of (2.1) and (2.4) yields for ψ the transformation law

$$\psi \longrightarrow D(h)\, \psi \tag{2.6}$$

where $h = h(\phi, g)$ is in general a non-linear structure which depends upon the preferred fields ϕ_a which parametrize L_ϕ .

The detailed form of the matrix $h(\phi, g)$ is dependent upon the parametrization scheme, i.e., upon which combinations of the components $(L_\phi)_{\alpha\beta}$ are taken as independent variables. Perhaps the simplest scheme is the one adopted by Coleman, Wess and Zumino[10) and, earlier, by Kibble[6),

$$L_\phi = e^{\phi \cdot A}$$

where A_a denotes the set of infinitesimal generators of G which are not contained in the algebra of H . Whatever the scheme chosen, one can discover the matrix $h(\phi, g)$ by referring the eq. (2.4) to a representation of G which contains a singlet of H . In such a representation there exists at least one column, χ , for which (2.4) takes the form

$$D(L_\phi)\chi \rightarrow D(L_{\phi'})\chi = D(g)\, D(L_\phi)\chi \quad , \tag{2.7}$$

i.e., for which $D(h)$ is represented by the identity. If the chosen parameters ϕ_a are expressed in terms of the components of the column which transforms according to (2. 7) then it is straightforward to compute the transformation law of these parameters. Having done this, one can compute the matrix h by comparison with (2. 2), i. e.,

$$h(\phi, g) = L_{\phi'}^{-1} \, g \, L_{\phi} \quad . \tag{2.8}$$

The method will be illustrated in the accompanying paper for the case of the conformal group.[11)]

Consider now the problem of defining a covariant derivative operator for the non-linear realization (2. 6). It is evident that the ordinary derivative is not covariant,

$$\partial_\mu \psi \rightarrow D(h) \, \partial_\mu \psi + \partial_\mu D(h) \psi \quad .$$

In order to be able to make covariant field equations it is essential that one defines a covariant operator resembling the derivative. This can be done in the following way.

Let us imbed ψ in some linear representation $D(g)$

$$\psi = D(L_\phi^{-1}) \Psi$$

and define, relative to it, the operator Δ_μ ,

$$\Delta_\mu \psi = D(L_\phi^{-1}) \, \partial_\mu \Psi \ , \tag{2.9}$$

which is clearly covariant. However, one should not adopt Δ_μ as the desired covariant derivative since it depends upon the imbedding representation $D(g)$. In order to remove this dependence it is necessary to analyse (2. 9) more closely. Write

$$\Delta_\mu \psi = \partial_\mu \psi + D(L_\phi^{-1}) \, \partial_\mu \, D(L_\phi) \psi \quad . \tag{2.10}$$

The matrix $D^{-1} \partial_\mu D$ can be simplified if use is made of the constraint $L_\phi \in G$ for all x . In particular it follows that the matrix

$$L_\phi^{-1}(x) \, L_\phi(x + \delta x) = 1 + \delta x_\mu \, L_\phi^{-1} \, \partial_\mu \, L_\phi + \ldots$$

is an infinitesimal transformation of G . In other words, the matrices $L_\phi^{-1} \partial_\mu L_\phi$ belong to the infinitesimal algebra of G . They can therefore be expanded in the form[11)]

$$L_\phi^{-1} \partial_\mu L_\phi = i (L_\phi^{-1} \partial_\mu L_\phi)_i s_i \tag{2.11}$$

where the matrices s_i constitute a basis of the algebra and the coefficients of the expansion are denoted by $(L_\phi^{-1} \partial_\mu L_\phi)_i$. In the representation D(g) where the infinitesimal generators s_i are represented by S_i , the expansion (2.11) takes the form

$$D(L_\phi^{-1}) \partial_\mu D(L_\phi) = i (L_\phi^{-1} \partial_\mu L_\phi)_i S_i . \tag{2.12}$$

The transformation behaviour of the coefficients in (2.11) is complicated by the presence of the derivative operator. From (2.2) one finds

$$L_\phi^{-1} \partial_\mu L_\phi \rightarrow h(L_\phi^{-1} \partial_\mu L_\phi) h^{-1} + h \partial_\mu h^{-1} . \tag{2.13}$$

That is, there is present, in general, an inhomogeneous term in the transformation law. Clearly, however, the inhomogeneity belongs to the algebra of H . This point is of crucial importance because it means that the fields $(L_\phi^{-1} \partial_\mu L_\phi)_i$ can be divided into two sets, one of which transforms covariantly while the other contains the inhomogeneity.

Let us suppose that the algebraic basis s_i has been chosen in such a way that it can split into two components[12)] , m_α and n_a , which transform independently under H . Suppose, moreover, that the m_α constitute a basis for the subalgebra H . In this basis the expansion (2.11) takes the form

$$\frac{1}{i} L_\phi^{-1} \partial_\mu L_\phi = \Gamma_{\mu\alpha} m_\alpha + \lambda D_\mu \phi_a n_a \tag{2.14}$$

which is to be looked upon as the definition of the field quantities $\Gamma_{\mu\alpha}$ and $D_\mu \phi_a$. The inhomogeneous term in the transformation (2.13) affects only the $\Gamma_{\mu\alpha}$, and the fields $D_\mu \phi_a$ therefore belong to a bona-fide non-linear realization of the group G . They are to be interpreted as the covariant derivatives of the preferred fields ϕ_a in terms of which the reducing matrix is parametrized. The real parameter

λ will be fixed by normalizing the kinetic energy term associated with ϕ_a.

