POLE DOMINANCE
AND A LOW-ENERGY THEOREM IN THE
RADIATIVE DECAYS OF CHARGED KAONS

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ABSTRACT

A low-energy theorem is derived for the structure-dependent axial vector form factor in the radiative decay $K \rightarrow \ell + \nu + \gamma$ in the soft kaon approximation. Corrections of the order of $(m_K^2/m_V^2)$ ($V = \rho, \omega, \phi$) are obtained in the pole-dominance approximation. In each approximation the model predictions of both (i) asymptotic SU(3) and (ii) current-mixing are investigated. The quantity $|\gamma_K| = |a_K(0)/F(0)|$ is calculated in both approximations and in both models. It is found that the soft kaon result is shifted upward by approximately 20%; the separation between the models in the two approximations is of the same order of magnitude.
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I. INTRODUCTION

The techniques of the current algebra have recently been used\textsuperscript{1,2} to study the radiative decays of charged pions. In particular a low-energy theorem for the structure-dependent axial-vector part of the radiative decay $\pi \rightarrow l + \nu + \gamma$ has been derived both in the soft-pion approximation\textsuperscript{1} and in the pole-dominance approximation (PDA).\textsuperscript{2} In this paper we consider the extension of such techniques to the analogous radiative decay $K \rightarrow l + \nu + \gamma$ where the theoretical situation is much less clear. One of the bases for our interest in such a calculation is the expectation that the PDA calculation of say,\textsuperscript{1} $dA/dv (q^2 = 0, \Delta^2 = 0)$, in this case, might prove substantially different from the SKA (soft-kaon approximation) result because of hardly negligible "correction terms" of the order of $(m_K^2/m_\gamma^2) (V = \rho, \omega, \phi)$. At the same time we are also not aware of any experimental data on this decay mode as yet, although such an experiment would appear to be feasible.\textsuperscript{3}

In the following section (Sec. II) we outline the soft kaon approach to this problem. There, two expressions for the sum rule for $a_K (\nu = 0) = dA/dv (0, 0)$ are derived, depending on whether results of asymptotic SU(3) obtained by Das, Mathur and Okubo,\textsuperscript{4} or results of the current-mixing model of Oakes and Sakurai\textsuperscript{5} are used.

In Sec. III we derive an expression for $A(q^2, \Delta^2)$ in PDA, after calculating the various form factors by the dispersion-theoretic technique of Das, Mathur and Okubo.\textsuperscript{6} The SKA results for $dA/dv (0, 0)$ are then merely the limiting values $(m_K^2 \rightarrow 0)$ of the corresponding PDA expressions. We find that although individually large, the aggregate effect of the correction terms on $| \gamma_K | \equiv | a_K (0)/F(0) |$ [$F(\nu)$ is the structure-dependent vector form factor] is just a shift to larger values of the order of 20% of the soft-kaon result in both models.\textsuperscript{4,5} Moreover, the differences in the model\textsuperscript{4,5} predictions are of the same order of magnitude. The remaining source of uncertainty in the numerical
estimates of $|\gamma_K|$ in the two models $^4, ^5$ resides, of course, in the calculation of $|F(0)|$; there we have chosen to relate the VVP couplings by nonet symmetry.$^7$

II. LOW-ENERGY THEOREM IN THE DECAY $K^+ \rightarrow \ell^+ + \nu + \gamma$:

SOFT-KAON APPROXIMATION (SKA)

Following Ref. 1 we write

$$M_{\mu\nu} = (2k_0)^2 \int d^4x \, e^{-i q \cdot x} \langle 0| T \{ V_{\mu}^{(\epsilon\nu)}(x), [V_{\nu}^{(\epsilon\mu)}(0) + A_{3\nu}(0)] \} |K^+(k)\rangle,$$

(1)

with the axial vector part of $M_{\mu\nu}$ (for arbitrary $q^2$) being given by

$$M_{\mu\nu}^{A}(q^2) = A(q^2) \delta_{\mu\nu} + B(q^2) q_\mu k_\nu + C(q^2) q_{\mu} q_{\nu} + D(q^2) q_{\nu} k_\mu + E(q^2) k_\mu k_\nu.$$

(2)

Using current algebra and taking the soft-kaon limit $k \rightarrow 0$, one finds

$$M_{\mu\nu}^{A}(q=0, q^2) = -\left[ \Delta_{\mu\nu}^{(A)} (q^2) - \Delta_{\mu\nu}^{(V)} (q^2) - \Delta_{\mu\nu}^{(S)} (q^2) \right] / F_K,$$

(3)

where the decay constant $F_K$ is defined by

$$\langle 0 | A_{3\nu}^{A}(0) | K^+(k) \rangle = -i k_\mu F_K / (2k_0)^2,$$

(4)

and

$$\Delta_{\mu\nu}^{A}(q^2) = i \int d^4x \, e^{-i q \cdot x} \langle 0 | T \{ A_{\mu}^{(A+5)}(x), A_{\nu}^{(A-5)}(0) \} | 0 \rangle,$$

