IRREDUCIBILITY
OF THE LADDER REPRESENTATIONS OF U(2, 2)
WHEN RESTRICTED TO ITS POINCARE SUBGROUP

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IRREDUCIBILITY OF THE LADDER REPRESENTATIONS OF U(2,2) WHEN RESTRICTED TO ITS POINCARÉ SUBGROUP

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ABSTRACT

It is shown that the most degenerate discrete series of unitary irreducible representations of $U(2,2)$, the so-called ladder representations, remain irreducible when restricted to representations of the Poincaré subgroup $ISL(2,\mathbb{C})$. They correspond to representations of this subgroup with mass zero and arbitrary integer or half-integer spin $j$ and helicity $\lambda = \pm j$. The basis vectors of the canonical basis are calculated as functions of a light-like 4-vector, which is formed by the simultaneous eigenvalues of the generators of the subgroup of translations.
IRREDUCIBILITY OF THE LADDER REPRESENTATIONS OF $U(2,2)$ WHEN RESTRICTED TO ITS POINCARE SUBGROUP

I. INTRODUCTION

The group of pseudo-unitary transformations $SU(2,2)$ has entered physics in various different ways. First, it has appeared as a covering group of the conformal group of space-time (including the Poincaré group, dilatations and special conformal transformations — see Appendix). With this interpretation, it has been considered as a broken symmetry which becomes exact in the limit of massless particles. Second, the Lie algebra of $SU(2,2)$ was considered as the spectrum-generating algebra for the (spinless) Hydrogen atom. Third, it was also used in hadron physics as an algebra acting on the space spanned by the states of an infinite multiplet of particles at rest, or acting on the indices of an infinite-component field. In this case, the so-called ladder representations of $SU(2,2)$ were used, which can be simply described in terms of creation and annihilation operators.

In the main part of the present note we show that all the ladder representations of $U(2,2)$ (or $SU(2,2)$) remain irreducible when restricted to representations of its Poincaré subgroup $ISL(2,C)$. They correspond to representations of this subgroup characterized by zero mass and arbitrary integer and half-integer spin $j$ and helicity $\lambda = \pm j$. The first-order Casimir operator of $U(2,2)$, which labels the irreducible representations of the ladder series, is linearly related to the helicity $\lambda$.

* The description of unitary representations of noncompact groups in terms of creation and annihilation operators was first given by KURŞUNOĞLU. It became popular after the paper of DOTHAN, GELL-MANN and NE'EMAN. The relation between the ladder representations of $U(p,q)$ and the GEL'FAND-GRAEV discrete series of representations of this group is described in Ref.
We also calculate the basis vectors of the canonical basis
as functions of a light-like 4-vector $\xi^\mu$, which is formed by the
simultaneous eigenvalues of the generators of the subgroup of
translations (eq. (IV.4)). The form of the generators of SU(2,2)
when acting on functions $f(\xi^\mu)$ is given in eq. (III.11).

It is amusing to find that one is led to the same
set of irreducible representations of SU(2,2) for all three
physical interpretations of this group mentioned above (sect. V). In
particular, the representation used for the group theoretic
description of the H-atom is equivalent to the one used for the
description of massless spin 0 particles.

An Appendix is added which deals with the conformal group
of space time in quantum field theory. A comment is included here
on the connection between the conformal structure of space time and
infinite-component field theories of the type investigated recently 8, 9.

II. THE LADDER REPRESENTATIONS OF U(2,2)

We start by recalling the definition of the ladder
representations. We present it here in a basis-independent way.
This is not only demanded by the canons of mathematical aesthetics
but is also convenient for practical computations as it allows one to
choose the most convenient basis for the treatment of any given
problem.

U(2,2) is defined as the group of linear transformations in
complex
the four-dimensional space $\mathbb{C}_4$ which preserves the hermitian form

$$
\overline{\psi} \beta \psi = \sum_{a=1}^{4} \overline{\psi}_a \nu^a
$$

(II.1)

Here and in the following a bar stands for complex conjugation. $\beta$
is a hermitian matrix with two positive and two negative eigenvalues*)
satisfying U(2,2)-invariant normalization $\det \beta = 1$.

By virtue of the invariance of eq. (II.1)

*) We remark that this property of $\beta$ is invariant because of the
inertia law of quadratic forms.
the generators \( J_{AB} = - J_{BA} = \gamma_{AB} \) of the defining representation of \( SU(2,2) \) obey

\[
\beta^* \gamma_{AB} \beta = \gamma_{AB}^* \quad (\text{II.2})
\]

They admit of the following commutation relations:

\[
\left[ J_{KL}, J_{MN} \right] = i \left( \delta_{KN} J_{LM} + \delta_{LM} J_{KN} - \delta_{MK} J_{LN} - \delta_{LK} J_{MN} \right) \quad (\text{II.3})
\]

where \( \delta_{KK} = (+ ---, - +) \) and \( \delta_{KM} = 0 \) otherwise. Capital roman letters \( J_{KL} \) ran over 0,1,2,3,5,6.

The remaining generator \( C_{1} \) of \( U(2,2) \) commutes with all the \( J_{AB} \) and is represented by unity in the defining representation.

