SIMPLY TRANSITIVE SUBGROUPS $G_4$

OF THE POINCARE GROUP:

A NEW FINDING

IN THE SPECIAL THEORY OF RELATIVITY

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1968

MIRAMARE - TRIESTE
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September 1968

* The subject of a seminar given at the ICTP on 20 August, 1968.
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1. Introduction

It is a generally acknowledged fact that the main results of the special theory of relativity, together with those of quantum theory, provide the underlying physical concepts and the basic mathematical terminology of the theory of elementary particles. Therefore, whenever it might happen that a new and significant finding is made in either of these two basic disciplines, it obviously would become very desirable to try to find out in what possible ways that new finding might lead to a deepening of the foundations, and to an extension of the predictive powers, of elementary particle theory. The present article puts forward a new finding of this kind in the special theory of relativity—a new finding which seems to be significant, partly because it firmly re-establishes the pre-relativity concept of absolute simultaneity on a new, Lorentz invariant basis, and partly because it appears to provide a possible new pathway to a deeper conceptual framework of description for elementary particle theory, in terms of new geometrical concepts of internal structure relating to space-time. In formal terms, our finding may be summarized in the following

In formal terms, our finding may be summarized in the following
Theorem I: Corresponding to every proper orthochronous Lorentz matrix $L$, there exists a distinct 1-parameter family of simply transitive non-Abelian 4-parameter subgroups $G$ of the Poincaré group $G_\text{lo}$, each $G$ being specifiable in terms of the characteristic values and characteristic vectors of $L$.

In the sequel, we shall prove this theorem, and, in the course of doing so, we shall examine some of its more immediate implications.
The proof of the foregoing theorem is greatly facilitated by making use of a certain parametrization\(^1\) of the set of all proper orthochronous Lorentz matrices \(L\), according to which any such Lorentz matrix \(L\) is expressible\(^2\) as the commuting product,

\[
L = L(\theta_r, \theta_i) = L_r(\theta_r) L_i(\theta_i) = L_i(\theta_i) L_r(\theta_r),
\]

(1)

of a certain pair of simpler Lorentz matrices \(L_r(\theta_r)\) and \(L_i(\theta_i)\), which are functions, respectively, of the real and imaginary parts of a certain complex parameter

\[
\theta = \theta_r + i\theta_i.
\]

(2)

In establishing this parametrization, however, one has to distinguish between two different kinds of Lorentz matrix, the first of which has been called by Synge\(^3\) nonsingular and by us\(^4\) nonminimal, and the second of which has been called by Synge\(^3\) singular and by us\(^4\) minimal. The parametrization then turns out to be such that \(L_r(\theta_r)\) rotates a certain invariant 2-flat\(^5\) \(\Pi_r\) of \(L\) into itself through an "angle" \(\theta_r\), and such that \(L_i(\theta_i)\) rotates a second invariant 2-flat \(\Pi_i\) of \(L\) into itself through an "angle" \(\theta_i\), the 2-flats \(\Pi_r\) and \(\Pi_i\) being always mutually orthogonal.\(^5\) In the nonminimal case, moreover, it always turns out that \(\Pi_r\) is spacelike\(^5\) and \(\Pi_i\) timelike;\(^5\) whereas, in the minimal case, both \(\Pi_r\) and \(\Pi_i\) are always null.\(^5\)
Now this parametrization of all proper orthochronous Lorentz matrices $L$ in terms of the complex parameter $\theta$ is essentially equivalent, in the nonminimal case, to the notion of a Lorentz $l$-screw as expounded by Synge. But even in the minimal case, as we have previously emphasized, one may speak of the "Lorentz $l$-screw $L(\theta_0, \theta_1)$" in essentially the same sense as in the nonminimal case. Accordingly, we shall henceforth attach to the term "$l$-screw" this broader meaning; and, as a result, we may now interpret Eqs. (1) to mean that every proper orthochronous Lorentz matrix $L$ (minimal as well as nonminimal) may be regarded as a Lorentz $l$-screw $L(\theta_0, \theta_1)$.

Consider now an arbitrary proper orthochronous homogeneous Lorentz transformation, operating actively upon the events of space-time, and leaving, in general, one and only one event fixed. If to this fixed event, which we shall call the center $C$ of the homogeneous Lorentz transformation, we assign space-time coordinates $x^k_0$ ($k = 0, 1, 2, \beta$), then the transformation under consideration here can be regarded, with the aid of Eqs. (1), as a mapping

$$x^k(0, 0) \rightarrow x^k(\theta_0, \theta_1)$$

(3)

of space-time onto itself, given by the equations

$$[x^k(\theta_0, \theta_1) - x^k_0] = L^k_j(\theta_0, \theta_1)[x^j(0, 0) - x^j_0],$$

(4)
where \( L^k_j(\theta, \phi) \) are the elements of the proper orthochronous Lorentz matrix

\[
L(\theta, \phi) = \begin{pmatrix}
\cosh \theta & 0 & 0 & \sinh \theta \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
\sinh \theta & 0 & 0 & \cosh \theta
\end{pmatrix}
\]  

which appears in Eqs. (1), the rows of \( L(\theta, \phi) \) corresponding to the values \( k = 0, 1, 2, 3 \), and the columns to the values \( j = 0, 1, 2, 3 \). We shall henceforth refer to this general type of mapping of space-time onto itself, given by Eqs. (4), as a Lorentz \( \theta \)-screw with center \( C \).