(5)

$$\Delta_{\mu\nu}^{V}(q^2) = i \int d^4x \, e^{-i q \cdot x} \langle 0 | T \{ V_{\mu}^{(A)}(x), V_{\nu}^{(A)}(0) \} | 0 \rangle.$$

(6)
Analogous to the result obtained in Ref. 1, one finds

\[ A(q^2) = -\frac{1}{F_K} \int \frac{\rho_{\text{As}}(m^2) - \rho_{\text{V}(3)}(m^2) - \rho_{\text{V}(8)}(m^2)}{q^2 + m^2} \, dq^2, \]

\[ C(q^2) = -\frac{1}{F_K} \int \frac{\rho_{\text{As}}(m^2) - \rho_{\text{V}(3)}(m^2) - \rho_{\text{V}(8)}(m^2)}{m^2(q^2 + m^2)} dq^2 + \frac{F_K}{q^2 + m_K^2}. \]

Vector meson dominance implies that

\[ A(0) = \left[ \int dq^2 \frac{\rho_{\text{V}(3)}(m^2) + \rho_{\text{V}(8)}(m^2) - \rho_{\text{As}}(m^2)}{m^2} \right] / F_K \]

\[ = \left( \frac{G_{\rho}^2}{m_{\rho}^2} + \frac{G_{\omega}^2}{m_{\omega}^2} + \frac{G_{K^*}^2}{m_{K^*}^2} - \frac{G_{K^0}^2}{m_{K^0}^2} \right) / F_K, \]

with

\[ \frac{G_{\rho}^2}{m_{\rho}^2} = \frac{G_{\rho}^2}{m_{\rho}^2} + \frac{G_{\omega}^2}{m_{\omega}^2}, \]

and Weinberg's first spectral function sum rule,

\[ 2 \frac{G_{\rho}^2}{m_{\rho}^2} - \frac{G_{K^0}^2}{m_{K^0}^2} = \frac{G_{K^*}^2}{m_{K^*}^2} - \frac{G_{K^0}^2}{m_{K^0}^2} = F_K^2. \]

implies that

\[ A(0,0) = F_K. \]

The sum rule which then emerges from the straightforward manipulation of the preceding material (as in Ref. 1) is
where the derivative of the kaon electromagnetic form factor $f_K(q^2)$ is given by

$$\frac{d^2A}{d\nu d\omega} (0,0) = -\left\{ \frac{1}{F_K} \int \frac{p_{A3}(m^2) - p_{V3}(m^2) - p_{V5}(m^2)}{m^4} dm^2 + \frac{1}{3} F_K \langle r^2 \rangle_K \right\},$$

(13)

Thus,

$$f_K'(0) = -\frac{4}{6} \langle r^2 \rangle_K^\omega \left[ \frac{G_{\rho} G_{\rho KK}}{m_\rho^4} + \frac{G_{\rho} G_{\rho KK}}{\sqrt{3} m_\rho^4} + \frac{G_{\omega} G_{\omega KK}}{\sqrt{3} m_\omega^4} \right],$$

(14)

and may be further reduced to

$$f_K'(0) = -\frac{4}{2} \left[ \frac{1}{m_\rho^2} + \frac{1}{m_\omega^2} + \frac{(m_\rho^2 G_{\rho}^2) (m_\omega^2 - m_\omega^4)}{m_\rho^2 m_\omega^2} \right].$$

(15)

Thus,

$$a_K(\nu=0) = -F_K \left\{ \frac{1}{F_K^2} \left( \frac{G_{KA}^2}{m_{KA}^4} - \frac{G_{\rho}^2}{m_\rho^4} + \frac{G_{\rho}^2}{m_\rho^4} - \frac{G_{\omega}^2}{m_\omega^4} \right) + \left[ \frac{1}{m_\rho^2} + \frac{1}{m_\omega^2} + \frac{(m_\rho^2 G_{\rho}^2) (m_\omega^2 - m_\omega^4)}{m_\rho^2 m_\omega^2} \right] \right\}.$$

(16)

