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ON THE ZERO ENERGY BEHAVIOUR OF REGGE POLES AND RESIDUES

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AND

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TRIESTE

A D D E N D U M

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On page 43, line 1, after the full stop, please insert the following:

The distinction drawn here between the E-E configuration on the one hand and the E-U and U-U configurations on the other needs examination. Although the leading term in the t-channel amplitudes is given by s^α in the E-E case, it happens that the s-channel amplitudes are dominated by $s^{\alpha-|j_0-|\lambda||}$. This results from some cancellations brought about by the crossing matrix which, in the E-E configuration at t = 0, is given by a product of rotation matrices $d^{S}_{\lambda\mu}(\pi/2)$. Since the residues are proportional to $d^{\alpha}_{S\mu j}(i\pi/2)$ at t = 0 one can use the property*)

$$\sum_\mu d^S_{\lambda\mu}(\pi/2)\, d^{kl}_{S\mu j}(i\pi/2) = 0$$

for

$$k+l-|k-l-|\lambda|| < j \le k+l \quad ,$$

to show that the s-channel amplitudes are dominated at t = 0 (E-E case) by the daughter $k = |j_0-|\lambda||$. Hence the leading term as $s\to\infty$ for s-channel amplitudes is given by

$$\frac{1+\tau e^{i\pi(\alpha-\lambda)}}{2\sin\pi(\alpha-j_0)}\,\beta_{(\lambda)}\, s^{\alpha-|j_0-|\lambda||} \quad , \qquad \text{E-E, E-U, and U-U .} \qquad (7.5)'$$

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*) This property is deduced by comparing the large ζ behaviour of the functions

$$d^{kl}_{s\lambda j}(\zeta) \sim (ch\,\zeta)^{k+l-|k-l-|\lambda||} \quad \text{and} \quad d^{J}_{\mu\nu}(i\zeta) \sim (ch\,\zeta)^{J} ,$$

which are related by the formula

$$d^{kl}_{s\lambda j}(\zeta) = \sum_{J\mu\nu} d^S_{\lambda\mu}(\pi/2)\, d^{kl}_{S\mu J}(i\pi/2)\, d^J_{\mu\nu}(i\zeta)\, d^{kl}_{J\nu j}(i\pi/2)\, d^j_{\nu\lambda}(-\pi/2) .$$

Clearly the largest value taken by J in the summation must be given by $J = k+l-|k-l-|\lambda||$.

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ABSTRACT

The behaviour of partial wave amplitudes in the neighbourhood of vanishing momentum is studied in the Born approximation. A set of phenomenological fields is introduced to assist in the construction of a kinematically correct Born term. It is found that the Born contribution to the partial wave amplitudes has a structure which, as a result of the kinematical properties of the phenomenological propagator and vertex parts, yields considerable information about the behaviour of Regge poles and residues at zero energy. Regge poles arrived at in this way group themselves into Toller families whose zero energy intercepts, slopes, etc., are severely constrained. Asymptotic formulae for forward scattering amplitudes are obtained.

ON THE ZERO ENERGY BEHAVIOUR OF REGGE POLES AND RESIDUES

1. INTRODUCTION

These notes are concerned with the structure of partial wave amplitudes in the neighbourhood of vanishing momentum transfer. If the process described involves particles with unequal masses then the amplitudes have a kinematic singularity at this point. The source of the singularity is not difficult to uncover. It arises because the cross-channel scattering angle $\theta_t(s)$ vanishes[1] identically at $t = 0$ which means that any functional relation of the type $A(s,t) = B(t, \theta_t)$ must be irregular there. In particular, the cross-channel partial wave expansion

$$A(s,t) = \sum_J (2J+1)\, f_J(t)\, d^J_{\lambda\mu}(\theta_t) \tag{1,1}$$

must fail in the limit $t \to 0$. It is clear that the coefficients $f_J(t)$ must diverge in some way to compensate the vanishing of θ_t at $t = 0$. However, the precise nature of the divergence is not at all clear.

Another way of describing this phenomenon is to remark that the partial wave expansion (1.1) can be appropriate only in kinematical circumstances which allow the classification of states into representations of O(3) - or its non-compact relative, O(2,1) - that is, into states with well-defined angular momentum. It is a well-known fact that the group O(3) is inappropriate for the classification of states with lightlike momentum. For such states it is impossible to define an intrinsic angular momentum and, consequently, their transition amplitudes cannot be labelled with J - hence the divergence of $f_J(t)$ at $t = 0$.

It is also well known that the correct group[2] for classifying states with lightlike momentum is the Euclidean group in two dimensions, E(2). The irreducible unitary representations of E(2) are labelled by a continuous positive parameter ρ^2 which must take the place of J. It can be shown that the correct analogue of the partial wave expansion (1.1) which must be used at $t = 0$ is given by

$$A(s,0) = \frac{1}{2}\int_0^\infty d\rho^2 \; g(\rho^2)\; J_{\lambda-\mu}(\rho\xi) \qquad (1.2)$$

where $J_{\lambda-\mu}$ denotes a Bessel function of the first kind and ξ is a positive quantity defined by

$$\xi^2 = \lim_{t\to 0}\left(\frac{\theta_t^2}{t}\right) \qquad (1.3)$$

which limit is a linear function of s. It is therefore clear that the expansion (1.1) must in some way go over into (1.2) as the limit $t\to 0$ is approached.

It is possible to gain some insight into the connection between the expansions (1.1) and (1.2) by means of the following qualitative argument. In view of the approximate relation $d^j_{\lambda-\mu}(\theta) \approx J_{\lambda-\mu}(\sqrt{j(j+1)}\,\theta)$ for $\theta \ll 1$, it appears that the combination $j(j+1)t$ manifests itself as ρ^2 in the limits $t\to 0$ and $j\to\infty$, that is,

$$d^j_{\lambda\mu}(\theta_t) \approx J_{\lambda-\mu}(\sqrt{j(j+1)t}\;\xi) \qquad (1.4)$$

near $t = 0$. By the sort of logic familiar from impact parameter methods one can see that the sum in (1.1) approximates more and more closely to the integral in (1.2) as $t\to 0$ if $g(\rho^2)$ is defined by the limit,

$$\lim_{J\to\infty}\left[\frac{J(J+1)}{\rho^2}\; f_J\left(\frac{\rho^2}{J(J+1)}\right)\right] = g(\rho^2) \quad . \qquad (1.5)$$

This condition can be otherwise expressed by

$$f_J(t) \sim t\;\; g(J(J+1)t) \qquad (1.6)$$

for $t\to 0$ and $J\to\infty$.

Unfortunately, the condition (1.5) is not sufficiently powerful to provide information about the singularity at $t = 0$ with J finite. Presumably this is because such behaviour is strongly dependent on dynamical effects. A more sensitive tool is needed for distinguishing these features. Such a tool can be found, we believe, by recourse to field theoretic arguments.

It is well known that the contributions to the partial wave amplitudes made by a field theoretic Born approximation must, by virtue of their origin, satisfy all the kinematic requirements demanded of a relativistic theory. In other words, field theory provides a useful guide to kinematical correctitude. It would be too optimistic, of course, to expect anything very far-reaching in the way of dynamical results to come out of field theoretic model calculations. However, a compromise in the nature of a phenomenological field theory might be usefully exploited in order to discover no more than a spectrum of kinematically allowable parametrizations of, for example, Regge poles and residues.

For this reason we shall adopt an ansatz based on analogies with field theory. We shall express the pole contributions to helicity amplitudes in the form of a field theoretic Born approximation, employing for this purpose a set of phenomenological fields, ϕ_A , and their corresponding currents, f_A . We shall assume, moreover, that these fields transform in a well-defined way under the operations of the complex Lorentz group as well as T, C, P and that they are local fields complying with the usual spin-statistics assumption.

With these assumptions the Born contribution to the process $1 + 2 \rightarrow 3 + 4$ can be expressed in the explicitly covariant form

$$\langle p_3\lambda_3, p_4\lambda_4 | T | p_1\lambda_1, p_2\lambda_2 \rangle_{Born} =$$

$$= \sum_{A,B} \langle p_3\lambda_3, p_4\lambda_4 | \bar{f}_A | 0 \rangle \; \Delta_{AB}(P) \langle 0 | f_B | p_1\lambda_1, p_2\lambda_2 \rangle \; , \qquad (1.7)$$

where

$$P = p_1 + p_2 = p_3 + p_4 \; .$$

The current f_A and its adjoint $\bar{f}_A$ belong to some representation of the (complex) Lorentz group, that is,

$$f_A \rightarrow U(\Lambda)\, f_A\, U(\Lambda^{-1}) = \sum_B D_{AB}(\Lambda^{-1})\, f_B \; , \qquad (1.8)$$

$$\bar{f}_A \rightarrow U(\Lambda)\, \bar{f}_A\, U(\Lambda^{-1}) = \sum_B \bar{f}_B\, D_{BA}(\Lambda) \; . \qquad (1.9)$$

Correspondingly, the propagator $\Delta_{AB}(P)$ must satisfy the covariance condition

$$\Delta_{AB}(P) = \sum_{A'B'} D_{AA'}(\Lambda^{-1})\, \Delta_{A'B'}(\Lambda P)\, D_{B'B}(\Lambda) \tag{1.10}$$

for any complex Λ. The content of the representation $D(\Lambda)$ need not be specified in detail for the present. This content, a direct sum of finite-dimensional representations, will be clarified when an explicit labelling is introduced in Secs. 2 and 3 where it is needed in order to remove from (1.7) the many redundant components appearing in the propagator.

The real aim of this work is to discover something of the behaviour of Regge poles and their residues in the neighbourhood of vanishing momentum transfer. We shall suppose that the poles of the S-matrix - including Regge poles - are contained in the Born term. The virtue of this ansatz lies in its clear separation of the dynamical singularities (poles), which are to occur in the propagator, from the kinematical ones which are confined to the vertex parts. In particular, the constraints which must be satisfied by $\Delta_{AB}(P)$ at $P^2 = 0$ can be formulated without reference to the external particles. Likewise, the kinematical singularities and constraints implicit in helicity amplitudes are properties of the vertices which can be considered without reference to the nature of the exchanged particles. The factorizability of residues is of course presupposed in the form (1.7). It is a stronger factorizability than that assumed in mass-shell S-matrix calculations. The S-matrix pole approximations are usually given in terms of the Poincaré invariants, mass and spin, and are not necessarily consistent except in the neighbourhood of the pole. The field theoretic Born approximation has, in addition to its dependence on mass and spin, a deeper structure expressed through representations of the homogeneous group. It is this structure which assures the consistency of the Born approximation even at zero momentum where the Poincaré classification fails.

It must be emphasised that we have no intention of employing a field theoretic model to calculate the functions appearing in (1.7).

Field theory is being invoked only in its most phenomenological sense in order to suggest the form (1.7) for the Born contribution. By exploiting the well-defined transformation properties of fields it is possible to isolate the kinematically independent components of the propagator and vertices and to separate the singularity at $P^2 = 0$ from the latter. A knowledge of these singularities is sufficient to fix uniquely the constraints which must be satisfied by the propagator at $P^2 = 0$ in order to make the over-all Born contribution (1.7) regular there. These constraints, it will be found, amount to nothing more than the classification of Regge trajectories into TOLLER families appropriate relations among their positions, slopes, etc., in the neighbourhood of $P^2 = 0$. Moreover, it is possible to do similar things with the vertex functions - particularly in the equal mass case - achieving thereby the most general kinematically allowable parametrization of Regge poles and residues.

Implicit in this continuation of a field theoretical Born term to complex values of J is the notion of an "analytic field", $\phi_{(k\,\ell)}$, which interpolates an infinite set of the familiar finite-dimensional fields with $k-\ell = j_0$, a fixed integer or half-integer,and $k+\ell = j_0, j_0+1$, ... That is, $k+\ell$ is complexified along with J. Such a "field" belongs to an infinite-dimensional (usually) non-hermitian representation of the Lorentz algebra. One should expect one-particle singularities of this field to comprise a Regge trajectory. Questions as to which of those properties - locality, TCP, etc. - usually taken as characteristic of fields can carry over to this object are not considered here.

The arrangement of the paper is as follows. Sec. 2 contains a summary of well-known properties of the finite-dimensional representations of the Lorentz group and the fields belonging to them. Sec. 3 considers the 2-point functions for these fields and defines the concept of the reduced propagator, $\Delta(W, j, \eta)$, a matrix consisting of scalar amplitudes (kinematically independent apart from constraints at $W = 0$) which incorporates the dynamical content of the usual propagator. The components of the reduced propagator are then expanded in a set of unconstrained amplitudes. Sec. 4 is devoted to the decomposition of the vertex parts into a set of scalar form factors from which the kinematical singularities at $W = 0$ and the constraints at the pseudo-threshold are then separated. In Sec. 5 the results of the preceding two sections are joined to give expressions for the partial wave projections of the Born term. It is then shown that, provided certain constraints are satisfied, the Born term is regular at zero momentum transfer in spite of the singularity of its partial wave projections. These projections, which are bilinear forms, are diagonalized to isolate particular pole contributions some properties of which are examined in the neighbourhood of vanishing momentum transfer. In Sec. 6, the continuation to complex angular momentum is made resulting in a number of properties of Regge poles and their residues which are enumerated. Sec. 7 includes some asymptotic formulae.

2. SOME PROPERTIES OF FIELDS

In order to fix the notation it is necessary to discuss the transformation properties of fields, that is, to make clear the meaning of the subscript A which serves to distinguish the components ϕ_A . This material is well known (see for example STREATER and WIGHTMAN [5]) but a restatement of it in concise form is in order here. Firstly, the finite-dimensional matrices which represent the homogeneous Lorentz transformations including reflections are defined in formulae (2.2), (2.8), (2.9) and (2.10). The behaviour of fields ϕ_A and their adjoints $\bar{\phi}_A$ (defined in (2.15)) are given by (2.13) and (2.14). The antiparticle conjugation operator C is defined by (2.17) and (2.18). Finally, some properties of the Clebsch-Gordan coefficients appropriate to the finite-dimensional representations of the Lorentz group are listed; (2.21) and (2.22). A fuller discussion of these coefficients is contained in Appendix I.

The finite-dimensional irreducible representations of the homogeneous proper Lorentz group are characterized by an ordered pair of integer or half-integer parameters (k, ℓ). The basis vectors of the irreducible representation $D^{k\ell}$ can be labelled by a parameter, j, taking the values $|k-\ell|$, $|k-\ell|+1$, ..., $k+\ell$ and by another parameter, m, taking for each j the values $-j, -j+1, \ldots, j$. This labelling is complete. It is possible in this basis to represent any proper Lorentz transformation, Λ, by the matrix[6)]

$$D^{k\ell}_{jm,\,j'm'}(\Lambda) = \sum_M D^j_{mM}(R)\; d^{k\ell}_{jMj'}(\alpha)\; D^{j'}_{Mm'}(R') \tag{2.1}$$

where R and R' denote ordinary three-dimensional rotations represented in the usual way by matrices $D^j(R)$ and $D^{j'}(R')$ of dimensionality $2j+1$ and $2j'+1$ respectively. The matrix $d^{k\ell}_{jMj'}(\alpha)$ represents a pure Lorentz transformation in the 03-plane through the hyperbolic angle α. It is defined by

$$d^{k\ell}_{jMj'}(\alpha) = \sum_{\kappa\lambda} \langle jM|k\kappa,\ell\lambda\rangle\; e^{-(\kappa-\lambda)\alpha}\; \langle k\kappa,\ell\lambda|j'M\rangle \tag{2.2}$$

where $\langle k\kappa,\ell\lambda|jM\rangle$ denotes a Clebsch-Gordan coefficient of the three-dimensional rotation group.

The improper transformations cannot be represented in the form (2.1) except in the subclass of representations with $k=\ell$. In general it is necessary to adjoin the representations $D^{k\ell}$ and $D^{\ell k}$ in order to incorporate the space-reflections. In terms of a set of basis vectors $|k\ell jm\rangle$ it is possible to represent the space-reflection operator, P, by

$$P|k\ell jm\rangle = |\ell kjm\rangle\, e^{i\pi j}, \quad k \neq \ell \tag{2.3}$$

or by

$$P|ffjm\rangle = \pm|ffjm\rangle\, e^{i\pi j}, \quad k = \ell = f \; . \tag{2.4}$$

The space-reflection operator can be diagonalized when $k \neq \ell$ by defining the new set of basis vectors

$$|(k\ell)\eta jm\rangle = \begin{cases} \frac{1}{\sqrt{2}}\,|k\ell jm\rangle + \frac{\eta}{\sqrt{2}}\,|\ell kjm\rangle\ , & k>\ell \\ \frac{\eta}{\sqrt{2}}\,|k\ell jm\rangle + \frac{1}{\sqrt{2}}\,|\ell kjm\rangle\ , & k<\ell \end{cases} \tag{2.5}$$

where $\eta = \pm 1$. On the states (2.5) we have

$$P\,|(k\ell)\,\eta jm\rangle = |(k\ell)\,\eta jm\rangle\ \eta\, e^{i\pi j} \quad . \tag{2.6}$$

The formula (2.4) can be expressed similarly,

$$P\,|(ff\eta)\,jm\rangle = |(ff\eta)jm\rangle\ \eta\, e^{i\pi j} \quad , \tag{2.7}$$

with the distinction that the parity type η need take only one value, $+1$ or -1.