One of the simplest possible examples of a (nonminimal) Lorentz \( \theta \)-screw with center \( C \) is afforded if we assign to the Lorentz matrix \( L(\theta, \phi) \), whose elements will enter into Eqs. (4), the specialized form:

\[
L = L(\theta, \phi) = \begin{pmatrix}
\cosh \theta & 0 & 0 & \sinh \theta \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
\sinh \theta & 0 & 0 & \cosh \theta
\end{pmatrix}
\]  

In this simple case, we than have
\[ L_{o}(\theta_{r}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{r} & \sin \theta_{r} & 0 \\ 0 & -\sin \theta_{r} & \cos \theta_{r} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \] (7)

and

\[ L_{i}(\theta_{i}) = \begin{bmatrix} \cosh \theta_{i} & 0 & 0 & \sinh \theta_{i} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \theta_{i} & 0 & 0 & \cosh \theta_{i} \end{bmatrix} \] (8)

and these two matrices obviously commute, as required by Eqs. (1).

Moreover, in this particular example, the general mapping of space-time onto itself given by Eqs. (2) is readily seen to decompose into the commuting product of two independent mappings, one of which is of the form

\[
\begin{align*}
\begin{bmatrix} x^1(\theta_{r}, \theta_{i}) - x^1_o \\ x^2(\theta_{r}, \theta_{i}) - x^2_o \\ 0 \\ 0 
\end{bmatrix} &= \begin{bmatrix} \cos \theta_{r} \sin \theta_{r} \\ -\sin \theta_{r} \cos \theta_{r} \end{bmatrix} \begin{bmatrix} x^1(0,0) - x^1_o \\ x^2(0,0) - x^2_o \\ 0 \\ 0 
\end{bmatrix},
\end{align*}
\] (9)

and the other of the form

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\[
\begin{bmatrix}
[x^0(\theta, \theta_1) - x^0_0] \\
[x^3(\theta, \theta_1) - x^3_0]
\end{bmatrix} = \begin{bmatrix}
\cosh \theta_1 \sinh \theta_1 \\
\sinh \theta_1 \cosh \theta_1
\end{bmatrix}
\begin{bmatrix}
[x^0(0,0) - x^0_0] \\
[x^3(0,0) - x^3_0]
\end{bmatrix}. \quad (10)
\]

Now the first of these two mappings, given by Eqs. (9), evidently operates in an identical manner upon each one of a double infinity of mutually parallel \textit{spacelike 2-flats} $\Pi_2(x^0, x^3)$, as shown schematically in Fig. 1. In each distinct 2-flat $\Pi_2(x^0, x^3)$ of this mutually parallel set, specified by any arbitrary pair of values of $x^0, x^3$, there is a pseudo-origin $0_2$ with coordinates $(x^0, 0, 0, x^3)$ and a pseudo-center $C_2$ with coordinates $(x^0, x^1, x^2, x^3)$; and, under the mapping given by Eqs. (9), every such 2-flat $\Pi_2(x^0, x^3)$ is rotated into itself through the same ordinary angle $\theta_2$ about its own pseudo-center $C_2$, the pseudo-origin $0_2$ being mapped into the event $P_2$ as shown in Fig. 1.

In a similar way, the second of the above two mappings, given by Eqs. (10), operates in an identical manner upon each one of a double infinity of mutually parallel \textit{timelike 2-flats} $\Pi_1(x^1, x^2)$, as shown schematically in Fig. 2. Again, in each distinct 2-flat $\Pi_1(x^1, x^2)$ of this mutually parallel set, specified by any arbitrary pair of values of $x^1, x^2$, there is a pseudo-origin $0_3$ with coordinates $(0, x^1, x^2, 0)$ and a pseudo-center $C_3$ with coordinates $(x^0, x^1, x^2, x^3_0)$; and, under the mapping given by Eqs. (10), every such 2-flat $\Pi_1(x^1, x^2)$ is rotated into itself through the same hyperbolic angle $\theta_1$ about its own pseudo-center $C_3$, the pseudo-origin $0_3$ being
Fig. 1
mapped into the event $P$ as shown in Fig. 2. But in contrast to the ordinary rotation of the spacelike 2-flats $\Pi_0(x^0, x^3)$, each of which is seen from Fig. 1 to rotate in its entirety as a single unit, the hyperbolic rotation of the timelike 2-flats $\Pi_1(x^1, x^2)$, on the other hand, is seen from Fig. 2 to be such that each of the latter has to be regarded as composed of four disconnected parts or quadrants, which separately "rotate" entirely into themselves as distinct sub-units. Indeed, as is evident from Fig. 2, in each of these four disconnected parts or quadrants, there exists a "reflected image" of the pseudo-origin $0_{03}$ (namely, the events $0'_0, 0''_0, 0'''_0$), and a corresponding "reflected image" of its mapping into the event $P_{03}$ (namely, the mappings $0'_0 \rightarrow P'_0, 0''_0 \rightarrow P''_0, 0'''_0 \rightarrow P'''_0$).