Expression (16) may be simplified somewhat if we accept (i) the proposal of Das, Mathur and Okubo that $\int dm^2 \rho_{\alpha\beta}(m^2)$ satisfy the Gell-Mann-Okubo formula, or (ii) the proposal of Oakes and Sakurai (OS) that $\int dm^2 m^{-4} \rho_{\alpha\beta}(m^2)$ satisfy the Gell-Mann-Okubo formula. In case (i) we find,
\[ a_K (\nu = 0) = - F_K \left\{ \left[ \frac{(m_{K^*}/m_{KA})}{(m_{K^*}^2 - m_{K^*}^2)} \right] - \frac{(F_{\pi}/F_K)^2}{m_{\rho}^2} \right\} \]

\[ - \frac{(F_{\pi}/F_K)^2}{3m_{\rho}^2m_{\omega}^2} \left\{ 3(m_{\rho}^2 + m_{\omega}^2) + m_{\rho}^2 - 4m_{K^*}^2 \right\} \]

\[ + \left\{ \frac{4}{m_{\rho}^2} + \frac{1}{m_{\omega}^2} + \frac{1}{m_{\rho}^2} - \frac{1}{3} \left( \frac{4m_{K^*}^2 - m_{\rho}^2}{m_{\rho}^2m_{\omega}^2} \right) \right\} , \]  

while in case (ii), where

\[ \frac{G_{\omega}^2}{m_{\omega}^4} + \frac{G_{\rho}^2}{m_{\rho}^4} = \frac{1}{3} \left[ 4 \left( \frac{1}{2} G_{K^*}^2 \right) \frac{1}{m_{K^*}^4} - \frac{G_{\rho}^2}{m_{\rho}^4} \right] , \]

we find

\[ \left[ a_K (0) \right]_{\text{OS}} = - F_K \left\{ \left[ \frac{(m_{K^*}/m_{KA})}{(m_{K^*}^2 - m_{K^*}^2)} \right] - \frac{(F_{\pi}/F_K)^2}{m_{\rho}^2} \right\} \]

\[ - \frac{(F_{\pi}/F_K)^2}{3m_{K^*}^2m_{\rho}^2} \left\{ 4m_{\rho}^2 - m_{K^*}^2 \right\} + \frac{2}{3} \left( \frac{1}{m_{\rho}^2} + \frac{2}{m_{K^*}^2} \right) \} . \]

In case (ii) we need the results,

\[ G_{\rho}^2 \approx \frac{m_{\rho}^4m_{\omega}^2}{m_{\rho}^2(m_{\omega}^2 - m_{\rho}^2)} \left( \frac{1}{3m_{K^*}^2} - \frac{1}{3m_{\rho}^2} - \frac{1}{m_{\omega}^2} \right) G_{\rho}^2 \approx 1.43 G_{\rho}^2 , \]

\[ G_{\omega}^2 \approx \frac{m_{\omega}^4m_{\rho}^2}{m_{\rho}^2(m_{\rho}^2 - m_{\omega}^2)} \left( \frac{1}{3m_{K^*}^2} - \frac{1}{3m_{\rho}^2} - \frac{1}{m_{\omega}^2} \right) G_{\rho}^2 \approx 0.24 G_{\rho}^2 . \]
We shall present numerical calculations of the ratio \( |\gamma_K| = |a_K(0)/F(0)| \) in the next section, where the "corrections" of the order of \( (m_K^2/m_\nu^2) \) (\( \nu = \rho, \omega, \varphi \)) in PDA are taken up.

III. POLE DOMINANCE APPROACH TO THE DECAY \( K^+ \to \ell^+ + \nu + \gamma \)

In the pole dominance approach to the decay \( K^+ \to \ell^+ + \nu + \gamma \) we are concerned with the retarded amplitude,

\[
M_{\mu\nu}^A = \int d^4x \ e^{-i q \cdot x} \langle 0 | \theta(x_0) [V_{\mu}^{(em)}(x), A_{3\nu}^4(0)] | K^+(k) \rangle (2k_0)^{\frac{A}{2}},
\]

with absorptive part,

\[
\text{Abs } M_{\mu\nu}^A = -i \pi \sum_{\lambda} \langle 0 | V_{\mu}^{(em)}(0) | \rho^0(q); \lambda \rangle \left[ 2 \left( m_\rho^2 + q^2 \right)^{\frac{A}{2}} \right]
\times \langle \rho^0(q); \lambda | A_{3\nu}^4(0) | K^+(k) \rangle (2k_0)^{\frac{A}{2}} \delta(q^2 + m_\rho^2)
\]

\[-i \pi \sum_{\lambda} \langle 0 | V_{\mu}^{(em)}(0) | \omega(q); \lambda \rangle \left[ 2 \left( m_\omega^2 + q^2 \right)^{\frac{A}{2}} \right] \langle \omega(q); \lambda | A_{3\nu}^4(0) | K^+(k) \rangle
\times (2k_0)^{\frac{A}{2}} \delta(q^2 + m_\omega^2)
\]

\[-i \pi \sum_{\lambda} \langle 0 | V_{\mu}^{(em)}(0) | \varphi(q); \lambda \rangle \left[ 2 \left( m_\varphi^2 + q^2 \right)^{\frac{A}{2}} \right] \langle \varphi(q); \lambda | A_{3\nu}^4(0) | K^+(k) \rangle
\times (2k_0)^{\frac{A}{2}} \delta(q^2 + m_\varphi^2)\]
\[ + i \pi \langle 0 | A_{3 \nu}^\dagger (0) | K^+(\Delta) \rangle 2 (m_{K^-} + \Delta^2)^{1 \over 2} \langle K^+(\Delta) | V^{(e\mu)}_{\mu}(0) | K^+(k) \rangle \]
\[ \times (2k_0)^{1 \over 2} \delta (\Delta^2 + m_{K^-}^2) \]
\[ + i \pi \sum_\lambda \langle 0 | A_{3 \nu}^\dagger (0) | K^+_A (\Delta; \lambda) \rangle 2 (m_{K_A} + \Delta^2)^{1 \over 2} \]
\[ \times (2k_0)^{1 \over 2} \delta (\Delta^2 + m_{K_A}^2), \]

