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ABSTRACT

We review work done on realization of broken symmetry under the conformal group of space-time in the framework of finite-component field theory. Topics discussed include: Most general transformation law of fields over Minkowski space. Consistent formulation of an orderly broken conformal symmetry in the framework of Lagrangian field theory; algebra of currents and their divergences. Manifestly conformally covariant fields and their couplings.

FINITE-COMPONENT FIELD-REPRESENTATIONS OF THE CONFORMAL GROUP

I. INTRODUCTION

The conformal symmetry of space-time as a possible generalization of Poincaré symmetry has provided a recurrent theme in particle physics.[1] The problems associated with conformal symmetry are i) its physical interpretation[4] and ii) the problems arising from its broken character and the precise manner of descent to Poincaré invariance.

In this paper we wish to concentrate on ii) and review work done on realization of conformal symmetry - and particularly of the algebra associated with the group - using field operators which satisfy Lagrangian equations of motion. The fields may be defined over the Minkowski space-time manifold x_μ or over a projective six-dimensional manifold η_A related to x_μ. We believe this approach to conformal symmetry offers the best hope of exploiting the symmetry physically in contrast to approaches based on a group theoretic treatment of state-vector spaces associated with the group.[5] This is essentially because in such an approach it is extremely difficult to see how to break the symmetry down to Poincaré invariance.

The plan of the paper is as follows: In Sec.II we give the most general transformation law of fields (defined over Minkowski space x_μ) for conformal symmetry, using the theory of induced representations, and also exhibit the field realizations of the generators of conformal algebra. In Sec.III we enumerate the Lagrangians (for particles of spin $\leqslant 1$) which are conformally invariant and describe some modes of symmetry breaking - in particular the physically interesting case of conformal symmetry breaking to the extent of breaking dilatation invariance only. (An explicit model is discussed in the Appendix.) Even where the formalism of Secs. II and III is conformally covariant it is not manifestly so. In Sec.IV we treat the manifestly covariant formulation of wave equations for quantized fields defined over a six-dimensional projective space. Spins higher than one can be more easily treated using this formalism. In Sec.V we review an attempt to understand V-A or V+A weak interaction theory as a conformally invariant theory of fundamental interactions. Not considered in this paper are representations of the conformal group which give rise to infinite component fields. They will be dealt with elsewhere.

II. TRANSFORMATION LAW OF FIELDS

The conformal group of space time consists of coordinate transformations as follows:

1. Dilatations $x'_\mu = \rho x_\mu \quad , \quad \rho > 0$

2. Special conformal transformations

$$x'_\mu = \sigma^{-1}(x)(x_\mu - c_\mu x^2) \qquad \text{(II.1)}$$

where $\sigma(x) = 1 - 2cx + c^2 x^2$

3. Inhomogeneous Lorentz **transformations**

Its generators D of dilatations, K_μ of special conformal transformations, and P_μ, $M_{\mu\nu}$ of the Poincaré **group** admit of the following commutation relations (C.R.):

$$\begin{aligned} [D,P_\mu] &= iP_\mu & [D,M_{\mu\nu}] &= 0 \\ [D,K_\mu] &= -iK_\mu & [K_\mu,K_\nu] &= 0 \\ [K_\mu,P_\nu] &= 2i(g_{\mu\nu}D - M_{\mu\nu}) & [K_\rho,M_{\mu\nu}] &= i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) \end{aligned} \qquad \text{(II.2)}$$

plus those of the Poincaré algebra. Parity must satisfy

$$\pi D \pi^{-1} = D \ , \quad \pi K_\mu \pi^{-1} = \pm K_\mu \quad , \quad \pi P_\mu \pi^{-1} = \pm P_\mu \qquad \text{(II.3)}$$

where the + sign stands for $\mu=0$, and the - sign for $\mu=1,2,3$.

The C.R. (II.2) can be brought into a form which exhibits the O(2,4) structure of the conformal group explicitly by defining[2], for $\mu,\nu = 0\ldots3$,

$$J_{\mu\nu} = M_{\mu\nu} \ , \ J_{56} = D \ , \ J_{\mu5} = \tfrac{1}{2}(P_\mu - K_\mu) \ , \ J_{\mu6} = \tfrac{1}{2}(P_\mu + K_\mu) \ .$$

Then

$$[J_{KL},J_{MN}] = i(g_{KN}J_{LM} + g_{LM}J_{KN} - g_{KM}J_{LN} - g_{LN}J_{KM}) \qquad \text{(II.4)}$$

where $g_{AA} = (+---,-+) \ , \ A = 0,1,2,3,5,6$.

Note that the special conformal transformations do not take momentum eigenstates into momentum eigenstates, as $[K_\mu,P_\nu]$ does not commute with the momenta P_ρ. We also notice the relation

$$e^{i\alpha D}\, P^2\, e^{-i\alpha D} = e^{-\alpha}\, P^2$$

Because of this relation, exact dilatation symmetry (with an integrable generator D that takes one-particle states into one-particle states) implies that the mass spectrum is either continuous or all masses are zero.[6] **This clearly implies that exact dilatation symmetry is physically unacceptable and one will** therefore have to make assumptions on the dynamics which specify how the conformal symmetry is broken. A theory of this type, which is in a sense analogous to the SU(3)xSU(3) current algebra with PCAC will be presented in the next section.

First we have to define, however, what we mean by an (infinitesimal) dilatation or special conformal transformation, as we want it to transform a physical system into another one that is realizable in nature (and not, e.g., a proton with mass m into some nonexistent particle with mass $\rho^{-1}m$, for arbitrary $\rho > 0$).

To do this we postulate[7,8] that there exist interpolating fields to every particle which transform according to a representation of the conformal algebra, i.e.

$$\left(T(g)\ \varphi\right)_{\alpha}(x) = S_{\alpha\beta}(g,x)\ \varphi_{\beta}(g^{-1}x) \qquad (II.5)$$

for infinitesimal $g \in O(2,4)$

where g acts on the coordinates x as indicated in eq.(II.1)

It follows from eq.(II.5) and the multiplication law for the representation matrices T(g) that

S(g,0) must be a representation of the stability subgroup of x = 0.

It is seen from eq.(II.1) that this subgroup (the little group in physical usage) which leaves x = 0 invariant is given by special conformal transformations, dilatations and homogeneous Lorentz transformations. From the C.R. eq.(II.2) one finds that the Lie algebra of this subgroup is isomorphic to a Poincaré algebra + dilatations, i.e. we have

$$(SO(3,1) \otimes \{D\}) \circledS T_4 \qquad (II.6)$$

The 4-dimensional translation subgroup T_4 corresponds to the special conformal transformations, and SO(3,1) is the spin part of the Lorentz group.

Given any representation S(g,0) of the little group (II.6) we can now determine, in accordance with the standard theory of induced representations[9], the complete action of the generators of the conformal group on the field $\varphi(x)$ as follows:

Let $\Sigma_{\mu\nu}$, δ, κ_{μ} be the infinitesimal generators of the little group (II.6) corresponding to Lorentz transformations, dilatations, and special conformal transformations, respectively. They satisfy

$$[\kappa_\mu, \kappa_\nu] = 0 \quad , \qquad [\delta, \kappa_\mu] = -i\kappa_\mu$$

$$[\kappa_\rho, \Sigma_{\mu\nu}] = i(g_{\rho\mu}\kappa_\nu - g_{\rho\nu}\kappa_\mu) \quad , \tag{II.7}$$

$$[\Sigma_{\rho\sigma}, \Sigma_{\mu\nu}] = i(g_{\sigma\mu}\Sigma_{\rho\nu} - g_{\rho\mu}\Sigma_{\sigma\nu} - g_{\sigma\nu}\Sigma_{\rho\mu} + g_{\rho\nu}\Sigma_{\sigma\mu})$$

Choose the basis in index space in such a way that space time translations do not act on the indices, i.e.

$$P_\mu \, \varphi_\alpha(x) = -i \frac{\partial}{\partial x^\mu} \varphi_\alpha(x) \quad . \tag{II.8}$$

It follows that for every element X of the conformal algebra

$$X \varphi(x) = \exp(+iP_\mu x^\mu) \, X' \varphi(0) \qquad \text{where}$$

$$X' = \exp(-iP_\mu x^\mu) \, X \exp(+iP_\mu x^\mu) \tag{II.9}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} x^{\nu_1} \dots x^{\nu_n} [P_{\nu_1}, [\dots [P_{\nu_n}, X] \dots]] \quad .$$

The important point is that the sum on the RHS. of eq.(II.9) is actually finite. From the C.R. eq.(II.2) it is found by inspection that there are at most three non-vanishing terms in this sum.[10) Evaluating the finite multiple commutators, e.g., for $X = K_\mu$, we get

$$\exp(-iP_\nu x^\nu) \, K_\mu \exp(+iP_\nu x^\nu) = K_\mu - 2x^\nu(g_{\mu\nu}D - M_{\mu\nu}) + 2x_\mu x^\nu P_\nu - x^2 P_\mu \, .$$

From this we now deduce the action of K_μ, D, $M_{\mu\nu}$ on $\varphi(x)$, since the action on $\varphi(0)$ is known by hypothesis; e.g. $K_\mu\varphi(0) = \kappa_\mu\varphi(0)$. The final results are

$$P_\mu \, \varphi(x) = -i\partial_\mu \varphi(x)$$

$$M_{\mu\nu} \, \varphi(x) = \{ i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu} \} \varphi(x) \tag{II.10}$$

$$D \, \varphi(x) = \{ -i \, x_\nu \partial^\nu + \delta \} \varphi(x)$$

$$K_\mu \, \varphi(x) = \{ -i(2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu + 2ix^\nu[g_{\mu\nu}\delta - \Sigma_{\mu\nu}]) + \kappa_\mu \} \varphi(x)$$

where the matrices $\Sigma_{\mu\nu}$, δ, κ_μ satisfy the C.R. (2.7).