Corresponding to the expansion (2.14) one has, in the representation $D(g)$,

$$\frac{1}{i} D(L_\phi^{-1}) \partial_\mu D(L_\phi) = \Gamma_{\mu\alpha} M_\alpha + \lambda D_\mu \phi_a N_a \quad , \tag{2.15}$$

which can be substituted into the expression (2.10) for Δ_μ,

$$\Delta_\mu \psi = \partial_\mu \psi + i \Gamma_{\mu\alpha} M_\alpha \psi + i \lambda D_\mu \phi_a N_a \psi \quad . \tag{2.16}$$

Since the left-hand side of (2.16) transforms covariantly, as does the third term on the right, therefore the sum of the remaining two terms must also be covariant. The latter part, denoted $D_\mu \psi$, has in addition the required property of being independent of the imbedding representation. Thus, the covariant derivative of ψ is given by

$$D_\mu \psi = \partial_\mu \psi + i \Gamma_{\mu\alpha} M_\alpha \psi . \tag{2.17}$$

Finally, consider the problems which arise when the transformations of the group G are made space-time dependent, i.e., when G is turned into a gauge group of the Yang-Mills type

$$\Psi(x) \rightarrow D(g) \Psi(x) \quad , \quad g = g(x) \in G \quad . \tag{2.18}$$

There is no need to alter the prescription (2.5) for extracting the non-linear realizations from Ψ. Indeed, the non-linear transformation law (2.6) is formally unchanged since the matrix $h(\phi, g)$ is defined for arbitrary $g(x) \in G$. The modifications are of course needed in the definition of covariant derivatives.

Now it is well known that the ordinary derivative is not covariant under space-time dependent transformations

$$\partial_\mu \Psi(x) \rightarrow D(g) \partial_\mu \Psi(x) + \partial_\mu D(g) \Psi(x) \quad . \tag{2.19}$$

With the basis s_i defined above one can write

$$\frac{1}{i}\, g\,\partial_\mu\, g^{-1} = (g\,\partial_\mu\, g^{-1})_i\, s_i \tag{2.20}$$

since the matrices $g^{-1}\,\partial_\mu\, g$ belong to the algebra of G. Therefore, (2.19) can be expressed in the form

$$\partial_\mu \Psi(x) \rightarrow D(g)\ (\partial_\mu + i\,(g\,\partial_\mu\, g^{-1})_i\, S_i)\ \Psi(x)\ . \tag{2.21}$$

In order to replace this with a covariant formula one must introduce a set of gauge fields

$$A_\mu = A_{\mu i}\, s_i \tag{2.22}$$

which transform according to the law

$$A_\mu \rightarrow g\, A_\mu\, g^{-1} + \frac{1}{if}\, g\,\partial_\mu\, g^{-1}\ . \tag{2.23}$$

The covariant derivative, for linear representations, is then defined by

$$\mathcal{D}_\mu \Psi = (\partial_\mu + if\, A_{\mu i}\, S_i)\,\Psi\ . \tag{2.24}$$

In the usual fashion the covariant derivative of the gauge field itself is contained in the antisymmetric tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + if\,[A_\mu, A_\nu]\ . \tag{2.25}$$

From the expressions (2.24) and (2.25) which are covariant in the linear sense, one can project out the generalized non-linear covariant derivatives. Firstly, if ψ is defined by (2.5) its covariant derivative must be contained in the operator

$$\Delta_\mu \psi = D(L_\phi^{-1})\ (\partial_\mu + if\, A_{\mu i}\, S_i)\,\Psi$$

which can be simplified to the form

$$\Delta_\mu\, \psi = (\partial_\mu + if\, B_{\mu i}\, S_i)\ \psi \tag{2.26}$$

where $B_{\mu i}$ is a modified gauge field defined by

$$B_\mu = L_\phi^{-1}\, A_\mu\, L_\phi + \frac{1}{if}\, L_\phi^{-1}\, \partial_\mu\, L_\phi\ . \tag{2.27}$$

It transforms under a general gauge transformation $g(x)$ according to

$$B_\mu \to h B_\mu h^{-1} + \frac{1}{if} h \partial_\mu h^{-1} \quad . \tag{2.28}$$

The most important feature of this transformation law is the fact that the inhomogeneous term belongs to the algebra of H . This means that the operator Δ_μ of (2.26) can be separated covariantly into two pieces,

$$\Delta_\mu \psi = D_\mu \psi + i\lambda (D_\mu \phi_a) N_a \psi$$

which defines the generalized covariant derivatives

$$D_\mu \psi = (\partial_\mu + if B_{\mu\alpha} M_\alpha) \psi \tag{2.29}$$

$$\lambda D_\mu \phi_a = f B_{\mu a} \quad . \tag{2.30}$$

These expressions are covariant against gauge transformations of the second kind. It remains only to find the covariant derivatives of the fields $B_{\mu\alpha}$ and $B_{\mu a}$. These are contained in the antisymmetric tensor

$$B_{\mu\nu} = L_\phi^{-1} F_{\mu\nu} L_\phi$$

which, in view of the definition (2.27), goes into the form

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + if [B_\mu, B_\nu] \quad . \tag{2.31}$$