Let the Dirac matrices (i.e., spin affinors \( \gamma_{\mu} = (\gamma_{\mu})^{\nu}_{\beta} \)) be defined in the usual way, satisfying covariant anticommutation relations for \( \mu, \nu = 0 \ldots 3 \):

\[
\{ \gamma_{\mu}, \gamma_{\nu} \} = 2 \delta_{\mu \nu}
\]

Then we may choose the matrices \( \gamma_{AB} \) of the defining representation as follows:

\[
\begin{align*}
\gamma_{\mu5} &= \frac{i}{2} \gamma_{\mu} \\
\gamma_{56} &= \frac{i}{2} \gamma_{5} \\
\gamma_{\mu6} &= \frac{i}{2} \gamma_{\mu}
\end{align*}
\]

They satisfy \( \text{eq. (II.3)} \). A matrix \( Q \) obeying \( \text{eq. (II.2)} \) always exists. Its precise form depends on the choice of basis for the \( \gamma \)-matrices \( \gamma \).

The ladder representations are now constructed as follows:

we define the operator valued 4-component spinor,

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* It has been conventional to choose \( \gamma_{6} \) and \( i \gamma_{5} \) hermitian, in that case \( \beta = \gamma_{6} \). This identification is however not invariant under an arbitrary change of basis, but only under those induced by a unitary transformation.
and $\varphi^\alpha$ and $\varphi^\alpha = (\varphi_\alpha^\gamma)$, $\alpha = 1,2,3,4$ and impose the canonical commutation relations

$$[\varphi^\alpha, \varphi^\beta] = \delta^\alpha_\beta . \quad (\text{II.5})$$

Each (star) representation of the canonical commutation relations for the $\varphi'$s gives rise to a unitary (infinite-dimensional) representation of $U(2,2)$ generated by

$$C_1 = \varphi^\star \varphi , \quad J_{AB} = \varphi^\star Y_{AB} \varphi . \quad (\text{II.6})$$

There are two important inequivalent realizations of the canonical commutation relations (II.5) and, correspondingly, two series of inequivalent ladder representations of $U(2,2)$. For the first, we define a $U(2) \otimes U(2)$ invariant vector $\psi_0$ by the equation

$$\Pi_+ \varphi \psi_0 = \frac{1}{2}(1 + \gamma_0) \varphi \psi_0 = 0 \quad ; \quad (\text{II.7})$$

for the second one we put

$$\Pi_- \varphi \psi_0 = \frac{1}{2}(1 - \gamma_0) \varphi \psi_0 = 0 \quad . \quad (\text{II.8})$$

To be consistent with the positivity of the metric in the representation space we have to assume in each case the positive definiteness of the $4 \times 4$ matrix:

$$\left( \psi_0 , (\varphi \Pi_+^*) (\varphi \Pi_-^*) \psi_0 \right) = \left( \psi_0 , (\beta \Pi_+ \varphi) (\varphi \Pi_- \psi_0 \right) = \beta \Pi_+ \quad (\text{II.9})$$

(the last equality is a consequence of eqs. (II.5) and (II.7) or (II.8)). This leads to a different sign of $\beta$ in the two cases. Taking a basis in which $\gamma_0$ is hermitian and $\gamma_j$ are antihermitian, we obtain $\beta = \gamma_0$ in the case (II.7) and $\beta = -\gamma_0$ in the case (II.8). In the first case we obtain the so-called $\mathcal{L}$-series of most degenerate representations of $U(2,2)$; in the second case we arrive at the $\mathcal{L}^*$ series (see Ref.12); the representations of these two series are conjugate to each other. Each series contains a
denumerable set of (unitary) irreducible representations of \( U(2,2) \), labelled by the (integer) value of the first order Casimir operator \( C_1 \). The canonical basis is defined in terms of the eigenvectors of the (maximal) set of commuting operators

\[
-\frac{1}{2} C_1 - 1, \ J_{06}, \ H_0^2 = J_{12}^2 + J_{23}^2 + J_{31}^2, \ M_3 = J_{12},
\]

with eigenvalues \( \lambda, n, s(s+1), m \), correspondingly

\[
(\lambda = 0, \pm \frac{1}{2}, \ldots; \pm n = |\lambda| + 1, |\lambda| + 2, \ldots) \ (\text{sign for the case (II.7) and sign for the case (II.8)}), \ s = |\lambda|, |\lambda| + 1, \ldots \ \text{for} \ s \leq m \leq s .
\]

The representation used for the group-theoretical description of the non-relativistic hydrogen atom is contained in the \( \mathcal{L} \)-series for \( \lambda = 0 \) (\( C_1 = -2 \)). All these representations are known to be integrable to unitary representations of the group. 13)

III. REDUCTION TO REPRESENTATIONS OF THE POINCARÉ SUBALGEBRA

It is convenient to introduce an alternative set of generators in which the Poincaré subalgebra is displayed explicitly. We define for \( \mu, \nu = 0 \ldots 3 \)

\[
M_{\mu\nu} = J_{\mu\nu} ; \ P_{\mu} = J_{\mu4} + J_{\mu5} ; \ K_{\mu} = J_{\mu4} - J_{\mu5} ; \ D = J_{56} . \quad (III.1)
\]
Their coordinates follow from eq. (II.3) and are given in eq. (A.2) of the Appendix. In particular, the generators $N_{\mu\nu}$ and $P_{\mu}$ satisfy the C.R. of the Poincaré algebra.

Let us choose the Dirac matrices such that $Y_5$ is diagonal, and the matrix $\mathcal{Q}$ satisfying eq. (II.2) as follows:

$$
Y_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad Y_5 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}
$$

$$
\mathcal{Q} = \tau_0 Y_0
$$

where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\tau_i$ are Pauli matrices. We shall take $\sigma_3$ diagonal.