The proper transformations can be represented in the basis (2.5) by a generalization of (2.1),

$$D^{(k\ell)}_{\eta jm,\eta' j'm'}(\Lambda) = \sum_M D^j_{mM}(R)\ d^{(k\ell)}_{\eta jMj'\eta'}(\chi)\ D^{j'}_{Mm'}(R') \tag{2.8}$$

where

$$\left.\begin{aligned} d^{(k\ell)}_{\eta jMj'\eta'}(\alpha) &= \frac{1}{2}\left[d^{k\ell}_{jMj'}(\alpha) + \eta\eta'\ d^{\ell k}_{jMj'}(\alpha)\right] \ , \quad k>\ell \\ d^{(ff)}_{\eta jMj'\eta'}(\alpha) &= \delta_{\eta\eta'}\ d^{ff}_{jMj'}(\alpha) \qquad , \quad k=\ell=f \ . \end{aligned}\right\} \tag{2.9}$$

The improper transformations can be represented by the product of (2.8) with the matrix representing space-reflection,

$$D^{(k\ell)}_{\eta jm,\eta' j'm'}(P) = \delta_{\eta\eta'}\ \delta_{jj'}\ \delta_{mm'}\ \eta\, e^{i\pi j} \quad . \tag{2.10}$$

The formulae (2.8) and (2.10) together with (2.2) and (2.9) complete the specification of the irreducible representations $D^{(k\ell)}$ and $D^{(ff\eta)}$. It may be mentioned that the matrices so defined can serve also to represent the complex Lorentz transformations.

Two properties of the matrices $D^{(k\ell)}$ which are important in the following are

a) reality:

$$D^{(k\ell)}_{\eta jm,\eta' j' m'}(\Lambda)^* = \eta(-)^{k+\ell+m}\, D^{(k\ell)}_{\eta j-m,\eta' j'-m'}(\Lambda^*)\,(-)^{k+\ell+m'}\,\eta' \tag{2.11}$$

b) pseudo-orthogonality:

$$D^{(k\ell)}_{\eta jm,\eta' j' m'}(\Lambda^{-1}) = (-)^{j-m}\, D^{(k\ell)}_{\eta' j'-m',\eta j-m}(\Lambda)\,(-)^{j'-m'} \quad . \tag{2.12}$$

They can be deduced from the definitions (2.2), (2.8), (2.9) and (2.10).

It will prove useful to employ two alternative notations to distinguish the representations $D^{(k\ell)}$ in the following discussion. We shall use $D^{j_0\sigma}$ where $j_0 = k-\ell$, $\sigma = k+\ell+1$ and, more simply, D^{α}.

The summary of properties of the finite-dimensional irreducible representations of the Lorentz group given above is sufficient for the purposes of this note. Henceforth the field ϕ_A is to be written $\phi_{r\alpha\eta jm}$ where the indices α, η, j, m specify the spacetime transformation properties and r denotes any additional (Lorentz invariant) labels which may be necessary. Thus, under a homogeneous Lorentz transformation Λ,

$$\begin{aligned}\phi_{r\alpha\eta jm}(x) \rightarrow\ & U(\Lambda)\,\phi_{r\alpha\eta jm}(x)\,U(\Lambda^{-1}) = \\ & = \sum_{\eta' j' m'} D^{\alpha}_{\eta jm,\eta' j' m'}(\Lambda^{-1})\,\phi_{r\alpha\eta' j' m'}(\Lambda x)\ .\end{aligned} \tag{2.13}$$

Similarly, the adjoint field $\bar{\phi}_{r\alpha\eta jm}$ transforms according to

$$\begin{aligned}\bar{\phi}_{r\alpha\eta jm}(x) \rightarrow\ & U(\Lambda)\,\bar{\phi}_{r\alpha\eta jm}(x)\,U(\Lambda^{-1}) = \\ & = \sum_{\eta' j' m'} \bar{\phi}_{r\alpha\eta' j' m'}(\Lambda x)\, D^{\alpha}_{\eta' j' m',\eta jm}(\Lambda)\end{aligned} \tag{2.14}$$

In view of the properties (2.11) and (2.12) of the transformation matrices $D^{\alpha}(\Lambda)$ it is possible to make an association between ϕ and $\bar{\phi}$, namely

$$\bar{\phi}_{r\alpha\eta jm}(x) = \eta\ (-)^{k+\ell-j}\ \phi_{r\alpha\eta jm}(x)^{\dagger} \tag{2.15}$$

where $\phi^{\dagger}$ denotes the hermitian conjugate of ϕ . It should perhaps be emphasised that we are working with non-unitary representations of the complex Lorentz group. The transformation laws (2.13) and (2.14) are consistent with (2.15) only if the operators $U(\Lambda)$ satisfy the "pseudo-unitarity" condition

$$U(\Lambda^{-1}) = U(\Lambda^{*})^{\dagger} \tag{2.16}$$

which, it is clear, implies unitarity only for real Lorentz transformations.

The antiparticle conjugation operator, C , can be defined in the usual way: it is a linear, unitary and Lorentz-invariant operator which connects ϕ and $\bar{\phi}$. It is defined by the relations

$$C\ \phi_{r\alpha\eta jm}(x)\ C^{-1} = \begin{cases} (-)^{j+m}\ \bar{\phi}_{r\alpha\eta j-m}(x)\ , & 2(k+\ell)\ \text{even} \\ (-)^{j+m}\ \bar{\phi}_{r\alpha-\eta j-m}(x)\ , & 2(k+\ell)\ \text{odd} \end{cases} \tag{2.17}$$

$$C\ \bar{\phi}_{r\alpha\eta jm}(x)\ C^{-1} = \begin{cases} (-)^{j-m}\ \phi_{r\alpha\eta j-m}(x)\ , & 2(k+\ell)\ \text{even} \\ (-)^{j-m}\ \phi_{r\alpha-\eta j-m}(x)\ , & 2(k+\ell)\ \text{odd.} \end{cases} \tag{2.18}$$

The distinction between fermion and boson fields implied by oddness or evenness of $2(k+\ell)$ is necessary here because, according to the definition (2.10), the former have imaginary parity and the latter real parity.

Some use will be made in the next section of the Clebsch-Gordan coefficients for the finite-dimensional representations of the Lorentz group[7]. These coefficients, which couple the direct product $D^{(k_1\ell_1)} \otimes D^{(k_2\ell_2)}$ to the irreducible representation $D^{(k\ell)}$ are denoted $\langle (k\ell)\eta jm \,|\, (k_1\ell_1)\eta_1 j_1 m_1, (k_2\ell_2)\eta_2 j_2 m_2 \rangle$. They must satisfy the invariance condition

$$\sum_{\eta' j' m'} D^{(k\ell)}_{\eta j m, \eta' j' m'}(\Lambda) \langle (k\ell)\eta' j' m' | (k_1\ell_1)\eta_1 j_1 m_1, (k_2\ell_2)\eta_2 j_2 m_2 \rangle =$$

$$= \sum_{\eta_1' j_1' m_1'} \sum_{\eta_2' j_2' m_2'} \langle (k\ell)\eta j m | (k_1\ell_1)\eta_1' j_1' m_1', (k_2\ell_2)\eta_2' j_2' m_2' \rangle \cdot$$

$$\cdot D^{(k_1\ell_1)}_{\eta_1' j_1' m_1', \eta_1 j_1 m_1}(\Lambda) D^{(k_2\ell_2)}_{\eta_2' j_2' m_2', \eta_2 j_2 m_2} \qquad (2.19)$$

for any Lorentz transformation Λ. They can be expressed in terms of a set of reduced coefficients by

$$\langle (k\ell)\eta j m | (k_1\ell_1)\eta_1 j_1 m_1, (k_2\ell_2)\eta_2 j_2 m_2 \rangle =$$

$$= \langle (k\ell)\eta j \| (k_1\ell_1)\eta_1 j_1, (k_2\ell_2)\eta_2 j_2 \rangle \langle jm | j_1 m_1, j_2 m_2 \rangle$$

(2.20)

where $\langle jm | j_1 m_1, j_2 m_2 \rangle$ denotes an ordinary SO(3) Clebsch-Gordan coefficient. The reduced coefficients are developed in some detail in Appendix I where they are given as linear combinations of 9-j symbols. If any one of the variables $k, \ell, j, \ldots$ should vanish it becomes possible to express them in terms of 6-j symbols.

Two symmetry properties of the reduced coefficients which will be needed in the next section are given by

$$\langle (k\ell)\eta j \| (k_1\ell_1)\eta_1 j_1, (k_2\ell_2)\eta_2 j_2 \rangle =$$

$$= \langle (k\ell)\eta j \| (k_2\ell_2)\eta_2 j_2, (k_1\ell_1)\eta_1 j_1 \rangle (-)^{k_1+\ell_1+j_1+k_2+\ell_2+j_2+k+\ell+j}$$

(2.21)

$$\langle (k\ell)\eta j \| (k_1\ell_1)\eta_1 j_1, (k_2\ell_2)\eta_2 j_2 \rangle =$$

$$= \langle (k\ell)\eta j, (k_1\ell_1) \pm \eta_1 j_1 \| (k_2\ell_2)\eta_2 j_2 \rangle \left[\frac{(2k+1)(2\ell+1)(2j_2+1)}{(2k_2+1)(2\ell_2+1)(2j+1)} \right]^{\frac{1}{2}} (-)^{2k_1}$$

(2.22)

where, if $D^{(k_1 \ell_1)}$ is a boson (fermion) representation, one must use $\eta_1 (-\eta_1)$.

3. PROPAGATORS

The propagator matrix $\Delta_{AB}(P)$ introduced in Sec. 1 has many redundant components. It is possible by means of the covariance condition (1.10) to eliminate the redundancy and to express $\Delta_{AB}(P)$ in terms of a set of scalar amplitudes, the reduced propagator matrix $\Delta(W, J, \eta)$ defined below in (3.3). Such expressions for $\Delta_{AB}(P)$ are given in this section by (3.6) and (3.13). The components of the reduced propagator, which retain the essential dynamical content of the full propagator, are themselves subject to constraints. Two conditions resulting from TCP-invariance and C-invariance, respectively, are given by (3.9) and (3.11). Finally, and most important, there is a set of constraints to be satisfied by the components of the reduced propagator in the neighbourhood of $W = 0$. It will be demonstrated in Sec. 5 that these constraints are essential for maintaining the regularity of the Born term at $W = 0$. An expansion of the reduced propagator into a set of unconstrained amplitudes $G(W^2, N)$ which facilitates the treatment of Sec. 5 is given by (3.16).

Expressed in the notation of Sec. 2 the covariance condition (1.10) reads

$$\Delta_{\tau\alpha\eta j m, \tau'\alpha'\eta' j' m}(P) = \sum_{\bar{\eta}\bar{j}\bar{m}, \bar{\eta}'\bar{j}'\bar{m}'} D^{\alpha}_{\eta j m, \bar{\eta}\bar{j}\bar{m}}(\Lambda^{-1})\, \Delta_{\tau\alpha\bar{\eta}\bar{j}\bar{m}, \tau'\alpha'\bar{\eta}'\bar{j}'\bar{m}'}(\Lambda P)\, D^{\alpha'}_{\bar{\eta}'\bar{j}'\bar{m}', \eta' j' m'} \tag{3.1}$$

for any complex Λ. Most of the kinematical redundancy in $\Delta(P)$ can be eliminated quite simply by exploiting the subgroup which leaves P_μ invariant. More particularly, when P_μ takes the standard form

$$\hat{P}_\mu = (W, 0, 0, 0) \tag{3.2}$$

this invariance subgroup corresponds to the ordinary three-dimensional rotations and reflections. One concludes immediately that $\Delta(\hat{P})$ takes the form

$$\Delta_{r\alpha\eta jm,\, r'\alpha'\eta' j' m'}(\hat{P}) = \delta_{\eta\eta'}\, \delta_{jj'}\, \delta_{mm'}\, \Delta_{r\alpha,\, r'\alpha'}(W, j, \eta) \quad . \tag{3.3}$$

The Lorentz-invariant amplitudes $\Delta(W, j, \eta)$ constitute the reduced propagator matrix. This matrix characterizes the propagation of particles of spin j and parity $\eta\, e^{i\pi j}$.

The formula (3.3) can be transformed to an arbitrary frame yielding thereby an expression for the general $\Delta(P)$ in terms of the reduced propagator. To this end it is useful to define the boost transformations L_P which serve to transform $\hat{P}$ into P , i.e.,

$$P_\mu = (L_P)_{\mu\nu}\, \hat{P}_\nu = (L_P)_{\mu 0}\, W. \tag{3.4}$$

Some arbitrariness of convention enters into the definition of the boosts in that there are many different matrices, L_P , L'_P , ..., all of which satisfy (3.4). However, one can show that any two boosts, L_P and L'_P, can always be connected by a three-dimensional rotation, R ,

$$L'_P = L_P R \text{ where } R\,\hat{P} = \hat{P}. \tag{3.5}$$

If now the substitution $\Lambda = L_P^{-1}$ is made in (3.1) it follows from (3.3) that $\Delta(P)$ takes the form

$$\Delta_{r\alpha\eta jm,\, r'\alpha'\eta' j' m'}(P) =$$

$$= \sum_{\bar{\eta}\bar{j}\bar{m}} D^{\alpha}_{\eta jm,\, \bar{\eta}\bar{j}\bar{m}}(L_P)\, \Delta_{r\alpha,\, r'\alpha'}(W, \bar{j}, \bar{\eta})\, D^{\alpha'}_{\bar{\eta}\bar{j}\bar{m},\, \eta' j' m'}(L_P^{-1}) \tag{3.6}$$

and, moreover, using (3.5) one can show that this form is invariant under the substitution $L_P \rightarrow L'_P$. It is independent of any particular boosting conventions. It appears, therefore, that the kinematically independent components of the matrix $\Delta(P)$ are contained within the reduced matrix $\Delta(W, j, \eta)$ the symmetries of which remain to be discovered.

The symmetry of $\Delta(W, j, \eta)$ which results from TCP-invariance can be arrived at by applying the complex Lorentz transformation, I_{st}, which reflects all the components of P_μ. It is represented by the matrices

$$D^{(k\ell)}_{\eta jm, \eta' j' m'}(I_{st}) = (-)^{2k} \delta_{mm'} \delta_{jj'} \delta_{\eta, \pm\eta'} \tag{3.7}$$

where again it is necessary to distinguish fermionic representations for which $\eta = -\eta'$ from bosonic for which $\eta = \eta'$. Substitution of the matrices (3.7) into the general covariance condition (3.1) yields the symmetry

$$\Delta_{r\alpha\eta jm, r'\alpha'\eta' j' m'}(-P) = (-)^{2k} \Delta_{r\alpha\pm\eta jm, r'\alpha'\pm\eta' j' m'}(p) (-)^{2k'} . \tag{3.8}$$

This formula can be interpreted in the rest frame (3.2) as

$$\Delta_{r\alpha, r'\alpha'}(-W, j, \eta) = (-)^{2k} \Delta_{r\alpha, r'\alpha'}(W, j, \pm\eta) (-)^{2k'} \tag{3.9}$$

where $+\eta$ is taken for bosons and $-\eta$ for fermions.

The remaining symmetry to be exploited is C-invariance. The implications of C-invariance can be derived by expressing $\Delta(P)$ as the Fourier transform of the vacuum expectation value of a time-ordered product in the usual way. One arrives at the formula

$$\Delta_{r\alpha\eta jm, r'\alpha'\eta' j' m'}(-P) = \epsilon (-)^{j'-m'} \Delta_{r'\alpha'\pm\eta' j' -m', r\alpha\pm j -m}(P) (-)^{j+m} \tag{3.10}$$

where $\epsilon = +1$ for boson fields and $\epsilon = -1$ for fermion fields (the spin-statistics relation). In the rest frame this gives

$$\Delta_{r\alpha, r'\alpha'}(-W, j, \eta) = \Delta_{r'\alpha', r\alpha}(W, j, \pm\eta) \tag{3.11}$$

where, on the right-hand side, one must take $+\eta$ for bosons and $-\eta$ for fermions.

The reduction of the propagator matrix into kinematically independent components is now complete, being summarized in the formulae

(3.6), (3.9) and (3.11).

It will be shown in Sec. 5 that the reduced propagator $\Delta(W, j, \eta)$ is subject to a set of constraints at $W = 0$ if the Born term is to be finite there. Therefore, it will be advantageous to develop a set of unconstrained amplitudes in terms of which the components of the reduced propagator can be expanded. Such a set can be found in the following way.