(22)

where \( \Delta_{\mu} = (k-q)_\mu \). On substituting the definitions,

\[ \langle 0 | V^{(e\mu)}_{\mu}(0) | P^0(q) \rangle = G_{\mu} \xi^{(\rho)}_{\mu} (q), \]
\[ \langle 0 | V^{(e\mu)}_{\mu}(0) | Q_{\omega}(q) \rangle = (1/\sqrt{3}) G_{\mu \omega} \xi^{(\rho)}_{\mu} (q), \]
\[ \langle 0 | A_{3 \nu}^\dagger (0) | K_A^+(\Delta) \rangle = - G_{K_A} \xi^{(K_A)}_{\nu} (\Delta), \]

(23)

and the relevant form factors tabulated in the Appendix in eq. (22), one finds

\[ \text{Abs} A(q^2, \Delta^2) = \pi G_{\mu} \mathcal{P}_1^{(\rho)} (\Delta^2) \delta (q^2 + m_{\rho}^2) + \pi G_{\mu \omega} \mathcal{P}_1^{(\omega)} (\Delta^2) \delta (q^2 + m_{\omega}^2) \]
\[ + \pi G_{K_A} \mathcal{Q}_1 (q^2) \delta (\Delta^2 + m_{K_A}^2). \]

(24)

From eq. (24) we first determine \( A(q^2, \Delta^2) \) up to "unknown" pole terms in \( q^2 \).
\[ A(q^2, \Delta^2) = -\frac{G_{KA} G_{41}(q^2)}{m_{KA}^2 + \Delta^2} + \sum_{V=\rho, \phi, \omega} \frac{C_V}{m_V^2 + q^2} \]

\[ = -\frac{(G_{KA} / F_K)}{m_{KA}^2 + \Delta^2} - \frac{G_{KA}}{m_{KA}^2 + \Delta^2} \sum_{V} \left( \frac{g_V^2 + m_{KA}^2}{g_V^2 + m_V^2} \right) \frac{G^{K,V}_V}{m_V^2 - m_{KA}^2} \]

\[ + \sum_{V} \frac{C_V}{m_V^2 + q^2}, \]

(25)

where

\[ g_V = G_\rho, \quad (1/\sqrt{3}) G_\phi, \omega, \]

(26)

and then fix the pole residues \( C_V \quad (V = \rho, \phi, \omega) \) by reference to the absorptive part of \( A \) as given by (24). Thus,

\[ C_\rho = \frac{G_\rho^2}{F_K} - \frac{G_\rho G_{KA} G_{S,K,\rho}}{m_{KA}^2 - m_\rho^2}, \]

(27)

\[ C_\phi, \omega = \frac{G_\phi, \omega}{F_K} - \frac{G_\phi, \omega G_{KA} G_{S,K,\phi,\omega}}{m_{KA}^2 - m_\phi, \omega^2}, \]

and

\[ A(q_0^2, \Delta^2) = -\frac{(G_{KA} / F_K)}{m_{KA}^2 + \Delta^2} - \sum_{V=\rho, \phi, \omega} \left( \frac{g_V^2 + m_{KA}^2}{g_V^2 + m_V^2} \right) \frac{1}{m_{KA}^2 - \Delta^2} \]

\[ \times \frac{G_{KA} G_{V} G_{S,K}^{K,V}}{(m_V^2 - m_{KA}^2)} + \sum_{V=\rho, \phi, \omega} \frac{(G_V^2 / F_K)}{m_V^2 + q^2} \]

\[ + \sum_{V=\rho, \phi, \omega} \frac{G_{KA} G_{V} G_{S,K}^{K,V}}{m_V^2 + q^2} \frac{1}{(m_V^2 - m_{KA}^2)}, \]