Consider now the Hilbert space of those complex functions $f(z_1, z_2)$ of two complex variables $z_1, z_2$ which have finite norm $(f, f)$, the scalar product being defined by

$$
(f, g) = \int \frac{f(z_1, z_2) g(z_4, z_2)}{d^2z_4} d^2z_2
$$

Here $d^2z = d\Re z \ d\Im z = \frac{i}{2} dz \ d\bar{z}$. The polynomials in $z, \bar{z}, \partial/\partial z, \partial/\partial \bar{z}$ form an irreducible set of operators in this Hilbert space. Furthermore, $(\partial/\partial z)^* = -\partial/\partial \bar{z}$ and $z^* = \bar{z}$. We may then proceed to writing down the operator valued 4-component spinor $\psi$ in the form

$$
\psi = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \\ i \frac{\partial}{\partial \bar{z}_1} \\ i \frac{\partial}{\partial \bar{z}_2} \end{pmatrix}
$$

and

$$
\tilde{\psi} = \begin{pmatrix} -\frac{\partial}{\partial \bar{z}_1} \\ -\frac{\partial}{\partial \bar{z}_2} \\ z_1 \\ z_2 \end{pmatrix}
$$

again the + sign referring to the case (II.7) and the - sign to the case (II.8). In both cases the normalized, $U(2) \otimes U(2)$ invariant vector is given by $\psi_0 = \frac{1}{\pi} e^{-z \bar{z}}$. In this form the ladder representation has been displayed in Ref. 14.
The explicit form of the generators is found to be

\[ P_{\mu} = z \sigma_{\mu} \bar{z} \]
\[ D = \frac{i}{2} \left( z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right) \]
\[ M_k = \frac{i}{2} \epsilon_{ijk} J_{i,j} = \frac{i}{2} \left( z \sigma_k \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \sigma_k \bar{z} \right) \]
\[ K_{\mu} = - \frac{\partial}{\partial \bar{z}} \sigma_{\mu} \frac{\partial}{\partial z} \]
\[ N_k = J_{0k} = - \frac{i}{2} \left( z \sigma_k \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \sigma_k \bar{z} \right) \]
\[ C_i = z \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \bar{z} \quad (III.5) \]

Here and in the following \( \sigma^\mu = \epsilon^{\mu \nu} \sigma_\nu \), and the summation convention for \( \mu, \nu = 0 \ldots 3 \) is adopted.

Using the identity

\[ (\sigma^\mu)_{ab} (\sigma_\nu)_{cd} = 2 \delta_{ac} \delta_{bd} \]
\[ \epsilon_{ij} = 4 \quad \text{if } i = j \]
\[ \epsilon_{ij} = 0 \quad \text{otherwise} \quad (III.6) \]

it is immediately found from eq. (III.5) that

\[ P^\mu P_\mu = 0 \quad (III.7) \]

i.e., the ladder representations contain only zero mass representations of the Poincaré group.\(^*\)

It remains to show that these representations remain actually irreducible when restricted to representations of the Poincaré group and to determine the helicities. Recall that the zero mass representations of the Poincaré group are labelled by the helicity \( \lambda \).

To carry out the reduction we re-express the generators as differential operators acting on functions of a light-like 4-vector \( \xi_{\mu} \) which is formed by the simultaneous eigenvalues of the generators \( P_\mu \) of translations.

For this aim we introduce a new set of four real independent variables \( \xi^1, \xi^2, \xi^3 \) and \( \alpha \), related to the complex variables \( z_1, z_2 \) by

\[ -\xi^j = z \sigma_j \bar{z} \quad j = 1, 2, 3 \quad -\infty < \xi^j < +\infty \]
\[ \alpha = \arg(z_1) + \arg(z_2) \quad 0 \leq \alpha < 4\pi \quad (III.8) \]

\(^*\) This observation has been made earlier, cf. ref. 15\).

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All points with $\xi \neq 0$ are identified, then the mapping is bijective.

$\arg(z)$ stands for the phase of $z$ as usual. It is convenient to introduce in addition

$$\xi^0 = \frac{1}{\xi} = z \sigma_0 \overline{z}, \quad \xi^\nu = \sigma_\nu \xi^\nu$$  \hspace{1cm} (III.9)

Let us now re-express the functions $f(z_1, z_2)$, which form our Hilbert space, as functions of the new variables $\xi^i, \alpha$

$$f(z_1, z_2) \equiv \varphi(\xi^1, \xi^2, \xi^3, \alpha)$$  \hspace{1cm} (III.10)

