CONTINUOUS-PARAMETER SUM RULES

M. O. TAHA

1968

MIRAMARE - TRIESTE
CONTINUOUS-PARAMETER SUM RULES *

M.O. Taha **

TRIESTE
June 1968

* To be submitted to "Physical Review".

** On leave of absence from the University of Khartoum, Sudan.
ABSTRACT

Using Laplace transform and Regge high-energy behaviour, continuous-parameter sum rules are found for strong interaction amplitudes. The finite-energy version of these sum rules is discussed and equations for the Regge parameters obtained. Forward \( \pi N \) scattering is considered as an example and numerical values are found for the parameters of \( \rho \) and \( \rho' \) trajectories. A mathematical appendix on the incomplete gamma function is added.
INTRODUCTION:

In this paper strong interaction continuous-parameter sum rules are introduced through the application of the asymptotic theorem on Laplace transforms and the assumption of Regge pole high-energy behaviour for the scattering amplitude. The resulting sum rules, (see eq. (2.3)) exhibit the contribution of the Regge terms even though the energy integral is not cut off. The continuous parameter that appears is the Laplace transform parameter, here denoted by \( \eta \). When \( \eta \) vanishes the sum rules lead to the usual superconvergence results.

The paper mainly deals with the finite-energy\(^{1-8}\) version of these sum rules, since these appear to be more directly useful. In this case the integral over the Regge poles introduces into the analysis the incomplete gamma-function \( \gamma(a,x) \) and the related analytic function \( \gamma^*(a,x) \) whose properties are given in the appendix. For the validity of the application of the asymptotic Laplace theorem the cut-off energy \( R \) must be taken large and the parameter \( \eta \) small, of order \( 1/R \).

The sum rules are introduced and discussed in Section 2. A result on some circumstances that may indicate the insufficiency of a single Regge pole is also given in this section.

In Section 3 the equations determining \( \chi(t) \), from these sum rules, are discussed. Both single and double pole dominance are considered. These equations involve a function \( F(a,x) \), simply related to \( \gamma^*(a,x) \), whose properties are also discussed in the appendix.

As an example, we consider forward \( \pi N \) charge-exchange scattering in Section 4. The finite-energy continuous-parameter sum rule for this case is numerically evaluated using a cut-off energy \( R = 30 \) pion masses (or 4.2 GeV/c) and varying \( \eta \) between 0 and .11. A fit with two Regge poles, \( \rho \) and \( \rho' \), for \( \chi_p(0) = .54 \) gives \( \chi_p(0) = -1 \pm .5 \), which is lower than the value normally assumed\(^{10}\). For the Regge residues \( \beta \) we get \( \beta_p(0) = 5.5^{+1.7}_{-1.9} \times 10^{-2} \mu^{-1} \) and \( \beta_{\rho'}(0) = -1.4^{+1.8}_{-2.3} \mu^{-1} \). The
value for $\beta_p(0)$ is in agreement with that experimentally determined from data above 6 GeV/c$^2$. $\beta_p(0)$ is again larger than is normally assumed$^{15}$, which is expected from the value found for $\alpha_p(0)$. We also set a limit on the ratio of the contributions $q_p$ and $q_p$ of the two Regge poles to the original sum rule with $\eta = 0$ : $0.6 < |q_p|^2 / |q_p|^2 < 4$. We finally observe that these figures hold up to $\alpha_p(0) = 0.60$.

It is important to emphasize that the main purpose of our work is to introduce this class of continuous-parameter sum rule and to propose the method of Laplace transformation as a tool for deriving sum rules from high-energy behaviour. It is also worth stating that this type of sum rule, like the usual Fubini-Furlan or finite-energy sum rules, requires knowledge of only the absorptive part of the amplitude, which gives a clear advantage over, for example, Olsson's$^5$ type of sum rule.