Consider the matrices, Π, defined by

$$\Pi_{\alpha\eta j m, \alpha'\eta' j' m'}(P, \bar{j}, \bar{\eta}) = \sum_{\bar{m}} D^{\alpha}_{\eta j m, \bar{\eta}\bar{j}\bar{m}}(L_P)\, D^{\alpha'}_{\bar{\eta}\bar{j}\bar{m}, \eta' j' m'}(L_P^{-1}) \tag{3.12}$$

in terms of which one could express the propagator, i.e.,

$$\Delta_{\tau\alpha\eta j m, \tau'\alpha'\eta' j' m'}(P) = \sum_{\bar{j}\bar{\eta}} \Delta_{\tau\alpha, \tau'\alpha'}(W, \bar{j}, \bar{\eta})\, \Pi_{\alpha\eta j m, \alpha'\eta' j' m'}(P, \bar{j}, \bar{\eta}) \ . \tag{3.13}$$

From (3.5) it is clear that Π does not depend on the details of the boosting convention. In fact it can be demonstrated that the components of $\Pi(P, j, \eta)$ are polynomials in P_μ/W which means that (3.13) can be regarded as an expansion of $\Delta(P)$ into polynomials in P_μ with invariant coefficients. Now the polynomials in P_μ can be grouped into symmetrical traceless tensors of rank N (i.e., belonging to the representation $D^{(\frac{N}{2}\frac{N}{2}+)}$ of the Lorentz group). A convenient notation for these tensor polynomials is given by [8]

$$W^N D^{\frac{N}{2}\frac{N}{2}}_{JM,00}(L_P) \ . \tag{3.14}$$

The desired reformulation of the reduction formula consists in a reordering of the polynomial expansion (3.13) into the form

$$\Delta_{\tau\alpha\eta j m, \tau'\alpha'\eta' j' m'}(P) =$$
$$= \sum_{NJM} G_{\tau\alpha, \tau'\alpha'}(W^2, N)\, \langle \alpha\eta j m | \alpha'\eta' j' m', (\tfrac{N}{2}\tfrac{N}{2}+) JM \rangle\, W^N D^{\frac{N}{2}\frac{N}{2}}_{JM,00}(L_P) \tag{3.15}$$

where the Clebsch-Gordan coefficient is clearly necessary in order that $\Delta(P)$ have the correct transformation properties. The connection between the invariant functions $G(W^2, N)$ and the reduced propagator $\Delta(W, j, \eta)$ emerges when (3.15) is referred to the rest frame (3.2) where $L_P = 1$. Thus

$$\Delta_{r\alpha, r'\alpha'}(W, j, \eta) = \sum_N W^N G_{r\alpha, r'\alpha'}(W^2, N) \langle \alpha\eta j \| \alpha'\eta j, (\tfrac{N}{2}\, \tfrac{N}{2}+)\, 0 \rangle \tag{3.16}$$

and it follows from the properties of the reduced Clebsch-Gordan coefficients (2.21) and (2.22) that G is, apart from a multiplier, symmetric

$$G_{r\alpha, r'\alpha'}(W^2, N) = G_{r'\alpha', r\alpha}(W^2, N) \left[\frac{(2k'+1)(2\ell'+1)}{(2k+1)(2\ell+1)}\right]^{\frac{1}{2}} (-)^{k+\ell-k'-\ell'} . \tag{3.17}$$

It will be demonstrated in Sec. 5 that the regularity of the Born term at $W = 0$ is assured in all cases if and only if the functions $G(W^2, N)$ are regular at $W^2 = 0$. This of course means that the components of Δ are constrained in accordance with (3.16) in the neighbourhood of $W = 0$.

To summarize, one can separate from the components of the propagator a kinematically independent set, the reduced propagator defined by (3.3). The reduced propagator must satisfy the symmetry conditions (3.9) and (3.11) resulting from TCP-invariance and C-invariance, respectively. The components of the reduced propagator are further constrained at $W = 0$ by the requirement that the Born term be regular there. These constraints are made explicit in the expansion (3.16) by requiring that the coefficients $G(W^2, N)$ appearing there be finite at $W^2 = 0$.

4. VERTEX PARTS

The aim of this section is twofold: a) to express the matrix elements of the current operator f_A in terms of a set of invariant form factors $G(W, J, \eta)$ and, b) to analyse the kinematical singularities of these form factors at $W = 0$. As a preliminary to this it is necessary to discuss briefly the definition of two-particle states and, more particularly, their continuation to complex values of the momenta (since $W = 0$ is generally an unphysical point). In order to do this meaningfully, some mention must be made of the irreducible representations of the complex Poincaré group. The essential features are summarized in formulae (4.1) - (4.8). A more detailed treatment is included in Appendix II. The matrix elements of the current operator between the vacuum and two-particle centre-of-mass states are expressed by (4.9) in terms of the form factors $G(W, J, \eta)$ which are defined in (4.10). These are in turn expressed in terms of a set of functions $F(W, J, \eta)$ which are regular at $W = 0$, (4.13), (4.14). The restrictions on these functions resulting from P and TCP invariance are given in (4.17) and (4.18), respectively. Finally, a scheme for expanding the regularized form factors $F(W, J, \eta)$ in powers of W is presented in (4.22), (4.24).

Helicity states must be defined relative to some boosting convention. Formally the one-particle helicity state $|p\lambda>$ can be defined by

$$|p\lambda> = U(L_p)\,|\lambda> \tag{4.1}$$

where L_p denotes a suitable boost transformation of the type introduced in Sec. 3 and $|\lambda>$ denotes a rest state with spin S and J_z-component λ. The usual convention, which we adhere to, is to define L_p in terms of three polar angles α, θ and φ according to

$$U(L_p) = e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{i\varphi J_{12}}\, e^{-i\alpha J_{03}} \tag{4.2}$$

where the (hermitian) operators $J_{\mu\nu}$ are generators of infinitesimal Lorentz transformations and where the components of p_μ are given by

$$p_\mu = m(\mathrm{ch}\,\alpha, \mathrm{sh}\,\alpha \sin\theta\cos\varphi, \mathrm{sh}\,\alpha \sin\theta\sin\varphi, \mathrm{sh}\,\alpha\cos\theta) \quad . \tag{4.3}$$

The angles can be expressed uniquely in terms of the components, p_μ, if suitable limits are imposed $\alpha \geqslant 0$, $0 \leqslant \theta \leqslant \pi$ and $-\pi < \varphi \leqslant \pi$. The states $|\,p\lambda\rangle$ defined in this way span an irreducible unitary representation of the real Poincaré group (at least if $m > 0$).

If the angles are allowed to take complex values in the respective domains

$$\begin{aligned} 0 \leqslant \mathrm{Re}\,\alpha &< \infty \quad , \quad -\pi \leqslant \mathrm{Im}\,\alpha \leqslant \pi \\ 0 \leqslant \mathrm{Re}\,\theta &\leqslant \pi \quad , \quad -\infty < \mathrm{Im}\,\theta < \infty \\ -\pi \leqslant \mathrm{Re}\,\varphi &\leqslant \pi \quad , \quad -\infty < \mathrm{Im}\,\varphi < \infty \end{aligned} \tag{4.4}$$

then, with suitable conventions about boundary values, they can be given as single-valued functions of the complex 4-momentum with the result that the states (4.1) span an irreducible but non-unitary[9)] representation of the complex Poincaré group.

The two-particle states can be defined in the same fashion, i.e.,

$$|\,p_1\lambda_1, p_2\lambda_2\rangle = U^{(1)}(L_{p_1})\, U^{(2)}(L_{p_2})\,|\,\lambda_1, \lambda_2\rangle \tag{4.5}$$

where $U^{(1)}$ and $U^{(2)}$ operate independently in the spaces of particles (1) and (2) respectively. Their corresponding infinitesimal generators, $J^{(1)}_{\mu\nu}$ and $J^{(2)}_{\mu\nu}$, commute. For most purposes it is sufficient to deal with the subset of states (4.5) for which the total momentum, $p_1 + p_2$, takes the form

$$p_1 + p_2 = (W, 0, 0, 0) \tag{4.6}$$

For such states the six polar angles are constrained by four relations and can therefore be expressed in terms of two independent angles φ and θ

in addition to W. The resulting form for the two-particle centre-of-mass states is then

$$|p_1\lambda_1, p_2\lambda_2\rangle = e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\alpha_1(W) J_{03}^{(1)} + i\alpha_2(W) J_{03}^{(2)}} \cdot$$

$$\cdot\, e^{i\pi J_{31}^{(2)}}\, |\lambda_1 \lambda_2\rangle\, e^{i(\lambda_1+\lambda_2)\varphi} \tag{4.7}$$

where $J_{\mu\nu} = J_{\mu\nu}^{(1)} + J_{\mu\nu}^{(2)}$ and the angles $\alpha_1(W)$, $\alpha_2(W)$ are given by

$$ch\alpha_1 = \frac{W^2 + m_1^2 - m_2^2}{2m_1 W}, \qquad ch\alpha_2 = \frac{W^2 - m_1^2 + m_2^2}{2m_2 W} \tag{4.8}$$

subject to the conditions $Re\alpha > 0$, $-\pi \leqslant Im\alpha < \pi$. The functions $\alpha_1(W)$ and $\alpha_2(W)$ are analytic in the W-plane with cuts as detailed in Appendix II. It should be remarked that the form (4.7) is a valid representation of the two-particle centre-of-mass states only in the region $|W| \geqslant |m_1^2 - m_2^2|^{\frac{1}{2}}$. This is because, on the boundary $|W| = |m_1^2 - m_2^2|^{\frac{1}{2}}$, either $Re\alpha_1 = 0$ if $m_1 > m_2$ or $Re\alpha_2 = 0$, if $m_1 < m_2$, which means that, on crossing the boundary, the real part of one of the angles changes sign and therefore leaves the region of definition (4.4). What this in fact means is that the state $|p_1\lambda_1, p_2\lambda_2\rangle$, $|W| > |m_1^2 - m_2^2|^{\frac{1}{2}}$, is the analytic continuation of a "flipped" state, for example $|p_1 -\lambda_1, p_2\lambda_2\rangle (-)^{S_1-\lambda_1} e^{i2\lambda_1\varphi}$, $|W| < |m_1^2 - m_2^2|^{\frac{1}{2}}$. A detailed analysis of this phenomenon is contained in Appendix II.

The vertex parts are defined as matrix elements of the current f_A between the vacuum and two-particle states. It is a simple matter to separate from these matrix elements a set of invariant form factors $G(W, j, \eta)$. Thus, using the rotational properties of the current, one can write

$$\langle 0| f_{\tau\alpha\eta jm} |p_1\lambda_1, p_2\lambda_2\rangle =$$

$$= \langle 0| f_{\tau\alpha\eta jm}\, e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\alpha_1 J_{03}^{(1)} + \alpha_2 J_{03}^{(2)}}\, e^{i\pi J_{31}^{(2)}} |\lambda_1\lambda_2\rangle\, e^{i(\lambda_1+\lambda_2)\varphi}$$

$$= e^{-im\varphi}\, d^{j}_{m\,\lambda_1-\lambda_2}(\theta)\, e^{i(\lambda_1+\lambda_2)\varphi}\, (0| G_{\tau\alpha}(W, j, \eta) |\lambda_1 \lambda_2) \tag{4.9}$$

where the form factors are defined by

$$(0|G_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2) = \langle 0| f_{\tau\alpha\eta j\lambda_1-\lambda_2}\, e^{-i\alpha_1(W)J_{03}^{(1)} + i\alpha_2(W)J_{03}^{(2)}}\, e^{i\pi J_{31}^{(2)}}|\lambda_1,\lambda_2\rangle\,. \tag{4.10}$$

The functions $G(W, j, \eta)$ have singularities in the W-plane some of which have their origin in singularities of the functions $\alpha_1(W)$ and $\alpha_2(W)$. These are usually termed kinematical. All others are dynamical singularities. Moreover there are kinematical constraints at threshold and pseudothreshold where $sh\alpha_1 = sh\alpha_2 = 0$. In the neighbourhoods of those points it is possible to perform multipole expansions in (4.10) obtaining in a straightforward way the behaviour

$$\sum_{\lambda_1\lambda_2} (0|G_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2)\langle S_1\lambda_1|S_2\lambda_2, S\lambda\rangle \sim (sh\alpha)^{|j-S|} \tag{4.11}$$

for $S = |S_1-S_2|, \ldots, S_1 + S_2$ where S_1 and S_2 denote the intrinsic spins of particles (1) and (2).

If the masses are unequal, $m_1 \neq m_2$, then the form factors (4.10) have a kinematical singularity at $W = 0$ the removal of which is one of our principal aims. The separation can be effected by rearranging the exponents in (4.10). Thus, if one writes

$$e^{-i\alpha_1 J_{03}^{(1)} + i\alpha_2 J_{03}^{(2)}} = e^{-\frac{i}{2}(\alpha_1-\alpha_2)J_{03} - \frac{i}{2}(\alpha_1+\alpha_2)(J_{03}^{(1)} - J_{03}^{(2)})} \tag{4.12}$$

then it follows from the behaviour of the current under pure Lorentz transformations that

$$(0|G_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2) = \sum_{\bar{j}\bar{\eta}} d^{\alpha}_{\eta j\lambda_1-\lambda_2\bar{j}\bar{\eta}}\left(\frac{\alpha_1-\alpha_2}{2}\right)(0|F_{\tau\alpha}(W,\bar{j},\bar{\eta})|\lambda_1\lambda_2) \tag{4.13}$$

where the regularized form factors, $F(W, j, \eta)$, are defined by

$$(0|F_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2) = \langle 0| f_{\tau\alpha\eta j\lambda_1-\lambda_2}\, e^{-\frac{i}{2}(\alpha_1+\alpha_2)(J_{03}^{(1)}-J_{03}^{(2)})}\, e^{i\pi J_{31}^{(2)}}|\lambda_1,\lambda_2\rangle\,. \tag{4.14}$$

The functions d^{α} appearing in (4.13) are of course the representation matrices defined by (2.2), (2.9). The angle $\alpha_1 + \alpha_2$ is defined by

$$ch(\alpha_1 + \alpha_2) = \frac{W^2 - m_1^2 - m_2^2}{2m_1 m_2} \tag{4.15}$$

and clearly has no singularity at $W = 0$. Hence the regularized form factors $F(W, j, \eta)$ have no kinematical singularity there. On the other hand, the angle $\alpha_1 - \alpha_2$ is given by

$$ch(\alpha_1 - \alpha_2) = \frac{m_1^2 + m_2^2}{2m_1 m_2} - \frac{(m_1^2 - m_2^2)^2}{2m_1 m_2} \frac{1}{W^2} \tag{4.16}$$

and is therefore singular at $W = 0$ if $m_1 \neq m_2$. The kinematical singularity of $G(W, j, \eta)$ is therefore confined to the known functions d^{α}. For the special case, $m_1 = m_2$, where $\alpha_1 = \alpha_2$ there is no singularity at $W = 0$.

The regularized form factors are not all independent. Invariance under space-reflections yields the symmetry

$$(0|F_{\tau\alpha}(W, j, \eta)|\lambda_1\lambda_2) = \eta\eta_1\eta_2 (0|F_{\tau\alpha}(W, j, \eta)|-\lambda_1, -\lambda_2) \tag{4.17}$$

where the intrinsic parities of particles (1) and (2) are represented in the form $\eta_1 e^{i\pi S_1}$ and $\eta_2 e^{i\pi S_2}$ respectively.

Invariance under TCP yields a further symmetry. This can be derived most directly by remarking that $\alpha(-W) = \alpha(W) - i\pi$ for $\mathrm{Im}W > 0$. Substituting this into (4.14) and using the transformation law (2.13) for currents, one finds

$$(0|F_{\tau\alpha}(-W, j, \eta)|\lambda_1\lambda_2) = (-)^{2k} (0|F_{\tau\alpha}(W, j, \pm\eta)|\lambda_1\lambda_2) e^{i\pi(\lambda_1 - \lambda_2)} \tag{4.18}$$

where, as before, $+\eta$ applies to boson currents and $-\eta$ to fermion currents.

To conclude this section on the structure of vertex parts we consider the feasibility of multipole expansions of the form factors. The functions $G(W, j, \eta)$ are constrained by (4.11) in the neighbourhoods of the threshold and the pseudothreshold and can therefore be expanded in multipoles about these points. However, if $m_1 \neq m_2$ then $G(W, j, \eta)$ is singular at $W = 0$ which point may lie near to the pseudothreshold thereby curtailing the convergence of a multipole series. With the regularized form factors, $F(W, j, \eta)$, this problem does not arise.

Consider therefore the expansion of $F(W, j, \eta)$ about the point $W = |m_1 - m_2|$. It can be seen from the analysis in Appendix II that

$$\lim_{W \to |m_1 - m_2| \pm i0} (\alpha_1(W) + \alpha_2(W)) = \pm i\pi \tag{4.19}$$

so that, if the expression (4.14) for $F(W, j, \eta)$ can be recast into a form involving $\alpha_1 + \alpha_2 \mp i\pi$ then it can be expanded in powers of this angle. One can make the rearrangement

$$-\frac{i}{2}(\alpha_1+\alpha_2)(J_{03}^{(1)} - J_{03}^{(2)}) = \frac{\pi}{2} J_{03} - \frac{i}{2}(\alpha_1 + \alpha_2 - i\pi)(J_{03}^{(1)} - J_{03}^{(2)}) - \pi J_{03}^{(2)} \tag{4.20}$$

and therefore put (4.14) into the form

$$\begin{aligned}(0|F_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2) &= \langle 0| f_{\tau\alpha\eta j\lambda}\, e^{\frac{\pi}{2}J_{03}}\, e^{-\frac{i}{2}(\alpha_1+\alpha_2-i\pi)(J_{03}^{(1)}-J_{03}^{(2)})} e^{-\pi J_{03}^{(2)}} e^{i\pi J_{31}^{(2)}} |\lambda_1,\lambda_2\rangle = \\ &= \sum_{\bar{j}\bar{\eta}} d^{\alpha}_{\eta j \lambda \bar{j}\bar{\eta}}(i\pi/2)\, \langle 0| f_{\tau\alpha\bar{\eta}\bar{j}\lambda}\, e^{-\frac{i}{2}(\alpha_1+\alpha_2-i\pi)(J_{03}^{(1)}-J_{03}^{(2)})} \cdot \\ &\quad \cdot e^{-\pi J_{03}^{(2)}}\, e^{i\pi J_{12}^{(2)}} |\lambda_1,-\lambda_2\rangle\, e^{-i\pi S_2}\end{aligned} \tag{4.21}$$

The operator, $e^{-\pi J_{03}^{(2)}} e^{i\pi J_{12}^{(2)}}$, is a scalar for bosons or a pseudoscalar for fermions and can be otherwise ignored. It follows that, in the neighbourhood of $W = |m_1 - m_2| + i0$ the regularized form factors can be represented by

$$\begin{aligned}(0|F_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2) &= \sum_{s\lambda\bar{\eta}\bar{j}} d^{\alpha}_{\eta j\lambda\bar{j}\bar{\eta}}(i\pi/2) \left(\frac{(m_1-m_2)^2 - W^2}{4m_1m_2}\right)^{\frac{1}{2}|\bar{j}-s|} \cdot \\ &\quad \cdot (0|H_{\tau\alpha}(W,\bar{j},\bar{\eta})|s\lambda) \langle s\lambda|s_1\lambda_1, s_2 -\lambda_2\rangle\end{aligned} \tag{4.22}$$

where $H(W, j, \eta)$ is regular at the pseudothreshold. The pseudothreshold factor in (4.22) results from the definition of $\alpha_1 + \alpha_2 - i\pi$,

$$\operatorname{sh} \frac{\alpha_1 + \alpha_2 - i\pi}{2} = \sqrt{\frac{(m_1 - m_2)^2 - W^2}{4 m_1 m_2}} \quad . \tag{4.23}$$

Thus it is established that the regularized form factors can be expanded in multipoles about the pseudothreshold. Such an expansion could be useful for representing these functions in the neighbourhood of W = 0 provided the mass difference is not large, i.e., provided

$$(m_1 - m_2)^2 \quad << \quad 4 m_1 m_2 \quad . \tag{4.24}$$

To summarize, the principal formulae of this section are: (4.9) and (4.10) which give the vertex parts in terms of a set of invariant form factors $G(W, j, \eta)$; (4.13) and (4.14) which serve to isolate the kinematical singularity at $W = 0$ of $G(W, j, \eta)$ and define a set of regularized form factors $F(W, j, \eta)$; (4.17) and (4.18) which give the P and TCP symmetries of the regularized form factors ; and (4.22) which exhibits the pseudothreshold behaviour. A completely analogous set of formulae could be given for the matrix elements of the adjoint current $\langle p_1\lambda_1, p_2\lambda_2 | \bar{f}_A | 0 \rangle$ and their associated form factors $\bar{G}(W, j, \eta)$ and $\bar{F}(W, j, \eta)$.