Making use of the chain rule

$$\frac{\partial}{\partial z_\alpha} = \sum_{\beta=1}^3 \frac{\partial \xi^\beta}{\partial z_\alpha} \frac{\partial}{\partial \xi^\beta} + \frac{\partial \alpha}{\partial z_\alpha} \frac{\partial}{\partial \alpha} = \sum_{\beta=1}^3 - \sigma_\alpha \frac{\partial}{\partial \xi^\beta} \frac{\partial}{\partial \xi^\beta} - \frac{1}{z_\alpha} \frac{\partial}{\partial \alpha}$$

and its analogue for $\partial / \partial \overline{z}$, one finds the form of the generators acting on the new functions $F$ as

$$+ \frac{P_\mu}{\mu} = \frac{\xi^\mu}{\mu} \quad (\text{+ for the } \chi \text{-series and } - \text{ for the } \chi^* \text{-series})$$

$$M_{13} = -i \left( \xi^2 \frac{\partial}{\partial \xi^2} - \xi^3 \frac{\partial}{\partial \xi^3} \right) + |\xi| \xi^1 \xi^2^{-2} \frac{\partial}{\partial \alpha}$$

$$M_{23} = -i \left( \xi^3 \frac{\partial}{\partial \xi^3} - \xi^1 \frac{\partial}{\partial \xi^1} \right) + |\xi| \xi^2 \xi^2^{-2} \frac{\partial}{\partial \alpha}$$

$$M_{33} = -i \left( \xi^1 \frac{\partial}{\partial \xi^1} - \xi^3 \frac{\partial}{\partial \xi^3} \right)$$

$$M_{01} = +i |\xi| \frac{\partial}{\partial \xi^1} - \xi^2 \xi^3 \xi^2^{-2} \frac{\partial}{\partial \alpha}$$

$$M_{02} = +i |\xi| \frac{\partial}{\partial \xi^2} + \xi^1 \xi^2 \xi^2^{-2} \frac{\partial}{\partial \alpha}$$

$$M_{03} = +i |\xi| \frac{\partial}{\partial \xi^3}$$  \hspace{1cm} (III.11)

$$D = i \left( 1 + \xi^i \frac{\partial}{\partial \xi^i} \right)$$
Here we have used the abbreviation

\[ \Delta_\xi^i = \sum_{j=1}^{3} \frac{\partial^2}{\partial \xi^j} \] and \[ \xi^2 = (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \]

Summation over repeated indices is to be carried out from 1 to 3.

The first-order Casimir operator \( C_1 \) of \( U(2,2) \) takes the form

\[ C_1 = -2 \frac{\partial}{\partial \alpha} - 2 \] (III.12)

Finally, one finds for the invariant scalar product from eq. (III.3)

\[ (\Phi, G) = \frac{1}{16} \int_{0}^{4\pi} d\xi \int_{0}^{\frac{3\pi}{2}} \frac{d\xi}{|\xi|} \Phi(\xi, \alpha) G(\xi, \alpha) \] (III.13)

For an irreducible representation of \( U(2,2) \), \( C_1 \) is diagonal and takes integer values. Consequently

\[ \frac{\partial}{\partial \alpha} = \lambda, \quad \lambda = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \ldots \] and \[ C_1 + 2 = -2\lambda \] (III.14)

Substituting this into eq. (III.11) we see that \( P_\lambda , M_{\mu \nu} \) are then generators of an irreducible representation of the Poincaré
Evidently the remaining generators $D_K$ act on the same irreducible representation space of the Poincaré group. The first-order Casimir operator $C_1$ is linearly related to the helicity $\lambda$ by eq. (III.13). This completes the proof of our statements.

As a final remark we note that the decomposition of an arbitrary vector $F$ into a sum of vectors that transform according to an irreducible representation of $U(2,2)$ is effected by carrying out a Fourier transform of $F$ with respect to the variable $\alpha$ over the interval $0 \ldots 4\pi$

$$\varphi(\xi, \alpha) = \frac{1}{\pi} \sum_{\lambda=0, \pm 1, \ldots} \varphi_\lambda(\xi) e^{-i\lambda \alpha}$$

Here the helicity $\lambda$ runs through all integer and half-integer numbers. The scalar product (III.13) may then be rewritten as

$$\langle \varphi, G \rangle = \sum_{\lambda} \frac{1}{4} \int \frac{d^3 \xi}{12} \varphi_\lambda(\xi) G_\lambda(\xi)$$

(III.15)

IV. THE CANONICAL BASIS

The canonical basis

$$\Psi_{\lambda, n, s, m}$$

introduced in sect. II is defined in terms of the eigenvectors of a complete set of commuting operators of the maximal compact subgroup $U(2) \times U(2)$

*) Apart from a sign error in Shirokov's formula for $M_{23} \equiv M_1$. 

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Upon introduction of the parametrization

\[
\begin{align*}
Z_1 &= \left( \frac{\rho}{2} \right)^{\frac{1}{2}} e^{\frac{1}{2} (\alpha + \varphi)} \sin \frac{\theta}{2} \quad ; \\
Z_2 &= -\left( \frac{\rho}{2} \right)^{\frac{1}{2}} e^{\frac{1}{2} (\alpha - \varphi)} \cos \frac{\theta}{2}
\end{align*}
\]

eq. (IV.1) reduce to the following system of ordinary differential equations

\[
\begin{align*}
( i \frac{3}{\alpha} - \lambda ) \psi_{\lambda, n, s, m} &= 0 \\
- \left( \frac{3}{\alpha} \frac{3}{\phi} - m \right) \lambda \psi_{\lambda, n, s, m} &= 0
\end{align*}
\]

\[
\begin{align*}
\left[ - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left( \lambda^2 + m^2 - 2m \lambda \cos \theta \right) - s(s+1) \right] \psi_{\lambda, n, s, m} &= 0
\end{align*}
\]

\[
\begin{align*}
\left[ \frac{\partial^2}{\partial \phi^2} + \frac{2}{\rho} \frac{\partial}{\partial \phi} - \left( \frac{1}{4} + \frac{s(s+1)}{\rho^2} - \frac{m}{\rho} \right) \right] \psi_{\lambda, n, s, m} &= 0
\end{align*}
\]