We have thus shown that all field theoretically admissible representations of the conformal algebra are induced by a representation of the algebra of the little group (II.6). Since this algebra has two non-trivial ideals (= invariant subalgebra) $\{D\} \circledS T_4 \supset T_4$ there arise the following types of representations:

I. finite-dimensional representations of the little group
   a) $\kappa_\mu = 0$
   b) $\kappa_\mu \neq 0$ but nilpotent

II. infinite-dimensional representations of the little group.

Regarding these representations and their physical uses, the following remarks are in order:

1) For case Ia) $\delta = i\ell\mathbb{1}$ by Schur's lemma if the $\Sigma_{\mu\nu}$ form an irreducible representation of the homogeneous Lorentz algebra.

2) In Sec.III it will be shown that the notion of a (broken) conformal symmetry admits of a perfectly consistent formulation in the framework of ordinary Lagrangian field theory. This theory makes use of finite-dimensional representations of the little group (II.6), with $\kappa_\mu = 0$ (type Ia). All generators will be hermitian.

3) For case Ib) the conclusion that all the κ_μ must be nilpotent follows from the well-known fact that in any finite-dimensional representation of the Poincaré algebra the generators of translations are nilpotent.[11)]

4) The possibility of using representations of type Ib) for physical purposes is interesting because it can give rise to spin multiplets. The representations induced in this way are not fully reducible however (and therefore not unitary representations; cf. theorem 1 of Sec.IV.2). Further discussion on this possibility will be given in section V, where Hepner's work[12)] on the use of these representations will be reviewed.

5) The possible use of infinite-dimensional representations of the little group will not be discussed in this paper. This would lead to the consideration of infinite-component field theories which will be discussed elsewhere.

III. LAGRANGIAN FIELD THEORY; APPLICATION TO STRONG AND ELECTROMAGNETIC INTERACTION

In the present section we shall show that the idea of an orderly broken symmetry under the conformal group of space time admits of a perfectly consistent formulation in the framework of ordinary Lagrangian field theory. The considerations presented here are an extension of an unpublished note[8] by one of the present authors. For simplicity we assume fields with spin $\leqslant 1$ and minimal couplings.[13] We shall show that:

1) There exist local conformal currents $k_{\nu\mu}$ and a dilatation current $\mathcal{D}_\mu$ such that the corresponding generators K_μ and D are hermitian and have C.R. with the particle fields as given in eq.(II.10), with $\delta = i\ell\mathbb{1}$, $\kappa_\mu = 0$ (type Ia). <u>This is true independently of whether the action $\int \mathcal{L} d^4x$ is invariant or not.</u>[8]

2) The kinetic energy term without mass is conformal invariant.[14] The same is true of all non-derivative couplings with dimensionless coupling constants and all couplings arising from (Yang-Mills type)[15] gauge field theories. This includes electromagnetism. It also includes weak interactions mediated by an intermediate boson, if this boson is associated with a gauge field associated with some internal group (e.g., for hadrons, the Cabbibo SU(2) subgroup[16] of one of the SU(3) ideals of chiral SU(3) ⊗ SU(3), or for the U(2) ⊗ U(2) group considered for leptons by Ward and Salam[17], which includes both EM and weak interactions.)

3) Besides the exact symmetry limit corresponding to massless particles only, the possibility also exists of a spontaneous breakdown of conformal symmetry. There, all particles can be massive except for $I = 0$, $J^P = 0^+$ massless Goldstone boson. An example of a corresponding Lagrangian (the σ model of Gell-Mann and Lévy[18]) is discussed in the Appendix.

Abstracting from Lagrangian field theory, a current algebra type scheme may be set up. It is composed of the C.R. of the currents with the particle fields, eqs.(III.1) and (III.6), and the relation(III.17) between the divergences of the currents. In addition, an algebraic relation between the divergence of the dilatation current and the axial vector currents of chiral SU(3)×SU(3) has been proposed elsewhere[7] and is given in eqs.(A.4) and (A.5) of the Appendix. Eq.(III.17) expresses the idea that the breaking of conformal symmetry is minimal in the sense that there is only as much breaking of the conformal symmetry as is induced by the breaking of dilatation symmetry alone.

1. The conformal currents

According to eq.(II.10) we want to transform the interpolating fields as follows

$$\mathcal{D}\varphi(x) = i\,(\ell - x_\nu \partial^\nu)\,\varphi(x) \qquad \text{(a)}$$

$$K_\mu \varphi(x) = -i\,(-2\ell x_\mu + 2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu - 2i\, x^\nu \Sigma_{\mu\nu})\,\varphi(x) \qquad \text{(b)}$$

$$M_{\mu\nu}\varphi(x) = i\,(x_\mu \partial_\nu - x_\nu \partial_\mu - i\,\Sigma_{\mu\nu})\,\varphi(x) \qquad \text{(c)} \quad \text{(III.1)}$$

If φ is the electromagnetic vector potential, we may postulate this transformation law only up to a gauge transformation. Eq.(III.1) is to be understood in this sense in the following. The well known[19] reason for this is that, for a massless particle, the vector potential is not a manifestly Lorentz covariant field in the ordinary sense.

Through the last equation, $\Sigma_{\mu\nu}$ is defined in terms of the spin of the particle:

when acting on a spin 0 field	$\Sigma_{\mu\nu} = 0$
when acting on a spin $\frac{1}{2}$ field	$\Sigma_{\mu\nu} = \frac{i}{4} [\gamma_\mu , \gamma_\nu]$
when acting on a spin 1 field	$(\Sigma_{\mu\nu} A)_\rho = i (g_{\mu\rho} A_\nu - g_{\nu\rho} A_\mu)$

The present theory does not lead to multiplets of particles with different spin. We fix the values of ℓ to be

$\ell = -1$ for scalar and vector fields $\qquad \ell = -\frac{3}{2}$ for spin $\frac{1}{2}$ fields (III.2)

This choice is necessary in order to obtain acceptable currents because only then the canonical equal time commutation relations of the fields are invariant under dilatations.[20] The values of ℓ in (III.2) agree with the actual dimension of length of the fields in question. Note that (III.1c) is of the form $\varphi'(x) = \rho^{\ell} \varphi(\rho^{-1} x)$ under $x_\mu \to \rho x_\mu$; $\rho = 1 + \epsilon$ so that the fields transform under dilatations according to their actual dimension of length.

We can now write down the following local currents:[21]

$$\mathcal{D}_\nu (x) = x^\rho T_{\nu\rho} - \sum_r \ell_r \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} \varphi_r$$

$$k_{\nu\mu} (x) = 2 x^\rho x_\mu T_{\nu\rho} - x^2 T_{\nu\mu} - \sum_r \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} (2\ell_r x_\mu + 2i\, x^\rho \Sigma_{\mu\rho}) \varphi_r \right) - \sum_{\substack{\text{spin 0} \\ \text{fields}}} g_{\nu\mu} \phi_r^+ \phi \qquad \text{(III.3)}$$

The angular momentum current has the usual form. The energy momentum tensor is defined by

$$T_{\nu\rho}(x) = -g_{\nu\rho}\mathcal{L} + \sum_{r} \frac{\partial \mathcal{L}}{\partial \partial^{\nu}\varphi_r} \partial_\rho \varphi_r$$

It is convenient to choose the kinetic energy term in the Lagrangian as:

$$\mathcal{L}_0 = \sum_{\text{spin } \frac{1}{2}} \bar{\psi}\left(\frac{i}{2}\gamma_\mu \overleftrightarrow{\partial}^\mu - M\right)\psi + \sum_{\text{spin } 0} \frac{1}{2}\left(\partial_\mu \varphi^+ \partial^\mu \varphi - \mu^2 \varphi^+ \varphi\right) -$$

$$- \sum_{\text{spin } 1} \left\{ \frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)^2 + \frac{1}{2} m^2 B_\nu B^\nu \right\} \qquad \text{(III.4)}$$

Since the zeroth components of the currents, $T_{0\mu}$, $\mathcal{D}_0$, $k_{0\mu}$ and $\mathcal{M}_{0\nu\lambda}$ are all hermitian, the corresponding generators, formally defined as the space integrals of the zeroth component of these currents, are also hermitian.