The realizations discussed in this section can be used in the construction of Lagrangians which are invariant with respect to the transformations of G . Firstly, a Lagrangian which is manifestly invariant with respect to the subgroup H can be modified so as to become invariant with respect to the space-time independent transformations of the larger group G . It is necessary only to replace the ordinary derivatives $\partial_\mu \psi$ by their covariant form $D_\mu \psi$ as given in (2.17) and to take account of the new zero-spin boson field ϕ_a , whose existence this implies, by adding a covariant kinetic energy term,

$$\frac{1}{2} (D_\mu \phi_a^*) (D_\mu \phi_a)$$

and, possibly, other derivative coupling terms using $D_\mu \phi_a$. It is not possible to construct any covariant object which contains a term like $\phi_a^* \phi_a$ and it therefore follows that the new bosons must be without mass.

The invariance of the Lagrangian can be further enlarged to include the space-time dependent transformations of G by introducing a gauge field B_μ, which transforms according to the reducible non-linear realization (2.28). The covariant derivatives $D_\mu \psi$ and $D_\mu \phi_a$ are given the new forms (2.29) and (2.30), respectively, while, for the gauge field, it is necessary to adjoin the kinetic energy term

$$\frac{1}{4} B_{\mu\nu} \cdot B_{\mu\nu}$$

where $B_{\mu\nu}$ is given by (2.31). The upshot of these final modifications is that the preferred field ϕ_a and its massless quanta have disappeared from the Lagrangian. They have been absorbed by a redefinition (2.27) of the gauge field. Not only has the multiplet of massless bosons ϕ_a disappeared: part of the gauge field, $B_{\mu a}$, has acquired a well-defined mass, f/λ. The other part, $B_{\mu\alpha}$, which enters the covariant derivatives, remains without mass.

This phenomenon, whereby the introduction of a gauge multiplet of vector particles causes the disappearance of the massless zero-spin particles, has been discussed by a number of authors[8)] in the context of spontaneous symmetry breaking. In Sec. 3 we show that this is precisely the context in which non-linear realizations have meaning. The massless zero-spin particles are indeed the Goldstone bosons.

3. SPONTANEOUS SYMMETRY BREAKING

A. The purpose of this section is to demonstrate that the formalism of non-linear realizations and effective Lagrangians provides a natural framework for treating intrinsically broken symmetries. The arguments given here parallel those of Kibble [6].

To discuss a system with spontaneous symmetry breaking one must have in mind a Lagrangian which is invariant with respect to the transformations of some continuous group G. Secondly, one must assume that the ground state or vacuum is not an invariant of G but only of some subgroup H. This property of the ground state is signalled by the non-vanishing expectation values of fields or combinations of fields which belong to non-trivial representations of G. Its consequences include symmetry breaking perturbations of the masses and couplings of physical particles and, in particular, the appearance of spin-zero massless bosons.

B. Consider a system of fields, fermions and bosons, denoted collectively by Ψ and, in addition, a spin-zero multiplet M (some components, M_a', of which will correspond to Goldstone particles) which transform according to the reducible linear representations

$$\begin{aligned} \Psi &\to D(g)\,\Psi \\ M &\to \mathbb{D}(g)\,M \end{aligned} \tag{3.1}$$

where g denotes an element of G and D, $\mathbb{D}$ are the matrices appropriate to the representations concerned. The Lagrangian of this system is supposed to be invariant under these transformations,

$$L(\Psi, \partial_\mu \Psi, M, \partial_\mu M) = L(D\Psi, D\partial_\mu \Psi, \mathbb{D}M, \mathbb{D}\,\partial_\mu M)\,. \tag{3.2}$$

This means that the system is classified into complete multiplets of G with the various couplings which are allowed by this symmetry. It may therefore be quite unlike the physical reality which reflects the ground state asymmetry. The mass splittings of the physical multiplets can be large and in fact so large that some of the multiplets may be regarded as incomplete. Likewise for the couplings.

In such a system it is known that Goldstone bosons must be present. Being massless, these particles cause a re-adjustment of the stable states of the system. In particular, the physical vacuum should contain an admixture of zero-energy Goldstone particles (which is just a way of saying that it is degenerate). This property of the Goldstone particles can be put formally by saying that they effect a redefinition of the "bare" masses and coupling constants which takes account of the ground state asymmetry. We wish to introduce a set of fields $\phi_a(x)$ to represent the Goldstone particles and to put the Lagrangian (3.2) into a form which, <u>though still invariant under the transformations of G</u>, shows explicitly, in its bare masses and coupling constants, the effects of the underlying asymmetry. As stated in the introduction, the method of non-linear realizations, with the fields $\phi_a(x)$ as the preferred fields, appears to be just the right construct to solve this problem.