The normalized simultaneous solution of this system of equations is given by

\[
\psi_{\lambda, n, s, m} = \left\{ \frac{(2s + 1)(n - s - 1)!}{(n + s)!} \right\}^{\frac{1}{2}} \times
\]

\[
\begin{align*}
&\times e^{\frac{1}{2} \frac{\phi}{\rho}} s \left( \frac{2s + 1}{\rho} \right) e^{-i \alpha \rho} d_{\lambda, m}^{(s)}(\varphi) e^{-i m \phi} d_{\lambda, m}^{(s)}(\theta) e^{-i \lambda \alpha}
\end{align*}
\]

Here \( d_{\lambda, m}^{(s)} \) are the well known generalized spherical functions.
associated with the $2s+1$ dimensional representation of $SU(2)$. They are simply related to the Jacobi Polynomials.\textsuperscript{17)

The parameters $p/2$, $Q$, $\phi$ are the polar coordinates of the vector $\xi$, (cf. eqs. (III.8) and (IV.2)),

$$\begin{align*}
\xi^1 &= \frac{p}{2} \sin \theta \cos \phi; \\
\xi^2 &= \frac{p}{2} \sin \theta \sin \phi; \\
\xi^3 &= \frac{p}{2} \cos \theta.
\end{align*}$$

$L^{(\alpha)}_n$ is the Laguerre polynomial defined by\textsuperscript{18)

$$L^{(\alpha)}_n(\phi) = \sum_{\tau=0}^{n} \frac{(n+\alpha)}{(n-\tau)} \frac{(-\phi)^{\tau}}{\tau!} = \frac{(\alpha+n)!}{\alpha! n!} \frac{\alpha^\tau}{\tau!} (-n, \alpha+n; \phi)$$

Eq. (IV.4) is the desired expression for the canonical basis.

V. TRANSFORMATION LAW OF LORENTZ COVARIANT FIELDS

In the present section we shall present proof that the ladder representations are unitarily equivalent to the representations of $SU(2,2)$ used in Refs. 1-6 for the description of massless particles (cf. Appendix). This will involve establishing the transformation law under $SU(2,2)$ of Lorentz covariant local massless free fields. As is well known, such a field is associated with a pair of zero mass representations of the Poincaré group with helicity $\lambda$ and $-\lambda$. The result is given in eq. (V.9) below.

Consider the Fock space $\mathcal{F}$ created from a conformal invariant "vacuum" $|0\rangle$ by applying polynomials in smeared creation operators $a^\dagger(p, \pm \lambda)$, which satisfy either the usual Bose or Fermi rules for integer or half odd integer $\lambda$, respectively:

$$[a(p, \lambda), a^\dagger(p', \lambda')] = \delta_{\lambda\lambda'} 2^{\frac{1}{2} (p \cdot p') - 2} \delta^3 (p - p')$$

$$a(p, \lambda) |0\rangle = 0 \quad \text{and} \quad \mathcal{J}_{AB} |0\rangle = 0 \quad (V.1)$$

*) particle and antiparticle states in physical usage.
We may identify the Hilbert space of wave functions $F^\lambda(p)$ considered in Sec. III with that subspace of $\mathcal{H}$ which consists of "one-particle states" with helicity $\lambda$, by virtue of the isometry

$$\mathcal{F}_\lambda \rightarrow \mathcal{F}_\lambda = \frac{i}{\hbar} \int \frac{d^3p}{2|p|} F^\lambda(p) \alpha^\ast(p,\lambda) |0\rangle \quad (V.2)$$

whence

$$\frac{i}{\hbar} F^\lambda(p) = \langle p,\lambda | \mathcal{F}_\lambda \rangle \quad \text{where} \quad \langle p,\lambda | = \langle 0 | \alpha(p,\lambda) \quad (V.3)$$

Let us now impose on $\alpha(p,\lambda)$ the following transformation law:

$$\left[ \alpha(p,\lambda), T_{AB} \right] = \partial^{(\lambda)}_{AB} \alpha(p,\lambda) \quad (V.4a)$$

$\partial^{(\lambda)}_{AB}$ are the differential operators, defined by the RHS of eq. (III.11), which implement the action of the generators $T_{AB}$ in the irreducible representation space of functions $F^\lambda(p)$ considered in Sec. III. The corresponding differential operator for a representation of the $J^+$-series will be denoted by $\partial^{(\lambda^\ast)}_{AB}$. Eq. (V.4a) defines $T_{AB} = J^+_{AB}$ as (unbounded) operators in $\mathcal{H}$, because of eq. (V.1). They form a representation of the algebra of $SU(2,2)$ and its restriction to the subspace of $\mathcal{H}$ consisting of vectors of the form (V.2) is unitarily equivalent to the $J^-$-representation with helicity $\lambda$ which was considered in Sec. III. This follows immediately from eq. (V.3).