From Fig. 2 it will be noticed, furthermore, that these four disconnected parts or quadrants are separated from one another by two null lines, which intersect one another in the pseudo-center $C_{03}$. Now, in the particular case of the invariant timelike 2-flat $\Pi_1(x^1, x^2)$ which also contains the pseudo-center $C_{03}$, and only in this case, these two null lines remain invariant (i.e., are mapped into themselves) under the overall mapping given by Eqs. (4) and (6). These two invariant null lines are the so-called axial null rays of $L(\theta_r, \theta_1)$.

Now the foregoing simple example of a (nonminimal) Lorentz 4-screw with center $C$ has the property that, for it, the "angle of opening" between its two axial null rays, as it appears in Fig. 2, is the maximum "angle of opening" that is possible for any Lorentz...
h-screw. In fact, as we have previously explained in detail,\textsuperscript{10} one may, by reducing this "angle of opening," go continuously, through all intermediate stages, to the limiting minimal case, in which these two axial null rays of $L(\theta_1, \theta_2)$ coalesce into one.\textsuperscript{10,11} In this limiting case, the invariant timelike 2-flat $\Pi_1$ and the invariant spacelike 2-flat $\Pi_2$ simultaneously become null 2-flats,\textsuperscript{11,12} both becoming tangent to the null cone with vertex at $C$, along the above mentioned single limiting axial null ray, which they then both contain in common. In this limiting minimal case, moreover, $L(\theta_1)$ and $L_1(\theta_2)$ still represent "rotations" into themselves of the limiting invariant 2-flats $\Pi_r$ and $\Pi_1$, respectively; but now these "rotations" are no longer of the usual kind,\textsuperscript{13} in either the ordinary or the hyperbolic sense.

It is thus seen, in short, that a Lorentz h-screw with center $C$ is simply an arbitrary proper orthochronous homogeneous Lorentz transformation, operating actively upon the events of space-time, and leaving, in general, one and only one event fixed—namely, its center $C$ with coordinates $x^k$. But it is also seen that a Lorentz h-screw is further characterized by the distinctive property that it is always expressible as a commuting product of two independent proper orthochronous homogeneous Lorentz transformations, which involve the respective parameters $\theta_1$ and $\theta_2$ in the manner described above, and which separately leave fixed the same center $C$.  

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We shall find it advantageous, next, to direct our attention upon the totality of 1-parameter subgroups \( G \) of the Poincaré group \( G \). There are \( 10 \) of these \( G \)'s; and they are all, by definition, motions of Minkowski space-time into itself, determined by the solutions of the ordinary differential equations

\[
\frac{dx^k}{dt} = \xi^k \quad (k = 0, 1, 2, 3),
\]

where \( t \) is a real parameter (the parameter of each \( G \)), and where the functions \( \xi^k = \xi^k(x) \) are solutions of Killing's equations, which take the simplified form

\[
\frac{\partial \xi_k}{\partial x^j} + \frac{\partial \xi_j}{\partial x^k} = 0
\]

in the present case when the metric is that of Minkowski space-time.

It is easy to verify that the solutions of Eqs. (12) are of the general form

\[
\xi^k = v^k + F^k_j x^j,
\]

where \( v^k \) and \( F^k_j \) are constants, subject only to the condition that

\[
F_{kj} + F_{jk} = 0.
\]
Accordingly, the $\xi^k$, and therefore also the corresponding $G_1^s$, depend upon ten independent constants (the four $v^k$ and the six independent constants $F_{kj}$ for $k > j$). The fact that the maximum number of independent constants available in Eqs. (13) is ten is, of course, equivalent to our previous statement that there are $\omega^{10}$ different $G_1^s$ of $G_{10}$.

Now, in addition to being motions of Minkowski space-time into itself, all of these $G_1^s$ are Born-type rigid motions; that is to say, between any two infinitely close trajectories of any $G_1$ of $G_{10}$, the normal distance (with respect to the Minkowski metric) remains unchanged during the corresponding motion. This latter property, of course, was the decisive one which Born originally chose as the basis for his relativistic generalization of the classical concept of a rigid motion. One peculiar characteristic of Born's generalization of this classical concept, however, is that it makes no use of the corresponding classical concept of absolute simultaneity of all the points of a rigid body. And indeed, as Born actually recognized from the outset, the theory of special relativity has always appeared to rule out the possibility of attaching a Lorentz invariant meaning to the very idea of absolute simultaneity of different space-time events.

In addition to being Born-type rigid motions, however, all of the $G_1^s$ of $G_{10}$ may be regarded, in a strictly formal sense, as relativistic generalizations of the 1-parameter subgroups $\Gamma_1$ of the Galilei group $\Gamma_6$ in ordinary 3-space. Since these $\Gamma_1^s$ of $\Gamma_6$...
comprehend all possible motions of free rigid bodies in the classical nonrelativistic sense, one might well be tempted to conjecture that their formal relativistic generalizations, the $G_i$'s of $G_1$, should similarly comprehend all possible motions of free rigid bodies in some corresponding relativistic sense. One indispensable feature of the $\Gamma_i$'s of $\Gamma_1$, however, is that, for each of them, the group parameter $t$ is, in effect, an absolute time. Accordingly, if the $G_i$'s of $G_1$ are to represent successfully the relativistic generalization of the motions of free rigid bodies, in the present strictly formal sense, it must then turn out to be true that, for each of these $G_i$'s, the group parameter $t$ is once again, in effect, an absolute time.