The subset M_a of M referred to earlier transforms under the subgroup H like the set of those generators n_a (cf. (2.14)) which correspond to the spontaneously broken symmetries of G. The remaining components of M which are not included among the set M_a will be labelled as M_A,

$$M = \begin{pmatrix} M_a \\ M_A \end{pmatrix} .$$

In such a case it is possible to invent a transformation $L_\phi(x) \in G$ which transforms away the components M_a in the sense that one can represent M in the form

$$M = \mathbb{D}(L_\phi)\, m \tag{3.3}$$

with $L_\phi(x)$ so determined that

$$m_a = (\mathbb{D}(L_\phi^{-1})\, M)_a = 0 . \tag{3.4}$$

The components of the matrix L_ϕ - obtained as non-linear functions of M by solving (3.4) - must satisfy various constraint conditions in order that L_ϕ belong to G but can be expressed in terms of a set of suitably chosen independent parameters ϕ_a equal in number to the m_a of (3.4). In principle, therefore, one can solve for these parameters ϕ_a (the preferred fields) in terms of the original set of fields M. (The number of m_a's being set equal to zero equals the number of ϕ_a's introduced.) Since the latter fields transform according to the given linear rule (3.1) one can determine the transformation law of the preferred set ϕ_a. This law is quite generally non-linear and, if the non-linearities are expanded in powers, inhomogeneous. It therefore follows, as has been emphasised in Sec. 1, that the representation (3.3) can be used only in theories with non-invariant vacua.[13)]

C. It may be that some of the fields Ψ and in particular M_A in (3.2) do not represent physical particles - or they represent particles which are so far removed in mass from their partners in the set M as to be irrelevant dynamically (the analogy of M_a is with π_a (a = 1, 2, 3) and of M_A with σ in the chiral model). The chief problem therefore is to exhibit the formalism in such a way that these can be removed from consideration; we do this by following the standard non-linear prescription of imposing constraints and the details of the method are as follows.

In the Lagrangian (3.2) substitute the expression (3.3) for the fields M and its analogue for Ψ to give

$$L(\Psi, \partial_\mu \Psi, M, \partial_\mu M) = L\big(D(L_\phi)\psi, \partial_\mu D(L_\phi)\psi, \mathbb{D}(L_\phi)m, \partial_\mu \mathbb{D}(L_\phi)m\big)$$

$$= L\big(\psi, D(L_\phi^{-1})\partial_\mu D(L_\phi)\psi, m, \mathbb{D}(L_\phi^{-1})\partial_\mu \mathbb{D}(L_\phi)m\big) \qquad (3.5)$$

where the validity of the second step depends upon the invariance of this Lagrangian under the transformations of G. The derivative terms in (3.5) involve the operator Δ_μ defined in Sec. 2,

$$D(L_\phi^{-1})\,\partial_\mu\, D(L_\phi)\,\psi = \Delta_\mu \psi = D_\mu \psi + i\lambda\, D_\mu \phi_a\, N_a \psi \quad , \qquad (3.6)$$

where $D_\mu \psi$ and $D_\mu \phi_a$ denote the covariant derivatives defined by (2.14) and (2.17). Similar relations hold for m. The fields ψ and m defined by (3.3) and (3.4) belong to a reducible non-linear realization of G which becomes linear with respect to the subgroup H . The number of components ψ , ϕ_a and m_A is equal to the number of original field components Ψ and M . However, it is clear that any subset of the fields ψ and m_A which spans a (linear) representation of H can be set equal to zero without doing violence to the invariance of L . Such a disappearance of some of the components of ψ is balanced by the appearance of algebraic constraints on Ψ . This can be seen by inverting the formulae (3. 3) and (3. 4) and expressing the components ψ and m_A as non-linear functions [14] of Ψ and M.

Thus one can express the Lagrangian (3. 2) in terms of the non-linear variables

$$L(\Psi, \partial_\mu \Psi, M, \partial_\mu M) = L(\psi, \Delta_\mu \psi, m_A, \Delta_\mu m_A) \qquad (3.7)$$

and feed in the realistic bare masses and coupling constants. Unwanted components of ψ and m_A can now be set equal to zero [15]: the only rule to be observed is the manifest invariance of the right-hand side of (3. 7) under the transformations of the subgroup H .

D. Contained in the Lagrangian (3. 7) there will in general be the term

$$\frac{1}{2} D_\mu \phi_a D_\mu \phi_a^* = \frac{1}{2} \partial_\mu \phi_a \partial_\mu \phi_a^* + \text{ interaction terms } .$$

The first term on the right-hand side of this is to be interpreted as the kinetic energy of the Goldstone bosons. These are massless bosons because, although ϕ_a appears elsewhere in the Lagrangian (in the covariant derivatives of ψ), it is always accompanied by $\partial_\mu \phi_a$. There is no mass term. That they are the Goldstone bosons is clear since they appear only when the symmetry is broken spontaneously by the imposition of constraints. It is clear from the above discussion that these particles could not even be defined if the vacuum were symmetric. On the other hand, if the Lagrangian were not symmetric they would acquire a mass since the matrix $D(L_\phi)$ would, in such a case, fail to be eliminated completely out of the right-hand side in (3. 5).