From eqs. (V.4) and (III.11) we find the further relation

$$\left[ a^\ast(p,\lambda), T_{AB} \right] = -\partial^{(-\lambda)}_{AB} a^\ast(p,\lambda) = \partial^{(\lambda^\ast)}_{AB} a^\ast(p,\lambda) \quad (V.4b)$$

The problem of constructing Lorentz covariant local free massless fields from creation and annihilation operators has been solved by WEINBERG 19,20. By definition, a Lorentz covariant (finite component) field $\chi^\sigma(x)$ transforms under Poincaré transformations $x^\mu \rightarrow \Lambda^\nu_{\mu} x^\nu + a^\mu$ according to

$$U[A,\alpha] \chi^\sigma(x) U[A,\alpha]^\dagger = \sum_{\nu} D_{\sigma\nu}[\Lambda^\nu] \chi^\nu(\Lambda x + \alpha) \quad (V.5)$$
where $D^{j_1,j_2}$ is some finite-dimensional representation of $SL(2,C)$.

We may restrict ourselves to irreducible representations $D^{j_1,j_2}$, the result will carry over to the general case immediately.

Weinberg shows that the only such massless fields (acting in a Hilbert space with positive definite metric) are fields transforming according to a representation $D^{j_1,j_2}$ with $j_2-j_1 = \lambda$. Moreover, he proves that all these fields may be written as suitable derivatives of fields transforming according to $(j,0)$ and $(0,j)$ for $\lambda = \pm j$, respectively. We may therefore restrict our attention to this case. Then the field is given by

$$\chi_\sigma(x) = (2\pi)^{\frac{3}{2}} \left\{ \int d^3 p \, \delta(p^2) \, \alpha_\sigma(p,\lambda) \left[ \alpha(p,\lambda) e^{-ipx} + \alpha^*(p,\lambda) e^{ipx} \right] \right\}$$

with

$$\alpha_\sigma(p,\lambda) = \left[ 2p_1 \right]^{\frac{1}{2}} D^{(j)}_{\sigma\lambda} \left[ R(\hat{z}) \right] = \left[ 2p_1 \right]^{\frac{1}{2}} D^{(j)}_{\sigma\lambda} (\varphi, \pi/2, \theta, 0)$$

\[ j = \frac{1}{2} \lambda \]  \hspace{1cm} (V.6)

$R(\hat{z})$ is a (suitably standardized) 3-rotation which takes the z-axis into the unit vector $\hat{p} = p/|p|$ with polar co-ordinates

$$\hat{r} = \sin \theta \cos \varphi, \quad \hat{r}^2 = \sin \theta \sin \varphi, \quad \hat{r}^3 = \cos \theta$$  \hspace{1cm} (V.7)

Its Euler angles may be chosen as $\varphi + \pi/2, \theta, 0$, whence the second equation for $\alpha_\sigma$ are the rotation functions for the $2j+1$ dimensional irreducible representation of $SU(2)$.

The field satisfies the equations

$$\left( \frac{d}{dx}^2 + \frac{1}{x^2} \right) \chi(x) = 0 \quad (a) \quad \Delta \chi(x) = 0 \quad (b) \quad (V.8)$$

$\hat{x}$ is the usual $(2j+1)$ dimensional representation of angular momentum

$$[\mathbf{J}_1 \pm i \mathbf{J}_2]_\sigma^\prime = \delta_{\sigma',\sigma} \left[ (\hat{q} \pm \sigma) (\hat{q} \pm \sigma + 1) \right]^\frac{1}{2}$$

$$[\mathbf{J}_3]_\sigma^\prime = \sigma C_{\sigma',\sigma}.$$

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The representations of SU(2,2) which have been used for the description of massless particles were defined by the following field transformation law:

\[
\begin{align*}
\left[ \chi_\sigma(x), \mathcal{P}_\mu \right] &= i \partial_\mu \chi_\sigma(x) \\
\left[ \chi_\sigma(x), M_{\mu \nu} \right] &= \left\{ i \left( x_\mu \partial_\nu - x_\nu \partial_\mu \right) \delta_{\sigma \sigma'} + \left[ \Sigma_{\mu \nu} \right]_{\sigma \sigma'} \right\} \chi_{\sigma'}(x) \\
\left[ \chi_\sigma(x), \mathcal{J}_\mu \right] &= i \left( \partial^\nu - x_\nu \partial^\nu \right) \chi_{\sigma}(x) \\
\left[ \chi_\sigma(x), \mathcal{K}_\mu \right] &= \left\{ i \left[ -2x_\mu x_\nu \partial^\nu - x_\nu \partial^\nu \right] \delta_{\sigma \sigma'} + 2x^\nu \left[ \Sigma_{\mu \nu} \right]_{\sigma \sigma'} \right\} \chi_{\sigma'}(x)
\end{align*}
\]

(V.9)

Here $\Sigma_{\mu \nu}$ is the generator of SL(2,C) in the $(j_1, j_2)$ representation, i.e., in the present case of $(0, j)$ and $(j, 0)$ fields:

\[
\Sigma_{ij} = \epsilon_{i1k} J_k, \quad \Sigma_{ok} = -i \frac{\lambda}{d} J_k
\]

(V.10)

And

\[
\lambda = -1 - j
\]

MACK and SALAM have shown that eq. (V.9) is in fact the most general local SU(2,2) transformation law for a field that transforms under Lorentz transformations according to a finite dimensional, irreducible representation $D^{(\ell, j)}$ of SL(2,C) (see Appendix).