One checks by using the canonical equal time C.R. of the fields

$$[\phi(x), \pi(x')]_\pm = i\delta^3(\underset{\sim}{x} - \underset{\sim}{x}') \quad \text{for} \quad x_0 = x_0' \qquad \text{(III.5)}$$

that

$$\mathcal{D}\varphi(x) \equiv \int_{v \ni \underset{\sim}{x}} d^3x \, [\mathcal{D}_0(x'), \varphi(x)]_{x_0 = x_0'}$$

$$K_\mu \varphi(x) \equiv \int_{v \ni \underset{\sim}{x}} d^3x \, [k_{0\mu}(x'), \varphi(x)]_{x_0 = x_0'} \qquad \text{(III.6)}$$

satisfy eqs.(III.1). This is independent of whether the Lagrangian is conformal invariant or not. (The integration in (III.4) goes over some volume v including $\underset{\sim}{x} = \underset{\sim}{x}'$.)

2. Divergence of the currents

We now turn to the discussion of the properties of the divergence of the dilatation current $\mathcal{D}_\nu$ and special conformal currents $K_{\nu\mu}$. The dynamical information of a symmetry (exact or broken) defined by the transformation law of the fields, lies in the properties of the divergence of the corresponding currents.

From eq. (III.3) it is seen that the dilatation current $\mathcal{D}_\nu(x)$ depends on x explicitly and can therefore not be coupled to the field of a vector particle. However, using energy momentum conservation $\partial^\nu T_{\nu\rho} = 0$, we find for its divergence

$$\partial^\nu \mathcal{D}_\nu(x) = T^\nu{}_\nu - \partial_\nu \Big\{ \sum_r \ell_r \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} \varphi_r \Big\} \qquad \text{(III.7)}$$

From this we see that:

The divergence of the dilatation current is a local $I = 0$, $J^P = 0^+$ field.

Using the Euler Lagrange equation of motion, (III.7) can be rewritten in the form[22]

$$\partial^\nu \mathcal{D}_\nu(x) = -4\mathcal{L} - \sum_r \Big(\ell_r \frac{\partial \mathcal{L}}{\partial \varphi_r} \varphi_r + (\ell_r - 1) \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r} \partial^\mu \varphi_r \Big) \qquad \text{(III.8)}$$

From eq. (III.8) it is seen that all pieces of the Lagrangian that do not involve constants with nonzero dimension of length give zero contribution. To see this, notice that the RHS of eq. (III.8) vanishes, since it is simply the Euler equation for a homogenous function (all terms having the same dimensionality). This establishes that the kinetic energy term without mass is dilatation invariant, and so are all couplings with a dimensionless coupling constant.

Next we turn to the divergence of the conformal currents. Using energy momentum conservation we find from eq. (III.3)

$$\partial^\nu k_{\nu\mu}(x) = 2x_\mu \left(T^\nu{}_\nu - \partial_\nu \left\{ \sum_r l_r \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} \varphi_r \right\}\right) - \qquad \text{(III.9)}$$
$$- \sum_r \left\{ 2 l_r \frac{\partial \mathcal{L}}{\partial \partial^\mu \varphi_r} \varphi_r + 2i \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} \Sigma_{\mu\nu} \varphi_r \right\} + \sum_{\text{spin } 0} \partial_\mu (\varphi^+ \varphi) +$$
$$+ 2x^\rho \left\{ T_{\mu\rho} - T_{\rho\mu} + 2i \partial_\nu \left(\sum_r \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} \Sigma_{\mu\nu} \varphi_r \right) \right\}$$

The last line is equal to $2x^\rho \partial^\nu \mathcal{M}_{\nu\mu\rho}$ and vanishes by angular momentum conservation. Hence

$$\partial^\nu k_{\nu\mu} = 2x_\mu \partial^\nu \mathcal{D}_\nu + V_\mu \qquad \text{(III.10)}$$

where V_μ is a local vector field defined by the second line on the RHS of eq.(III.9). From eq.(III.4) one checks by explicit calculation that the kinetic energy term without mass gives zero contribution to V_μ. Since it gives no contribution to $\partial^\nu \mathcal{D}_\nu$ either, we see that the kinetic energy term without mass is fully conformal invariant. Furthermore, from (III.10) the condition that an interaction Lagrangean $\mathcal{L}_I$ be fully conformal invariant is found to be the following

1) It is dilatation invariant, i.e., has a dimensionless coupling constant

$$2) \sum_r \left(l_r \frac{\partial \mathcal{L}_I}{\partial \partial^\mu \varphi_r} \varphi_r + i \frac{\partial \mathcal{L}_I}{\partial \partial_\nu \varphi_r} \Sigma_{\mu\nu} \varphi_r \right) = 0; \mu = 0 \ldots 3 \qquad \text{(III.11)}$$

Condition 2) is independent from condition 1) as is seen from the example $\mathcal{L}_I = g A^\mu \pi(\partial_\mu \sigma)$ which satisfies 1) but not 2). An example which satisfies neither 1) nor 2) is the derivative pion nucleon coupling $\bar{N}\gamma_5\gamma_\mu N \partial^\mu \pi$. For nonderivative couplings, condition 2) is trivially satisfied.

3. Yang-Mills theory

Let us now turn to the question of the conformal invariance of the coupling of vector gauge fields in a Yang-Mills type gauge field theory.[15] Let $\underline{A}$ be the internal n-parameter symmetry algebra, and B_μ^a, $a = 1 \ldots n$, the corresponding gauge vector fields. Under an infinitesimal transformation with constant infinitesimal parameter ϵ^a, all fields transform according to

$$\phi_A' = \phi_A + \delta\phi_A \quad , \quad \delta\phi_A = i\epsilon^a T_{a,A}{}^B \phi_B \tag{III.12}$$

where the matrices T_a form a hermitian representation of the algebra $\underset{\sim}{A}$. Hermiticity reads

$$(T_{a,A}{}^B)^* = T_{a,B}{}^A \tag{III.13}$$

As is well known[15, 23], all couplings of the vector fields B_μ^a are completely determined from the postulates of a Yang-Mills type theory and are obtained by the substitution

$$\partial_\mu \phi_A \rightarrow \partial_\mu \phi_A - ig\, T_{a,A}{}^B B_\mu^a \phi_B \tag{III.14}$$

Here, g is a dimensionless real coupling constant. To test for conformal invariance we see that condition 1) above is always fulfilled, while condition 2) is also satisfied because the only derivative couplings are the couplings of mesons, which have the form

$$\mathcal{L}_I = -\tfrac{i}{2} g \left\{ (\partial^\mu \phi^A)^* (T_{a,A}{}^B)^* \phi_B - (\partial^\mu \phi^A) T_{a,A}{}^B \phi_B^* \right\} B_\mu^a \tag{III.15}$$

for spin 0 fields ϕ_A, and

$$\mathcal{L}_I = \tfrac{i}{2} g \left\{ (\partial^\mu \varphi^{\nu A})^* (T_{a,A}{}^B)^* (\varphi_{\nu B} B_\mu^a - \varphi_{\mu B} B_\nu^a) - h.c. \right\} \tag{III.16}$$

for spin 1 fields φ_A^ν. Inserting into eq.(III.11) and making use of eqs.(III.13) and (III.2) one finds that condition 2) is indeed satisfied.

It is now tempting to speculate that in physics there is only as much breaking of conformal symmetry as is induced by the breaking of dilatation symmetry. In other words there should be a remainder of conformal symmetry in the sense that all couplings satisfy condition 2) above. This is equivalent to the algebraic condition

$$\partial^\nu k_{\nu\mu} = 2x_\mu \partial^\nu \mathcal{D}_\nu \tag{III.17}$$

The virtue of this restriction is that it still allows for a breaking of the symmetry by the mass terms in the Lagrangian.

IV. MANIFESTLY O(2,4) COVARIANT FIELD TRANSFORMATION LAW

General experience from the history of elementary particle physics may lead one to the opinion the "the only good covariance is a manifest covariance". This is the motivation for the present section. It is mainly pedagogical in character and much of the material presented may be found in the literature for special cases, but is presented here in a unified way. Manifestly conformal invariant free field equations were first discussed by Dirac.[24] Invariant interactions were given by Kastrup.[25]

The problems to be solved are the following:

1) Write down manifestly conformal invariant transformation laws for fields.