E. It is of interest to see what happens if a long-range gauge field is present. Let us therefore replace the Lagrangian (3.2) by one which is invariant under transformations of the second kind. This means introducing a set of gauge fields, i.e.,

$$L(\Psi, \partial_\mu \Psi) \rightarrow L_1 = L(\Psi, \partial_\mu \Psi + if\, A_{\mu i}\, S_i\, \Psi) + \frac{1}{4} F_{\mu\nu i}\, F_{\mu\nu i} \tag{3.8}$$

in the notation of Sec. 2. If the expression (3.3) for Ψ in terms of the non-linear realization ψ is substituted in (3.8) one finds

$$L_1 = L(\psi,\ \partial_\mu \psi + if\, B_{\mu i}\, S_i\, \psi) + \frac{1}{4} B_{\mu\nu i}\, B_{\mu\nu i} \tag{3.9}$$

where B_μ is defined in terms of the gauge fields A_μ and the reducing matrix L_ϕ by (2.27). It transforms according to the non-linear rule (2.28). The covariant derivative of B_μ is contained in the expression (2.31) for $B_{\mu\nu}$. The Goldstone particles, represented by L_ϕ, have been absorbed in the redefined gauge fields B_μ. They no longer exist as independent particles. The fields ϕ_a still appear <u>implicitly</u> in the "non-linear" transformation laws of ψ and B_μ, but they no longer have any dynamical significance. Moreover, the Lagrangian L_1, being independent of the ϕ_a, is not of the non-linear variety.

Since the gauge fields $B_{\mu\alpha}$ transform inhomogeneously (2.28) it is essential for the preservation of gauge invariance of the second kind that there should be no mass term $m^2 B^2_{\mu\alpha}$. However, it is possible to maintain gauge invariance of the first kind in the presence of a term like

$$\frac{1}{2} m^2 \left(B_{\mu\alpha} - \frac{1}{if} (L_\phi^{-1}\, \partial_\mu\, L_\phi)_\alpha\right)^2 \tag{3.10}$$

or, in other words, if the Goldstone particles are revived. This means that intrinsic symmetry breaking in the presence of a gauge field of <u>finite range</u> requires the presence of Goldstone particles.

F. We conclude this section with the remark that the Goldstone fields ϕ_a will become massive if and only if there is introduced in (3.2) an explicit symmetry breaker. If that is done, it is clear that the reducing matrix L_ϕ must appear explicitly in the transformed Lagrangian

(3.5) and not merely in the covariant derivatives. This means that ϕ_a is no longer everywhere accompanied by $\partial_\mu \phi_a$ and so, by expanding L_ϕ in powers of ϕ, one can always find a term proportional to ϕ^2. For example, one could add to the Lagrangian (3.5) an explicit symmetry breaker of the form

$$\kappa^2 \mathbb{D}_{11} (L_\phi) \tag{3.11}$$

where $\mathbb{D}(g)$ denotes some chosen (self-conjugate) irreducible representation of G and $\mathbb{D}_{11}(g)$ indicates a matrix element of $\mathbb{D}(g)$ between states which are singlets of H, i.e.,

$$\mathbb{D}_{11}(hg) = \mathbb{D}_{11}(gh) = D_{11}(g) \ . \tag{3.12}$$

The presence of a term like (3.11) in the Lagrangian therefore does not violate the symmetry under H.

This is the approach advocated by Weinberg [16)]. A still more satisfactory Lagrangian, fully invariant under G but still producing a mass for the ϕ-particles, could be

$$L = (L - \frac{1}{2} m^2 \phi^2) + \frac{1}{2} m^2 \phi^2$$

where m^2 is computed self-consistently by setting up an interaction representation and computing the self-mass of the ϕ-particle which is then put equal to the physical mass, i.e., its bare mass is zero. Whether this self-consistency procedure will introduce other Goldstone particles into the theory is an open question.

4. A SIMPLE EXAMPLE

In order to illustrate the techniques presented in Sec. 2 we consider here the non-linear realizations of SU(3) which become linear with respect to the subgroup $SU(2)_I \times U(1)_Y$. This example is sufficiently complicated to exhibit the main features of the non-linear formalism and, moreover, it has some physical relevance in that the breaking of SU(3) symmetry may well be, at least in part, intrinsic. In addition, as can easily be seen, only very little effort will be needed to extend the formulae given here to the case of chiral SU(3) x SU(3) broken spontaneously to chiral SU(2) x SU(2).

The first stage in establishing the non-linear realizations is the construction of a reducing matrix $(L_K)_\alpha^\beta \in SU(3)$. This matrix must transform according to

$$L_K \rightarrow g\, L_K\, h^{-1}(K, g) \tag{4.1}$$

where $g \in SU(3)$ and $h \in SU(2) \times U(1)$. Since $SU(2) \times U(1)$ is a four-parameter group while SU(3) has eight parameters one expects that there should be a set of four preferred fields, K and $\bar{K}$, with which to parametrize L_K . These fields correspond to the hypercharge changing transformations of SU(3). Out of all the possible parametrizations we shall pick one that does not involve square roots (and is therefore the nearest in spirit to Weinberg's treatment of chiral SU(2) x SU(2)). It is given by

$$L_K = \begin{bmatrix} \left(1 - \frac{K\bar{K}}{\bar{K}K}\right) + \frac{1 - \lambda^2 \bar{K}K}{1 + \lambda^2 \bar{K}K} \frac{K\bar{K}}{\bar{K}K} & \frac{2\lambda K}{1 + \lambda^2 \bar{K}K} \\ -\frac{2\lambda \bar{K}}{1 + \lambda^2 \bar{K}K} & \frac{1 - \lambda^2 \bar{K}K}{1 + \lambda^2 \bar{K}K} \end{bmatrix} \tag{4.2}$$

where λ denotes a real parameter to be fixed later. The field K is a two-component column vector and $\bar{K}$ denotes its hermitian adjoint, a two-component row vector.