To prove our statements we have to show that the transformation law, eq. (V.9), agrees with (V.4) for a field defined by eq. (V.6), i.e., that

\[
\left[ \chi_\sigma(x), J_{AB} \right] = (2\pi)^{\frac{1}{2}} \epsilon_{AB} \sum_{(\lambda)} \alpha_{(P, \lambda)} \left[ e^{-iP \cdot x} \partial_{AB}^{(\lambda)} \alpha(\lambda) + e^{iP \cdot x} \partial_{AB}^{* (\lambda)} \alpha^*(\lambda) \right]
\]

(V.11)

where the LHS is defined by eqs. (V.9) and (III.1). For the generators $\mathcal{P}_\mu$, eq. (V.11) is obviously true. It suffices then to show eq. (V.11) for the generator $K_0 = J_{06} - J_{05}$, for all other
generators can be expressed in terms of multiple commutators of these. The proof is then an elementary, though somewhat tedious, exercise in properties of rotation functions. For the benefit of the reader let us sketch the calculation for the annihilation part $\chi^{(\pm)}_{\sigma}$ of a field, for positive helicity $\lambda = + j$. From eq. (V.9) we find, using well-known properties of the $\delta$-distribution

$$\left[ \chi^{(\pm)}_{\sigma}(x), K_{\sigma} \right] = (2\pi)^{-\frac{3}{2}} \int d\mathbf{\beta} \delta(\mathbf{p}^2) e^{-i\mathbf{p} \cdot \mathbf{x}} \left\{ -i \frac{\mathbf{p} \cdot \Delta - \frac{2\lambda}{\mathbf{p}^2}}{\mathbf{p}^2} \right\} \delta(\sigma) \cdot +$$

$$+ 2 \left[ J_{\mathbf{p}} \right]_{\sigma, \xi} \left( \frac{\partial}{\partial \mathbf{p}^2} \right) \chi^{(\pm)}_{\sigma}(\mathbf{p}, \mathbf{x}) \cdot a(\mathbf{p}, x).$$

As a consequence of their definition (V.6), $\chi^{(\pm)}_{\sigma}$ satisfy

$$\left( j \mathbf{p} - j \mathbf{p}^1 \right) \chi^{(\pm)}_{\sigma}(\mathbf{p}, x) = 0, \quad \left( \mathbf{p}^2 \frac{\partial}{\partial \mathbf{p}^2} - j \right) \chi^{(\pm)}_{\sigma}(\mathbf{p}, x) = 0.$$

Using these relations, it is readily established that

$$\chi^{(\pm)}_{\sigma}(x), K_{\sigma} \right] = (2\pi)^{-\frac{3}{2}} \int d\mathbf{\beta} \delta(\mathbf{p}^2) e^{-i\mathbf{p} \cdot \mathbf{x}} \left\{ -i \frac{\mathbf{p} \cdot \Delta + 2i \frac{\cos \theta}{\mathbf{p} \cdot \mathbf{n} \cdot \mathbf{p}} \frac{\partial}{\partial \mathbf{q}} + \frac{1}{\mathbf{p}^2 |\mathbf{n}|^2 \cos \theta} \mathbf{\beta}^2 \right\} \chi^{(\pm)}_{\sigma}(\mathbf{p}, x).$$

The differential operator in $\{ \}$ agrees indeed with eq. (III.11) upon inserting (III.14); q.e.d.

* Actually they are elementary functions, the use of eq. (V.12) is, however, more convenient.
Finally let us say a word about the vector potential $A_\mu$ for a massless spin 1 particle. As discussed in detail by Weinberg, it is not a Lorentz covariant field in the sense used above, but its Lorentz transformation law differs from (V.5) by a gauge transformation. The vector potential $A_\mu(x)$ is defined in terms of the field strengths, up to a gauge transformation, by

$$\partial_\nu A_\mu - \partial_\mu A_\nu = F_{\mu\nu}$$

(V.13)

$F_{\mu\nu}$ is expressed in terms of the electric and magnetic field strengths $E$ and $B$ in the usual way. $E - iB$ and $E + iB$ are the massless fields with helicity $\lambda = -1$ and $\lambda = +1$, respectively, as were considered above; and eq. (V.8a) are Maxwell's equations. From eq. (V.13) and the transformation law (V.9) for $F_{\mu\nu}$ one checks that $A_\mu(x)$ also transforms according to eq. (V.9) up to a gauge transformation, with

$$\lambda = -1$$

for the vector potential.

$\Sigma_{\mu\nu}$ are then the generators of $SL(2,\mathbb{C})$ in the $(1,1)$ representation.

An analogous statement holds for the spin 2 (gravitational) potential, again $\lambda = -1$. We refer the reader to Weinberg's paper for a detailed discussion of the gravitational potential.

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The conformal group of space time in quantum field theory

The conformal group of space time is compounded from co-ordinate transformations as follows: 1) **)

1) inhomogeneous Lorentz transformations
2) dilatation: \( x_\mu = \rho x_\mu, \rho > 0 \) (A.1)
3) special conformal transformations **)

\[ x_\mu = \sigma^{-1}(x)(x_\mu - c_\mu x^2) \]

where \( \sigma(x) = 1 - 2cx + c^2 x^2 \)

This is the largest continuous group which leaves the light cone invariant. The generators \( D \) of dilatations, \( K_\mu \), of special conformal transformations and \( P_\mu, M_{\mu\nu} \) of the Poincaré group admit of the following C.R.