(28)
On substituting the expressions for $G_s$, $V$ derived in the Appendix [eqs. (A. 29a) and (A. 30a)] into eq. (29), one finds

$$a_K(0) = - F_K \left\{ \frac{1}{F_K} \left[ \frac{G_{K_A}^2}{m_{K_A}^4} \left( 2 - \frac{3 \eta_\rho}{\eta_\rho + \eta_{K_A}} + \frac{\eta_\omega}{\eta_\omega + \eta_{K_A}} \left( \frac{G_{\omega}^2 m_\omega^2}{G_\rho^2 m_\rho^2} \right) \right) + \frac{\eta_\rho}{\eta_\rho + \eta_{K_A}} \left( \frac{2 \eta_{K_A}}{\eta_\rho + \eta_{K_A}} \right) - \frac{\eta_\omega}{\eta_\omega + \eta_{K_A}} \left( \frac{2 \eta_{K_A}}{\eta_\omega + \eta_{K_A}} \right) \right] - \frac{G_{\omega}^2}{m_{\omega}^4} \left( \frac{2 \eta_\omega}{\eta_\omega + \eta_{K_A}} \right) \right\} + 4 \left\{ \frac{\eta_{K_A}}{\eta_\rho + \eta_{K_A}} \frac{G_{\rho K_A} G_\rho}{m_\rho^4} + \frac{\eta_\omega}{\eta_\omega + \eta_{K_A}} \frac{G_{\omega K_A} G_\omega}{\eta_\rho + \eta_{K_A} \sqrt{3} m_\omega^4} \right\} ,$$

where

$$\eta_\alpha = 1 - \frac{m_{K_A}^2}{m_\alpha^2} , \quad (\alpha = \rho, \omega, K_A) \quad (31)$$

Of course, in the limit $m_{K_A}^2 \to 0$, we recover the soft-kaon result, eq. (16).

In order to make some estimate of say, $|\gamma_K|$, it is necessary to derive an expression for $F(\nu)$, the structure-dependent vector form factor, which is given by
\[
M_{\mu\nu}(\eta = 0, \Delta^2) = \left\{ d^4 x e^{-i q \cdot x} \langle 0 | T \left\{ V_{\mu}(x), V_{3\nu}(0) \right\} | K^+(k) \rangle \right\} (q^2 = 0)
\]

\[
= F(\nu) \epsilon_{\mu\nu\lambda\sigma} q^\lambda k^\sigma
\]

In the pole approximation one finds,

\[
|F(\nu)| = \left| \sum_{V = \rho, \phi, \omega} \frac{2 g_V G_{K^*K} G_{K^*K^*V}}{m^2_V (m_{K^*K^*}^2 - m_{K}^2 - 2\nu)} \right| .
\]

We evaluate the unknown couplings \( G_{K^*K^*} \) in (33) by assuming nonet symmetry, so that \( G_{K^*K^*} = (1/\sqrt{2}) \) \( G_{K^*K^*} = G_{K^*K^*} = (1/2) G_{K^*K^*} \) with \( m_{\pi}^2 G_{\rho\omega}^2 / (4\pi) = 0.40 \). Note that because of the conventions of Ref. 14, where

\[
\begin{align*}
\varphi &= -\cos \theta \omega_8 + \sin \theta \omega_1, \\
\omega &= \sin \theta \omega_8 + \cos \theta \omega_1,
\end{align*}
\]

one has rather

\[
|F(0)| = G_{\rho\pi\omega} \frac{G_{K^*K^*}}{(m_{K^*K^*}^2 - m_{K}^2)} \left[ \frac{G_{\rho\omega}}{m_{\rho}^2 - \frac{1}{3} \frac{G_{\omega}}{m_{\omega}^2}} \left( \frac{2}{3} \right) ^\frac{1}{2} \frac{G_{\omega}}{m_{\omega}^2} \right] .
\]

The SKA and PDA predictions for \( |\gamma_K| \) are compared in Table I for both models of interest.

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APPENDIX

We give here the pole dominant solutions for the matrix elements occurring in \( \mathcal{M}_{\mu \nu}^A \) [eq. (14)] after the method of Ref. 5. Thus,

\[
\langle p^o(q) | A^A_{\mu}(0) | K^+(k) \rangle = i \varepsilon^{(p)}(q) \left[ \mathcal{P}_1^{(p)}(\Delta^2) \delta_{\mu \nu} 
+ \mathcal{P}_2^{(p)}(\Delta^2) k_{\mu} (k+q)_{\nu} + \mathcal{P}_3^{(p)}(\Delta^2) k_{\mu} (q-k)_{\nu} \right],
\]

(A.1)

\[
\mathcal{P}_1^{(p)}(\Delta^2) = \frac{G_{p K}}{F_K} - \left( \frac{\Delta^2 + m_p^2}{\Delta^2 + m_K^2} \right) \frac{G_{K A} G_{d K}}{m_K^2 (m_K^2 - m_{p}^2)},
\]

(A.2)

\[
\mathcal{P}_2^{(p)}(\Delta^2) = \frac{-G_{K A} G_{d K}^{K/p}}{2(\Delta^2 + m_K^2)},
\]