5. THE BORN CONTRIBUTION

Having isolated the kinematical properties of the propagator and vertex parts, we are now in a position to express the Born contribution to the partial wave amplitudes in terms of the invariant components of the reduced propagator and form factors defined in Secs. 3 and 4. This is given by (5.5). Next it is demonstrated that the finiteness at $W = 0$ of the regularized form factors $F(W, j, \eta)$ and propagator components $G(W^2, N)$ is sufficient to assure the finiteness of the full Born term - the sum over partial waves. Following this the reduced propagator matrix is diagonalized (5.15) in order to isolate distinct pole contributions.

The components of the reduced propagator are constrained by the requirement that $G(W^2, N)$ be regular at $W = 0$. These constraints are exhibited in (5.21), (5.22). On the basis of these constraints it is possible to compute power series expansions (5.25) and (5.26) of the eigenvalues, $D_\alpha^{-1}(W, j, \eta)$, and eigenfunctions, $X_{\alpha\beta}(W, j, \eta)$, of the reduced propagator. The constraints imply that the coefficients, $\lim_{W\to 0}(\partial^N D_\alpha(W, j, \eta)/\partial W^N)$, are rational functions of j and are independent of η for $N < 2(k-\ell)$ if $k \neq \ell$ (in the notation $(k, \ell) = \alpha$). It has been verified for $N = 0, 1, 2$ that these coefficients are polynomials of order N in j, (5.31), and we conjecture this to be true for $N > 2$. The matrix, $X_{\alpha\alpha'}(W, j, \eta)$, which diagonalizes the reduced propagator is found to satisfy the constraint $\lim_{W\to 0}(\partial^N X_{\alpha\alpha'}(W, j, \eta)/\partial W^N) = 0$ for $N < |k + \ell - k' - \ell'| + |k - \ell - k' + \ell'|$, (5.32). This property allows one to predict the behaviour near $W = 0$ of the vertex functions, $\Gamma = XG$, which appear in the diagonalized form of the Born contribution, (5.33). The result is given by (5.39).

The centre-of-mass frame for the process $1 + 2 \to 3 + 4$ is defined in the usual way by specifying the momenta as follows:

$$
\begin{aligned}
p_1 &= m_1 \left(ch\alpha_1 ,\ 0 ,\ 0 ,\ sh\alpha_1 \right) \\
p_2 &= m_2 \left(ch\alpha_2 ,\ 0 ,\ 0 ,\ -sh\alpha_2 \right) \\
p_3 &= m_3 \left(ch\alpha_3 ,\ sh\alpha_3 \sin\theta ,\ 0 ,\ sh\alpha_3 \cos\theta \right) \\
p_4 &= m_4 \left(ch\alpha_4 ,\ -sh\alpha_4 \sin\theta ,\ 0 ,\ -sh\alpha_4 \cos\theta \right)
\end{aligned}
\qquad (5.1)
$$

where, if the angles, $\alpha_1, \ldots, \alpha_4$ are given by formulae of the type (4.8), then

$$p_1 + p_2 = p_3 + p_4 = (W,\ 0\ ,\ 0\ ,\ 0)\,. \qquad (5.2)$$

In the centre-of-mass frame the Born term (1.7) takes the form

$$\langle p_3\lambda_3, p_4\lambda_4|T|p_1\lambda_1, p_2\lambda_2\rangle_{Born} = \sum_{\substack{\tau\alpha\eta j m\\ \tau'\alpha'\eta' j' m'}} \langle p_3\lambda_3, p_4\lambda_4|\bar{f}_{\tau'\alpha'\eta' j' m'}|0\rangle \times$$

$$\times\ \Delta_{\tau'\alpha'\eta' j' m', \tau\alpha\eta j m}(P)\ \langle 0|f_{\tau\alpha\eta j m}|p_1\lambda_1, p_2\lambda_2\rangle =$$

$$= \sum_{\eta j} \sum_{\tau'\alpha'\tau\alpha} (\lambda_3\lambda_4|\bar{G}(W,j,\eta)|0)\ d^j_{\lambda'\lambda}(-\theta)\Delta_{\tau'\alpha',\tau\alpha}(W,j,\eta) \times$$

$$\times\ (0|G_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2) \tag{5.3}$$

where $\lambda = \lambda_1 - \lambda_2$ and $\lambda' = \lambda_3 - \lambda_4$. Use has been made of (3.3) and (4.9) in deriving (5.3). The expression (5.3) is to be compared with the standard partial wave expansion

$$\langle p_3\lambda_3, p_4\lambda_4|T|p_1\lambda_1, p_2\lambda_2\rangle = \sum_{\eta j} (2j+1)(\lambda_3\lambda_4|f(W,j,\eta)|\lambda_1\lambda_2)\ d^j_{\lambda'\lambda}(-\theta). \tag{5.4}$$

It is clear that the Born contribution to the parity-conserving amplitudes is given by

$$(2j+1)(\lambda_3\lambda_4|f(W,j,\eta)|\lambda_1\lambda_2)_{Born} =$$

$$= \sum_{\tau\alpha\tau'\alpha'} (\lambda_3\lambda_4|\bar{G}_{\tau'\alpha'}(W,j,\eta)|0)\ \Delta_{\tau'\alpha',\tau\alpha}(W,j,\eta)\,(0|G_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2). \tag{5.5}$$

It follows from (4.17) that the amplitudes (5.5) satisfy the usual parity constraints,

$$\begin{aligned}(\lambda_3\lambda_4|f(W,j,\eta)|\lambda_1\lambda_2) &= \eta\eta_3\eta_4\,(-\lambda_3-\lambda_4|f(W,j,\eta)|\lambda_1\lambda_2)\\ &= \eta\eta_1\eta_2\,(\lambda_3\lambda_4|f(W,j,\eta)|-\lambda_1-\lambda_2)\end{aligned} \tag{5.6}$$

and, from (4.18) and (3.9), the MacDowell symmetry,

$$(\lambda_3\lambda_4|f(W,j,\eta)|\lambda_1\lambda_2) = (-)^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}(\lambda_3\lambda_4|f(-W,j,\pm\eta)|\lambda_1\lambda_2)\,. \tag{5.7}$$

where $+\eta(-\eta)$ must be used for boson (fermion) channels.

The amplitudes defined in this way exhibit a singularity at W = 0 resulting from the singular behaviour of the form factors (for unequal masses). This much was to be expected from the discussion of Sec. 1. It remains to be shown under what circumstances the complete Born term (5.3) remains finite at W = 0.

To deal with this aspect of the problem it is necessary to replace the propagator $\Delta(W, j, \eta)$ by its expansion in terms of the amplitudes $G(W^2, N)$ as given in Sec. 3, (3.16). The form factors $G(W, j, \eta)$, moreover, must be replaced by their expansions, (4.13), in terms of the regularized form factors $F(W, j, \eta)$. The Born term, (5.3), then takes the form

$$\sum_{\substack{\tau\alpha, \tau'\alpha' \\ N}} G_{\tau'\alpha', \tau\alpha}(W^2, N) \sum_{\eta j, \eta' j'} (\lambda_3 \lambda_4 | \bar{F}_{\tau'\alpha'}(W, j', \eta') | 0)(\alpha' \eta' j' \lambda' | K_N(W, \theta) | \alpha \eta j \lambda) \cdot \\ \cdot (0| F_{\tau\alpha}(W, j, \eta) | \lambda_1 \lambda_2) \tag{5.8}$$

where the matrix $K_N(W, \theta)$ is a purely kinematical construction defined by

$$(\alpha' \eta' j' \lambda' | K_N(W, \theta) | \alpha \eta j \lambda) = \\ = W^N \sum_{\bar{\eta} \bar{j}} d^{\alpha'}_{\eta' j' \lambda' \bar{j} \bar{\eta}} \left(-\frac{\alpha_3 - \alpha_4}{2}\right) d^{\bar{j}}_{\lambda' \lambda}(-\theta) \quad \times \\ \times \langle \alpha' \bar{\eta} \bar{j} \| \alpha \bar{\eta} \bar{j}, (\tfrac{N}{2} \tfrac{N}{2} +) 0 \rangle \, d^{\alpha}_{\bar{\eta} \bar{j} \lambda j \eta} \left(\frac{\alpha_1 - \alpha_2}{2}\right) \quad . \tag{5.9}$$

The singularities of the Born term are thereby collected into the functions of d^{α} and $d^{\alpha'}$ in (5.9) where, as will now be shown, they cancel one another.

Consider the case $m_1 > m_2$, $m_3 > m_4$ which is typical of the so-called unequal-unequal configuration. (The treatments of the various possible mass configurations differ in detail but will not be considered explicitly here.) As $W \to 0$ both $\alpha_1 - \alpha_2$ and $\alpha_3 - \alpha_4$ tend to $-\infty$. On the other hand, $\theta \to 0$ as has been mentioned in Sec. 1. Although each term in the sum (5.9) individually diverges in this limit it can be shown by rearranging the summation that the total effect is finite. The necessary rearrangement can be effected by using the properties of Clebsch-Gordan coefficients which enable one to write

$$(\alpha'\eta'\jmath'\lambda'|K_N(W,\theta)|\alpha\eta\jmath\lambda) =$$

$$= \sum_{\bar{\eta}\bar{\jmath}s} D^{\alpha'}_{\eta'\jmath'\lambda',\bar{\eta}\bar{\jmath}\lambda}(\Lambda) \langle\alpha'\bar{\eta}\bar{\jmath}\|\alpha\eta\jmath,(N/2\, N/2\, +)s\rangle\; d^{N/2\, N/2}_{s00}\left(\frac{\alpha_1-\alpha_2}{2}\right) W^N, \tag{5.10}$$

where the transformation, Λ, is defined by

$$U(\Lambda) = e^{\frac{i}{2}(\alpha_3-\alpha_4)J_{03}}\; e^{i\theta J_{31}}\; e^{-\frac{i}{2}(\alpha_1-\alpha_2)J_{03}} \quad . \tag{5.11}$$

The parts of (5.10) can be dealt with piecemeal. Firstly, the transformation Λ can be brought into the standard form,

$$U(\Lambda) = e^{-i\phi J_{31}}\; e^{-i\zeta J_{03}}\; e^{-i\psi J_{31}}, \tag{5.12}$$

where the angles ϕ, ζ and ψ are given by

$$\tan\phi = \frac{\operatorname{sh}\frac{1}{2}(\alpha_1-\alpha_2)\sin\theta}{\operatorname{ch}\frac{1}{2}(\alpha_1-\alpha_2)\operatorname{sh}\frac{1}{2}(\alpha_3-\alpha_4) - \operatorname{sh}\frac{1}{2}(\alpha_1-\alpha_2)\operatorname{ch}\frac{1}{2}(\alpha_3-\alpha_4)\cos\theta}$$

$$\operatorname{ch}\zeta = \operatorname{ch}\tfrac{1}{2}(\alpha_1-\alpha_2)\operatorname{ch}\tfrac{1}{2}(\alpha_3-\alpha_4) - \operatorname{sh}\tfrac{1}{2}(\alpha_1-\alpha_2)\operatorname{sh}\tfrac{1}{2}(\alpha_3-\alpha_4)\cos\theta$$

$$\tan\psi = \frac{\operatorname{sh}\frac{1}{2}(\alpha_3-\alpha_4)\sin\theta}{\operatorname{ch}\frac{1}{2}(\alpha_3-\alpha_4)\operatorname{sh}\frac{1}{2}(\alpha_1-\alpha_2) - \operatorname{sh}\frac{1}{2}(\alpha_3-\alpha_4)\operatorname{ch}\frac{1}{2}(\alpha_1-\alpha_2)\cos\theta} \;. \tag{5.13}$$

From (4.16) it follows that near W = 0 the angles α_1-α_2 and α_3-α_4 diverge logarithmically,

$$\begin{aligned} \alpha_1 - \alpha_2 &= -\ln(1/W^2) + \dots \\ \alpha_3 - \alpha_4 &= -\ln(1/W^2) + \dots \quad , \end{aligned} \tag{5.14}$$

the terms represented by dots being finite. It is a simple matter to prove, using $\theta \sim W$, that the angles ϕ, ζ and ψ, and hence the transformation Λ, are all finite in the limit $W \to 0$. (This result could have been guessed by substituting (5.14) into (5.11) and setting $\theta = 0$). The other factor in (5.10), $W^N d^{\frac{N}{2}\frac{N}{2}}_{soo}$, can be represented as a polynomial of degree N in $W\,ch(\alpha_1-\alpha_2)/2$ and $W\,sh(\alpha_1-\alpha_2)/2$ and, again from (5.14), this is seen to be finite at $W = 0$. Thus all of the terms entering (5.10), and therefore K_N itself, are finite at $W = 0$.

The conclusion of this analysis is that the Born term (5.3) is regular at $W = 0$ provided the propagator components, $G(W^2, N)$, and the form factors, $F(W, j, \eta)$, are regular there. (One must of course assume in addition that none of the summations in (5.18) diverges. It has only been shown that each term of the summation is non-singular). It should be remarked that the special case $m_1 = m_2$ and $m_3 = m_4$ shows the additional result that only the $N = 0$ term contributes to the Born term at $W = 0$.

In keeping with the viewpoint set out in Sec. 1, the poles of the scattering amplitudes are presumed to reside in the propagator. A discussion of their properties is facilitated by diagonalizing the propagator. Conveniently, as it happens, the reduced propagator matrix, $\Delta(W, j, \eta)$ is symmetric apart from a diagonal multiplier. If its infinite dimensionality is assumed not to be a serious complication it can therefore be diagonalized by an orthogonal transformation. That is, one can express it in the form

$$\Delta_{\tau\alpha,\tau'\alpha'}(W, j, \eta) = (-)^{2k} \sum_{\tau''\alpha''} \frac{X_{\tau''\alpha'',\tau\alpha}(W, j, \eta)\, X_{\tau''\alpha'',\tau'\alpha'}(W, j, \eta)}{D_{\tau''\alpha''}(W, j, \eta)} \tag{5.15}$$

where $X(W, j, \eta)$ denotes an orthogonal matrix and $D^{-1}_{\tau\alpha}(W, j, \eta)$ the set of eigenvalues. The sign factor $(-)^{2k}$ is needed because not Δ itself but $(-)^{2k}\Delta$ is the symmetric matrix, as can be seen from a comparison of (3.9) and (3.11). The poles of the propagator are given by the solutions of the equations

$$D_{\tau\alpha}(W, j, \eta) = 0 \quad . \tag{5.16}$$

Field systems for which any one of the functions $D_{\tau\alpha}$ or $D^{-1}_{\tau\alpha}$ vanishes identically in W must be regarded as inadmissible. It will be assumed throughout the following discussion that the propagator $\Delta(P)$ can be inverted except for isolated values of P^2.

The functions $D_{\tau\alpha}(W, j, \eta)$ are defined only for those $\alpha = (k, \ell)$ corresponding to irreducible representations of the Lorentz group which contain states with spin j and parity $\eta e^{i\pi j}$. In Sec. 6, however, it will be supposed that there exist functions, meromorphic in the j-plane, which interpolate these physical values.

In general terms the picture advocated here is a simple one. Each physical particle of spin j and parity $\eta e^{i\pi j}$ occurs as a zero of one or other of the functions $D_{\tau\alpha}(W, j, \eta)$ which fixes its mass, $W = m_{\tau\alpha}(j, \eta)$. In a given model, of course, many of the D-functions may have no zero: such is the case with systems of free fields whose equations are usually set up in such a way as to produce only one particle - all of the D-functions but one being constants. The orthogonal matrix $X(W, j, \eta)$ which occurs in the residue can be looked upon as a set of mixing angles. These angles specify the mixture of the fields which goes to make up a particle of spin j and parity $\eta e^{i\pi j}$.