2) Determine the relation between the old fields $\varphi_\alpha(x)$ occurring in the transformation law eq.(II.10) and the new fields which are transformed manifestly covariantly.

3) Write down manifestly invariant free field equations and interactions.

The new fields will be multispinor functions on the four-dimensional surface in a five-dimensional projective space rather than Minkowski space. Their physical interpretation will nevertheless be guaranteed by correspondence with ordinary fields $\varphi_\alpha(x)$ over Minkowski space discussed in Sec.II. This correspondence also allows one to consider questions of unitarity and quantization by reference to Minkowski space.

1. Manifestly covariant transformation law for fields

A manifestly O(2,4) covariant transformation law may be written down for multispinor functions $\chi_\rho(\eta)$ defined on the five-dimensional hypersurfaces of $\mathbb{R}^6$ given by

$$\eta_B \eta^B = L^2 \tag{IV.1}$$

and satisfying $\chi_\rho(\eta) = \chi_\beta(-\eta)$.
Summation over B is over 1,2,3,5,6 with metric (+ ---; -+). There are three essentially different surfaces, corresponding to $L^2 = \pm 1, 0$.

Suppose that $\frac{1}{2}\gamma_{AB}$ is any representation of the algebra of O(2,4) ($\approx$ SU(2,2)) acting on the indices of $\chi_\beta(\eta)$ only. Then a manifestly covariant transformation law including an orbital part is given by (cf. eq.(II.4))

$$J_{AB}\,\chi(\eta) = (L_{AB} + \tfrac{1}{2}\gamma_{AB})\,\chi(\eta) \tag{IV.2}$$

where

$$L_{AB} = i(\eta_A \partial_B - \eta_B \partial_A), \qquad \partial_B \equiv \frac{\partial}{\partial \eta^B}$$

Clearly L_{AB} and $\frac{1}{2}\gamma_{AB}$ commute with each other and satisfy the C.R. of O(2,4) separately. It is important to notice that L_{AB} is a well defined operator when acting on functions that are only defined on the hypersurface (IV.1).[24] This is so because L_{AB} corresponds to an infinitesimal coordinate transformation which is a pseudorotation of the hypersurface (V.1) into itself.

The cone $$\eta_B \eta^B = 0 \tag{IV.3}$$

is also left invariant by the coordinate transformation $\eta_B \to \lambda\eta_B$, $\lambda > 0$. Moreover, this transformation commutes with the O(2,4) rotations. Therefore we may require the fields to be homogeneous functions on the cone (IV.3)

$$\chi(\lambda\eta) = \lambda^n \chi(\eta), \quad \text{i.e.} \quad \eta^B \partial_B \chi(\eta) = n\,\chi(\eta) \tag{IV.4}$$

These homogeneous functions then depend arbitrarily only on 4 of the 5 coordinates which determine a point on the cone (V.3), i.e., just as many as there are coordinates in Minkowski space.[26) We shall restrict our attention to this case in the following.

2. <u>Mathematical preliminaries</u>

Before proceeding we need to know a few mathematical lemmas.

<u>Lemma 1</u>: A set of commuting, nilpotent, finite-dimensional matrices κ_μ can simultaneously be brought to triangular form with zeros on the diagonal by a suitable choice of basis, i.e.,

$$\kappa_\mu = \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & * & \ddots & \\ & & & 0 \end{pmatrix} \qquad \text{for all } \mu \text{ simultaneously} \tag{IV.5}$$

This is a corollary of Engel's theorem which may be found in standard textbooks.[28] Recall that nilpotency of a matrix κ_μ means that there exists a positive integer m such that

$$(\kappa_\mu)^m = 0 \tag{IV.6}$$

Suppose that we are given a representation of the form (II.10) induced by a finite-dimensional representation $\kappa_\mu, \Sigma_{\mu\nu}, \delta$ of the algebra of the little group (II.6). As we have seen in (II.10)f, there arise in this way two types of induced representations. $\kappa_\mu = 0$ (type Ia), and $\kappa_\mu \neq 0$ but nilpotent (type Ib). By virtue of lemma 1 we may assume in the latter case that the four matrices κ_μ are all of the triangular form (IV.5) without loss of generality.

<u>Lemma 2</u>: Induced representations of type Ib have an invariant non-empty subspace $\mathcal{H}'$ on which an induced representation of type Ia is realized. This invariant subspace is spanned by those components of the field $\varphi(x)$ which satisfy

$$\kappa_\mu \, \varphi(x) = 0 \quad \text{for all } \mu = 0 \ldots 3 \tag{IV.7}$$

There is, however, no invariant complement to this invariant subspace.

The fact that the subspace defined by (IV.7) is non-empty follows from (IV.5) because the top row of all the matrices κ_μ is identically zero. The subspace (IV.7) is invariant by virtue of eq.(II.10) and the C.R. (II.7). Finally, let $\varphi(x)$ be such that, for some fixed μ, $\kappa_\mu \varphi(x) \neq 0$ but $\kappa_\nu \kappa_\mu \varphi(x) = 0$ for all ν. Such a φ exists by virtue of (IV.5). Clearly $\varphi \notin \mathcal{H}'$. Consider now $K'_\nu = \exp(-iP_\mu x^\mu)\, K_\nu \exp(iP_\mu x^\mu)$. This is an element of the conformal algebra for arbitrary x (cf. Sec.II). We have $K'_\mu \varphi(x) = \kappa_\mu \varphi(x) \in \mathcal{H}'$. Hence there cannot exist an invariant complement of $\mathcal{H}'$.

<u>Theorem 1</u>. The induced representations of type Ib as desribed in section II are not fully reducible.

At a heuristic level this is a corollary of the last statement in lemma 2.

We also need some properties of the finite dimensional representations of the algebra of O(2.4).

<u>Theorem 2</u>. All finite dimensional representations of the algebra of O(2,4) without parity can be obtained by reducing out tensor products of the two inequivalent four-dimensional representations $\Delta^{(+)}$ and $\Delta^{(-)}$ given by matrices $\frac{1}{2}\gamma_{AB}$ as follows:

$\Delta^{(+)}$

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \quad , \quad \gamma_{\mu 5} = i\gamma_\mu\gamma_5$$

$$\gamma_{56} = -\gamma_5 \quad , \quad \gamma_{\mu 6} = -\gamma_\mu$$

$\Delta^{(-)}$

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \quad , \quad \gamma_{\mu 5} = i\gamma_\mu\gamma_5$$

$$\gamma_{56} = +\gamma_5 \quad ; \quad \gamma_{\mu 6} = +\gamma_\mu$$

All matrices satisfy
$$\gamma_0 \gamma_{AB}^{+} \gamma_0 = \gamma_{AB} \qquad \text{(IV.8)}$$

Parity transforms $\Delta^{(+)}$ into $\Delta^{(-)}$ and vice versa [29]. γ_μ are Dirac matrices and $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$.

For a proof of this theorem see, e.g., ref. 2. The theorem essentially states that all finite dimensional representations can be constructed out of fundamental representations. In the above, two fundamental representations are used, one corresponding to right-handed spinors and the other to left handed spinors.

The simplest non-trivial representation of O(2,4) with parity is eight dimensional and unique up to a choice of basis. It is given by $\Delta = \begin{pmatrix} \Delta^{(+)} & 0 \\ 0 & \Delta^{(-)} \end{pmatrix}$. It is, however, convenient to make a basis transformation so that

$$\gamma_{\mu\nu} = \frac{i}{2}\,[\gamma_\mu , \gamma_\nu] \quad ; \quad \gamma_{\mu 5} = i\,\gamma_\mu \tau_2$$

$$\gamma_{56} = -i\,\tau_3 \quad ; \quad \gamma_{\mu 6} = \gamma_\mu \tau_1$$

$$\pi = \gamma_0 \tag{IV.9}$$

Parity is represented by γ_0 here. τ_i are Pauli matrices. In this form the eight-dimensional representation has been given by Murai.[30] For this representation a Clifford algebra β_A exists such that[24,25]

$$\frac{i}{2}\,[\beta_A , \beta_B] = \gamma_{AB} \quad ; \quad \{\beta_A , \beta_B\} = 2\, g_{AB} \tag{IV.10}$$

These matrices β_A transform as a 6-vector under $\frac{1}{2}\gamma_{AB}$. This is important for constructing invariant couplings. Explicitly, the β_A may be given by

$$\beta_\mu = \gamma_\mu \tau_3 \quad ; \quad \beta_5 = -i\,\tau_1 \quad ; \quad \beta_6 = -\tau_2 \tag{IV.11}$$

There exists[25] also a conformal pseudoscalar β_7

$$\beta_7 = -i\,\beta_0 \beta_1 \beta_2 \beta_3 \beta_5 \beta_6 = i\,\gamma_5 \tau_3 \quad ; \quad \pi \beta_7 \pi = -\beta_7$$

All matrices (V.9) satisfy

$$A\, \gamma_{AB}^{+}\, A = \gamma_{AB} \quad \text{for} \quad A = \gamma_0 \tau_1 \tag{IV.12}$$

Finally we need the following corollary of theorem 2.