(A.3)

\[
\mathcal{P}_3^{(p)}(\Delta^2) = \frac{2G_{p K K} F_K}{\Delta^2 + m_K^2} - \frac{G_{K A}}{m_K^2 (\Delta^2 + m_K^2)} \left[ G_{d K}^{K/p} 
- \frac{1}{2} \left( m_p^2 - m_K^2 \right) G_{d K}^{K/p} \right],
\]

(A.4)

and, for \( V = \varphi, \omega \),

\[
\langle V(q) | A^A_{\mu}(0) | K^+(k) \rangle = i \varepsilon^{(V)}(q) \left[ \mathcal{P}_1^{(V)}(\Delta^2) \delta_{\mu \nu} 
+ \mathcal{P}_2^{(V)}(\Delta^2) k_{\mu} (k+q)_{\nu} + \mathcal{P}_3^{(V)}(\Delta^2) k_{\mu} (q-k)_{\nu} \right],
\]

(A.5)
with the $P_i(\omega)$ being given by the simple replacement $\varphi \rightarrow \omega$ in the $P_i(\varphi)$ [eqs. (A.6)-(A.8)]; also,

$$\langle K_A^+(q)\Gamma_{\mu}^{(e\mu)}(0)K^+(k) \rangle = i\varepsilon_\nu^{(K_A)}(q) \left[ Q_1(\Delta^2) \delta_{\mu\nu} \right. $$

$$ + Q_2(\Delta^2) k_\nu (k+q)_\mu + Q_3(\Delta^2) k_\mu (q-k)_\mu \right], \quad (A.9)$$

$$Q_1(\Delta^2) = \frac{G_{K\varphi}}{F_K} + \sum_{V=\rho,\varphi,\omega} \left( \frac{\Delta^2 + m_{K_A}^2}{\Delta^2 + m_V^2} \right) \frac{g_V G_{K,\varphi}}{m_{K_A}^2 - m_V^2}, \quad (A.10)$$

$$Q_2(\Delta^2) = -\sum_{V=\rho,\varphi,\omega} \frac{g_V G_{d,\varphi}}{2(\Delta^2 + m_V^2)}, \quad (A.11)$$

$$P_1(\varphi)(\Delta^2) = \frac{\sqrt{3}G_{K\varphi}}{F_K} - \left( \frac{\Delta^2 + m_{K_A}^2}{\Delta^2 + m_{K_A}^2} \right) \frac{G_{K,\varphi} G_{S,\varphi}}{m_{K_A}^2 - m_{q}^2}, \quad (A.6)$$

$$P_2(\varphi)(\Delta^2) = \frac{G_{K,\varphi} G_{d,\varphi}}{2(\Delta^2 + m_{K_A}^2)}, \quad (A.7)$$

$$P_3(\varphi)(\Delta^2) = \frac{2G_{K,\varphi} G_{d,\varphi} F_K}{\Delta^2 + m_{K_A}^2} - \frac{G_{K,\varphi}}{m_{K_A}^2(\Delta^2 + m_{K_A}^2)} \left[ G_{S,\varphi}^{-1}(\Delta^2) \right.$$

$$- \frac{1}{2} \left( m_{K_A}^2 - m_{q}^2 \right) G_{d,\varphi}^{-1}(\Delta^2) \left. \right], \quad (A.8)$$
\[ \Phi_3(\Delta^3) = \sum_{V=\rho,\omega} g_v \frac{g_V}{m_V^2(\Delta^2+m_V^2)} \left[ G_S K_{1,V} - \frac{1}{2} (am_{K_A}^2 - am_K^2) G_d K_{1,V} \right]. \]  
(A.12)

To the matrix elements of (A.1), (A.5), (A.9), we add those of the octet vector current \[ \langle K_A^+(q) | V^{(8)}_\mu(0) | K^+(k) \rangle \], and of the unitary-singlet vector current \[ \langle K_A^+(q) | V^{(0)}_\mu(0) | K^+(k) \rangle \].

\[ \langle K_A^+(q) | V^{(8)}_\mu(0) | K^+(k) \rangle = i \epsilon^{(K_A)}(q) \left[ R_1(\Delta^3) \delta_{\nu\mu} + \ldots \right], \]  
(A.13)

\[ R_1(\Delta^3) = \frac{\sqrt{3}}{2} \frac{G_{K_A}}{F_K} + \sum_{V=\rho,\omega} \left( \frac{\Delta^2 + am_{K_A}^2}{\Delta^2 + m_V^2} \right) \frac{G_V G_S K_{1,V}}{2(\Delta^2 + m_V^2)}, \]  
(A.14)

\[ R_2(\Delta^3) = \sum_{V=\rho,\omega} \frac{G_V G_d K_{1,V}}{2(\Delta^2 + m_V^2)}, \]  
(A.15)

\[ R_3(\Delta^3) = \sum_{V=\rho,\omega} \frac{G_V}{m_V^2(\Delta^2 + m_V^2)} \left[ G_S K_{1,V} - \frac{1}{2} (am_{K_A}^2 - am_K^2) G_d K_{1,V} \right]. \]  
(A.16)