A very simple example which exhibits this mixing phenomenon is provided by the Proca model of a vector particle. There are two fields A_μ and $F_{\mu\nu}$ satisfying the equations

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &= m F_{\mu\nu} \\ -\partial_\mu F_{\mu\nu} &= m A_\nu \end{aligned} \tag{5.17}$$

on the basis of which one can construct the invariant components of the reduced propagator matrix,

$$\begin{aligned} \Delta(W, 0, +) &= 1/m \\ \Delta(W, 1, -) &= 1/m \\ \Delta(W, 1, +) &= \frac{1}{W^2 - m^2}\begin{pmatrix} -m & -W \\ W & m \end{pmatrix} \end{aligned} , \tag{5.18}$$

The vector particle is contained in both A_μ and $F_{\mu\nu}$, hence its propagator is a 2 x 2 matrix. This can be diagonalized and put into the standard form (5.15),

$$\Delta(W,1,+) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} (W-m)^{-1} & 0 \\ 0 & -(W+m)^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (5.19)$$

showing explicitly the pole at m and its antiparticle at -m. The vector particle evidently contains A_μ and $F_{\mu\nu}$ in equal proportions, $X = \pm 1/\sqrt{2}$. Unfortunately this model is too simple to illustrate some of the more interesting features. The mixing angles are constants, for example, and do not vanish at W = 0. This is because the eigenvalues of $(-)^{2k}\Delta(0,1,+)$ are degenerate.

The conditions formulated above and in Sec. 3 which assure the regularity of the Born term can be applied to the components of $\Delta^{-1}(P)$ as well as to those of $\Delta(P)$. They imply very powerful constraints on the structure of the functions $D_{\tau\alpha}(W,j,\eta)$ and $X_{\tau\alpha,\tau'\alpha'}(W,j,\eta)$ in the neighbourhood of W = 0. One can write, analogously to (3.16),

$$\Delta^{-1}_{\tau\alpha,\tau\alpha'}(W,j,\eta) = \sum_N W^N A_{\tau\alpha,\tau\alpha'}(W^2,N) \langle \alpha\eta j \| \alpha'\eta j, (N/2\ N/2)\, 0\rangle \quad (5.20)$$

and be assured that $A(W^2,N)$ is regular at $W^2 = 0$. From this it follows that

$$\lim_{W\to 0}\left(\frac{\partial^N \Delta^{-1}_{\tau\alpha,\tau'\alpha'}(W,j,\eta)}{\partial W^N}\right) = \begin{cases} 0\ , & N < N_0 \\ A_{\tau\alpha,\tau'\alpha'}(0,N_0)\langle\alpha\eta j\|\alpha'\eta j, (N_0/2\ \ N_0/2+)\, 0\rangle, & N = N_0 \end{cases} \quad (5.21)$$

where N_0 denotes the minimum value of N for which the Clebsch-Gordan coefficient in (5.20) is non-vanishing.

$$N_0 = |k+\ell - k'-\ell'| + |k-\ell - k'+\ell'| \quad (5.22)$$

The constraints (5.21) are of crucial importance for the discussion of Sec. 6, where application is made of the Born term model to the problem of classifying Regge trajectories with particular reference to their behaviour in the neighbourhood of W = 0.

Taking N = 0 in (5.21) one finds

$$\Delta^{-1}_{r\alpha, r'\alpha'}(0, j, \eta) = \delta_{\alpha\alpha'} A_{r\alpha, r'\alpha}(0,0) . \tag{5.23}$$

The limit of the propagator is diagonal in the Lorentz indices or, in other words, the representation mixing vanishes [10] at W = 0. Even more important, it no longer depends explictly on j and η. The eigenvalues at W = 0 are functions $D_{r\alpha}(0)$ which depend only implicitly on j and η in the sense of being defined only for those α which correspond to representations containing the states j, η.

It is feasible to use perturbation methods to discover the behaviour of the functions $D_{r\alpha}(W, j, \eta)$ in the neighbourhood of W = 0. For simplicity suppose that the index, r, is absent. Expand the inverse propagator in powers of W,

$$(-)^{2k} \Delta^{-1}_{\alpha\alpha'}(W, j, \eta) = \delta_{\alpha\alpha'} D^{0}_{\alpha} + W D^{1}_{\alpha\alpha'} + W^{2} D^{2}_{\alpha\alpha'} + \cdots \quad . \tag{5.24}$$

A well-known formula of conventional perturbation theory gives the corresponding power series expansions of the eigenvalues and eigen - functions,

$$D_{\alpha}(W, j, \eta) = D^{0}_{\alpha} + W D^{1}_{\alpha\alpha} + W^{2}\left\{ D^{2}_{\alpha\alpha} + \sum_{\beta \neq \alpha} \frac{D^{1}_{\alpha\beta} D^{1}_{\beta\alpha}}{D^{0}_{\alpha} - D^{0}_{\beta}} \right\} + \cdots \tag{5.25}$$

$$X_{\alpha\alpha'}(W, j, \eta) = \begin{cases} 1 - \dfrac{W^{2}}{2} \displaystyle\sum_{\beta \neq \alpha} \frac{D^{1}_{\alpha\beta} D^{1}_{\beta\alpha}}{(D^{0}_{\alpha} - D^{0}_{\beta})^{2}} + \cdots , & \alpha' = \alpha \\[2ex] W \dfrac{D^{1}_{\alpha\alpha'}}{D^{0}_{\alpha} - D^{0}_{\alpha'}} + W^{2}\left\{ \dfrac{D^{2}_{\alpha\alpha'}}{D^{0}_{\alpha} - D^{0}_{\alpha'}} - \dfrac{D^{1}_{\alpha\alpha'}(D^{1}_{\alpha\alpha} - D^{1}_{\alpha'\alpha'})}{(D^{0}_{\alpha} - D^{0}_{\alpha'})^{2}} + \right. & \\[2ex] \left. + \displaystyle\sum_{\beta \neq \alpha, \alpha'} \frac{D^{1}_{\alpha\beta} D^{1}_{\beta\alpha'}}{(D^{0}_{\alpha} - D^{0}_{\beta})(D^{0}_{\alpha} - D^{0}_{\alpha'})} \right\} + \cdots , & \alpha \neq \alpha' \end{cases} \tag{5.26}$$

provided the zeroth order terms, D^0_α, are non-degenerate. A more elaborate treatment would be needed in the event of degeneracy occurring or if the suppressed index r must play a role. To employ the formulae (5.25) and (5.26) in a manner consistent with the kinematical constraints (5.21) it is necessary only to express the quantities $D^1_{\alpha\beta}, D^2_{\alpha\beta}, \ldots,$ in terms of a set of independent parameters. These can be read off from (5.20) or its more convenient modification,

$$(-)^{2k} \Delta^{-1}_{\alpha\alpha'}(W, j, \eta) = \sum_N W^N B_{\alpha\alpha'}(W^2, N) \frac{\langle \alpha\eta j, \alpha' \pm \eta j \| (N/2\, N/2+) 0 \rangle (-)^{k-l}}{\sqrt{2j+1}}, \tag{5.27}$$

by expanding the (symmetric) matrices $B_{\alpha\alpha'}(W^2, N)$ in powers of W^2.

Some general conclusions can be based upon the power series expansions (5.25) and (5.26) with the help of Appendix I. These are:

1) <u>Symmetries under $W \to -W$</u>

$$D_\alpha(W, j, \eta) = D_\alpha(-W, j, \mp\eta) \tag{5.28}$$

$$X_{\alpha\alpha'}(W, j, \eta) = (-)^{2k} X_{\alpha\alpha'}(-W, j, \pm\eta)(-)^{2k'} \tag{5.29}$$

where $\eta(-\eta)$ is used for bosons (fermions).

2) <u>Constraints at $W = 0$</u>

The coefficients $D^N_\alpha(j,\eta) = \lim_{W\to 0}(\partial^N D_\alpha(W, j, \eta) / \partial W^N)$ have two important properties which are deduced from a detailed examination of the structure of the Clebsch-Gordan coefficients in (5.20).

2a) $$D^N_\alpha(j, \eta) = D^N_\alpha(j, -\eta) \text{ for } N < 2(k-l), \quad k \neq l \tag{5.30}$$

2b) $D^N_\alpha(j, \eta)$ = a rational function of j.

It has been verified for N = 0, 1, 2 that $D^N_\alpha(j,\eta)$ is a polynomial of order N in j, explicitly

$$D^0_\alpha(j,\eta) = a_0(\alpha)$$

$$D^1_\alpha(j,\eta) = \eta(j+\tfrac{1}{2}) \quad a_1(\alpha)\ \delta_{k-\ell,\frac{1}{2}}$$

$$D^2_\alpha(j,\eta) = a_2(\alpha) + j(j+1)\ b_2(\alpha) + \eta j(j+1)\ c_2(\alpha)\,\delta_{k-\ell,1}\,. \quad (5.31)$$

We might conjecture that the polynomial form persists for $N > 2$. It follows from (5.28) that only even powers of W contribute to the boson functions $D_\alpha(W,j,\eta)$ and from (5.30) that the only odd powers which can contribute to the fermion functions have $N \geq 2(k-\ell)$.

The diagonalizing matrix $X(W,j,\eta)$ satisfies the constraints

$$\lim_{W\to 0}\left[\partial^N X_{\alpha\alpha'}(W,j,\eta)/\partial W^N\right] = 0 \qquad \text{for} \quad N < N_0 \qquad (5.32)$$

where N_0 is the number defined by (5.22). This property which, like the others (5.28), ..., (5.31), depends upon the non-degeneracy assumption, will prove important in the discussion of the behaviour of residues near $W = 0$.

By employing the diagonalized form of the propagator (5.15) one can express the Born contribution to the parity-conserving partial wave amplitudes (5.5) in a correspondingly diagonalized form,

$$(2j+1)\,(\lambda_3\lambda_4|f(W,j,\eta)|\lambda_1\lambda_2)_{Born} =$$
$$= \sum_{\tau\alpha} (\lambda_3\lambda_4|\bar{\Gamma}_{\tau\alpha}(W,j,\eta)|0)\ D^{-1}_{\tau\alpha}(W,j,\eta)\ (0|\Gamma_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2). \quad (5.33)$$

The vertex functions $\Gamma(W,j,\eta)$ are related to the various form factors defined in Sec. 4 as follows:

$$(0|\Gamma_{\tau\alpha}(W,j,\eta)|\lambda_1\lambda_2) = \sum_{\tau'\alpha'} X_{\tau\alpha,\tau'\alpha'}(W,j,\eta)\ (0|G_{\tau'\alpha'}(W,j,\eta)|\lambda_1\lambda_2) =$$

$$= \sum_{r\alpha'\eta' j'} X_{r\alpha, r\alpha'}(W, j, \eta)\; d^{\alpha'}_{\eta j \lambda j' \eta'}\left(\frac{\alpha_1 - \alpha_2}{2}\right) (0| F_{r\alpha'}(W, j', \eta')|\lambda_1 \lambda_2)$$

$$= \sum_{r\alpha'\eta' j' S} X_{r\alpha, r\alpha'}(W, j, \eta)\; d^{\alpha'}_{\eta j \lambda j' \eta'}\left(\frac{\alpha_1 - \alpha_2 + i\pi}{2}\right) \left(sh\, \frac{\alpha_1 + \alpha_2 - i\pi}{2}\right)^{|j' - S|} \times$$

$$\times (0| H_{r\alpha'}(W, j', \eta')|s\lambda)\langle s\lambda | S_1\lambda_1, S_2 - \lambda_2\rangle\,, \tag{5.34}$$

where $H_{r'\alpha'}(W, j, \eta)$ is regular at $W = 0$ and unconstrained at the pseudothreshold. This formula with $\alpha_1 = \alpha_2$ applies also to the equal mass case, $m_1 = m_2$, where the pseudothreshold coincides with $W = 0$.

For the vertex $\overline{\Gamma}(W, j, \eta)$ it is possible to write formulae completely analogous to (5.34) with the exception that the eigenfactor $(-)^{2k}$ from (5.15) must be included,

$$(\lambda_3 \lambda_4 | \overline{\Gamma}_{r\alpha}(W, j, \eta)|0) = \sum_{r'\alpha'} (\lambda_3 \lambda_4 | \overline{G}_{r'\alpha'}(W, j, \eta)|0)\, (-)^{2k'}\, X^{-1}_{r'\alpha', r\alpha}(W, j, \eta)$$
$$= \cdots \quad \text{etc.} \tag{5.35}$$

The vertex functions $\Gamma(W, j, \eta)$ are singular at $W = 0$ if $m_1 \neq m_2$ as has been discussed in Sec. 4. This singularity is confined entirely to the functions $d^{\alpha'}(\frac{\alpha_1 - \alpha_2}{2})$ whose asymptotic behaviour is given by

$$d^{\alpha'}_{\eta j \lambda j' \eta'}(\zeta) \;\sim\; e^{(k' + \ell' - |k' - \ell' - |\lambda||)|\zeta|}\,, \quad \zeta \to \pm\infty\,. \tag{5.36}$$

Therefore, on using (5.14)

$$d^{\alpha'}_{\eta j \lambda j' \eta'}\left(\frac{\alpha_1 - \alpha_2}{2}\right) \sim \left(\frac{1}{W}\right)^{k' + \ell' - |k' - \ell' - |\lambda||}\,, \qquad W \to 0\,. \tag{5.37}$$

On the other hand, the off-diagonal elements of $X_{\alpha\alpha'}$ (again neglecting the index r) vanish according to (5.32),

$$X_{\alpha\alpha'}(W, j, \eta) \;\sim\; (W)^{|k + \ell - k' - \ell'| + |k - \ell - k' + \ell'|} \tag{5.38}$$

It is now a simple matter to pick out the most singular terms from the sums (5.34). Assuming that these do not cancel fortuitously - or diverge even more drastically if there should be an infinite number of them - one obtains the following behaviour:

$$\Gamma(W, j, \eta) \sim \left(\frac{1}{W}\right)^{k+\ell-|k-\ell-|\lambda||} \tag{5.39}$$

This result is independent of j and η. It is independent, moreover, of the range of $(k'\ell')$ values summed over although, of course, if this range should be infinite the argument is not a mathematically respectable one.

To summarize, the principal formulae of this section are: (5.5) which specifies the Born contribution to the parity-conserving partial wave amplitudes; (5.8), . . . , (5.14) which contain the proof of the fact that the Born term is regular at W = 0 in spite of the divergence of the individual partial waves; (5.15) which expresses the reduced propagator in diagonal form, thereby isolating distinct poles defined as the solutions of (5.16); (5.21) and (5.22) which give the constraints to which the reduced propagator must be subjected; (5.30), (5.31) and (5.32), giving some general properties of the eigenvalues and eigenfunctions of the reduced propagator resulting from the constraints; (5.33) which gives the Born contribution in diagonal form involving the eigenvalues of the reduced propagator and a new set of vertex functions, $\Gamma_{\tau\alpha}(W, j, \eta)$ defined by (5.34) and, finally, (5.39) which gives the behaviour of these functions near W = 0, a result which will prove important for the discussion of Regge residues in the next section.

6. REGGE FAMILIES

The central idea in this work has been the simple notion that the poles of the scattering amplitude are contained in the Born term. A natural extension of this would be to require that the poles of reggeized partial wave amplitudes are contained also in the Born contribution. It is proposed, in fact, that the knowledge gained of the behaviour near W = 0 of the Born contribution to the partial wave amplitudes, $f(W, j, \eta)_{\mathrm{Born}}$, for physical values of j can be extrapolated without significant modification to the domain of complex j. That is, we wish to embed the Regge poles in structures like (5. 5) and (5. 33) which must therefore be continued into the complex j-plane. The upshot of such an effort will then be a formulation of Regge poles which embodies a set of kinematical constraints which are known to be consistent with Lorentz invariance. One is led quite generally to the so-called "conspiratorial solution" of the constraint problem: infinite families of daughter trajectories whose residues, intercepts, slopes, etc., are subject to various constraints.

The first question to settle in such a programme is, what to do about the bounds $k-\ell \leqslant j \leqslant k+\ell$? For fixed (integer or half-integer) values of k and ℓ there are only a finite number of values open to j and there can of course be no uniqueness in the choice of analytic functions which interpolate only a finite set of points. This difficulty can be circumvented by setting

$$k - \ell = j_0 \qquad \text{and} \qquad k + \ell = j + \kappa \tag{6.1}$$

where $2j_0$ and κ are non-negative integers. Functions of k, ℓ and j can then be looked upon as functions of j_0, κ and j with j taking the values $j_0, j_0+1, \dots$, and κ taking the values $0, 1, 2, \dots$, both sequences increasing to infinity. We shall therefore postulate the existence of meromorphic functions $D^{\tau}_{j_0 j+\kappa}(W, j, \eta)$ and $\Gamma^{\tau}_{j_0 j+\kappa}(W, j, \eta)$ which interpolate the physical j values of the functions $D_{k\ell}(W, j, \eta)$ and $\Gamma_{k\ell}(W, j, \eta)$, respectively [11]. The index $\tau = \pm 1$ represents "Lorentz signature", the amplitudes of signature τ being supposed to interpolate the points $(-)^{2\ell} = \tau$.

The emergence of Lorentz signature appears natural in this scheme when it is recalled, (4.18) (5. 34), that the vertex functions $\Gamma_{k\ell}(W, j, \eta)$ have a definite symmetry under the transformation $W \to -W$ at physical values of j,

$$(0|\Gamma_{k\ell}(W, j, \eta)|\lambda_1\lambda_2) = (-)^{2k} (0|\Gamma_{k\ell}(-W, j, \pm\eta)|\lambda_1\lambda_2) e^{i\pi(\lambda_1-\lambda_2)} \tag{6.2}$$

This property, reflecting TCP-invariance, is clearly worth retaining in the complex j-plane. We therefore require

$$(0|\Gamma^{\tau}_{j_0\, j+\kappa}(W,j,\eta)|\lambda_1\lambda_2) = \tau(-)^{2j_0}(0|\Gamma^{\tau}_{j_0\, j+\kappa}(-W,j,\pm\eta)|\lambda_1\lambda_2)\, e^{i\pi(\lambda_1-\lambda_2)} . \tag{6.3}$$

At physical values of j the function $\Gamma_{j_0\, j+\kappa}(W, j, \eta)$ coincides with $\Gamma_{k\ell}(W, j, \eta)\ (1+\tau(-)^{2\ell})/2$ where $2k = j+\kappa+j_0$ and $2\ell = j+\kappa-j_0$.