Now, contrary to the previously mentioned universally held view that the theory of special relativity rules out the possibility of attaching a Lorentz invariant meaning to the idea of an absolute time, we shall presently find that, as a consequence of Theorem I stated above, the group parameter $t$ of any $G_i$ of $G_1$ can indeed be identified, in a Lorentz invariant sense, as an absolute time. Accordingly, we may conclude that the $G_i$'s of $G_1$ can represent successfully the relativistic generalization of the motions of free rigid bodies in the strictly formal sense just mentioned. And, furthermore, we may therefore anticipate that, as a potential quantum mechanical corollary to this result, it may now be possible to arrive at a satisfactory formulation, in terms of the $G_i$'s of $G_1$, of the notion of a relativistic free elementary particle state, or resonance state, with spatial extension.
4. "Time-Varying" Lorentz \( l \)-Screws and the \( G \)'s of \( G \)

If, in Eqs. (4), we set

\[
\begin{align*}
\theta_r &= 2Kt, \\
\theta_i &= 2\lambda t,
\end{align*}
\]

(15a) \hspace{1cm} (15b)

where \( t \) is a real parameter and \( K, \lambda \) are two arbitrary real constants, and if we let \( t \) play the role of a "time variable" (in the same sense that the corresponding parameter \( t \) of any \( G_1 \) of \( G \) might be regarded as playing such a role), then we may interpret Eqs. (4), in combination with Eqs. (15), as defining what we may call a "time-varying" Lorentz \( l \)-screw about the center \( C \). Correspondingly, we may now rewrite Eqs. (4) in the form

\[
[x^k(t) - x^k_0] = L^k_j(t)[x^j(0) - x^j_0],
\]

(16)

a form which emphasizes the fact that there is only one independent parameter for a "time-varying" Lorentz \( l \)-screw: namely, the "time variable" \( t \).

On differentiation with respect to \( t \), Eqs. (16) yield the differential equations

\[
(d/dt)[x^k(t) - x^k_0] = G^k_j[t^j(t) - x^j_0],
\]

(17)

where the quantities \( G^k_j \) are constants defined by the equations\(^{15}\).
\[
\frac{dL_j^k(t)}{dt} = G_j^k L_j^l(t) = L_j^k(t)G_j^l.
\] (18)

On comparing Eqs. (17) with Eqs. (11) and (13), we now find that these two sets of equations can be identified as the same, provided that we set

\[
F_j^k = G_j^k,
\] (19)

and also provided that we establish the connection

\[
\nu^k = -F_j^k x_j^0.
\] (20)

Thus, every "time-varying" Lorentz 4-screw, about any center \( C \), is in actuality a \( G_0 \) of \( G_1 \), and its "time variable" \( t \) is the group parameter of the corresponding \( G_1 \). Indeed, there are \( \omega^{10} \) different "time-varying" Lorentz 4-screws, just as there are \( \omega^{10} \) different \( G \)'s of \( G_1 \); for, as may be seen from Eqs. (18), there are \( \omega^6 \) different Lorentz matrix functions \( L(t) = \|L_j^k(t)\| \) available for use in Eqs. (16), while there are evidently \( \omega^4 \) different centers \( C \) which may be used in these equations.

The only \( G \)'s of \( G_1 \) which apparently are not "time-varying" Lorentz 4-screws are the translations, for which

\[
F_j^k = 0.
\] (21)
But even the translations may be regarded as limits of sequences of "time-varying" Lorentz 4-screws, in the following sense: If, initially, we assume certain finite values for both \( F_j^k \) and \( x_o^j \), then we have to do with a \( G_{10} \) of \( G \) which is a "time-varying" Lorentz 4-screw, but which is not a translation. Consider now the sequence of "time-varying" Lorentz 4-screws which corresponds to the sequence of values \( F'_j^k = \rho F_j^k \) and \( x'_o^j = \rho^{-1} x_o^j \), where \( \rho \) takes on the continuous sequence of real values between one and zero.

In the limit as \( \rho \to 0 \), we will have \( F'_j^k \to 0 \) and (for at least one \( j \) value) \( x'_o^j \to \infty \), while, as may be seen from Eqs. (20), the finite constants \( v_k^j \) will remain fixed. Thus, in this limiting case, the corresponding limiting \( G_{10} \) is a translation, while the corresponding limiting form of Eqs. (16) is also that of a translation. Accordingly, we may summarize the present results in the following

**Theorem II:** Every subgroup \( G_{10} \) of \( G \) is a "time-varying" Lorentz 4-screw or a limit of a sequence of them.
5. The Invariant Null Hyperplanes of a Subgroup $G$ of $G$

From our general discussion in section 2, it is readily seen that every nonminimal "time-varying" Lorentz $l$-screw, and therefore every nonminimal subgroup $G$ of $G$, possesses two distinct axial null rays, which always lie in the invariant timelike 2-flat $\Pi_1$ that contains the center $C$ of $G$, and which, in addition, always intersect one another in $C$. Similarly, it is seen from that discussion that every minimal "time-varying" Lorentz $l$-screw, and therefore every minimal subgroup $G$ of $G$, possesses only one axial null ray, which, again, always lies in the invariant null 2-flat $\Pi_1$ that contains the center $C$ of $G$, and which, in addition, always traverses $C$.