2. THE SUM RULES

We start by writing the contribution of a single Regge pole to the imaginary part $a(s,t)$ of a strong interaction amplitude in the form

$$a(s,t) \propto \frac{\beta(t) s^{\alpha(t)}}{\Gamma(x+1)} \quad \text{for } s \text{ large} \quad . \quad (2.1)$$

To (2.1) we apply the theorem on the asymptotic behaviour of a function and its Laplace transform. This theorem$^{11}$ applies for conditions under which the Laplace transforms in question exist and may be stated in the form: the behaviour of the Laplace transform $g(s)$ of the asymptotic form $G(t)$ of a function $F(t)$ for large $t$, in the neighbourhood of an extreme singularity $s_0$ determines the behaviour of $f(s) \equiv L\{F(t)\}$ near $s_0$. Thus

$$F(t) \sim G(t) \quad \text{for large } t \Rightarrow f(s) \sim g(s) \quad \text{for } s - s_0 \text{ small} \quad . \quad (2.2)$$

With $t$ fixed below its threshold value, we assume $a(s,t)$ to be - apart from $\delta$-functions at the bound state positions - a continuous function of $s$ of exponential order zero, so that its Laplace transform

$$\int_{0}^{\infty} e^{-s s} a(s,t) ds$$
exists for all \( \eta > 0 \). The theorem (2.2) may then be applied, resulting in the sum rule:

\[
\int_{0}^{\infty} e^{-\eta s} a(s,t) \, ds \approx \frac{\beta}{\eta^{\alpha+1}} \quad \text{for } \eta \text{ small, } \eta > 0 . \tag{2.3}
\]

This is a continuous-parameter infinite-energy sum rule, from which the usual superconvergence relation may be obtained by letting \( \eta \to 0 \):

\[
\int_{0}^{\infty} a(s,t) \, ds = \begin{cases} 0 & \alpha + 1 < 0 \\ \beta & \alpha + 1 = 0 \\ \infty & \alpha + 1 > 0 \end{cases} . \tag{2.4}
\]

It has, however, two rather obvious advantages over the relations (2.4):

(i) It smoothes the behaviour on the right-hand side as \( \alpha(t) \) passes through \(-1\), so that the discrete jumps from \( \alpha \) to \( \beta \) to \( \infty \) are avoided.

(ii) The dependence on the continuous parameter \( \eta \) enables one to make use of the same experimental data several times, thus providing an accurate determination of the Regge parameters within the experimental error.

The question of how small \( \eta \) should be for (2.3) to hold, is of course linked with the extent to which the single Regge term in (2.1) dominates the high-energy behaviour. If we include a second pole, the right-hand side of (2.3) becomes

\[
\frac{\beta}{\eta^{\alpha+1}} + \frac{\beta'}{\eta^{\alpha'+1}} .
\]

It is thus clear that we may keep the leading term only as long as we take \( \eta \) small enough so that

\[
\left| \frac{\beta'}{\beta} \right| \eta^{\alpha - \alpha'} \ll 1 . \tag{2.5}
\]

Thus a larger separation of the leading trajectory or a bigger ratio of its residue encourages one to take a larger \( \eta \) \((<1)\).

We now proceed to the finite-energy sum rule version of (2.3). This is obtained by rewriting (2.3) in the form

\[
\int_{0}^{\infty} e^{-\eta s} \left\{ a(s,t) - \frac{\beta}{\Gamma(\alpha+1)} s^{\alpha} \right\} \, ds \approx 0 \quad \text{for small } \eta > 0 ,
\]

and further approximating.
\[ a(s) \approx \frac{\beta}{\Gamma(\alpha+\eta)} s^\alpha \quad \text{for } S > R, \quad (2.6) \]

so that we have
\[ \int_0^R e^{-\eta s} a(s,t) ds \approx \frac{\beta}{\Gamma(\alpha+\eta)} \int_0^R e^{-\eta s} s^\alpha ds. \]

The integral on the right-hand side is related, by a change of variable, to the incomplete gamma function \( \gamma(\alpha,x) \), and the result may be written in terms of the entire function \( \gamma^*(\alpha,x) \) in the form
\[ \int_0^R e^{-\eta s} a(s,t) ds \approx \beta R^{\alpha+1} \gamma^*(\alpha+1, \eta R) \quad R, \frac{1}{\eta} \text{ large}. \quad (2.7) \]

This finite-energy continuous-parameter sum rule is our main relation. Some of the properties of the function \( \gamma^*(\alpha,x) \) are described in the appendix. It is a standard tabulated function, single-valued and analytic for all values of \( \alpha \) and \( x \) and is real for their real values.