It is necessary to demonstrate that the parametrization (4.2) is consistent with the transformation requirements (4.1). This can be done by setting up explicit transformation rules for the preferred fields K and $\bar{K}$. Firstly, under the subgroup $SU(2) \times U(1)$, since $h(K, g) = g$, it is clear that K and $\bar{K}$ transform like $I = 1/2$ fields with $Y = +1$ and $Y = -1$, respectively. To discover their behaviour under the hypercharge changing transformations it is necessary to use the fact that $h^{-1}(K, g)$ multiplies the third column of L_K by a phase factor but does not mix into it the first two columns. Let us consider the infinitesimal hypercharge changing transformation

$$g = \begin{pmatrix} 1 & i\epsilon \\ i\bar{\epsilon} & 1 \end{pmatrix} , \tag{4.3}$$

where ϵ denotes a two-component infinitesimal quantity and $\bar{\epsilon}$ its hermitian adjoint. The components of the column $(L_K)^3_\alpha$ are transformed according to

$$\delta \left(\frac{2\lambda K}{1 + \lambda^2 \bar{K}K}\right) = i\epsilon \, \frac{1 - \lambda^2 \bar{K}K}{1 + \lambda^2 \bar{K}K} + i\delta\varphi \, \frac{2\lambda K}{1 + \lambda^2 \bar{K}K}$$

$$\delta \left(\frac{1 - \lambda^2 \bar{K}K}{1 + \lambda^2 \bar{K}K}\right) = i \, \frac{2\lambda \, \bar{\epsilon} \, K}{1 + \lambda^2 \bar{K}K} + i\delta\varphi \, \frac{1 - \lambda^2 \bar{K}K}{1 + \lambda^2 \bar{K}K}$$

where $\delta\varphi = \delta\varphi(K, \epsilon)$ is a non-linear effect coming from h^{-1}. It can be eliminated from these formulae which then yield the transformation law

$$\delta \left(\frac{2\lambda K}{1 - \lambda^2 \bar{K}K}\right) = i\epsilon - \frac{2\lambda K}{1 - \lambda^2 \bar{K}K} \, \frac{2\lambda (i\,\bar{\epsilon} \cdot K)}{1 - \lambda^2 \bar{K}K} \, . \tag{4.4}$$

From this formula and its hermitian adjoint it is a simple matter to extract the result

$$\frac{1}{i} \, \delta K = \left[\frac{1}{2\lambda} \, (1 - \lambda^2 \bar{K} \cdot K) \, \epsilon - \lambda \, \bar{\epsilon} \cdot KK \right] - \frac{\lambda}{2} \, \frac{1 + \lambda^2 \bar{K}K}{1 - \lambda^2 \bar{K}K} \, (\bar{\epsilon} \cdot K + \bar{K} \cdot \epsilon) K \tag{4.5}$$

which is, therefore, the transformation law implied by (4.1) in the parametrization (4.2).

The infinitesimal form of the transformation h(K, g) is determined by the method of Sec. 2,

$$1 + \delta h = (L_K^{-1} + \delta L_K^{-1})(1 + \delta g)\, L_K \qquad (4.6)$$

where δL_K is obtained by using (4.5) in conjunction with the form (4.2). The result is

$$\frac{1}{i}\,\delta h = \begin{bmatrix} -\lambda(K\bar{\epsilon}+\epsilon\bar{K}) - \lambda^3 \dfrac{K(\bar{K}\cdot\epsilon+\bar{\epsilon}\cdot K)\bar{K}}{1-\lambda^2\bar{K}K} & 0 \\ 0 & \lambda\,\dfrac{\bar{K}\cdot\epsilon+\bar{\epsilon}\cdot K}{1-\lambda^2\bar{K}K} \end{bmatrix} \qquad (4.7)$$

which clearly belongs to SU(2) x U(1). This matrix controls all of the non-linear realizations with the exception of the preferred one (4.5). Suppose ψ transforms under SU(2) x U(1) according to a linear irreducible representation which is generated by the isospin and hypercharge matrices $\underline{I}$ and Y. These are defined, in the context of SU(3), by

$$\delta\psi = \delta h_\alpha^\beta F_\beta^\alpha \psi = (\delta h_a^b (I_b^a + \tfrac{1}{2}\delta_b^a Y) - \delta h_3^3 Y)\psi$$

$$= (\delta h_a^b I_b^a + \tfrac{1}{2}(\delta h_1^1 + \delta h_2^2 - 2\delta h_3^3)Y)\psi$$

$$= (\mathrm{tr}(\delta h \underline{\tau})\cdot\underline{I} - \tfrac{3}{2}\delta h_3^3 Y)\psi$$