Corollary: Let $\Sigma_{\mu\nu}$ be any finite-dimensional irreducible representation of the algebra of SL(2,C) extended to a representation of the algebra (II.7) by choosing $\delta = i\ell\, \mathbb{1}$, $\kappa_\mu = 0$.

Then there exists a finite-dimensional representation $\frac{1}{2}\gamma_{AB}$ of the algebra of O(2,4) with the following property: There exists a subspace $\mathcal{H}'$ of the representation space on which

$$\frac{1}{2}\gamma_{\mu\nu} e = \Sigma_{\mu\nu} e \quad (a) \quad ; \quad \frac{1}{i}\gamma_{56} e = (\delta + im) e \quad (b)$$

$$\frac{1}{i}(\gamma_{\mu 6} - \gamma_{\mu 5}) e = \kappa_\mu e \equiv 0 \quad (c) \qquad (IV.13)$$

for all $e \in \mathcal{H}'$ and some suitable real m.

This corollary guarantees that every finite-dimensional representation of the algebra (II.7) with $\kappa_\mu = 0$ can be extended to a representation of the O(2,4) algebra by enlarging the representation space. Note that a true enlargement is always necessary, unless $\Sigma_{\mu\nu} \equiv 0$, since no generator κ_μ of a simple Lie algebra can be represented by 0 in a non-trivial representation.

Our representation $\frac{1}{2}\gamma_{AB}$ (whose existence is guaranteed by the corollary) can be constructed as follows: As is well known, all finite-dimensional representations of SL(2,C) can be constructed from left-handed and right-handed spinors. Let the representation space of $\Sigma_{\mu\nu}$ be constructed in terms of Lorentz two-component spinors. Then one obtains the desired representation space for $\frac{1}{2}\gamma_{AB}$ simply by substituting conformally transforming 4-component spinors for the Lorentz spinors. The desired subspace is as defined by (IV.13c).

3. Relation between manifestly covariant fields and fields over Minkowski space.

Suppose we are given a field with indices, $\chi(\eta)$, over the cone (IV.3) which satisfies the homogeneity condition (IV.4) and transforms according to eq.(IV.2), with $\frac{1}{2}\gamma_{AB}$ being a finitedimensional representation of the O(2,4) algebra.

We want to obtain from this a field $\varphi(x)$ over Minkowski space which transforms according to eq.(II.10). As we have seen, $\chi(\eta)$ depends arbitrarily on 4 coordinates, i.e. as many as there are space-time coordinates x_μ. We will proceed in three steps:

1. coordinatetransformation $\eta \to x$.
2. x-dependent basis transformation in index space to transform away the intrinsic part of the translation operator, i.e. ensure eq.(II.8).
3. Project out unphysical components if necessary.

By step 3 we mean the following:

After having carried out steps 1 and 2 we shall already have arrived at a field over Minkowski space which transforms according to eq.(II.10). If we start with a finitedimensional representation $\frac{1}{2}\gamma_{AB} \neq 0$ this will be a representation with $\varkappa_\mu = \frac{1}{2}(\gamma_{\mu 6}-\gamma_{\mu 5}) \neq 0$. If we want a representation with $\varkappa_\mu = 0$ (i.e. type Ia) we must project onto the invariant subspace on which this is true, i.e. we keep as physical components only those which satisfy

$$\varkappa_\mu \varphi(x) = 0 \quad \text{for } \mu=0\ldots3 \text{ , with } \varkappa_\mu \equiv \frac{1}{2}(\gamma_{\mu 6}-\gamma_{\mu 5}) \qquad \text{(IV.14)}$$

This subspace is nonempty and invariant by lemma 2 of section IV.2. Eq.(IV.14) may also be read as a subsidiary condition which makes the unphysical components equal to zero. It may be necessary to emphasize the conformal invariance of eq.(IV.14). It does not break down the symmetry but is a necessary condition for the irreducibility of the representation.

Step 1 has been described in great detail by Dirac[24]. For the spin $\frac{1}{2}$ case, step 2 has also been carried out by Dirac, and later described in greater detail by Hepner.[12] We will give a unified treatment for general spin.

Step 1. Define

$$\tilde{\varphi}_\alpha(x) = (\eta_5 + \eta_6)^{-n}\, \chi_\alpha(\eta) \qquad \text{where} \qquad x_\mu = \frac{\eta_\mu}{\eta_5 + \eta_6} \tag{IV.15}$$

The function $\tilde{\varphi}_\alpha$ defined in this way does indeed only depend on x_μ and not on $\eta_5 + \eta_6$, by virtue of the homogeneity condition (IV.4). $\eta_6 - \eta_5$ is not an independent parameter anyway because of eq. (IV.3).

A conformal transformation (IV.2) of χ induces on $\tilde{\varphi}$ a transformation of the form

$$J_{AB}\, \tilde{\varphi}(x) = (\tilde{L}_{AB} + \tfrac{1}{2}\gamma_{AB})\, \tilde{\varphi}(x)$$

where

$$\begin{aligned} \tilde{L}_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ \tilde{L}_{\mu 6} - \tilde{L}_{\mu 5} &= -i(-2nx_\mu + 2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu) \\ \tilde{L}_{\mu 6} - \tilde{L}_{\mu 5} &= -i\partial_\mu \\ \tilde{L}_{56} &= i(n - x_\nu \partial^\nu) \end{aligned} \tag{IV.16}$$

In particular we have then

$$P_\mu\, \tilde{\varphi}(x) = (-i\partial_\mu + \gamma_\mu^{(+)})\, \tilde{\varphi}(x) \quad \text{where} \quad \gamma_\mu^{(+)} = \tfrac{1}{2}(\gamma_{\mu 6} + \gamma_{\mu 5})$$

Step 2. Define [12, 31)]

$$\varphi_\alpha(x) = V_{\alpha\beta}\, \tilde{\varphi}_\beta(x) \qquad \text{where} \quad V = \exp(ix^\nu \gamma_\nu^{(+)}) \tag{IV.17}$$

The matrix V exists because we assumed the γ_{AB} finite dimensional Moreover, V is always a finite polynomial in x because the $\gamma_\mu^{(+)}$ are also nilpotent. It could therefore be worked out explicitly in each case. In practice such straightforward but sometimes tedious calculations can usually be avoided by using translation invariance and the fact that V = 1 at x = 0.

Because all $\gamma_\mu^{(+)}$ commute, V has an inverse given by

$$V^{-1} = \exp(-ix^\nu \gamma_\nu^{(+)}) \tag{IV.18}$$

Furthermore

$$V(-i\partial_\mu)V^{-1} = -i\partial_\mu - \gamma_\mu^{(+)} \tag{IV.19}$$

Using eqs. (IV.19) and the C.R. of the matrices $\frac{1}{2}\gamma_{AB}$ as given by eq. (II.4) one may check that the components of the field φ in the new basis do indeed transform according to eq. (II.10) with

$$\Sigma_{\mu\nu} = \frac{1}{i}\gamma_{\mu\nu} \quad ; \quad \varkappa_\mu = \frac{1}{i}(\gamma_{\mu 6} - \gamma_{\mu 5}); \quad \delta = in + \frac{1}{i}\gamma_{56} \qquad \text{(IV.20)}$$

n is given by eq.(IV.4). The remaining matrices have disappeared from the transformation law.

Summing up, the sought-for relation between the fields $\varphi(x)$ and $X(\eta)$ is given by eqs.(IV.15) and (IV.17). This establishes our claim that, after having carried out steps 1 and 2, we arrive at a field which transforms according to eq.(II.10), with $\varkappa_\mu \neq 0$ unless $\gamma_{AB} \equiv 0$.

As explained above we may then, as our step 3, proceed to dropping the unphysical components which do not satisfy eq.(IV.14) The "dimension of length" arising in eq.(III.1) is related to the degree of homogeneity n by

$$\ell = n - i \times (\text{eigenvalue of } \tfrac{1}{i}\gamma_{56} \text{ in the subspace (IV.14)}). \qquad \text{(IV.21)}$$

4. Invariant wave equations and interactions.

With manifestly covariant fields it is straightforward to write down manifestly invariant wave equations and interactions. The following examples are due to Kastrup.[25] γ_{AB} is finite dimensional and all fields are to correspond to fields over Minkowski space which transform according to eq.(II.10) with $\kappa_\mu = 0$, as do the fields employed in section III.