If in (2.7) we let \( \eta \to 0 \), we get
\[ \int_0^R a(s,t) ds \approx \beta R^{\alpha+1} \gamma^*(\alpha+1,0) = \frac{\beta R^{\alpha+1}}{(\alpha+1) \Gamma(\alpha+1)} \quad (2.8) \]

which is the usual finite-energy sum rule. The continuous-parameter sum rule (2.7) is thus seen to be a generalization of (2.8). Sum rules involving a moment \( n \) may be obtained either by starting with the function \( s^n a(s,t) \) or by differentiating (2.7) with respect to \( \eta \).

For the validity of the sum rule (2.7), one of course needs \( R \) large enough so that (2.6) holds. Similar to (2.5) this may be stated in the form
\[ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha'+1)} \left| \frac{\beta'}{\beta} \right| \frac{1}{R^{\alpha' - \alpha}} << 1. \quad (2.9) \]

We shall therefore take \( \eta \sim O(\frac{1}{R}) \). In the numerical example discussed later the values for \( \eta \) are in fact taken in the interval \( (0, \frac{3}{R}) \).

It is clear that (2.7) immediately generalizes to the case of \( n \) Regge poles.
Since $\eta$ is a continuous parameter, it is in principle possible to determine all $2n$ Regge parameters on the right-hand side of (2.10) by simply taking $2n$ values for $\eta$ in the region $(0, \frac{1}{R})$ say. For this, however, one needs a spacing of order $\frac{R}{2n-1}$ in the values of $\eta$ chosen, which for large $n$ may leave the integrands in (2.10) experimentally indistinguishable for neighbouring values of $\eta$. Thus, more refined measurement of the same data, up to the same energy $R$, will enable one to determine more Regge parameters on using (2.10).

One final remark on (2.7) may be of interest. It is an observation that may help one see whether more than one Regge pole is needed to dominate the amplitude at the value of $R$ (and $t$) used. This stems from the fact\footnote{14} that $\gamma^*(a,x)$ has no zeros for $a > 0$. From (2.7) one may therefore extract the following statement:

Any amplitude $A(s,t)$ for which the integral in (2.7) vanishes at some value $\eta \in (0, \eta_0)$, $\eta_0 = \frac{1}{R}$, is not sufficiently dominated at that value of $R$ (and $t$) by the single Regge exchange for the sum rule (2.7) to hold, except when $\gamma(t) < -1$. This indicator is made use of in the numerical example of Section 4.

3. EVALUATION OF $\alpha$:

For the case of a single Regge pole, one needs two values of $\eta$ to determine both $\alpha$ and $\beta$ at a fixed momentum transfer $t$. These we shall take as $\eta = 0$ and another value $\eta$. On dividing the sum rule for arbitrary $\eta$ by that for $\eta = 0$ (with the same value of $R$) one gets

$$F(\alpha+1, R) = \frac{I(\eta, R)}{I(0, R)}$$

where

$$F(\alpha+1, R) = (\alpha+1) \Gamma(\alpha+1) \gamma^\ast(\alpha+1, \eta R)$$
The value of $\alpha$ may thus be directly calculated from eq. (3.1), on using the experimental value of the right-hand side and a graphical representation of $F(a, x)$ - for fixed $x$ - easily constructed as in (A.15) of the appendix. If a single Regge pole sufficiently dominates, the value of $\alpha$ thus calculated should not be sensitive to variations in $\eta$ in the neighbourhood of the chosen value. This may again serve to indicate the inadequacy of a single Regge pole.