We may therefore assert that, for any given nonminimal $G$, there always exist two distinct 1-parameter families of null vectors, $n_\pm(\varphi)$, which remain invariant in direction under that $G$, each family of null vectors being constrained to lie along one of the two axial null rays of $G$. In fact, these two families of null vectors, $n_\pm(\varphi)$, are just the characteristic null vectors of the corresponding nonminimal Lorentz matrix $L(t)$, with components

$$n_\pm^k(\varphi) = x_\pm^k(\varphi) - x_0^k,$$  \hspace{1cm} (22)

where $\varphi$ is a continuously variable real parameter with range $-\infty \leq \varphi \leq +\infty$; for they satisfy the equations
\[ L(t)n_\pm(\varphi) = e^{\pm \lambda t}n_\pm(\varphi), \quad (23a) \]

\[ = n_\pm(\varphi + 2\lambda t). \quad (23b) \]

Thus, for the simple example given by Eqs. (6) and (15), we may take \( n_\pm(\varphi) \) to be, for each value of \( \varphi \), the two null vectors with components \( (e^{\pm \varphi}, 0, 0, \pm e^{\pm \varphi}) \) and the two null vectors with components \( (-e^{\pm \varphi}, 0, 0, \mp e^{\pm \varphi}) \), the upper sign yielding a pair of equal and opposite null vectors \( n_+(\varphi) \), and the lower sign yielding another pair of equal and opposite null vectors \( n_-(\varphi) \).

Now, in general, each of the two 1-parameter families of characteristic null vectors of \( L(t) \), with components \( n_\pm^k(\varphi) \), determines a single null hyperplane \( \Sigma_\pm \), the equation of which is intrinsically the same for all values of \( \varphi \), and is of the form

\[ n_{\pm k}(\varphi)(x^k - x_0^k) = 0. \quad (24) \]

Moreover, each of these two null hyperplanes \( \Sigma_\pm \) has the distinctive property that it not only contains, but it is also orthogonal to, all of the corresponding characteristic null vectors \( n_\pm(\varphi) \) which determine it. Furthermore, both of these two null hyperplanes \( \Sigma_\pm \) have the additional property that they contain the invariant spacelike 2-flat \( \Pi_r \). Accordingly, each of the two null hyperplanes \( \Sigma_\pm \) is a 3-dimensional subspace in 4-dimensional space-time which remains invariant under \( G_1 \). When \( G_1 \) is nonminimal, as we have just assumed, the two invariant null hyperplanes \( \Sigma_\pm \) are
distinct; but in the limiting minimal case they coalesce into one limiting null hyperplane $\Sigma$.

In the case of the simple example given by Eqs. (6) and (15), the defining Eqs. (24) for $\Sigma_\pm$ are

$$e^{\pm \psi}(x^0 - x_0^0) \pm (x^3 - x_0^3) = 0.$$  \hspace{1cm} (25)

Moreover, in this case, any event contained in $\Sigma_\pm$ must have coordinates $x^k (k = 0, 1, 2, 3)$ which satisfy the equations

\begin{align*}
x^0 - x_0^0 &= n_0^0(\psi) = \mp e^{\pm \psi}, \hspace{1cm} (26a) \\
x^1 - x_0^1 &= a^1, \hspace{1cm} (26b) \\
x^2 - x_0^2 &= a^2, \hspace{1cm} (26c) \\
x^3 - x_0^3 &= n_3^0(\psi) = \pm e^{\pm \psi} \text{ or } \mp e^{\pm \psi}, \hspace{1cm} (26d)
\end{align*}

where $\psi, a^1, a^2$ are three independent parameters, corresponding to the three dimensions of $\Sigma_\pm$.

Thus, we may summarize these results in the following

**Theorem III**: Every nonminimal (minimal) subgroup $G$ of $G$ leaves invariant two (one) null hyperplanes determined by the two (one) 1-parameter families of characteristic null vectors of the corresponding Lorentz matrix $L(t)$. 