and

\[ \langle K_A^+(q) | V^{(0)}_\mu(0) | K^+(k) \rangle = i \epsilon^{(K_A)}(q) \left[ S_1(\Delta^3) \delta_{\nu\mu} + \ldots \right], \]  
(A.17)

\[ S_1(\Delta^3) = \sum_{V=\rho,\omega} \frac{\left( \frac{\Delta^2 + am_{K_A}^2}{\Delta^2 + m_V^2} \right) G_V G_{K_{1,V}}}{m_{K_A}^2 - m_V^2}, \]  
(A.18)
The divergence conditions, $V = \rho^0, \varphi, \omega$

\[
S_2(\Delta^2) = -\sum_{V=\rho^0, \varphi, \omega} \frac{\sigma_V G_d K', V}{m_V^2 (\Delta^2 + m_V^2)} - \frac{1}{2} \left( m^2_{k^2} - m^2_{K^2} \right) G_d K', V^2 \right]. (A.20)
\]

The divergence conditions for $V = \rho^0, \varphi, \omega$

\[
\langle V(\varphi) \partial \Lambda^A_{3Y}(0) \mid K^+(k) \rangle = -\frac{2F_{K^0} m_{K^0}}{\Delta^2} \Gamma_{Y,K} (\gamma, \delta, 0)(0) \mid K^+(k) \rangle = 0; (A.21)
\]

\[
\langle K_A^+(\varphi) \partial \Lambda^A_{3Y}(0) \mid K^+(k) \rangle = 0. (A.22)
\]

With

\[
\sigma_{\varphi} = -\frac{G_{\varphi}}{G_\varphi} \left( \frac{m_{\rho^0}}{m_{\varphi}} \right), (A.23)
\]

are then sufficient to determine the three $s$-wave and three $d$-wave couplings, $G_{s,K,V}$, $G_{d,K,V}$, $(V = \rho^0, \varphi, \omega)$. These conditions are

\[
-\frac{G_{\rho^0}}{F_K} + \frac{G_{K^0} G_{s,K^0}}{m_{K^0}^2 - m_{\rho^0}^2} + 2G_{\rho^0} F_K - \frac{G_{K^0}}{m_{K^0}^2} \left[ G_{s,K^0} \right] - \frac{1}{2} \left( m_{K^0}^2 - m_{K^2}^2 \right) G_{d,K^0} \rho^0^2 \right] = 0; (A.24)
\]

\[
\frac{G_{K^0}}{F_K} + \sum_{V=\rho^0, \varphi, \omega} \frac{m_{K^0}^2 G_{s,V} G_{d,V} K^0}{m_V^2 (m_V^2 - m_{K^0}^2)} + \frac{1}{2} \left( m_{K^0}^2 - m_{K^2}^2 \right) \times \sum_{V=\rho^0, \varphi, \omega} \frac{g_{V} G_{d,V} K^0}{m_V^2} = 0; (A.25)
\]
\[-\sqrt{3} \frac{G_V}{F_K} + \frac{G_{KA}G_S}{(m_{KA}^2 - m_S^2)} + 2G_{VKK} F_K - \frac{G_{KA}}{m_{KA}^2} [G_{K,V}^f] = 0,\]

\[(V = \varphi, \omega)\] (A. 26)

\[\frac{\sqrt{3}G_{KA}}{2F_K} + \sum_{V=\varphi,\omega} \frac{G_V G_S}{(m_{KA}^2 - m_V^2)} + \sum_{V=\varphi,\omega} \frac{G_V}{m_V^2} [G_{S,V}^f] = 0,\]

\[(A. 27)\]

\[G_{\omega} G_{S,K}^f \varphi \frac{m_{KA}^2}{m_{KA}^2 - m_{\omega}^2} - G_{\varphi} G_{S,K}^f \omega \frac{m_{KA}^2}{m_{KA}^2 - m_{\varphi}^2}\]

\[-\frac{1}{2} (m_{KA}^2 - m_{K}^2) (G_{\omega} G_{d,K}^f \varphi - G_{\varphi} G_{d,K}^f \omega) = 0,\]

(A. 28)

with solutions,

\[G_{S,K}^f = \left(2 - \frac{m_{K}^2}{m_\rho^2} - \frac{m_{\omega}^2}{m_{KA}^2}\right)^{-1} \left(m_{\rho}^2 - m_{KA}^2\right) \left(1 - \frac{m_{K}^2}{m_{\rho}^2}\right) \frac{m_{KA}^2}{m_{\omega}^2} \frac{G_{KA}}{2 F_K},\]

\[X \left(- \frac{G_{\rho}}{F_K} + 2G_{\rho KK} F_K\right) + \left(1 - \frac{m_{K}^2}{m_\rho^2}\right) m_{\rho}^2 \frac{m_{KA}^2}{m_{\omega}^2} \frac{G_{\rho}}{2 F_K^2} = \left(\frac{3 G_{KA}}{2 F_K}\right)\]