When two Regge poles are considered, the four sum rules needed (which we again take with the same value $R$) may be written in the form:

$$I(\eta_i, R) = g F(\alpha + 1, \eta_i R) + g' F(\alpha' + 1, \eta_i R)$$

where

$$g = \frac{\beta R^{\alpha + 1}}{(\alpha + 1) \pi (\alpha + 1)}$$

with a similar expression for $g'$. Taking $\eta_0 = 0$, the first equation is

$$I(0, R) = g + g'$$

since $F(a, 0) = 1$. Eliminating $g$, the sum rules (3.4) give:

$$G_r (\alpha, \alpha') = g' \quad r = 1, 2, 3$$

where

$$G_r = \frac{I(\eta_r, R) - I(0, R) F(\alpha + 1, \eta_r R)}{F(\alpha' + 1, \eta_r R) - F(\alpha + 1, \eta_r R)}$$

The equations (3.7) can be graphically solved for $\alpha, \alpha'$ and $g'$ by plotting $G_r$ in the $($\alpha, \alpha'$)-plane for different contour values $g'$. One must, of course, always make sure that the experimental error allows the values of $I(\eta_r, R), r = 0, 1, 2, 3,$ to be clearly distinct.
An alternative computational procedure consists in constructing the functions

\[ H_{ij}(\alpha, \alpha') = \frac{F(\alpha' + l, \eta_i R) - F(\alpha + l, \eta_i R)}{F(\alpha' + l, \eta_j R) - F(\alpha + l, \eta_j R)} \]  
\[ K_{ij}(\alpha) = \frac{I(\eta_i R) - I(0, R) F(\alpha + l, \eta_i R)}{I(\eta_j R) - I(0, R) F(\alpha + l, \eta_j R)} \]  

so that the equations for determining \( \alpha \) and \( \alpha' \) become

\[ H_{12}(\alpha, \alpha') = K_{12}(\alpha) \]  
\[ H_{13}(\alpha, \alpha') = K_{13}(\alpha) \]  

From graphs for \( H_{ij}(\alpha, \alpha') \) at fixed \( \alpha \), one can solve (3.11) for \( \alpha' \), using a given value for \( \alpha \). These values of \( \alpha \) and \( \alpha' \) are then inserted in (3.12) to see how well it is satisfied. A series of successive approximations to \( \alpha \) then converges to a solution.

Two observations must now be made:

(i) A system of equations of the type (3.7) or (3.11), (3.12), will in general have many solutions. This ambiguity may, however, be overcome by first finding an approximate value for the dominant \( \alpha \) from (3.1) using a single pole form. A solution is then sought with one of the \( \alpha's, \alpha \) say, varying in the neighbourhood of this value. Such a solution may still be multivalued in \( \alpha' \). Suppose it is \((\alpha, \alpha', \alpha), r = 1, 2, ... \) The desired solution is then \((\alpha_0, \alpha_0', \alpha_r)\) where \( \alpha_r \) is the largest \( \alpha_r \) such that \( \alpha_0 > \alpha_r \). This is the required solution since it gives a high-energy behaviour consistent with the data and power-dominant over the neglected solutions.

(ii) When choosing the possible \( \eta_i \) values over which \( \eta \) varies in (3.8) or (3.10), it is necessary to make sure that the smallest non-zero value, \( \eta_i \), say, is such that the corresponding sum rule is not dominated by the first single pole \( \alpha \). For, in such a case, \( g_1 \) (or \( K_{ij} \)) vanishes. Thus \( g' \) vanishes and there can be no restriction on the second trajectory \( \alpha' \).
This may at first seem contradictory since the value of $g'$ is independent of $\eta$. In fact $G_1$ does not vanish identically but assumes a value, $\pm x$ say, consistent with both zero and the range of $g'$ determined otherwise.