The spin 0 field (scalar or pseudoscalar) corresponds to a conformal scalar $A(\eta)$ with degree of homogeneity $n = -1$, i.e.

$$\eta^B \partial_B A(\eta) = -A(\eta), \quad \gamma_{AB} A(\eta) \equiv 0 \qquad \text{(IV.22)}$$

The free wave equation is

$$\Box_6 A(\eta) = 0 \qquad \text{where} \quad \Box_6 = \partial^B \partial_B \qquad \text{(IV.23)}$$

As discussed by Dirac,[24] this is a well defined equation for $A(\eta)$ defined on the cone $\eta^2 = 0$ only, if and only if $n = -1$ as we assume. By eq(IV.12), $\ell = n = -1$ in agreement with the discussion in section III.
The scalar field in Minkowsi space is then given by

$$a(x) = (\eta_5 + \eta_6)^{+1} A(\eta) \qquad \text{(IV.24)}$$

and satisfies

$$\partial^\mu \partial_\mu a(x) = 0 \quad . \qquad \text{(IV.25)}$$

The spin ½ field is an 8-component spinor $\chi(\eta)$ of degree of homogeneity $n = -2$

$$\eta^B \partial_B \chi(\eta) = -2\, \chi(\eta) \;, \quad \gamma_{AB} \text{ given by eq.(IV.9)} \qquad \text{(IV.26)}$$

The adjoint is defined by

$$\bar{\chi} = \chi^+ \gamma_o \tau_1 \qquad \text{(IV.27)}$$

cf. eq.(IV.12)

The corresponding 8-spinor over Minkowski space is again given by eqs.(IV.15) and (IV.17) which reads

$$\Psi(x) = (\eta_5 + \eta_6)^{+2} (1 + i x^\mu \gamma_\mu \tau^+) \chi(\eta) \quad \text{where} \quad \tau^+ = \tfrac{1}{2}(\tau_1 + i\tau_2) \qquad \text{(IV.28)}$$

Its physical components are given by eq.(IV.14) which takes the simple form

$$(1 + \tau_3)\, \Psi(x) = 0 \qquad \text{(IV.29)}$$

In the basis where τ_3 has the usual diagonal form, these are just the lowest four components. From eqs.(IV.9) and (IV.21) one finds $\ell = n + \frac{4}{2} = -\frac{3}{2}$, as was assumed in section III because of unitarity requirements.

The free wave equation is

$$-i(\gamma^{AB} L_{AB} + 4)\,\chi(\eta) = 0 \qquad L_{AB} = i(\eta_A \partial_B - \eta_B \partial_A) \tag{IV.30}$$

This amounts to diagonalizing the second-order Casimir operator $\frac{1}{2} J^{AB} J_{AB}$ (Hepner and Murai).[12,30]

The spin 1 gauge fields are 6-vectors $A_B(\eta)$ of degree of homogeneity $n = -1$.

$$\eta^B \partial_B A_C(\eta) = -A_C(\eta) \;;\; (\tfrac{i}{2}\gamma_{AB} A)_C = i(g_{AC} A_B - g_{BC} A_A) \tag{IV.31}$$

satisfying the subsidiary condition $\eta^B A_B(\eta) = 0$. (IV.32)
If we impose in addition the generalized Lorentz condition

$$\partial^C A_C(\eta) = 0$$

then the admissible gauge transformations are, for the electromagnetic potential,

$$A_C(\eta) \rightarrow A_C(\eta) + \partial_C S(\eta) \tag{IV.33}$$

where the gauge function S must be specified on a whole neighbourhood of the cone $\eta^2 = 0$, and satisfy there

$$\eta^B \partial_B S(\eta) = 0 \quad ; \quad \Box_6 S(\eta) = 0 \quad .$$

The free field equation is then

$$\Box_6 A_C(\eta) = 0 \tag{IV.34}$$

Again, the choice of n=-1 makes this into a well-defined equation for $A_C(\eta)$ defined on the cone $\eta^2 = 0$ only.

The corresponding field $a_B(x)$ is again given by eqs.(IV.15) and (IV.17). For the first four components this takes the explicit form

$$a_\mu(x) = (\eta_5 + \eta_6)^{+1} \left[A_\mu(\eta) - x_\mu \{ A_5(\eta) - A_6(\eta) \} \right] \qquad \mu = 0 \ldots 3 \tag{IV.35}$$

and the subsidiary condition eq.(IV.32) reads

$$a_6(x) - a_5(x) = 0 \tag{IV.36}$$

This can be seen in the following way: For $x_\mu = 0$, we have $\eta_\mu = 0$ and $\eta_5 = \eta_6$. Therefore eq.(IV.36) is clearly true at this point. Now eq.(IV.32) is conformal invariant and therefore, in particular, translation invariant. However, by construction, all $a_B(x)$ transform under translations according to eq.(II.10). Thus, by translation invariance the validity of eq.(IV.36) for arbitrary x_μ follows from its validity at $x_\mu=0$.

The first four components of $a_B(x)$ are the physical ones as they satisfy eq.(IV.14) by virtue of the subsidiary condition eq.(IV.36). Namely

$$(\kappa_\mu a)_\nu(x) = \frac{1}{i}(\gamma_{\mu 6} - \gamma_{\mu 5})\, a_\nu = i\, g_{\mu\nu}(a_6 - a_5) = 0 \qquad \nu = 0\ldots 3 \qquad \text{(IV.37)}$$

<u>Invariant couplings</u>:

Following Kastrup, it is easy to see that a conformal invariant coupling between a pseudoscalar field $A(\eta)$ and the spin $\frac{1}{2}$ field $\chi(\eta)$ is given by the following wave equation:

$$\Box_6\, A = -g\, \eta^C \bar{\chi}\, \beta_C \beta_7 \chi$$
$$-i(\gamma^{AB} L_{AB} + 4)\, \chi = g\, \eta^C \beta_C \beta_7 \chi\, A \qquad \text{(IV.38)}$$

and the coupling of the electromagnetic field to the spin $\frac{1}{2}$ field is given by

$$\gamma^{AB}\left[\eta_A(\partial_B - iq\, A_B) - \eta_B(\partial_A - iq\, A_A) - 4i\right]\chi = 0$$
$$\Box_6\, A_C = q\, j_C(\eta) \qquad \text{(IV.39)}$$

where
$$j_C(\eta) = \eta^B \bar{\chi}\, \gamma_{BC}\, \chi$$

The β-matrices are given by eqs.(IV.10) and (IV.11).

As is seen, the electromagnetic coupling is obtained by making the gauge-invariant substitution $\partial_C \to \partial_C - iq A_C$. In this form it can be immediately generalized to arbitrary sets of gauge fields $A_C{}^a(\eta)$. Let T_a be the representation matrices of the relevent group as discussed in section III; then the general rule is to substitute

$$\partial_C \to \partial_C - iq\, A_C{}^a\, T_a \qquad \text{(IV.40)}$$

in the free field equations. Summation over a is understood. In this way one obtains couplings which are both conformal invariant and gauge invariant.

Finally there also exists a quadrilinear conformal invariant spin 0 boson coupling. A corresponding wave equation would be, e.g.,

$$\Box_6\, A(\eta) = g\, [A(\eta)]^3 \qquad \text{(IV.41)}$$

Of course all couplings mentioned above can occur simultaneously.

A point we want to stress is the following: Not all manifestly covariant looking couplings are physically acceptable. They must satisfy the following additional requirements:

1). The field equations must have solutions which are homogeneous functions. This requires that all terms in a certain field equation must have the same degree of homogeneity. The degree of homogeneity of such a term is calculated by counting each explicit coordinate η with +1, each derivative $\partial/\partial\eta$ with -1, and each field with the appropriate number n (e.g. -1 for bosons and -2 for fermions in the cases discussed above).

2). The interaction terms must not couple unphysical field components to physical ones. This turns out to be a strong restriction in practice.

The couplings given above do satisfy this condition, while, e.g., a coupling $\bar{\chi}\chi A$ would not.

There is an easy way to check whether condition 2 is satisfied without going through the tedious transformations of section IV.3. Because of translation invariance, it is sufficient to check that the condition is satisfied at $x_\mu = 0$. This corresponds to $\eta_\mu = 0$ and $\eta_5 = \eta_6$. At this point the boost operator V in eq.(IV.17) is simply unity: $V_{\alpha\beta} = \delta_{\alpha\beta}$. Therefore we have in general

$$\varphi_\alpha(0) = (\eta_5+\eta_6)^{-n}\chi_\alpha(\underset{\sim}{Q}) \qquad \text{(IV.42)}$$

where the abbreviation $\chi(\underset{\sim}{Q}) = \chi(\eta_\mu = 0, \eta_5 = \eta_6)$ has been used. The physical components of a field at this point are then simply those satisfying

$$\varkappa_\mu \chi(\underset{\sim}{Q}) = \tfrac{1}{2}(\gamma_{\mu 6} - \gamma_{\mu 5})\chi(\underset{\sim}{Q}) = 0 \quad , \qquad \text{(IV.43)}$$

i.e. for a spin $\frac{1}{2}$ field $\qquad (1+\tau_3)\chi(\underset{\sim}{Q}) = 0 \qquad$ (IV.44)

and for the electromagnetic field the four components $A_\mu(0)$, $\mu = 0 \ldots 3$.