(A. 29a)
\[ G_{dK,V} = (2 - \frac{m_K^2}{m_{KA}^2} - \frac{m_K^2}{m_V^2})^{-1} \left\{ \frac{2m_K^2}{m_V^2 G_{KA}} \left( -\frac{G_{d}}{F_K} + 2G_{VKK}F_K \right) \right. \\
+ \left. \frac{2m_K^2}{m_{KA}^2 G_{d}} \right\} \]

\[ G_{sK,V} = (2 - \frac{m_K^2}{m_{KA}^2} - \frac{m_K^2}{m_V^2})^{-1} \left( m_{KA}^2 - m_V^2 \right) \left\{ (1 - \frac{m_K^2}{m_{KA}^2}) \times \frac{m_{KA}^2}{m_V^2 G_{KA}} \left( -2G_{VKK}F_K + \sqrt{3} \frac{G_{V}}{F_K} \right) \right. \\
+ \left. (1 - \frac{m_K^2}{m_V^2}) \frac{m_K^2}{m_{KA}^2 G_{d}} \left( -\frac{\sqrt{3}}{2} \frac{G_{KA}G_{V}}{F_K G_{d}} \right) \right\} \quad (V = \varphi, \omega) \]

\[ G_{dK,V} = (2 - \frac{m_K^2}{m_{KA}^2} - \frac{m_K^2}{m_V^2})^{-1} \left\{ \frac{2m_K^2}{m_V^2 G_{KA}} \left( \frac{\sqrt{3} G_{d}}{F_K} \\
- 2G_{VKK}F_K \right) + \frac{2m_K^2}{m_{KA}^2 G_{d}} \left( \frac{\sqrt{3} G_{KA}G_{V}}{2F_K G_{d}} \right) \right\} \quad (V = \varphi, \omega) \]
REFERENCES AND FOOTNOTES

3. We wish to thank Professor S. L. Meyer for some discussion on this point.
8. S. Weinberg, Phys. Rev. Letters 18, 507 (1967). Note that in this paper $G^2$ is defined (as in Ref. 4) by

$$G^2 = \frac{G_{\rho}}{\left(Q^2/2 \rho_{\pi}^2\right)^{\frac{1}{2}}} = \frac{\langle 0 | V_{\rho}^{(3)}(0) | \rho_{\pi}^{(1)}(q) \rangle}{\left(-1/\sqrt{2}\right) \langle 0 | V_{\pi}^{(1)}(0) | \rho_{\pi}^{(1)}(q) \rangle},$$

with $(G^2, m^2) = F^2_{\pi}$; also $\int dm^2 \rho_4(m^2) = \frac{1}{2} G^2_{K^*}$, as in the first, but not the second, of Refs. 4.

9. For calculational purposes we do not assume the existence of the $\kappa$ meson in this paper. [However, see K. Kang, ANL preprint (1968), for arguments on the other side.] Thus from Ref. 4 we have, for example,

$$G^{2}_{K^*} = G^{2}_{K_A},$$

and

$$G^{2}_{K_A} \left(1 - \frac{m^{2}_{K^*}}{m^{2}_{K_A}}\right) = \frac{G^2}{2},$$

where it is assumed that $m^{2}_{K_A} = 1320$ MeV.

10. As in case (i) we assume the validity of the second spectral sum rule in the subgroup $SU(2) \times SU(2)$.
11. Our notation follows that of the second of Refs. 4.

12. In view of the experimental situation we confine ourselves here solely to numerical estimates of this quantity.

13. We now dispense with kinematical factors (e.g., \(2(m_i^2 + \Delta^2)^{\frac{1}{2}}\)) and polarization sums, for simplicity.

14. We use the conventions of Yellin [J. Yellin, Phys. Rev. 147, 1080 (1966)].


16. Note that in Ref. 4

\[
\begin{align*}
G_\phi &= G_\epsilon \cos \theta + G_\mu \sin \theta \\
G_\omega &= -G_\epsilon \sin \theta + G_\mu \cos \theta.
\end{align*}
\]

We also use \((G_\varphi / G_\rho) = \cos \theta\), and \((G_\omega / G_\rho) = -\sin \theta\).

17. We have taken \((F^-_K / F^-_\tau) = 1.1\) in our computations.

18. We shall consider the application of some of these relations in a forthcoming work on the \(K^+ - K^0\) mass difference in collaboration with K. Kang.
Table I

|                | $|\gamma_K|_{(SKA)}$ | $|\gamma_K|_{(PDA)}$ |
|----------------|---------------------|---------------------|
| DMO model      | 0.19                | 0.22                |
| OS model       | 0.22                | 0.27                |

Table Caption

Table I Values of the parameter $|\gamma_K|$ in the DMO and OS models for soft and hard kaons.