Finally, it is clear that these equations, which represent some discrete points chosen from a continuous range of $\eta$, may be replaced by a fit to $I(\eta, R)$ plotted as a function of $\eta$. The equations (3.7) or (3.11), (3.12) can then be used to indicate the range of variation of the parameters consistent with possible fits. In the example discussed in the next section, the function $I(\eta, R)$ is plotted as a continuous curve against $\eta$ and a fit, for a fixed $\alpha$, is sought by varying the other parameters and using eq. (3.4), when an indication of the range of variation of these parameters has already been obtained from eqs. (3.11), (3.12).

4. EXAMPLE: FORWARD $\pi N$ SCATTERING:

Consider $\pi p$ helicity non-flip forward scattering amplitude $A(\omega, 0)$, where $\omega$ is the lab. energy, whose imaginary part is given by

$$a(\omega) = -2\pi f^2 \delta (\omega - \omega_B) + \frac{1}{4\pi} (\omega^2 - m^2)^2 \left\{ \sigma_{T} - \sigma_{T+N}(\omega) \right\}$$

(4.1)

where $\omega_B = \frac{m^2}{2\pi}$, $f^2 = \frac{C^2}{4\pi}\left(\frac{m}{2\pi}\right)^2 = .061 \pm .002$,

and $m, M$ the pion and nucleon masses, respectively.

Taking units in which the pion mass is unity, the general sum rule (2.10) gives

$$-\frac{1}{8\pi^2} \int_0^\infty d\omega \int_0^R \left(\omega^2 - 1\right)^2 \{\sigma_{T} - \sigma_{T+N}(\omega) \} d\omega = \frac{1}{2\pi^2} \sum_i \beta_i R^i G^i (\alpha_i, \eta, R)$$

(4.2)

A numerical evaluation of the left-hand side $I(\eta, R)$ of this sum rule was carried out using the data given in ref. 1, with $R = 30$ (i.e. $4.2$ GeV/c) and $\eta$ varying between 0 and .11. The calculated function $I(\eta, 30)$ was found to possess a zero at the end point $\eta = .11$ and dominance by a single pole, at this value of $R$ throughout the chosen
range of $\eta$, was therefore ruled out. Assuming dominance by two Regge poles, $\rho$ and $\rho'$, (4.2) takes the form

$$I(\eta, 30) = \varepsilon_\rho F(\alpha_\rho(0) + 1, 30 \eta) + \varepsilon_{\rho'} F(\alpha_{\rho'}(0) + 1, 30 \eta)$$

(4.3)

where

$$\frac{\varepsilon_i}{\beta_i} = \frac{1}{2\pi} \frac{\beta_i(0) \Gamma(\alpha_i(0)+1)}{\Gamma(\alpha_i(0)+2)}$$

(4.4)

A direct fit to (4.3), with $\alpha_{\rho'}(0)$ fixed at .54, resulted in the following values:

$$\alpha_{\rho}(0) = .54 \quad \varepsilon_{\rho} = 1.2^{+1.4}_{-1.2} \quad \alpha_{\rho'}(0) = -1 \pm .5 \quad \varepsilon_{\rho'} = -.23^{+1.7}_{-4}$$

(4.5)

with $\frac{\varepsilon_{\rho'}}{\varepsilon_{\rho}} = -2^{+1.4}_{-0.2}$

Some remarks are relevant:

(i) The value obtained for $\alpha_{\rho}(0)$ is below that normally assumed\(^{10}\) ($\leq -\frac{1}{2}$). This is, in turn, reflected in a stronger coupling for the $\rho'$. In fact (4.5) gives

$$\beta_{\rho'}(0) = 5.5^{+1.7}_{-0.9} \times 10^{-2} \mu^{-1} \quad \beta_{\rho'}(0) = -1.4^{+1.8}_{-2.3} \mu^{-1}$$

(4.6)

This value for $\beta_{\rho'}(0)$ agrees with the experimental value, 5.98 ± .47 x 10^{-2}, obtained from data above 6 GeV/c. Note that $\beta_{\rho}(0)$ is not consistent with zero.