It is now easy to check whether there is a coupling of physical to unphysical components or not. For example, in the case of the pseudoscalar coupling (IV.38) we find from eqs.(IV.9)...(IV.12)

$$\eta^C \bar{\chi} B_C B_7 \chi A = \tfrac{1}{2}(\eta_5+\eta_6)\chi^\dagger \gamma_0 (1-\tau_3)\gamma_5 \chi A \qquad \text{at } \eta_\mu = 0$$

Thus we see that the pseudoscalar field A is only coupled to the physical components of $\chi(\underset{\sim}{Q})$ which satisfy eq.(IV.44). The one and only component of A(0) is clearly physical; it satisfies eq.(IV.43) trivially. One can also check in this way, from eqs.(IV.42) and (IV.44), that the wave equations (IV.38) and (IV.39) do indeed correspond to the Dirac, Klein-Gordon and Maxwell equations for the physical field components in Minkowski space, with minimal electromagnetic interaction and nonderivative pseudoscalar pion-nucleon interaction.

In section III an alternative characterization of all (physically acceptable) conformal invariant couplings for spin ≤ 1 has been given. If all vector mesons are assumed to be gauge fields, then the manifestly conformal invariant couplings given above, and their obvious generalization to the case of several fields of the same spin, exhaust all possibilities for spin ≤ 1. This may be checked by enumerating all possibilities, as there are only a few types of couplings with dimensionless coupling constant, and the gauge field couplings are fixed in their form.

V. REPRESENTATIONS INDUCED BY FINITE-DIMENSIONAL REPRESENTATIONS OF THE LITTLE GROUP (II.7) WITH $\kappa_\mu \neq 0$.

As has been mentioned at the end of Sec.II, these representations are interesting in principle because they could give rise to spin multiplets, but they are not fully reducible (and therefore not unitary representations) The only author who has recognized the power of representations of this type is Hepner who uses conformal invariance to generate uniquely[32] the V-A (or V+A) weak interaction.[12] Assume we are working with a four-component spinor ψ (quark, μ or e-field). If we postulate that a four-Fermi interaction be conformal invariant, then there exist two possible interactions

$$g \; \bar{\psi}_1 \psi_2 \bar{\psi}_3 \psi_4$$

$$g \; (\bar{\psi}_1 \gamma^{(-)}_\mu \psi_2)(\bar{\psi}_3 \gamma^{\mu(-)} \psi_4) \qquad (V.1)$$

corresponding to $\bar{\chi}\chi\bar{\chi}\chi$ and $\eta_A \bar{\chi}\gamma^{AB}\chi \;\; \eta^C \bar{\chi}\gamma_{CB}\chi$ in the six-dimensional language of Sec.IV.

Here $\frac{1}{\kappa}\gamma^{(-)}_\mu \cdot \kappa_\mu =$ either $-\frac{1}{2}\gamma_\mu(1+i\gamma_5)$ or $\frac{1}{2}\gamma_\mu(1-i\gamma_5)$ depending on which of the two inequivalent 4-dimensional representations $\Delta^{(\pm)}$ of the algebra of the index-O(2,4) is chosen (cf. Sec. IV eq.(IV.8)). We will set $\kappa = 1$ numerically. This can always be achieved by a basis transformation with the matrix $\exp(i\alpha\gamma_5)$, for suitable α. The expression (V.1) is just the familiar V-A or V+A coupling.

An invariant wave equation would be

$$\left(i\gamma^{(-)}_\mu \partial^\mu + i\left(\frac{n}{2}-1\right)\gamma_5 + m\right)\psi = g\,\gamma^{(-)}_\mu \psi \,\bar{\psi}\, \gamma^{\mu(-)} \psi \qquad (V.2)$$

The quantity n appearing here and in eq.(IV.20) must be a solution of $n^3 - n + 2 = 0$ in order that (V.2) be invariant. (This is related to the homogeneity requirement discussed at the end of Sec.IV.)

We now break the conformal symmetry and descend to Poincaré invariance by adding the symmetry-breaking term $i\gamma_\mu^{(+)}\partial^\mu - i(\frac{m}{2} - 1)\gamma_5$; eq.(V.2) goes over into the usual Dirac equation with weak interactions.

The point of view regarding weak interactions taken here is different from that of Sec.III. The theory of Sec.III has the advantage that the kinetic energy term without mass does not break the conformal invariance so that the canonical equal time C.R. of the fields are conformal invariant and one can write down hermitian generators.

APPENDIX

THE σ-MODEL AS AN ILLUSTRATION OF IDEAS IN SEC.III

Consider the Lagrangian of the σ-model of Gell-Mann and Lévy [18]

$$\mathcal{L} = \bar{N}(i\not{\partial} - M)N + ig\bar{N}\gamma_5 \underline{\tau} N \underline{\pi} + \tfrac{1}{2}(\partial_\mu \underline{\pi}\partial^\mu \underline{\pi} - \mu^2\pi^2)$$
$$+ \tfrac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - \{\mu^2 + \tfrac{2\lambda}{f^2}\}\sigma^2) - \lambda(\{\pi^2+\sigma^2\} - \tfrac{2}{f}\,\sigma\{\pi^2+\sigma^2\}) \,. \qquad \text{(A.1)}$$

Here N is the nucleon field, π is the pion field, and σ is the field of a I=0, $J^P = 0^+$ meson. $f = g/2M$. Let us choose the free parameter λ to be

$$\lambda = g^2 \frac{\mu^2}{4M^2}$$

Reexpressing the Lagrangian in terms of the field $\sigma' = \sigma - (2f)^{-1}$, and calculating the dilatation current and conformal currents from eq.(III.3), one finds for their divergences [34]

$$\partial^\mu \mathcal{D}_\mu = g^{-1} M m_\sigma^2\ \sigma(x) \quad \text{(a)}, \quad \partial^\nu \mathcal{K}_{\nu\mu} = 2x_\mu\ \partial^\nu \mathcal{D}_\nu \quad \text{(b)} \qquad \text{(A.2)}$$

The last equation follows directly from eq.(III.10)ff since the present Lagrangian does not involve any derivative couplings. m_σ is the (bare) σ-mass. We see that in the limit of a massless boson σ both currents are conserved, and we have a spontaneous breakdown of conformal symmetry.

With the usual definition of the axial vector current $\mathcal{A}_j^{\ \mu}$ for this model, one finds that generally, also for $m_\sigma \neq 0$

$$\left[F_i^{\ 5}(x_o), \partial_\mu \mathcal{A}_j^{\ \mu}(x)\right] = -i\,\delta_{ij}\,\tfrac{1}{3}\,(\partial^\mu \mathcal{D}_\mu - \tfrac{3}{4}\,\mu^2 f^{-2}) \quad \text{for} \quad i,j = 1,2,3$$

where $F_i^{\ 5} = \int d^3x\, \mathcal{A}_i^{\ o}$. (A.3)

Elsewhere it has been proposed to generalize this formula to chiral SU(3) x SU(3) in the following form: [7]

$$\partial^\mu \mathcal{D}_\mu = \alpha_o u_o(x) + \alpha_8 u_8(x) - \langle 0|\{\alpha_o u_o + \alpha_8 u_8\}|0\rangle \qquad \text{(A.4)}$$

with $\quad \alpha_o + \frac{1}{\sqrt{2}}\alpha_8 = -3\sqrt{\frac{3}{2}}\,.$

α_8/α_0 is a measure of the breaking of the eightfold way. The u_i must satisfy the C.R. of (integrated) scalar densities with vector and axial vector currents as proposed by Gell-Mann $(i,j,k = 0...8)$ [33]

$$\left[F_i^{\,5},\ v_j(x)\right]_{eq.t.} = i\, d_{ijk}\, u_k(x)$$

$$\left[F_i^{\,5},\ u_j(x)\right]_{eq.t.} = -i\, d_{ijk}\, v_k(x) \qquad (A.5)$$

with

$$v_j(x) = \partial^\mu \mathfrak{a}_\mu^{\,j}(x) \qquad \text{for } j=1,2,3.$$

The matrix elements of $u_o + \frac{1}{\sqrt{2}} u_8$ are known in current algebra calculations as "σ-terms". A method to calculate them on the basis of eq.(A.4) and eq. (III.1) has been outlined in Ref. 7.