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6. Subgroups $G_4$ of $G_{10}$

According to Theorem II, the finite equations of every subgroup $G$ of $G$ are expressible in the form of Eqs. (16), or as the limit of a sequence of such equations. If, then, using Eqs. (16), we fix the Lorentz matrix function $L(t) = \|l_j^k(t)\|$ and the coordinates $x^k_0$ of the center $C$, we will have picked out a definite subgroup $G_1$ of $G_{10}$. If this $G_1$ is nonminimal, its center $C$ will be an event which is common not only to both axial null rays of $G_1$, but also to the invariant spacelike 2-flat $\Pi_r$ of $G_{10}$. Hence it follows from our immediately preceding discussion that $G$ will be an event common to both invariant null hyperplanes $\Sigma_\pm$ of $G_{10}$. Let us then see what are the possible motions of space-time into itself which will always leave $C$ either in $\Sigma_+$ or in $\Sigma_-$. First, it is evident that, in either case, any two linearly independent translations of $\Pi_r$ into itself are always admissible. Secondly, according to whether $C$ is to remain in $\Sigma_+$ or in $\Sigma_-$, a third translation is always possible, provided that it carries the appropriate one of the two axial null rays of $G_1$ into itself. Thus we find that, for any given nonminimal subgroup $G_1$ of $G_{10}$, there exist two distinct (Abelian) 3-parameter translation subgroups $G_{3\pm}$ of $G_{10}$, each of which transforms into itself a corresponding one of the two invariant null hyperplanes $\Sigma_\pm$ of $G_{10}$.

When one now considers the product of either translation subgroup $G_{3\pm}$ with the originally given nonminimal subgroup $G_1$ of $G_{10}$,
one finds that the resulting transformations constitute a definite non-Abelian 4-parameter subgroup $G_{4 \pm 10}$ of $G_{10}$. It is evident, moreover, that the corresponding null hyperplane $\Sigma_{4 \pm}$ remains invariant under all transformations of $G_{4 \pm}$. Indeed, each null hyperplane $\Sigma_{4 \pm}$ may be referred to as an isolated invariant variety of the corresponding nonminimal subgroup $G_{4 \pm 10}$ of $G_{10}$; and the originally given nonminimal subgroup $G_{4 \pm 10}$ of $G_{10}$ may be referred to as a subgroup of stability of each of the two corresponding nonminimal subgroups $G_{4 \pm 10}$ of $G_{10}$.

In the limiting case when the originally given subgroup $G_{10}$ becomes minimal, the two distinct translation subgroups $G_{3 \pm}$ coalesce into a single limiting translation subgroup $G_{3}$; and, correspondingly, the two resulting non-Abelian subgroups $G_{4 \pm 10}$ of $G_{10}$ coalesce into a single limiting non-Abelian minimal subgroup $G_{4 \pm 10}$ of $G_{10}$.

In the case of the simple example given by Eqs. (6) and (15), the two translation subgroups $G_{3 \pm}$ are evidently defined by the respective sets of equations

$$
\begin{align*}
x^0 & \rightarrow x^0 + a^0, \\
x^1 & \rightarrow x^1 + a^1, \\
x^2 & \rightarrow x^2 + a^2, \\
x^3 & \rightarrow x^3 \pm a^0,
\end{align*}
$$

(27)

where $a^0$, $a^1$, $a^2$ are three independent parameters.

Thus, we may summarize these results in the following

\textbf{Theorem IV:} For any given nonminimal subgroup $G_{10}$, each of the two associated invariant null hyperplanes $\Sigma_{4 \pm}$ determines a corresponding non-Abelian 4-parameter subgroup $G_{4 \pm 10}$ of $G_{10}$, which
contains $G$ as a subgroup of stability. Each null hyperplane $\Sigma_\pm$ is thus the defining isolated invariant variety of the corresponding nonminimal $G_{\pm}$, and it is also the locus of all centers $C$ for $G_{4\pm}$.

In the limiting case when $G$ is minimal, the two null hyperplanes $\Sigma_\pm$ coalesce into a limiting single null hyperplane $\Sigma$, and the two subgroups $G_{\pm}$ which they define coalesce into a limiting single minimal subgroup $G$ of $G_{10\pm}$.

We may now re-express the essential content of Theorem IV in the following alternative way, which leads us, finally, to the desired proof of the validity of Theorem I. Corresponding to every nonminimal (minimal) Lorentz matrix function $L(t)$, there exist $\omega^4$ distinct nonminimal (minimal) subgroups $G$ of $G_{10}$ with different centers $C$. But $\omega^3$ of these $G$'s have the same invariant null hyperplane $\Sigma_\pm$ (or limiting $\Sigma$), and these $\omega^3$ $G$'s therefore all belong to the same subgroup $G$ (or limiting $G$). Thus, corresponding to every nonminimal (minimal) Lorentz matrix function $L(t)$, there exists a 1-parameter family of different subgroups $G$ (or limiting $G$'s) of $G_{10}$, each of which is specifiable in terms of the corresponding invariant null hyperplane $\Sigma_{\pm}$ (or limiting $\Sigma$). Now this result is essentially equivalent to Theorem I as originally stated, and therefore demonstrates the validity of the latter, except for the fact that we have not yet shown that every such $G$ (or limiting $G$) is simply transitive. This property, however, will be discussed in the next section.
7. Absolute Simultaneity as a Necessary Consequence of the Existence of Simply Transitive Subgroups $G_4$ of $G_{10}$