No such simple argument can be used to justify appending the label τ to the interpolating functions for $D_{k\ell}(W, j, \eta)$ which possess the more straightforward symmetry

$$D_{k\ell}(W, j, \eta) = D_{k\ell}(-W, j, \pm\eta) . \tag{6.4}$$

However, in any reasonable model some account must be taken of unitarity and one would expect the absorptive part of D to receive contributions from $\bar{\Gamma}\ \Gamma$ and if these are τ dependent the same must be true of D. It will therefore be assumed that even and odd values of 2ℓ must be interpolated by independent functions $D^{\tau}_{j_0\, j+\kappa}(W, j, \eta)$, $\tau = \pm 1$.

With these assumptions about interpolating functions, the diagonalized Born contribution (5.33) takes the form

$$(2j+1)(\lambda_3\lambda_4|f(W,j,\eta)|\lambda_1\lambda_2)_{Born} =$$

$$= \sum \frac{1+\tau(-)^{\kappa}e^{i\pi(j-j_0)}}{2} \frac{(\lambda_3\lambda_4|\bar{\Gamma}^{\tau}_{j_0\, j+\kappa}(W,j,\eta)|0)(0|\Gamma^{\tau}_{j_0 j+\kappa}(W,j,\eta)|\lambda_1\lambda_2)}{D^{\tau}_{j_0 j+\kappa}(W,j,\eta)} . \tag{6.5}$$

for complex j. The poles of (6.5) correspond to the zeros of $D^{\tau}_{j_0\, j+\kappa}(W, j, \eta)$ which are given by formulae of the type

$$j = \alpha^{\tau}_{j_0\kappa}(W, \eta) . \tag{6.6}$$

It now remains to discuss those properties of the trajectory functions $\alpha^{\tau}_{j_0\kappa}(W,\eta)$ which are consequent upon the constraints set out in Sec. 5.

The terms up to order (W^2) in $D(W, j, \eta)$ were listed in (5.31).

They can be recast in the notation (6.1) to read

$$
\begin{aligned}
D^{\tau}_{j_0 j+\kappa}(0,j,\eta) &= a^{\tau}_0(j_0, j+\kappa) \\
\partial D^{\tau}_{j_0 j+\kappa}(0,j,\eta)/\partial W &= \eta(j+\tfrac{1}{2})\, a^{\tau}_1(j+\kappa)\, \delta_{j_0 \frac{1}{2}} \\
\partial^2 D^{\tau}_{j_0 j+\kappa}(0,j,\eta)/\partial W^2 &= a^{\tau}_2(j_0, j+\kappa) + j(j+1)\, b^{\tau}_2(j_0, j+\kappa) + \eta j(j+1)\, c^{\tau}_2(j+\kappa)\, \delta_{j_0 1}
\end{aligned}
\tag{6.7}
$$

It is a simple matter, using (6.7), to solve the equations, D = 0, up to order (W^2) with the following results:

$$
\begin{aligned}
\alpha^{\tau}_{j_0 \kappa}(0,\eta) &= \alpha^{\tau}(j_0) - \kappa\,, \\
\partial \alpha^{\tau}_{j_0 \kappa}(0,\eta)/\partial W &= \eta\,(A^{\tau}_1 + \kappa B^{\tau}_1)\, \delta_{j_0 \frac{1}{2}}\,, \\
\partial^2 \alpha^{\tau}_{j_0 \kappa}(0,\eta)/\partial W^2 &= A^{\tau}_2(j_0) + (\alpha^{\tau}-\kappa)(\alpha^{\tau}-\kappa+1)\, B^{\tau}_2(j_0) + \\
&\quad + \eta\,(\alpha^{\tau}-\kappa)(\alpha^{\tau}-\kappa+1)\, C^{\tau}_2\, \delta_{j_0 1} + \\
&\quad + (\alpha^{\tau}-\kappa+\tfrac{1}{2})\left[D^{\tau}_2 + (\alpha^{\tau}-\kappa+\tfrac{1}{2})E^{\tau}_2\right] \delta_{j_0 \frac{1}{2}}\,.
\end{aligned}
\tag{6.8}
$$

where A, B, C,....., are expressible in terms of the parameters a, b, c, . . . , of (6.7).

One is thus led to the following conclusions[12]:

1) Regge trajectories occur in families labelled by two quantum numbers j_0 and τ. The members of a family, or daughters, are labelled by $\kappa = 0, 1, 2, \ldots$, and have alternating signatures $\tau(-)^{\kappa}$. If $j_0 = 0$ all members of a family have the same parity type, η. Otherwise both types occur. (The quantum number j_0 labels eigenvalues of the reduced propagator and must be distinguished from the j_0 label occurring on the fields which labels rows and columns of the reduced propagator.)

2) In a family the intercepts at $W = 0$ are separated by integers and do not depend on η if $j_0 \neq 0$.

3) Only if $j_0 = 1/2$ can the trajectories have a linear term in W. These linear terms are expressed - for an entire family - by two independent parameters.

4) If $j_0 \neq 1/2, 1$, the quadratic terms are expressed by two parameters; if $j_0 = 1/2, 1$, by three parameters. Only if $j_0 = 1$ can the quadratic term resolve the parity degeneracy.

5) In families with parity doubling ($j_0 \neq 0$) the lowest term in $\alpha^{\tau}_{j_0 \kappa}(W, \eta)$ which can depend on η is of order W^{2j_0}. This suggests that observed parity doubling effects may be explained by assigning $j_0 > 1$ to the particles in question.

Consider now the properties of Regge residues. They contain as factors the vertex functions $\Gamma(W, j, \eta)$ defined in Sec. 5. The defining formula (5.34) must be continued to complex values of j. It is convenient to replace the helicity labels λ_1 and λ_2 by total spin, S, and helicity, $\lambda = \lambda_1 - \lambda_2$, by multiplying in the appropriate Clebsch-Gordan coefficient which leads to the formula

$$(0|\Gamma^{\tau}_{j_0 j+\kappa}(W, j, \eta)|S\lambda) =$$

$$= \sum_{j_0' \kappa' \eta' j'} X^{\tau}_{j_0 j+\kappa, j_0' j+\kappa'}(W, j, \eta)\, d^{j_0' j+\kappa}_{\eta j \lambda j' \eta'}\left(\frac{\alpha_1 - \alpha_2 + i\pi}{2}\right)\left(\mathrm{sh}\, \frac{\alpha_1 + \alpha_2 - i\pi}{2}\right)^{|j' - S|} \times$$

$$\times \quad (0|H^{\tau}_{j_0' j+\kappa'}(W, j', \eta')|S\lambda) \tag{6.9}$$

where j' is summed over the values j_0', $j_0' + 1, \ldots$. It is therefore necessary to suppose that the (unconstrained) form factors, $H_{k\ell}(W, j, \eta)$, can be continued analytically in the variable $k + \ell$ while keeping fixed both $k - \ell = j_0$ and $j = j_0, j_0 + 1, \ldots$. With such an assumption it is possible to draw some inferences about the small W behaviour of Regge residues.

Firstly, with equal masses, $m_1 = m_2 = m$, one has

$$\alpha_1 = \alpha_2 \qquad \text{and} \qquad \mathrm{sh}\,\frac{\alpha_1+\alpha_2-i\pi}{2} = \frac{W}{2m} \tag{6.10}$$

so that (6.9) reduces to the form

$$(0|\Gamma^{\tau}_{j_0 j+\kappa}(W,j,\eta)|S\lambda) =$$
$$= \sum X^{\tau}_{j_0 j+\kappa,\, j_0' j+\kappa'}(W,j,\eta)\, d^{j_0 j+\kappa}_{\eta j\lambda j'\eta'}\left(\frac{i\pi}{2}\right)\left(\frac{W}{2m}\right)^{|j'-S|}(0|H^{\tau}_{j_0' j+\kappa}(W,j,\eta')|S\lambda) \tag{6.11}$$

and therefore, taking account of the damping behaviour of X near W = 0, i.e. $X \sim W^{|j_0-j_0'|+|\kappa-\kappa'|}$, one finds in the limit 13),

$$(0|\Gamma^{\tau}_{j_0 j+\kappa}(0,j,\eta)|S\lambda) = d^{j_0 j+\kappa}_{\eta j\lambda S\eta}\left(i\frac{\pi}{2}\right)\ (0|H^{\tau}_{j_0 j+\kappa}(0,S,\eta)|S\lambda)$$

which means that the residue at $j = \alpha^{\tau}_{j_0} - \kappa$ is fixed in terms of the residue at $j = \alpha^{\tau}_{j_0}$. This is just the result of Toller. Clearly it should be possible, at least in principle, to uncover the form factors $H(W,j,\eta)$ by measurements of $\Gamma(W,j,\eta)$ in the neighbourhood of W = 0. Perhaps more interesting would be the measurement of the mixing angles $X(W,j,\eta)$ which should be independent of the particular external particles used.

Unequal mass vertex functions cannot be analysed in such detail. This is because the singular behaviour of $d\left(\frac{\alpha_1-\alpha_2+i\pi}{2}\right)$ compensates the damping in X for arbitrarily large κ' which must therefore be summed to ∞. The small W behaviour of the sum is, however, barring dynamical accidents, the same as that given in (5.39), viz 14)

$$(0|\Gamma^{\tau}_{j_0 j+\kappa}(W,j,\eta)|S\lambda) \sim \left(\frac{1}{W}\right)^{j+\kappa-|j_0-|\lambda||} \tag{6.12}$$

It could even happen that this vertex vanishes at W = 0, i.e.,

$$j+\kappa < |j_0 - |\lambda|| \tag{6.13}$$

Such an eventuality has been postulated by MANDELSTAM 15) for the pion trajectory in order to make it decouple at W = 0.

7. ASYMPTOTIC FORMULAE

The results obtained in Sec. 6 can be employed in the derivation of high-energy formulae for scattering amplitudes, it being assumed that the Born term dominates. The three mass configurations, $m_1 = m_2$ and $m_3 = m_4$ (E-E), $m_1 = m_2$ and $m_3 \neq m_4$ or $m_1 \neq m_2$ and $m_3 = m_4$ (E-U), $m_1 \neq m_2$ and $m_3 \neq m_4$ (U-U) behave differently and must be treated separately. In addition the limits[16] $s \to \infty$, t fixed, and $s \to \infty$, θ_s fixed, are quite distinct and it is necessary to specify which is being considered. For simplicity of illustration we shall confine our attention to the particular limit $s \to \infty$, $\theta_s = 0$.

It is a well-known fact that the condition $\theta_s = 0$ implies $\theta_t = 0$ (or π) except in the E-E configuration where θ_t varies with s. This means that the functions, $d^j_{\lambda'\lambda}(-\theta_t)$, can contribute to the large s behaviour of the partial wave expansion only in the E-E case. On the other hand, in the E-U and U-U configurations at least one of the vertex functions $\Gamma(t^{\frac{1}{2}}, j, \eta)$ is singular at $t = 0$. But it is just for these cases that the equation $\theta_s = 0$ maps $t = 0$ into $s = \infty$. In fact, the value of t for which θ_s vanishes is given by

$$t = \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{s} + O\left(\frac{1}{s^2}\right) \tag{7.1}$$

for large s. In the E-E configuration $\theta_s = 0$ of course gives $t = 0$. At $\theta_s = 0$, therefore, the vertex function gives, according to (6.12), the asymptotic factor

$$\Gamma_{j_0 j+\kappa}(t^{\frac{1}{2}}, j, \eta) \sim \left(\frac{1}{t^{\frac{1}{2}}}\right)^{j+\kappa-|j_0-|\lambda||} \sim \begin{cases} s^{j+\kappa-|j_0-|\lambda||} & , \ \text{E-U} \\ (s^{\frac{1}{2}})^{j+\kappa-|j_0-|\lambda||} & , \ \text{U-U} \end{cases} \tag{7.2}$$

The familiar Regge behaviour arises as a result of the operation of two distinct mechanisms: in the E-E configuration the angle θ_t increases with s so that the functions $d^j(\theta_t)$ become asymptotically proportional to s^j while the vertex functions tend to constants; in the E-U configuration $\theta_t = 0$, one of the vertex functions tends to a constant while the other (at the unequal mass vertex) yields a factor $s^{j+\kappa-|j_0-|\lambda||}$

in the U-U configuration $\theta_t = 0$ and each of the vertices yields a factor $(s^{\frac{1}{2}})^{j+\kappa - |j_0 - |\lambda||}$.

The first step in obtaining the dominant high-energy term is the standard one of replacing the sum over partial waves by a contour integral in the complex j-plane. The diagonalized Born term (6.5) then takes the form

$$\langle p_3\lambda_3, p_4\lambda_4 | T | p_1\lambda_1, p_2\lambda_2\rangle_{Born} =$$

$$= \sum_{\eta\tau j_0\kappa} \int dj \; \frac{1+\tau\, e^{i\pi(j+\kappa-\lambda')}}{2\sin\pi(j-j_0)} \; d^{j}_{-\lambda',\lambda}(\pi-\theta_t) \times$$

$$\times \; \frac{(\lambda_3\lambda_4|\Gamma^{\tau}_{j_0 j+\kappa}(t^{\frac{1}{2}}, j, \eta)|0)(0|\Gamma^{\tau}_{j_0 j+\kappa}(t^{\frac{1}{2}}, j, \eta)|\lambda_1\lambda_2)}{D^{\tau}_{j_0 j+\kappa}(t^{\frac{1}{2}}, j, \eta)} \tag{7.3}$$

where $\lambda = \lambda_1 - \lambda_2$ and $\lambda' = \lambda_3 - \lambda_4$. (The sign factor $(-)^{j-j_0}$ has been absorbed by writing $(-)^{j-j_0}\, d^j_{\lambda'\lambda}(-\theta_t) = (-)^{j_0-\lambda'}\, d^j_{-\lambda',\lambda}(\pi-\theta_t)$.)

The next step is to open up the contour and retain only the pole contributions. These poles are given by formulae of the type

$$j = \alpha - \kappa + O(t^{\frac{1}{2}}) \tag{7.4}$$

where α, denoting the intercept at $t = 0$ of the parent trajectory, depends only on j_0 and τ.

It follows that the leading term as $s \to \infty$ corresponding to a Regge family with given j_0 and τ is given by

$$\frac{1+\tau e^{i\pi(\alpha-\lambda)}}{2\sin\pi(\alpha-j_0)} \; \beta_{(\lambda)} \times \begin{cases} s^{\alpha} & , \quad E-E \\ s^{\alpha-|j_0-|\lambda||} & , \quad E-U \text{ and } U-U \end{cases} \tag{7.5}$$

Since $\theta_t = 0$ in the E-U and U-U configurations the residues $\beta_{(\lambda)}$ must vanish for these cases unless $\lambda = \lambda'$. The forms (7.5) were obtained first by TOLLER [3], COSENZA, SCIARRINO and TOLLER [7] and by SAWYER [17].

It must be emphasised that the results (7.5) exemplify asymptotic forms at the fixed angle, $\theta_s = 0$. In the E-U and U-U configurations they have the eccentric feature that all members of a Regge family contribute to the leading term. In the E-E configuration only the parent ($\kappa = 0$) contributes.

Different asymptotic forms can be obtained by taking the limit $s \to \infty$ with t fixed and then, in the leading terms so obtained, letting t vanish. Here the s-dependence comes out of the functions

$$d^{j}_{\lambda'\lambda}(-\theta_t) \sim \left\{ \frac{2st}{\left[(t-(m_1-m_2)^2)(t-(m_1+m_2)^2)(t-(m_3-m_4)^2)(t-(m_3+m_4)^2)\right]^{1/2}} \right\}^{j}, \quad s \to \infty \tag{7.6}$$

which, when the limit t = 0 is approached, take the forms

$$d^{j}_{\lambda\lambda'}(-\theta_t) \sim \begin{cases} s^{j} & , \quad \text{E - E} \\ (st^{\frac{1}{2}})^{j} & , \quad \text{E - U} \\ (st)^{j} & , \quad \text{U - U} \end{cases} \tag{7.7}$$

Before the limit t = 0 is taken the dominant term is governed in all cases by the parent trajectory which yields the factor s^{α}. However, in the E-U and U-U configurations this factor is accompanied by a corresponding $t^{\alpha/2}$ or t^{α} which diminishes its importance as t is made to vanish. On the other hand, the unequal mass vertex functions contribute factors $(t^{\frac{1}{2}})^{-\alpha+|j_0-|\lambda||}$ which must be taken into account. The contribution of the daughter, κ, to the asymptotic form thus contains the factor

$$s^{\alpha-\kappa}\, t^{|j_0-|\lambda||-\kappa} \quad \text{or} \quad s^{\alpha-\kappa}\, t^{|j_0-|\lambda'||-\kappa}, \qquad \text{E - U}$$

$$s^{\alpha-\kappa}\, t^{\frac{1}{2}|j_0-|\lambda||+\frac{1}{2}|j_0-|\lambda'||-\kappa}, \qquad \text{U - U} \tag{7.8}$$

which is seen either to vanish or become singular at t = 0 except for a particular daughter given by

$$\kappa = |j_0 - |\lambda|| \quad \text{or} \quad |j_0 - |\lambda'|| \;, \qquad E - U$$

$$\kappa = \tfrac{1}{2}|j_0 - |\lambda|| + \tfrac{1}{2}|j_0 - |\lambda'|| \;, \qquad U - U \,. \tag{7.9}$$

Thus it would appear that the contributions of the parent and a certain number of daughters are extinguished at t = 0. The singularities of the contributions coming from the lower daughters can be ignored since these must be compensated in view of the regularity demonstrated in Sec. 5. Hence the leading term in this limit is given by

$$s^{\alpha - |j_0 - |\lambda||} \quad \text{or} \quad s^{\alpha - |j_0 - |\lambda'||} \;, \qquad E - U$$

$$s^{\alpha - \frac{1}{2}|j_0 - |\lambda|| - \frac{1}{2}|j_0 - |\lambda'||} \;, \qquad U - U \tag{7.10}$$

being in each case the contribution of a particular daughter.(In the U-U configuration if the κ specified by (7.9) should be half-integral then one must make the replacement $\kappa \to \kappa + \frac{1}{2}$ since the most singular part of one of the vertex functions cannot be operative in this case. A corresponding modification of (7.10) is implied).