(ii) $\varepsilon_{\rho}$ and $\varepsilon_{\rho'}$ are the respective contributions to the original sum rule with $\eta = 0$:

$$-\frac{g^2}{8\pi^2} \int_{-1}^{30} (\omega^2 - 1) \left\{ \sigma_{\eta^-}(\omega) - \sigma_{\eta^+}(\omega) \right\} d\omega = g_{\rho} + g_{\rho'}$$

(4.7)

The fit obtained above then shows that the absolute ratio of these contributions satisfies:

$$0.06 \leq \left| \frac{g_{\rho'}}{g_{\rho}} \right| \leq 0.4$$

(4.8)
As $\eta$ increases the integral $I(\eta, R)$ in (4.2) gets increasing contributions from the $\rho'$ term. At $\eta = .11$ the two contributions balance each other and the integral vanishes.

(iii) The same values for $\varphi_p(O)$, $\varepsilon_p$, and $\sigma_p$ as in (4.5) are obtained if one fixes $\varphi_p(O)$ at 0.60. Departures from these values become significant when $\varphi_p(O)$ is taken above this.

ACKNOWLEDGEMENTS

I would like to thank Professor B.J. Squires for helpful discussions and for reading the manuscript. I am indebted to the Institute for Advanced Study, Princeton, for a fellowship held during the early stages of this work and to the International Centre for Theoretical Physics for an associateship that enabled me to complete it. Thanks are also due to the IAEA, Professor Abdus Salam and Professor P. Budini for hospitality at the Centre.
In this appendix we list some of the properties of the function \( \gamma^*(a,x) \), related to the incomplete gamma function \( Y(a,x) \) by

\[
\gamma^*(a,x) = \frac{x^{-a}}{\Gamma(a)} Y(a,x)
\]  

(A.1)

where \( Y(a,x) \) is defined by

\[
Y(a,x) = \int_0^x e^{-t} t^{a-1} dt
\]  

(A.2)

so that \( Y(a,\infty) = \Gamma(a) \).

\( \gamma^*(a,x) \) is an analytic function of \( a \) and \( x \), with no finite singularities. It has the series development

\[
\gamma^*(a,x) = \frac{1}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(-x)^n}{(a+n)n!}
\]  

(A.3)

converging for all \( x \), and is related to Kummer's confluent hypergeometric function \( M(a,b,x) \) by

\[
\gamma^*(a,x) = \frac{e^{-x}}{\Gamma(a+1)} M(1,1+a,x)
\]  

(A.4)

Use may be made of (A.4), to translate standard properties of \( M \) into corresponding ones for \( \gamma^*(a,x) \). In particular one has the following integral representation:

\[
\gamma^*(a,x) = \frac{\Gamma(-a)-x}{\pi} \mathcal{R} \left\{ \frac{e^{-x} \pi}{\cos (a\theta+x\sin \theta)} \cos \theta \right\}
\]  

(A.5)

The differential and recurrence equations satisfied by \( \gamma^*(a,x) \) are

\[
x \frac{\partial^2 \gamma^*}{\partial x^2} + (a+1+x) \frac{\partial \gamma^*}{\partial x} + a \gamma^* = 0
\]  

(A.6)
and
\[ \gamma^*(a-1, x) = x \gamma^*(a, x) + \frac{e^{-x}}{\Gamma(a)} \]  \hspace{1cm} (A.7)

As special values we have
\[ \gamma^*(-n, x) = x^n, \]  \hspace{1cm} (A.8)

and
\[ \gamma^*(\frac{1}{2}, x^2) = x^{-1} \operatorname{erf}(x). \]  \hspace{1cm} (A.9)

Numerical values for \( \gamma^*(a, x) \) may be found from Pearson's "Tables of the Incomplete Gamma Function", which give the function \( I(u, p) \), related to \( \gamma^*(a, x) \) by
\[ \gamma^*(a, x) = x^{-a} I(a \frac{1}{2} x, a-1) \] \hspace{1cm} (A.10)

Alternatively, one may use tables for \( \gamma(a, b, x) \), together with (A.4).