Knowing now about the existence of the subgroups $G_4$ of $G_{10}$ just discussed, let us next examine some of their properties. In the first place, we know that, for each such $G_4$, there must be four linearly independent symbols, which we may write down in the form

$$A_a = A_a^k \frac{\partial f}{\partial x^k} \quad (a = 0,1,2,3). \quad (28)$$

In Eqs. (28), the quantities $A_a^k$ may be spoken of as the components $(k = 0,1,2,3)$ of the four vectors $(a = 0,1,2,3)$ of $G_4$. Under a suitable choice of coordinates $x^k$ $(k = 0,1,2,3)$, it turns out that the components of these four vectors may be expressed in the form

$$A_a^k = \delta_a^k + F_{ja}^k x^j, \quad (29)$$

where

$$F_{kja} + F_{jka} = 0. \quad (30)$$

In the case of the simple example given by Eqs. (6) and (15), the four matrices $[F^k_{ja}]$ turn out to have the form

$$\begin{bmatrix}
0 & 0 & 0 & 2\lambda_a \\
0 & 0 & -2\lambda_a & 0 \\
0 & 2\lambda_a & 0 & 0 \\
2\lambda_a & 0 & 0 & 0
\end{bmatrix} \quad (a = 0,1,2,3), \quad (31)$$
where

\[ K = K = \lambda = \lambda = 0 \]  

(32)

and

\[(K_0)^2 - (K_3)^2 = (\lambda_0)^2 - (\lambda_3)^2 = K_0 \lambda_0 - K_3 \lambda_3 = 0.\]  

(33)

An important quantity in the theory of these \( G_4 \) s is the determinant, \( \Delta \), of the matrix \( \|A_k^0\| \). In the case of the simple example given by Eqs. (6) and (15), this determinant turns out to have the value

\[ \Delta = 1 + 2\lambda x^0 + 2\lambda x^3, \]  

(34)

as may be verified with the aid of Eqs. (29) and (31). Since, for generic values of the \( x^k \) \((k = 0,1,2,3)\), \( \Delta \) does not vanish, it follows that \( G_4 \) (in the case of the present simple example) is simply transitive. This property, as it turns out, is a general one: every \( G_4 \) of the kind whose existence we have established is simply transitive.

Moreover, for every such \( G_4 \), the equation

\[ \Delta = 0 \]  

(35)

is the equation of its defining isolated invariant variety. Thus, in the case of the simple (nonminimal) example given by Eqs. (6) and (15), Eqs. (34) and (35), taken together, are readily found to be equivalent to Eqs. (25) for either \( \Sigma_+ \) or \( \Sigma_- \), depending upon the choice of sign of the ratio.
\[
\lambda_0 / \lambda_3 = \pm 1. \quad (36)
\]

Finally, it turns out that, for every such \( G \), the group parameter \( t \) for every subgroup of stability \( G \) of \( G \) is necessarily a definite function of position in space-time. In the case of the simple example given by Eqs. (6) and (15), the functional dependence of the group parameter \( t \) upon position in space-time turns out to be given by either one of the two equivalent equations

\[
\theta_t = 2Kt = (K / \lambda_0) \ln |\Delta|, \quad (37a)
\]
\[
\theta_1 = 2\lambda t = (\lambda / \lambda_0) \ln |\Delta|. \quad (37b)
\]

Now it is most important, for our purpose, to note that Eqs. (37) make it possible to regard \( t \) as a Lorentz invariant universal time, common to all subgroups of stability \( G \) of \( G \). It will be noted, moreover, that every pair of null hyperplanes \( \Delta = \pm |\Delta| \) (|\Delta| = const.) constitutes, with respect to this universal time \( t \), a 3-space of absolute simultaneity

\[
t = \text{const.} \quad (38)
\]

A schematic representation of the resulting sequence of these (double-sheeted) 3-spaces of absolute simultaneity is given in Fig. 3, where the intersections of these 3-spaces with the invariant 2-flat \( \Pi_i(x^1, x^2) \) are shown.
Fig. 3
Through this critically important identification of the successive values of the parameter $t$ with a continuous succession of 3-spaces of absolute simultaneity, we are enabled to interpret the foregoing Eqs. (16) for an arbitrary subgroup of stability $G_1$ of $G_4$ as equations of motion which describe the gradual unfolding, in the course of the universal time $t$, of this succession of 3-spaces of absolute simultaneity embedded in 4-dimensional space-time. Thus, for any given 3-space of absolute simultaneity $t = \text{const.}$, acted on by an arbitrary $G_1$, the instantaneous "motion," in the sense in which we are employing this term here, is its actual group motion, under $G_1$, into its successor, the 3-space of absolute simultaneity $t + dt = \text{const.}$ Note, however, that the isolated invariant variety $Δ = 0$ of $G_4$ (for which $|t| = \infty$) differs from all other 3-spaces of absolute simultaneity $t = \text{const.}$ in that it is its own successor under every $G_1$.

While our present conclusions are based upon the simple example of a (nonminimal) $G_4$ given by Eqs. (6) and (15), their general validity can be established by employing considerations of the type with which we shall be concerned in the next section.
8. Change of Coordinates for a \( G \) and Change of Basis for a \( G \)

and the Distinction Between Their Physical Meanings.