It may be noted that the forms (7.10) with $\lambda = \lambda'$ reduce to (7.5) which correspond to a different limiting procedure.

APPENDIX I

THE CLEBSCH-GORDAN COEFFICIENTS

Since the finite-dimensional representations of O(3,1) can be made to correspond with the unitary representations of O(4) by means of the Weyl trick and since SO(4) is isomorphic to SO(3) x SO(3), it is possible to construct the Clebsch-Gordan coefficients of O(3,1) out of the well-known SO(3) ones. The proper subgroup SO(3,1) corresponds to SO(3) x SO(3) and can be dealt with quite easily. The incorporation of space-reflections complicates the problem somewhat and will be considered afterwards.

The basis vectors of the product of two irreducible SO(3) representations couple according to the rule

$$|j_1 m_1, j_2 m_2\rangle = \sum_{jm} |j_1 j_2; j m\rangle \langle jm | j_1 m_1, j_2 m_2\rangle \tag{I. 1}$$

where $\langle jm|\, j_1 m_1 ,\, j_2 m_2 \rangle$ denotes one of the usual SO(3) Clebsch-Gordan coefficients. Likewise the basis vectors of the product of two irreducible SO(3) x SO(3) representations must couple according to the rule

$$\begin{aligned} &|k_1 \ell_1 \kappa_1 \lambda_1 ,\, k_2 \ell_2 \kappa_2 \lambda_2 \rangle = \\ &= \epsilon \sum_{k\kappa\ell\lambda} |k_1 \ell_1 k_2 \ell_2 ; k \ell \kappa \lambda\rangle \langle k\kappa | k_1\kappa_1, k_2 \kappa_2\rangle \langle \ell\lambda | \ell_1 \lambda_1, \ell_2 \lambda_2\rangle \end{aligned} \tag{I. 2}$$

where the labels k, κ (ℓ, λ) are the $\underline{J}^2$ and J_z labels referring to the left (right) factors of SO(3) x SO(3). The phase factor ϵ will be determined when space-reflections are brought in. It is more useful to employ the SO(3,1) basis $|\, k\ell jm\rangle$ defined by

$$|k\ell jm\rangle = \sum_{\kappa\lambda} |k\ell\kappa\lambda\rangle \langle k\kappa, \ell\lambda | jm\rangle \quad . \tag{I. 3}$$

Transforming (I. 2) to the new basis one finds

$$|k_1 \ell_1 j_1 m_1, k_2 \ell_2 j_2 m_2\rangle = \epsilon \sum |k_1 \ell_1 k_2 \ell_2 ; k\ell jm\rangle \langle k\ell jm | k_1 \ell_1 j_1 m_1, k_2 \ell_2 j_2 m_2 \rangle \tag{I. 4}$$

where the Clebsch-Gordan coefficient $\langle k\ell jm | k_1 \ell_1 j_1 m_1 ,\, k_2 \ell_2 j_2 m_2 \rangle$ is given by the sum

$$\langle k\ell j m \mid k_1 \ell_1 j_1 m_1 , k_2 \ell_2 j_2 m_2 \rangle =$$

$$= \epsilon \sum_{\kappa, \lambda, \ldots} \langle j m \mid k\kappa, \ell\lambda \rangle \langle k\kappa \mid k_1\kappa_1, k_2\kappa_2 \rangle \langle \ell\lambda \mid \ell_1\lambda_1, \ell_2\lambda_2 \rangle \times$$
$$\langle k_1\kappa_1, \ell_1\lambda_1 \mid j_1 m_1 \rangle \langle k_2\kappa_2, \ell_2\lambda_2 \mid j_2 m_2 \rangle$$

$$= \langle k\ell j \parallel k_1 \ell_1 j_1 , k_2 \ell_2 j_2 \rangle \langle j m \mid j_1 m_1 , j_2 m_2 \rangle \qquad \text{(I. 5)}$$

The reduced Clebsch-Gordan coefficient $\langle k\ell j \parallel k_1 \ell_1 j_1 , k_2 \ell_2 j_2 \rangle$ defined here can be expressed in terms of a 9-j symbol,

$$\langle k\lambda j \parallel k_1 \ell_1 j_1 , k_2 \ell_2 j_2 \rangle = \epsilon \left[(2k+1)(2\ell+1)(2j_1+1)(2j_2+1) \right]^{\frac{1}{2}} \begin{Bmatrix} k_1 & \ell_1 & j_1 \\ k_2 & \ell_2 & j_2 \\ k & \ell & j \end{Bmatrix} \qquad \text{(I. 6)}$$

Of particular interest for the purposes of Sec. 3 are the coefficients with $k_2 = \ell_2 = N/2$ and $j_2 = 0$. They reduce the 6-j symbols because

$$\begin{Bmatrix} k_1 & \ell_1 & J \\ N/2 & N/2 & 0 \\ k & \ell & J \end{Bmatrix} = \frac{(-)^{k_1 + \ell + J + N/2}}{\left[(2J+1)(N+1) \right]^{1/2}} \begin{Bmatrix} k_1 & \ell_1 & J \\ \ell & k & N/2 \end{Bmatrix} \qquad \text{(I. 7)}$$

The formulae (I. 5) and (I. 6) summarize the coupling properties of the finite-dimensional irreducible representations of SO(3,1). Consider now what must be added to them in order to assure invariance under the improper transformations. It is sufficient to deal with the space-reflection operator P defined by

$$P \mid k \ell j m \rangle = \mid \ell k j m \rangle \, e^{i\pi j} \quad , \quad k \neq \ell \qquad \text{(I. 8)}$$

and

$$P \mid \tfrac{n}{2} \tfrac{n}{2} j m \rangle = \pm \mid \tfrac{n}{2} \tfrac{n}{2} j m \rangle \, e^{i\pi j} \quad , \quad k = \ell = \tfrac{n}{2} \quad , \qquad \text{(I. 9)}$$

since any improper transformation can be expressed as the product of P with a proper one. The representations of SO(3,1) with $k = \ell = n/2$ carry an intrinsic parity, $\pm$, and require no extension in order to represent the full group O(3, 1). The representations with $k \neq \ell$ must be doubled in order to represent O(3, 1). It is convenient for some purposes to employ a basis which diagonalizes P. One such is defined by

$$\sqrt{2}\,|(k\ell)\eta j m\rangle \doteq \begin{cases} |k\ell j m\rangle + \eta\,|\ell k j m\rangle, & k>\ell \\ \eta|k\ell j m\rangle + |\ell k j m\rangle, & k<\ell \end{cases} \tag{I.10}$$

where $\eta = \pm 1$. The states so defined are symmetric in (k, ℓ) and it is helpful to adopt the convention $k > \ell$. On the states (I. 10) one finds

$$P\,|(k\ell)\eta j m\rangle = |(k\ell)\eta j m\rangle\, \eta\, e^{i\pi j} \quad . \tag{I.11}$$

If the Clebsch-Gordan coefficient (I. 5) is to be invariant under improper transformations as well as proper ones, it must satisfy the condition

$$\langle k\ell j m|k_1\ell_1 j_1 m_1, k_2\ell_2 j_2 m_2\rangle = (-)^{j_1+j_2-j}\langle \ell k j m|\ell_1 k_1 j_1 m_1, \ell_2 k_2 j_2 m_2\rangle \tag{I.12}$$

if $k > \ell$, $k_1 > \ell_1$, $k_2 > \ell_2$, while if, say, $k_2 = \ell_2 = n/2$ it must satisfy

$$\langle k\ell j m|k_1\ell_1 j_1 m_1, (\tfrac{n}{2}\tfrac{n}{2}\pm)j_2 m_2\rangle = \pm(-)^{j_1+j_2-j}\langle \ell k j m|\ell_1 k_1 j_1 m_1, (\tfrac{n}{2}\tfrac{n}{2}\pm)j_2 m_2\rangle \tag{I.13}$$

where $\pm$ is the intrinsic parity type of the representation $D^{(\frac{n}{2}\frac{n}{2}\pm)}$. The reduced coefficients defined by (I. 5) are subject to the same conditions. Using the symmetry of 9-j symbols

$$\begin{Bmatrix} k_1 & \ell_1 & j_1 \\ k_2 & \ell_2 & j_2 \\ k & \ell & j \end{Bmatrix} = \begin{Bmatrix} \ell_1 & k_1 & j_1 \\ \ell_2 & k_2 & j_2 \\ \ell & k & j \end{Bmatrix} (-)^{k_1+k_2+k+\ell_1+\ell_2+\ell+j_1+j_2+j} \quad , \tag{I.14}$$

one can deduce that the phase factors $\epsilon(k\ell\, k_1\ell_1 k_2\ell_2)$ entering the definition (I. 6) must satisfy

$$\epsilon(k\ell\, k_1\ell_1 k_2\ell_2) = \epsilon(\ell k\, \ell_1 k_1 \ell_2 k_2)\,(-)^{k_1+k_2-k+\ell_1+\ell_2-\ell} \tag{I.15}$$

corresponding to (I. 12) or

$$\epsilon\,(k\,\ell\,k_1\ell_1\,\tfrac{n_2}{2}\tfrac{n_2}{2}\,\pm) = \pm\,\epsilon\,(\ell\,k\,\ell_1 k_1\,\tfrac{n_2}{2}\tfrac{n_2}{2}\,\pm)\,(-)^{k_1-k+\ell_1-\ell+n} \tag{I.16}$$

corresponding to (I.13), and similarly for the other possible situations. There are many ways of choosing the phases to satisfy (I.15), (I.16), etc., but in the absence of any deeper criteria we shall adopt the following scheme which is the simplest we can think of.

<u>Case 1. $k \neq \ell$, $k_1 \neq \ell_1$; $k_2 \neq \ell_2$</u>

$$\epsilon(k\ell k_1\ell_1\, k_2\ell_2) = (-)^{k_1+k_2-k} \tag{I.17}$$

<u>Case 2. $k \neq \ell$, $k_1 \neq \ell_1$, $k_2 = \ell_2 = n/2$</u>

$$\epsilon\left(k\,\ell\,k_1\,\ell_1\,\frac{n_2}{2}\,\frac{n_2}{2}\,\pm\right) = \begin{cases} (-)^{k_1+\frac{n_2}{2}-k} & , k > \ell \\ \pm\,(-)^{k_1+\frac{n_2}{2}-k} & , k < \ell \end{cases} \tag{I.18}$$

<u>Case 3. $k \neq \ell$, $k_1 = \ell_1 = n_1/2$, $k_2 = \ell_2 = n_2/2$</u>

$$\epsilon\left(k\,\ell\,\tfrac{n_1}{2}\tfrac{n_1}{2}\eta_1,\tfrac{n_2}{2}\tfrac{n_2}{2}\eta_2\right) = \begin{cases} (-)^{n_1/2+n_2/2-k} & , \quad k > \ell \\ \eta_1\eta_2(-)^{n_1/2+n_2/2-k} & , \quad k < \ell \end{cases} \tag{I.19}$$

<u>Case 4. $k = \ell = n/2$, $k_1 = \ell_1 = n_1/2$, $k_2 = \ell_2 = n_2/2$</u>

$$\epsilon\left(\frac{n}{2}\,\frac{n}{2}\,\eta\,,\,\frac{n_1}{2}\,\frac{n_1}{2}\,\eta_1\,,\,\frac{n_2}{2}\,\frac{n_2}{2}\,\eta_2\right) = \begin{cases} (-)^{\frac{n_1+n_2-n}{2}} & , \eta = \eta_1\eta_2 \\ 0 & , \eta = -\eta_1\eta_2 \end{cases} \tag{I.20}$$

The Clebsch-Gordan coefficients have the symmetry

$$\langle k\,\ell\,j\,m\,|\,k_1\ell_1 j_1 m_1\,,\,k_2\,\ell_2\,j_2\,m_2\rangle = \langle k\,\ell\,j\,m\,|\,k_2\ell_2 j_2 m_2\,,\,k_1\,\ell_1\,j_1\,m_1\rangle\,(-)^{k_1+k_2-k+\ell_1+\ell_2-\ell} \tag{I.21}$$

which can be deduced from the properties of 9-j symbols and O(3) Clebsch-Gordan coefficients. For the reduced coefficient this becomes

$$\langle k\ell j \| k_1\ell_1 j_1, k_2\ell_2 j_2\rangle = \langle k\ell j \| k_2\ell_2 j_2, k_1\ell_1 j_1\rangle \, (-)^{k_1+k_2+k+\ell_1+\ell_2+\ell+j_1+j_2+j} . \tag{I. 22}$$

Another useful symmetry is given by

$$\langle k\ell j \| k_1\ell_1 j_1, k_2\ell_2 j_2\rangle = \langle k\ell j, k_1\ell_1 j_1 \| k_2\ell_2 j_2\rangle \times$$

$$\times \left[\frac{(2k+1)(2\ell+1)(2j_2+1)}{(2k_2+1)(2\ell_2+1)(2j+1)}\right]^{1/2} \frac{\epsilon(k\ell\, k_1\ell_1\, k_2\ell_2)}{\epsilon(k_2\ell_2\, k\ell\, k_1\ell_1)} \tag{I. 23}$$

All of these coefficients are of course real.

The Clebsch-Gordan coefficients employed in Sec. 3 refer to the basis (I. 10) which diagonalizes the parity operator. The reduced coefficients are defined by

$$\langle (k\ell)\eta j m \,|\, (k_1\ell_1)\eta_1 j_1 m_1, (k_2\ell_2)\eta_2 j_2 m_2\rangle =$$

$$= \langle (k\ell)\eta j \| (k_1\ell_1)\eta_1 j_1, (k_2\ell_2)\eta_2 j_2\rangle \langle jm | j_1 m_1, j_2 m_2\rangle . \tag{I. 24}$$

They can be expressed as linear combinations of the 9-j symbols. The various cases must be considered in turn. Since the coefficients are symmetric in $k\,\ell$, ..., it will be sufficient to exhibit them for $k \geqslant \ell$, $k_1 \geqslant \ell_1$, $k_2 \geqslant \ell_2$.

Case 1. $k > \ell$, $k_1 > \ell_1$, $k_2 > \ell_2$

Using the formulae (I. 10) one finds

$$\sqrt{2}\,\langle (k\ell)\eta j \| (k_1\ell_1)\eta_1 j_1,\ (k_2\ell_2)\eta_2 j_2\rangle =$$

$$= \langle k\ell j \| k_1\ell_1 j_1, k_2\ell_2 j_2\rangle + \eta\,\langle \ell k j \| k_1\ell_1 j_1, k_2\ell_2 j_2\rangle$$

$$+ \eta_1 \langle k\ell j \| \ell_1 k_1 j_1, k_2\ell_2 j_2\rangle + \eta_2\,\langle k\ell j \| k_1\ell_1 j_1, \ell_2 k_2 j_2\rangle$$

where use has been made of the fact that they vanish unless

$$\eta\,\eta_1\eta_2\,(-)^{j_1+j_2-j} = 1 \quad . \tag{I. 25}$$

In terms of 9-j symbols then

$$\langle (k\ell)\eta j \| (k_1\ell_1)\eta_1 j_1,\ (k_2\ell_2)\eta_2 j_2\rangle =$$

$$= \frac{1}{\sqrt{2}}\left[(2k+1)(2\ell+1)(2j_1+1)(2j_2+1)\right]^{1/2}\left[(-)^{k_1+k_2-k}\begin{Bmatrix} k_1 & \ell_1 & j_1\\ k_2 & \ell_2 & j_2\\ k & \ell & j\end{Bmatrix} + \right.$$

$$\left. + \eta\,(-)^{k_1+k_2-\ell}\begin{Bmatrix} k_1 & \ell_1 & j_1\\ k_2 & \ell_2 & j_2\\ \ell & k & j\end{Bmatrix} + \eta_1(-)^{\ell_1+k_2-k}\begin{Bmatrix} \ell_1 & k_1 & j_1\\ k_2 & \ell_2 & j_2\\ k & \ell & j\end{Bmatrix} + \eta_2\,(-)^{k_1+\ell_2-k}\begin{Bmatrix} k_1 & \ell_1 & j_1\\ \ell_2 & k_2 & j_2\\ k & \ell & j\end{Bmatrix}\right] \tag{I. 26}$$

Case 2. $k > \ell\ ,\ k_1 > \ell_1\ ,\ k_2 = \ell_2 = n/2$

$$\langle (k\ell)\eta j \| (k_1\ell_1)\eta_1 j_1,\ (\tfrac{n}{2}\tfrac{n}{2}\eta_2) j_2\rangle =$$

$$= \langle k\ell j \| k_1\ell_1 j_1, \tfrac{n}{2}\tfrac{n}{2}\eta_2 j_2\rangle + \eta_1\,\langle k\ell j \| \ell_1 k_1 j_1, \tfrac{n}{2}\tfrac{n}{2}\eta_2 j_2\rangle$$

$$= \left[(2k+1)(2\ell+1)(2j_1+1)\right]^{\frac{1}{2}}\left[(-)^{k_1+\frac{n}{2}-k}\begin{Bmatrix} k_1 & \ell_1 & j_1\\ \frac{n}{2} & \frac{n}{2} & j_2\\ k & \ell & j\end{Bmatrix} + \eta_1\,(-)^{\ell_1+\frac{n}{2}-k}\begin{Bmatrix} \ell_1 & k_1 & j_1\\ \frac{n}{2} & \frac{n}{2} & j_2\\ k & \ell & j\end{Bmatrix}\right] \tag{I. 27}$$

Case 3. $k > \ell$, $k_1 = \ell_1 = n_1/2$, $k_2 = \ell_2 = n_2/2$

$$\langle (k\ell)\eta j \| (\tfrac{n_1}{2}\tfrac{n_1}{2}\eta_1)j_1, (\tfrac{n_2}{2}\tfrac{n_2}{2}\eta_2)j_2 \rangle =$$

$$= \frac{1}{\sqrt{2}} \langle k\ell j \| \tfrac{n_1}{2}\tfrac{n_1}{2}\eta_1 j_1, \tfrac{n_2}{2}\tfrac{n_2}{2}\eta_2 j_2 \rangle$$

$$= \frac{1}{\sqrt{2}} \left[(2k+1)(2\ell+1)(2j_1+1)(2j_2+1) \right]^{1/2} (-)^{\frac{n_1}{2}+\frac{n_2}{2}-k} \times$$

$$\times \begin{Bmatrix} \frac{n_1}{2} & \frac{n_1}{2} & j_1 \\ \frac{n_2}{2} & \frac{n_2}{2} & j_2 \\ k & \ell & j \end{Bmatrix} \qquad \text{(I. 28)}$$

Case 4. $k = \ell = n/2$, $k_1 = \ell_1 = n_1/2$, $k_2 = \ell_2 = n_2/2$

$$\langle \tfrac{n}{2}\tfrac{n}{2}\eta j \| \tfrac{n_1}{2}\tfrac{n_1}{2}\eta_1 j_1, \tfrac{n_2}{2}\tfrac{n_2}{2}\eta_2 j_2 \rangle = \left[(n+1)^2 (2j_1+1)(2j_2+1) \right] (-)^{\frac{n_1+n_2-n}{2}} \begin{Bmatrix} \frac{n_1}{2} & \frac{n_1}{2} & j_1 \\ \frac{n_2}{2} & \frac{n_2}{2} & j_2 \\ \frac{n}{2} & \frac{n}{2} & j \end{Bmatrix}$$

(I. 29)

It is of course assumed that the parity condition (I. 25) is satisfied in each case. Otherwise the coefficients vanish. For case (4) there is a stronger condition. The 9-j symbol in (I. 29) vanishes unless $(-)^{j_1+j_2-j} = 1$ and therefore the couplings (I. 29) can be made only if $\eta\eta_1\eta_2 = 1$.