The function \( F(a, x) \) is defined in the text by
\[ F(a, x) = a \Gamma(a) \gamma^*(a, x). \]  \hspace{1cm} (A.11)

It is a single-valued analytic function of both \( a \) and \( x \), except for the points \( a = -n - 1, n = 0, 1, 2, \ldots \), where it has simple poles with residue
\[ \frac{(-1)^n}{n+1} x^{n+1} \] \hspace{1cm} (A.12)

From (A.7), \( F(a, x) \) is seen to satisfy the recurrence relation
\[ F(a, x) = \frac{a}{x} \left\{ F(a-1, x) - e^{-x} \right\}. \] \hspace{1cm} (A.13)

This recurrence relation may be used with the special values
\[ F(0, x) = 1 \]
\[ F(\frac{1}{2}, x) = \frac{1}{2} \sqrt{\frac{\pi}{x}} \operatorname{erf}(\sqrt{x}) \] \hspace{1cm} (A.14)
to provide an easy and practical computation scheme for the numerical values of the function:

\[
\begin{align*}
P(0, x) &= 1 \\
P(\frac{1}{2}, x) &= -\sqrt{\frac{\pi}{x}} \text{erf}(\sqrt{x}) \\
P(1, x) &= \frac{1}{x} \left[ 1 - e^{-x} \right] \\
P(\frac{3}{2}, x) &= \frac{3}{2x} \left[ P\left(\frac{1}{2}, x\right) - e^{-x} \right]
\end{align*}
\] (A.15)

For \( a < 0 \), a first approximation to the \( r \)th zero of \( F(a, x) \) is

\[
\chi_{0}^{(r)} \approx -\frac{\pi^2 (r + \frac{1}{2} a - \frac{1}{4})^2}{\lambda (a - 1)} \left( 1 + O\left( \frac{1}{2|a|} \right) \right), \quad r = 1, 2. \tag{A.16}
\]

For a fixed negative value of \( a \), \( F(a, x) \) has in fact only one or two zeros as a function of \( x \), and it has no zeros for positive values of \( a \).
REFERENCES


5) M.G. OLSSON, Phys. Letters, 26B, 310 (1968)


8) A. DELLA SELVA, L. MASPERI and R. ODORICO, Nuovo Cimento (to be published)

9) Many standard mathematical texts treat these functions. See, for example:
   ABRAMOWITZ and STEGUN, "Handbook of Mathematical Functions," Dover (1965)

   For detailed systematic study see:
   F.G. TRICOMI, Ann. Mat. Pura Appl. IV, 28 (1951)
   F.G. TRICOMI, J. d'Analyse Math. I, 209 (1951)

10) See, for example, OLSSON ref.5) whose assumed value $\alpha_p(0) = -0.5$ is just about consistent with our determination; and R.K. LOGAN, J. BEAUPRé and L. SERTORIO, Phys. Rev. Letters 18, 259 (1967), whose value $\alpha_p(0) = 0.17$ is also inconsistent with the sum rule of ref.1). This latter is in fact the case $\eta = 0$ of the sum rule (4.2).
11) See, for example,

12) This observation is also made on finite-energy sum rules in ref.3).


15) V.S. BARASHENKOV and V.M. MAL'TSEV, Fortschr. Phys., Sonderband 2, 549 (1961);
    M.J. LONGO and B.J. MOYER, UCRL-10174 (1962);
    A. CITRON, W. GALBRAITH, T.P. KYCIA, B.A. LEONTIC, R.H. PHILLIPS,

16) For example,
   L.J. SLATER, "Confluent Hypergeometric Functions", Cambridge (1960)