where the F_α^β are defined on the SU(3) quark by $F_\alpha^\beta \psi_\gamma = \delta_\gamma^\beta \psi_\alpha - \frac{1}{3}\delta_\alpha^\beta \psi_\gamma$. Then, corresponding to the hypercharge changing transformation (4.3), one finds, using (4.7) in (2.6),

$$\frac{1}{i}\,\delta\psi = -\lambda\left[(\bar{\epsilon}\underline{\tau}K + \bar{K}\underline{\tau}\epsilon)\cdot\underline{I} + \frac{\bar{\epsilon}\cdot K + \bar{K}\cdot\epsilon}{1-\lambda^2\bar{K}K}(\lambda^2\bar{K}\underline{\tau}K\cdot\underline{I} + \tfrac{3}{2}Y)\right]\psi \qquad (4.8)$$

where $\underline{\tau}$ denotes the Pauli matrices.

The covariant derivatives are determined in the parametrization (4.2) by the matrix

$$L_K^{-1}\partial L_K = \left(\begin{array}{c|c} \dfrac{2\lambda^2(K\bar{K}_\mu - K_\mu\bar{K})}{1+\lambda^2\bar{K}K} - \dfrac{2\lambda^4 K(\bar{K}_\mu\cdot K - \bar{K}\cdot K_\mu)\bar{K}}{(1+\lambda^2\bar{K}K)^2} & \dfrac{2\lambda K_\mu}{1+\lambda^2\bar{K}K} + 2\lambda^3 K\dfrac{\bar{K}_\mu\cdot K - \bar{K}\cdot K_\mu}{(1+\lambda^2\bar{K}K)^2} \\ & \\ -\dfrac{2\lambda\bar{K}_\mu}{1+\lambda^2\bar{K}K} + 2\lambda^3\dfrac{\bar{K}_\mu\cdot K - \bar{K}\cdot K_\mu}{(1+\lambda^2\bar{K}K)^2}\bar{K} & 2\lambda^2\dfrac{\bar{K}\cdot K_\mu - \bar{K}_\mu\cdot K}{(1+\lambda^2\bar{K}K)^2} \end{array}\right)$$

(4. 9)

where $K_\mu = \partial_\mu K$ and $\bar{K}_\mu = \partial_\mu \bar{K}$. This matrix, being antihermitian and traceless, belongs to the algebra of SU(3). The part which belongs to the algebra of SU(2) x U(1) can be separated out and used in the construction of the covariant derivative of ψ ,

$$D_\mu\psi = \partial_\mu\psi + 2\lambda^2\left\{\frac{\bar{K}_\mu\underline{\tau}K - \bar{K}\underline{\tau}K_\mu}{1+\lambda^2\bar{K}K}\cdot\underline{I} - \frac{\bar{K}_\mu K - \bar{K}K_\mu}{(1+\lambda^2\bar{K}K)^2}\left(\lambda^2\bar{K}\underline{\tau}K\cdot\underline{I} + \frac{3}{2}Y\right)\right\}\psi \quad ,$$

(4. 10)

and the remainder is used to define the covariant derivatives of K and $\bar{K}$,

$$D_\mu K = \frac{K_\mu}{1+\lambda^2\bar{K}K} + \lambda^2 K\frac{\bar{K}_\mu K - \bar{K}K_\mu}{(1+\lambda^2\bar{K}K)^2}$$

$$D_\mu\bar{K} = \frac{\bar{K}_\mu}{1+\lambda^2\bar{K}K} - \lambda^2\frac{\bar{K}_\mu K - \bar{K}K_\mu}{(1+\lambda^2\bar{K}K)^2}\bar{K} \quad . \qquad (4.11)$$

Any Lagrangian made out of ψ and $\bar{\psi}$ and their covariant derivatives together with $D_\mu K$ and $D_\mu\bar{K}$ will be SU(3) invariant if it conserves isospin and hypercharge. The vacuum state corresponding to such a Lagrangian will not be invariant under the hypercharge changing transformations and the resulting Goldstone bosons are characterized by the fields K and $\bar{K}$.

The interaction of, for example, nucleons N and hyperons Λ with the Goldstone fields K and $\bar{K}$ could be described by the SU(3)-invariant Lagrangian

$$L = \bar{N}(i\,\gamma_\mu\, D_\mu - m_N)N + \bar{\Lambda}(i\,\gamma_\mu\, D_\mu - m_\Lambda)\Lambda + D_\mu\, \bar{K}\, D_\mu\, K +$$

$$+ g_{\Lambda KN}\, \bar{N}\, \gamma_\mu\, \Lambda\, D_\mu\, K + \ldots \tag{4.12}$$

where $D_\mu \Lambda = \partial_\mu \Lambda$ and $D_\mu N$ is given by (4.10) with $\underline{I} = \underline{\tau}/2$ and $Y = 1$. To the invariant part (4.12) one might add an explicit symmetry breaking term in order to give mass to K and $\bar{K}$. This term could take the form of a matrix element between SU(2) x U(1) singlets of the transformation $D(L_K)$ in the manner outlined in Sec. 3. The simplest representation, D, which can serve in this role is of course the octet. The octet symmetry breaker is given by