\[
\begin{align*}
[D, P_\mu] &= iP_\mu & [D, M_{\mu\nu}] &= 0 \\
[D, K_\mu] &= -iK_\mu & [K_\mu, K_\nu] &= 0 \\
[K_\mu, P_\nu] &= 2i(g_{\mu\nu}D - M_{\mu\nu}) & [K_\lambda, M_{\mu\nu}] &= i(g_{\lambda\mu}K_\nu - g_{\lambda\nu}K_\mu)
\end{align*}
\]

\[
\begin{align*}
[P_\lambda, M_{\mu\nu}] &= i(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu) \\
[M_{\lambda\mu}, M_{\nu\lambda}] &= i(g_{\lambda\mu}M_{\nu\lambda} - g_{\lambda\nu}M_{\mu\lambda} - g_{\lambda\nu}M_{\mu\lambda} + g_{\lambda\mu}M_{\nu\lambda})
\end{align*}
\]

(A.2)

Note that the special conformal transformations do not transform momentum eigenstates into momentum eigenstates, since \([K_\mu, P_\nu]\) does not commute with the momenta \( P_\rho \). The relation

**) They may be written as \( x'_\mu = RT(c)R x_\mu \), where \( R \) is the transformation by reciprocal radii, \( R x_\mu = -x_\mu/x^2 \), and \( T(c) \) stands for a translation \( T(c)x_\mu = x_\mu + c_\mu \). We stress however that \( R \) does not belong to the proper conformal group.
implies that the mass squared spectrum contained in a unitary representation of \( SU(2,2) \) either covers (at least) a whole real semiaxis or consists of the zero point only.

Let us now consider a (quantum) field which transforms according to a representation of the conformal group, i.e.,

\[
\left( \Gamma \chi \right)_\alpha (x) = S_{\beta} (q \cdot x) \chi_{\rho} (q' \cdot x)
\]

\[ (A.4) \]

Here \( \Gamma \) acts on \( x \) as indicated in eq. (A.1).

The little group which leaves \( x = 0 \) invariant is given by dilatations, special conformal transformations and homogeneous Lorentz transformations. As is seen from eq. (A.2), it is isomorphic to an inhomogeneous Lorentz group plus dilatations, i.e.,

\[
\left( \mathfrak{sl}(2,\mathbb{C}) \oplus \{ \mathfrak{d} \} \right), T_4
\]

\[ (A.5) \]

The "translations" \( T_4 \) are the image of the special conformal transformations. Let the generators of this little group be denoted by \( \delta, \kappa_\mu \) and \( \Sigma_{\mu\nu} \), respectively. By the standard theory of induced representations one finds

\[
P_\mu \chi(x) = \iota \partial_\mu \chi(x)
\]

\[
M_{\mu\nu} \chi(x) = \{ \imath (x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu} \} \chi(x)
\]

\[
\mathfrak{D} \chi(x) = \{ -\imath x^\nu \partial_\nu + \delta \} \chi(x)
\]

\[
\kappa_\mu \chi(x) = \{ \imath (2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu) + 2\imath x^\nu [q_{\mu\nu} \delta - \Sigma_{\mu\nu}] \} \chi(x)
\]

\[ (A.6) \]
Two choices of the representation of the little group (A.5) seem to be of particular interest:

1) finite-dimensional representations with $\kappa_\mu = 0$ and $\delta = i \mathbb{1}$;
2) infinite-dimensional unitary representations with $\kappa_\mu \neq 0$.

We wish to make a comment on case 2 first. In this case $\chi(x)$ is an infinite component field. Recently, infinite-component field theories have been investigated\(^8,9\) which have the following property: for fixed space-time co-ordinate $x$, $x = 0$ say, the components of the field span an irreducible representation space of the algebra of $SU(2,2)$. The representations used are the ladder representations. The auxiliary (index-) $SU(2,2)$ is, of course, not identical with the conformal group of space-time, as it does not act on the space-time co-ordinates. On the other hand, let us assume that the conformal structure of space-time reflects itself in the fact that the fields $\chi(x)$ form a representation space for the algebra of the conformal group of space-time. The resulting little group which acts on the indices only is then smaller than $SU(2,2)$ and is given by (A.5). However, the result of the present paper tells us that the ladder representations of the above-mentioned index- $SU(2,2)$ remain irreducible when restricted to the subgroup (A.5), because this group contains an inhomogeneous Lorentz group. Thus the conformal structure of space-time provides a new motivation for using fields that transform according to a reducible representation of the "spin" $SL(2,\mathbb{C})$. Moreover, for a special choice of representations of the little group (A.5) (corresponding to $\kappa_\mu \kappa^\mu = 0$) we arrive at precisely the same reducible representations of the "spin" $SL(2,\mathbb{C})$ as have been used previously with the H-atom as motivation.

Case 1, has been investigated in some detail\(^1-6\). Here we have to do with ordinary finite-component fields. This is the case which we discussed in Sect.V. It is seen from eq. (A.6) and the C.R. of $\Sigma_{\mu\nu}, \kappa_\mu, \delta$ that we must have $\delta = c \mathbb{1}, \kappa_\mu = 0$ if $\Sigma_{\mu\nu}$ form an irreducible representation of $SL(2,\mathbb{C})$ algebra. This is a consequence of Schur's lemma.

\(^{6}\) They are, however, not quite the only representations of $SU(2,2)$ with this property. CASTELL has shown\(^{24}\) that the same is true for some other discrete degenerate representations. They belong to $\kappa_\mu \kappa^\mu \neq 0$ and spin 0. The unitary ray representations of the group (A.5) have been investigated by OTTOSON\(^{22}\).
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