FOOTNOTES AND REFERENCES

1) A historical survey including an extensive list of literature prior to 1962 is found in Refs. 2 and 3 below and in the review article by T. Fulton, R. Rohrlich and L. Witten, Rev. Mod. Phys. 34, 442 (1962).

2) H.A. Kastrup, Ann. Physik 7, 388 (1962).

3) F. Gürsey, Nuovo Cimento 3, 988 (1956).

4) Discussions on the physical interpretation of the conformal group of space-time are found in:

A. Gamba and G. Luzatto, Nuovo Cimento 18, 1086 (1960);

E.C. Zeeman, J. Math. Phys. 5, 490 (1964);

T. Fulton, R. Rohrlich and L. Witten, Nuovo Cimento 26, 652 (1962) and Ref. 1;

L. Castell, Nucl. Phys. B4, 343 (1967); B5, 601 (1968); Nuovo Cimento 46A, 1 (1966).

J. Rosen, Brown University preprints NYO-2262 TA-151, NYO-2262 TA-161 (1967);

H.A. Kastrup, Phys. Rev. 142, 1060 (1966); 143, 1041 (1066); 150, 1189 (1966); Nucl. Phys. 58, 561 (1964) and Ref. 2,

plus a list of publications prior to 1962 reviewed in Ref. 1 and 2.

Without further discussion of the usefulness and consistency of possible different interpretations, we adopt for the purpose of the present paper the following point of view, following essentially Kastrup:

1. Exact dilatation symmetry would imply that to every physical system in any given state <u>sub specie aeternitatis</u> and for arbitrary $\rho > 0$ another one exists which is realizable in nature and differs from the first one only in that every physical observable is changed by a factor ρ^{ℓ}; where ℓ is the dimension of length of the observable in question.

2. Special conformal transformations may be interpreted as space-time dependent dilatations.

3. We consider only infinitesimal transformations. Then no problem arises with causality, in particular the equal time C.R. of fields will be invariant. (Sec. III.)

5) Unitary irreducible representations of SU(2, 2) ($\approx$ O(2, 4))have been constructed by:
Y. Murai, Progr. Theoret. Phys. (Kyoto) 9, 147 (1953);
A. Esteve and P.G. Sona, Nuovo Cimento 32, 473 (1964);
I.M. Gel'fand and M.I. Graev, Izv. Akad. Nauk SSSR, Ser. Mat. 29, 1329 (1965) and "Proceedings of the international spring school for theoretical physics", Yalta (1966);
A. Kihlberg, V.F. Müller and F. Halbwachs, Commun. Math. Phys. 3, 194 (1966);
I.T. Todorov, ICTP, Trieste, preprint IC/66/71;
R. Raçzka, N. Limić and J. Niederle, J.Math. Phys. 7, 1861, 2026 (1966); 8, 1079 (1967);
T. Yao, J. Math. Phys. 8, 1931 (1967).

The problem of reduction with respect to the Poincaré subalgebra has been considered for a few special cases in:
R.L. Ingraham, Nuovo Cimento 12, 825 (1954);
L. Gross, J. Math. Phys. 5, 687 (1964);
L. Castell (1967) loc. cit.[4];
B. Kurşunoglu, J. Math. Phys. 8, 1694 (1967).

6) J.E. Wess, Nuovo Cimento 18, 1086 (1960).

7) G. Mack, Nucl. Phys. B5, 499 (1968).

8) G. Mack, Ph.D. thesis, Berne, Switzerland (1967).

9) G.W. Mackey, Bull. Am. Math. Soc. 69, 628 (1963);
R. Hermann, "Lie group for physicists", ed. W.A. Benjamin, New York (1966),ch. 9.

10) The finiteness of this sum follows also from a general theorem of O'Raifeartaigh: Let $\underset{\sim}{G}$ be any finite-dimensional Lie algebra containing the Poincaré algebra $P = SO(3,1) \circledS T$. Then

there exists a finite number n_0 such that

$$[T, \underbrace{[T, [\ldots[T,X]\ldots]]]}_{n \text{ times}} = 0 \quad \text{for } n \geq n_0 \text{ and any } X \in G .$$

In the present case $n_0 = 3$.

L. O'Raifeartaigh, Phys. Rev. Letters 14, 575 (1965).

11) N. Jacobson, "Lie algebras", Interscience Publishers, New York (1962), p. 45, corollary 2.
Recall that nilpotency of a matrix $\varkappa$ means that there exists a positive integer m such that $\varkappa^m = 0$.

12) W.A. Hepner, Nuovo Cimento 26, 352 (1962).

13) Minimal couplings are those involving no derivatives of fermion and up to first-order derivatives of boson fields; they have the virtue of not affecting the canonical equal time C.R. of the fields.

14) The conformal invariance of the free Maxwell equations and the free massless Klein-Gordon, Dirac and Weyl equation has been known for a long time, cf. Refs. 1 to 3.
A conformal invariant spinor equation with a non-linear interaction $(\bar{\psi}\psi)^{2/3}$ has been considered by Gürsey[1].
Free wave equations in Minkowski space for massless particles with arbitrary spin have been considered by:
J.A. McLennan, Nuovo Cimento 3, 1360 (1956); 5, 640 (1957);
L. Gross, loc. cit.[5]
Gross also presents a proof that the solutions of these equations - and in particular the Maxwell equations - span a Hilbert space on which a unitary representation of the conformal algebra of space-time is realized.

15) C.N. Yang and R.L. Mills, Phys. Rev. 96, 191 (1954).

16) N. Cabibbo, Phys. Rev. Letters 10, 531 (1963).

17) Abdus Salam and J.C. Ward, Phys. Letters 13, 168 (1964).

18) M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

19) S. Weinberg, Phys. Rev. 134, B882 (1964); 138, B988 (1965); see also J.E. Wess, loc. cit.[6)]

20) E.g., for baryons this is seen as follows: If the generators are hermitian, then the transformation law for ψ implies that for ψ^+. Then

$$\{\psi^{+\prime}(x), \psi^{\prime}(y)\}_{+,\,eq.t.} = \rho^{2\ell}\{\psi^+(\rho^{-1}x), \psi(\rho^{-1}y)\}_+ = \rho^{2\ell}\,\delta(\rho^{-1}\underset{\sim}{x} - \rho^{-1}\underset{\sim}{y})$$
$$= \rho^{2\ell+3}\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) .$$

This is of the form (III.5) only if $\ell = -\frac{3}{2}$. If a different value for ℓ is chosen, there exist no hermitian generators that induce (III.1). The value $\ell = -\frac{3}{2}$ also follows from the requirement that the free massless Dirac equation be conformal invariant. Similarly the free massless Klein-Gordon equation for a spinless field is only conformal invariant if $\ell = -1$.

21) For the free field case with exact symmetry, similar currents have been written down by :
J. McLennan, loc. cit.[14)]; and
J.E. Wess, loc. cit.[6)]

22) D.M. Greenberger, Ann. Phys. (N.Y.) 25, 290 (1963).

23) R. Utiyama, Phys. Rev. 101, 1597 (1956). Our matrices T_a differ from his by a factor of i .

24) P.A.M. Dirac, Ann. Math. 37, 429 (1936).

25) H.A. Kastrup, Phys. Rev. 150, 1186 (1966).

26) A subtle point here is that the points where at least one coordinate x_μ is infinite may also correspond to finite coordinates on the cone, see for example Ref. 27. This is, however, not important here as we consider only infinitesimal transformations.

27) L. Castell (1967) loc. cit.[4)]

28) S. Helgason, "Differential geometry and symmetric spaces", Academic Press, New York (1962), p. 135;
N. Jacobson, loc. cit.[11)], p. 36.

29) To avoid misunderstanding it is important to notice that the parity transformation must satisfy (II. 3) for our interpretation of O(2, 4). It is <u>not</u> an inner automorphism and cannot be represented by γ_0 in the representations $\Delta^{(+)}$ and $\Delta^{(-)}$ since this would violate the C. R. (II. 3).

30) Y. Murai, Nucl. Phys. 6, 489 (1958).

31) R. L. Ingraham, loc. cit. [5)]

32) Hepner's treatment has a number of errors which have been corrected, e. g., the interaction he designates as conformal invariant $(\overline{\psi}_1 \gamma_\mu^{(+)} \psi_2) \cdot (\overline{\psi}_2 \gamma^{\mu(+)} \psi_3)$ is in fact not so. Hepner also does not consider the significant rôle of n in (IV. 4) and (V. 2).

33) M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

34) Note that the σ-"meson" plays two different rôles here. Firstly it is a manifestation of the breaking of dilatation symmetry, eq. (A. 2a). Secondly, it provides an attractive πN force which is necessary to cancel some of the big s-wave repulsion inherent in the non-derivative πN coupling, which is not observed experimentally. Recall that requirement of validity of eq. (A. 2b) does not allow for a derivative πN coupling. (Sec. III.)

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