In the case of the simple example of a (nonminimal) \( G \) given by Eqs. (6) and (15), we have already chosen coordinates \( x^k (k = 0,1,2,3) \) in such a way that the four vectors of \( G \) with components \( A^k \) are given by Eqs. (29). If, now, we were to adopt new coordinates \( x'{}^k \) for \( G \), related to the original coordinates \( x^k \) through an arbitrary constant proper orthochronous homogeneous Lorentz transformation

\[
x^k - x'{}^k = \xi_j^{k} x^j, \quad (j,k = 0,1,2,3)
\]

we would then say that we were not changing the observed events of space-time, but that, rather, we were changing the coordinate axes of the observer.

On the other hand, we could, in the case of this same simple example, make an analogous change of basis for the group \( G \); that is to say, we could adopt new symbols \( A'_a \) which differ from the original symbols \( A_a \) of \( G \) given by Eqs. (28). In order to make clear the significance of such a change of basis, let us first note that Eqs. (11) and (13), which determine an arbitrary subgroup \( G_1 \) of \( G \) and are equivalent to Eqs. (17), (19), and (20), can now be rewritten in the alternative form

\[
dx^k/dt = e^a A^k_a
\]
where \( e^a (a = 0, 1, 2, 3) \) are arbitrary real constants and the repeated index \( a \) indicates Einstein summation over the four values \( a = 0, 1, 2, 3 \), and where the \( G \) in question is now to be regarded as a subgroup of \( G_{4\pm} \).

The solutions of Eqs. (1+0) are, of course, again expressible in the form of Eqs. (16); but, nevertheless, we must now regard these solutions as functions of certain so-called canonical parameters,

\[
u^a = e^a t \quad (a = 0, 1, 2, 3), \tag{41}\]

each canonical parameter \( u^a \) being proportional to a corresponding arbitrary constant \( e^a \), and the selection of the values of these four constants \( e^a \) being equivalent to the selection of a given subgroup \( G \) of \( G_{4\pm} \).

Now, in analogy with the coordinate transformation (39), there arise similar transformations involving the canonical parameters, of the same numerical form,

\[
u^a \rightarrow u'^a = L_b^a u^b \quad (a, b = 0, 1, 2, 3), \tag{42}\]

and preserving the same Minkowski metric in the "space-time" of these canonical parameters; and it is these transformation Eqs. (42) which effect a change of basis for the group \( G \) of the kind that we have just mentioned above, in connection with the introduction of new symbols \( A'^a \) for \( G_{4\pm} \).
The physical significance of these transformation Eqs. (42) is entirely different from that of the coordinate transformation Eqs. (39). For, when we adopt new canonical parameters $u'^a$, related to the original canonical parameters $u^a$ through an arbitrary constant proper orthochronous homogeneous Lorentz transformation given by Eqs. (42), we must say that now we are not changing the coordinate axes of the observer, but we are changing the observed events of space-time.

This distinction between the physical meanings of the respective Eqs. (39) and (42) has been emphasized by us previously, and in fact has been considered by us as a possible way of distinguishing, by spatiotemporal means, between "external" attributes of elementary particles, such as ordinary spin, and "internal" attributes of elementary particles, such as isospin.
Acknowledgments.

The question as to the possible existence of simply transitive 4-parameter subgroups $G$ of $G_{10}$, other than the translation subgroup, first engaged my attention during the period 1965-6, while I was a guest of the Theoretical Study Division of CERN, on sabbatical leave of absence from the Naval Research Laboratory. It is a pleasure to acknowledge, here, the hospitality and interest of Professors J. Prentki and J. S. Bell during that period. In 1967, and again in 1968, while a guest of the International Centre for Theoretical Physics in Trieste, on leave of absence from NRL, I was enabled to pursue a detailed and systematic investigation of this question, and to prepare the present preliminary and partial account of the results. I am especially pleased to express thanks to Professor Abdus Salam, Director of the Centre, for his kind hospitality and personal interest on both of these occasions.
References and footnotes

1 See E. J. Schremp, NRL Quarterly on Nuclear Science and Technology, January 1963, pp. 1-22. In particular, see p. 17, Eqs. (3.140).

2 See Ref. 1, p. 14, Eqs. (3.101).


4 See Ref. 1, p. 2, footnote 18 and related text.

5 See Ref. 3, p. 60.

6 See Ref. 3, p. 89.

7 See Ref. 1, p. 14.

8 By this we mean that the observed events of space-time, and not the coordinate axes of the observer, are changed by the transformation. In this connection, see Ref. 3, p. 75.

9 See Ref. 3, p. 91.

10 See Ref. 1, p. 10.

11 See Ref. 3, p. 436.

12 See Ref. 1, p. 11.

13 For a general discussion of these "rotations" in the minimal case, see Ref. 1, pp. 13-14.


15 The fact that Eqs. (18) are identically satisfied by the Lorentz matrix \( L(t) = \frac{1}{2} L_j^k(t) \) of any "time-varying" Lorentz 4-screw can readily be seen to follow from the canonical or exponential form of \( L(t) \), as given by Eqs. (3.47) on p. 5 of Ref. 1.

16 See Ref. 1, p. 4, footnotes 21 and 22.

By this we mean that the observed events of space-time, and not the coordinate axes of the observer, are changed by the transformation. In this connection, see Ref. 3, p. 75.