$$\frac{M^2}{12\lambda^2}\ \mathrm{tr}\,(L_K\, \lambda_8\, L_K^{-1}\, \lambda_8) = \frac{M^2}{6\lambda^2}\ \frac{1 - 4\lambda^2 \bar{K}K + \lambda^4 (\bar{K}K)^2}{1 + 2\lambda^2 \bar{K}K + \lambda^4 (\bar{K}K)^2}$$

$$= \frac{M^2}{6\lambda^2} - M^2\, \bar{K}K + 2\, M^2\, \lambda^2\, (\bar{K}K)^2 + \ldots \tag{4.13}$$

where the normalization has been adapted to give the kaons mass M . There is no necessity to stop at octet breaking. It would be a fairly straightforward calculation to derive the explicit form of a symmetry breaker belonging to any higher representation. In such calculations it is useful to define the field

$$\Phi = L_K\, \lambda_8\, L_K^{-1} \tag{4.14}$$

which transforms like an ordinary (linear) octet,

$$\Phi \rightarrow g\, \Phi\, g^{-1}\ , \quad g \in SU(3)\ .$$

but satisfies the algebraic constraints,

$$\Phi(\Phi + 1) = 2\ . \tag{4.15}$$

The $I = Y = 0$ component of Φ is proportional to the octet symmetry breaker (4.13). The generalized symmetry breakers, belonging to higher representations of SU(3), could be expressed in terms of the $I = Y = 0$ components of the appropriate irreducible tensor polynomials in Φ .

REFERENCES AND FOOTNOTES

1) For a recent compilation of the very numerous references to non-linear realizations see the review by S. Weinberg, 14th International Conference on High-Energy Physics, Vienna (1968).

2) J. Goldstone, Nuovo Cimento 19, 154 (1961);
J. Goldstone, Abdus Salam and S. Weinberg, Phys. Rev. 127, 965 (1962);
T.W.B. Kibble, Proceedings of the 1967 Conference on Particles and Fields, Rochester (Interscience).

3) We wish to emphasise that this paper is NOT about chiral symmetries; these are discussed in this section as an illustration only. Our interest is with the general method of non-linear realizations of which the chiral realizations are one particular example.

4) J. Schwinger, Phys. Letters 24B, 473 (1967).

5) S. Gasiorowicz and D. Geffen in their excellent review of chiral symmetries (Effective Lagrangians", Argonne Report ANL/HEP 6809) make this point though they do not stress enough how vital it is for the dynamical relevance of the whole non-linear method.

6) T.W.B. Kibble, Phys. Rev. 155, 1554 (1967).

7) In a given model it might be possible to compute this quantity by a self-consistent method. That is, substitute $\chi = \langle\chi\rangle + \chi'$ in the Lagrangian and set equal to zero the perturbation corrections to $\langle\chi'\rangle$.

8) P.W. Anderson, Phys. Rev. 130, 439 (1963);
P.W. Higgs, Phys. Letters 12, 132 (1964) and Phys. Rev. 145, 1156 (1966);
G.S. Guralnik, C.R. Hagen and T.W.B. Kibble, Phys. Rev. Letters 13, 585 (1964);
T.W.B. Kibble, Ref. 6.

9) E.P. Wigner, Ann. of Math. 40, 149 (1939).

10) S. Coleman, J. Wess and B. Zumino, Harvard preprint, 1968; C.G. Callan, S. Coleman, J. Wess and B. Zumino, Harvard preprint 1968.

11) Part II of this paper (ICTP, Trieste, preprint, IC/68/107).

12) A summation over the repeated subscript i is implied. On occasion it will be useful to divide the sum into two parts, one over the subalgebra H indicated by repeated greek subscripts $\alpha, \beta, \ldots$ and the remainder indicated by repeated latin subscripts from the beginning of the alphabet, a, b, ...

13) We do not have a general method which defines the transformation L_ϕ in cases where the Lagrangian contains explicitly no Goldstone boson fields with the requisite quantum numbers. For the above method to work, it is necessary to introduce extra fields with the requisite quantum numbers perhaps by some self-consistent method which we have not investigated.

14) To illustrate, take the case of $M_a = \pi_a$, a = 1, 2, 3, and $M_A = \sigma$. One can introduce (following Higgs and Kibble) a field $\chi(x) = \sqrt{\underline{\pi}^2 + \sigma^2}$. For the non-linear method to apply it is essential that $\chi_0 = \langle\chi(x)\rangle$ should not vanish. Writing $\chi = \chi_0 + \chi'$ where χ' is the quantized field, what we are saying in the text is that χ' can be set equal to zero. This is the facility which the non-linear method gives us over the considerations of Higgs and Kibble.

15) We have of course set the components $m_a = 0$ by way of defining the preferred set ϕ_a. This does not reduce the number of independent variables nor does it imply any constraints upon the M. When constraints are introduced, then m_A can also be set equal to zero so that all m's are out of (3. 7). The Goldstone bosons, however, remain, being described by the ϕ_a's which appear in the definitions of Δ_μ. The formula (3. 7) is analogous to Kibble's formula (10). Our formulation, being

completely general, is valid for any symmetry group, any representation.

16) Weinberg actually writes out the expression $\mathbb{D}_{11}(L_{\phi})$ as a power series and it is hard to recognize in his way of setting it out that it can be so compactly expressed.