The cases of particular relevance in Sec. 3 are those for which $k_2 = \ell_2 = N/2$, $j_2 = 0$ and $j = j_1 = J$ where the 9-j symbols reduced to 6-j symbols as in (I. 7). The corresponding simplified versions of (I. 27), (I. 28) and (I. 29) are as follows:

Case 2. $k > \ell$, $k_1 > \ell_1$

$$\langle (k\ell)\eta J \| (k_1\ell_1)\eta J, (\tfrac{N}{2}\tfrac{N}{2}+).0 \rangle =$$

$$= \left[\frac{(2k+1)(2\ell+1)}{N+1} \right]^{\frac{1}{2}} (-)^{k+\ell+J} \left[\begin{Bmatrix} k_1 & \ell_1 & J \\ \ell & k & \frac{N}{2} \end{Bmatrix} + \eta \begin{Bmatrix} \ell_1 & k_1 & J \\ \ell & k & \frac{N}{2} \end{Bmatrix} \right] \qquad \text{(I. 30)}$$

Case 3. $k > \ell$, $k_1 = \ell_1 = n_1/2$

$$\langle (k\ell)\eta J \| (\tfrac{n_1}{2}\tfrac{n_1}{2})\eta J, (\tfrac{N}{2}\tfrac{N}{2}) + 0 \rangle = \left[\frac{(2k+1)(2\ell+1)}{2(N+1)} \right]^{\frac{1}{2}} (-)^{k+\ell+J} \begin{Bmatrix} \frac{n_1}{2} & \frac{n_1}{2} & J \\ \ell & k & \frac{N}{2} \end{Bmatrix}$$

(I. 31)

Case 4. $k = \ell = n/2$, $k_1 = \ell_1 = n_1/2$

$$\langle (\tfrac{n}{2}\tfrac{n}{2}\eta) J \| (\tfrac{n_1}{2}\tfrac{n_1}{2}\eta) J, (\tfrac{N}{2}\tfrac{N}{2}+) 0 \rangle = \frac{n+1}{\sqrt{N+1}} (-)^{n+J} \begin{Bmatrix} \frac{n_1}{2} & \frac{n_1}{2} & J \\ \frac{n}{2} & \frac{n}{2} & \frac{N}{2} \end{Bmatrix} \quad \text{(I. 32)}$$

APPENDIX II

CONTINUATION OF TWO-PARTICLE STATES

The two-particle states are characterized in general by six polar angles $\alpha_1 \theta_1 \varphi_1$ and $\alpha_2 \theta_2 \varphi_2$ which are related to the components of p_1 and p_2 by formulae of the type (4. 3). The subset of centre-of-mass states which are subject to the conditions

$$p_1 + p_2 = (W , 0 , 0 , 0) \tag{II. 1}$$

are characterized by two angles and by W. The six polar angles are constrained by four relations

$$W = m_1 \, ch\alpha_1 + m_2 \, ch\alpha_2 \tag{II. 2}$$

$$0 = m_1 \, sh\alpha_1 \cos\theta_1 + m_2 \, sh\alpha_2 \cos\theta_2 \tag{II. 3}$$

$$0 = m_1 \, sh\alpha_1 \sin\theta_1 \cos\varphi_1 + m_2 \, sh\alpha_2 \sin\theta_2 \cos\varphi_2 \tag{II. 4}$$

$$0 = m_1 \, sh\alpha_1 \sin\theta_1 \sin\varphi_1 + m_2 \, sh\alpha_2 \sin\theta_2 \sin\varphi_2 \tag{II. 5}$$

which can be solved in many ways. The simplest approach is to take

$$\theta = \theta_1 = \pi - \theta_2 \tag{II. 6}$$

$$\varphi = \varphi_1 = \varphi_2 \pm \pi \tag{II. 7}$$

as two independent variables thereby replacing (II. 3), (II. 4) and(II. 5) by the condition

$$0 = m_1 \, sh\alpha_1 - m_2 \, sh\alpha_2 \tag{II. 8}$$

It is then possible to solve (II. 2) and (II. 8) for $\alpha_1(W)$ and $\alpha_2(W)$ up to multiples of $2\pi i$ and an overall sign. The solutions can be given in the form

$$\operatorname{ch}\alpha_1 = \frac{W^2 + m_1^2 - m_2^2}{2m_1 W}$$

$$\operatorname{ch}\alpha_2 = \frac{W^2 - m_1^2 + m_2^2}{2m_2 W} \tag{II. 9}$$

Of the conditions (4. 4) necessary for the unique specification of a boost

$$-\pi \leq \operatorname{Im}\alpha \leq \pi \tag{II. 10}$$

$$0 \leq \operatorname{Re}\alpha < \infty \tag{II. 11}$$

It is always possible to impose the restriction (II. 10) on solutions of (II. 9). However it is possible to impose (II. 11) only for $|W| \geq |m_1^2 - m_2^2|^{\frac{1}{2}}$. The real part of α_1 (α_2) vanishes on this circle if $m_1 > m_2$ ($m_1 < m_2$) and becomes negative inside it. This effect necessitates the exercise of some care in the construction of boosts.

The boost operators appropriate for two-particle centre-of-mass states are given as follows:

$$U(L_{p_1}) = \begin{cases} e^{-i\varphi J_{12}} \; e^{-i\theta J_{31}} \; e^{i\varphi J_{12}} \; e^{-i\alpha_1(W) J_{03}} & , \operatorname{Re}\alpha_1 \geq 0 \\ e^{-i(\varphi\pm\pi)} \; e^{-i(\pi-\theta)J_{31}} \; e^{i(\varphi\pm\pi)J_{12}} \; e^{i\alpha_1(W)J_{03}} & , \operatorname{Re}\alpha_1 < 0 \end{cases} \tag{II. 12}$$

$$U(L_{p_2}) = \begin{cases} e^{-i(\varphi\pm\pi)J_{12}} \; e^{-i(\pi-\theta)J_{31}} \; e^{i(\varphi\pm\pi)J_{12}} \; e^{-\alpha_2(W)J_{03}} & , \operatorname{Re}\alpha_2 \geq 0 \\ e^{-i\varphi J_{12}} \; e^{-i\theta J_{31}} \; e^{i\varphi J_{12}} \; e^{i\alpha_2(W)J_{03}} & , \operatorname{Re}\alpha_2 < 0 \end{cases} \tag{II. 13}$$

in terms of the angles defined in (II. 6) and (II. 7). Corresponding to these boosts the centre-of-mass states take the form

$$|p_1\lambda_1 \,,\, p_2\lambda_2\rangle = U^{(1)}(L_{p_1})\, U^{(2)}(L_{p_2})\, |\lambda_1\lambda_2\rangle =$$

$$= \begin{cases} e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\alpha_1 J_{03}^{(1)} + i\alpha_2 J_{03}^{(2)}}\, e^{i\pi J_{31}^{(2)}}\, e^{i\varphi J_{12}} |\lambda_1,\lambda_2\rangle, & \mathrm{Re}\,\alpha_1 \geq 0,\ \mathrm{Re}\,\alpha_2 \geq 0 \\ e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\alpha_1 J_{03}^{(1)} + i\alpha_2 J_{03}^{(2)}}\, e^{i\pi J_{31}}\, e^{i\varphi J_{12}} |\lambda_1,\lambda_2\rangle, & \mathrm{Re}\,\alpha_1 < 0,\ \mathrm{Re}\,\alpha_2 \geq 0 \\ e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\alpha_1 J_{03}^{(1)} + i\alpha_2 J_{03}^{(2)}}\, e^{i\varphi J_{12}} |\lambda_1,\lambda_2\rangle, & \mathrm{Re}\,\alpha_1 \geq 0,\ \mathrm{Re}\,\alpha_2 < 0 \end{cases} \tag{II. 14}$$

Defining the quasi-states $|W\theta\varphi\,\lambda_1\lambda_2\rangle$ over the entire W-plane by

$$|W\theta\varphi\lambda_1\lambda_2\rangle = e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\alpha_1 J_{03}^{(1)} + \alpha_2 J_{03}^{(2)}}\, e^{i\pi J_{31}^{(2)}} |\lambda_1\lambda_2\rangle\, e^{i(\lambda_1+\lambda_2)\varphi} \tag{II. 15}$$

one can express the formulae (II. 14) in the form

$$|p_1\lambda_1, p_2\lambda_2\rangle = |W\theta\varphi\lambda_1\lambda_2\rangle \quad , \quad |W| \geq |m_1^2 - m_2^2|^{\frac{1}{2}}$$

$$|p_1\lambda_1, p_2\lambda_2\rangle = |W\theta\varphi{-\lambda_1}\lambda_2\rangle\, (-)^{s_1+\lambda_1}\, e^{2i\lambda_1\varphi}, \quad |W| < |m_1^2 - m_2^2|^{\frac{1}{2}},\ m_1 > m_2$$

$$|p_1\lambda_1, p_2\lambda_2\rangle = |W\theta\varphi\lambda_1{-\lambda_2}\rangle\, (-)^{s_1-\lambda_1}\, e^{2i\lambda_2\varphi}, \quad |W| < |m_1^2 - m_2^2|^{\frac{1}{2}},\ m_1 < m_2 \ .$$

(II. 16)

The quasi-states are by definition continuable in W and therefore the expressions (II. 16) constitute the rules for continuing two-particle centre-of-mass states across the boundary $|W| = |m_1^2 - m_2^2|^{\frac{1}{2}}$.

1) For the t-channel process $1 + 2 \to 3 + 4$ the angle $\theta_t(s)$ vanishes at $t = 0$ if $m_1 \neq m_2$ <u>and</u> $m_3 \neq m_4$. On the other hand, if $m_1 = m_2$ <u>or</u> $m_3 = m_4$ then $\theta_t(s) = \pm \pi/2$ at $t = 0$. If $m_1 = m_2$ <u>and</u> $m_3 = m_4$ then $\theta_t(s)$ depends on s at $t = 0$.

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8) Using the fact that $L_{\Lambda P} = \Lambda L_P R$ where R is a three-dimensional rotation whose precise form depends on the boosting conventions, one finds that the expressions (3.14) satisfy

$$W^N D^{\frac{N}{2}\frac{N}{2}}_{JM,00}(L_{\Lambda P}) = \sum_{J'M'} D^{\frac{N}{2}\frac{N}{2}}_{JM,J'M'}(\Lambda)\; W^N D^{\frac{N}{2}\frac{N}{2}}_{J'M',00}(L_P)$$

which transformation behaviour means that they are indeed proportional to symmetrical traceless tensors of rank N.

9) The non-unitary representation of the complex group so constructed contains just two unitary representations of the real subgroup, one with positive energies corresponding to particle states and the other with negative energies corresponding to antiparticle states. It is this characteristic which makes these representations important in the discussion of TCP.

10) If any of the eigenvalues should be degenerate at W = 0 then, as was found in the Proca example, the limit $W \to 0$ may produce non-vanishing mixing angles. We shall ignore this possibility in future

11) Conceivably one could postulate the existence of "analytic" fields, $\phi^{\tau}_{(j_0 j+\kappa)\eta jm}$, defined for complex j and $\tau = \pm 1$ interpolating the points $j = j_0, j_0 + 1, \ldots, (-)^{2\ell} = \tau$. However, for our present purposes where the concept of field is used only in its kinematical aspects - as a guide to formulating constraints - it is not necessary to make explicit use of such a radical invention.

12) Conclusions (1) and (2) were arrived at by Toller (Ref. 3). Similar conclusions to (3) - (5) have been obtained on the basis of a Bethe-Salpeter model by Domokos and Surànji (Ref. 4).

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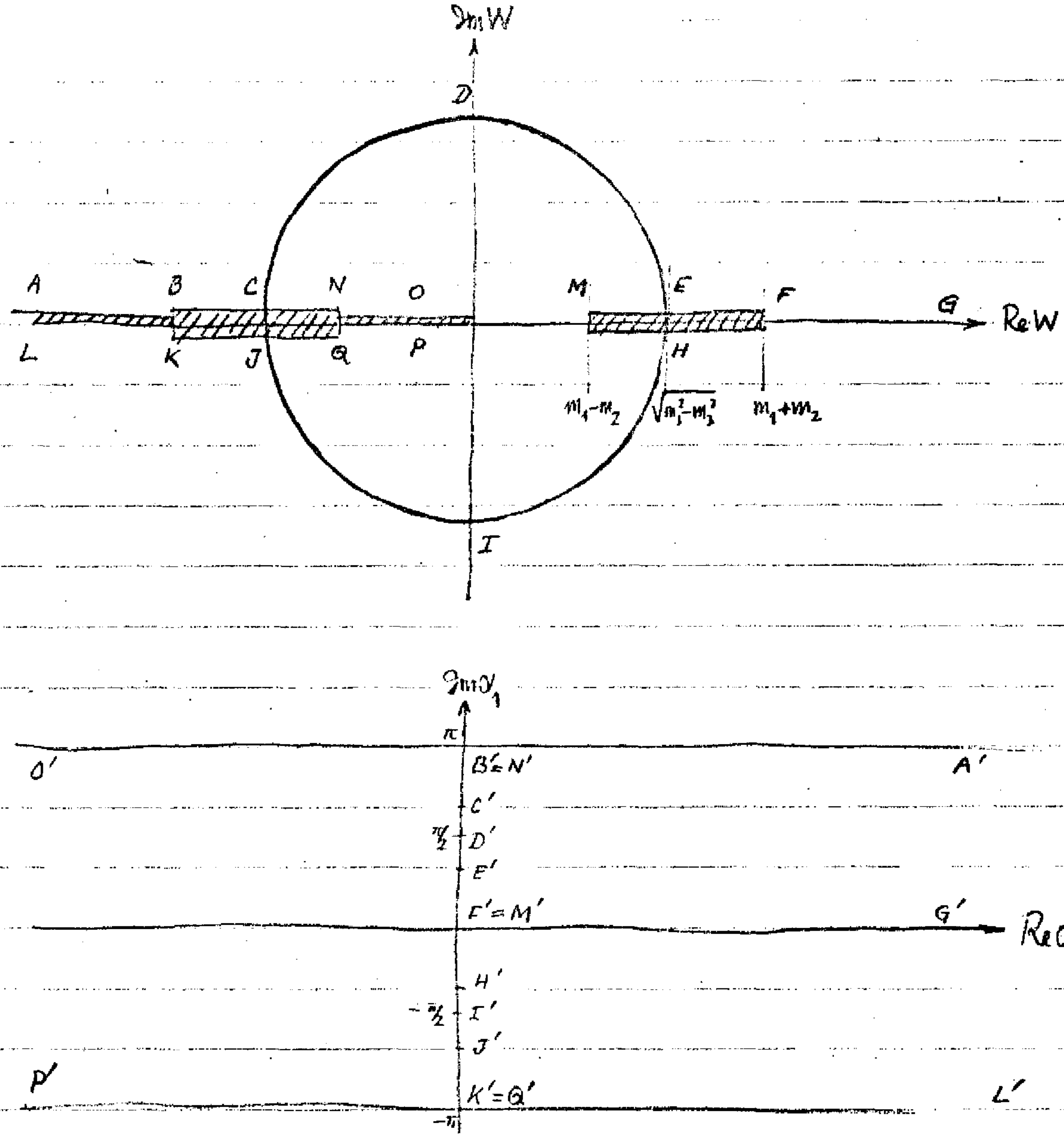

Fig. 1

The mapping $W \to \alpha_1(W)$, $m_1 > m_2$.

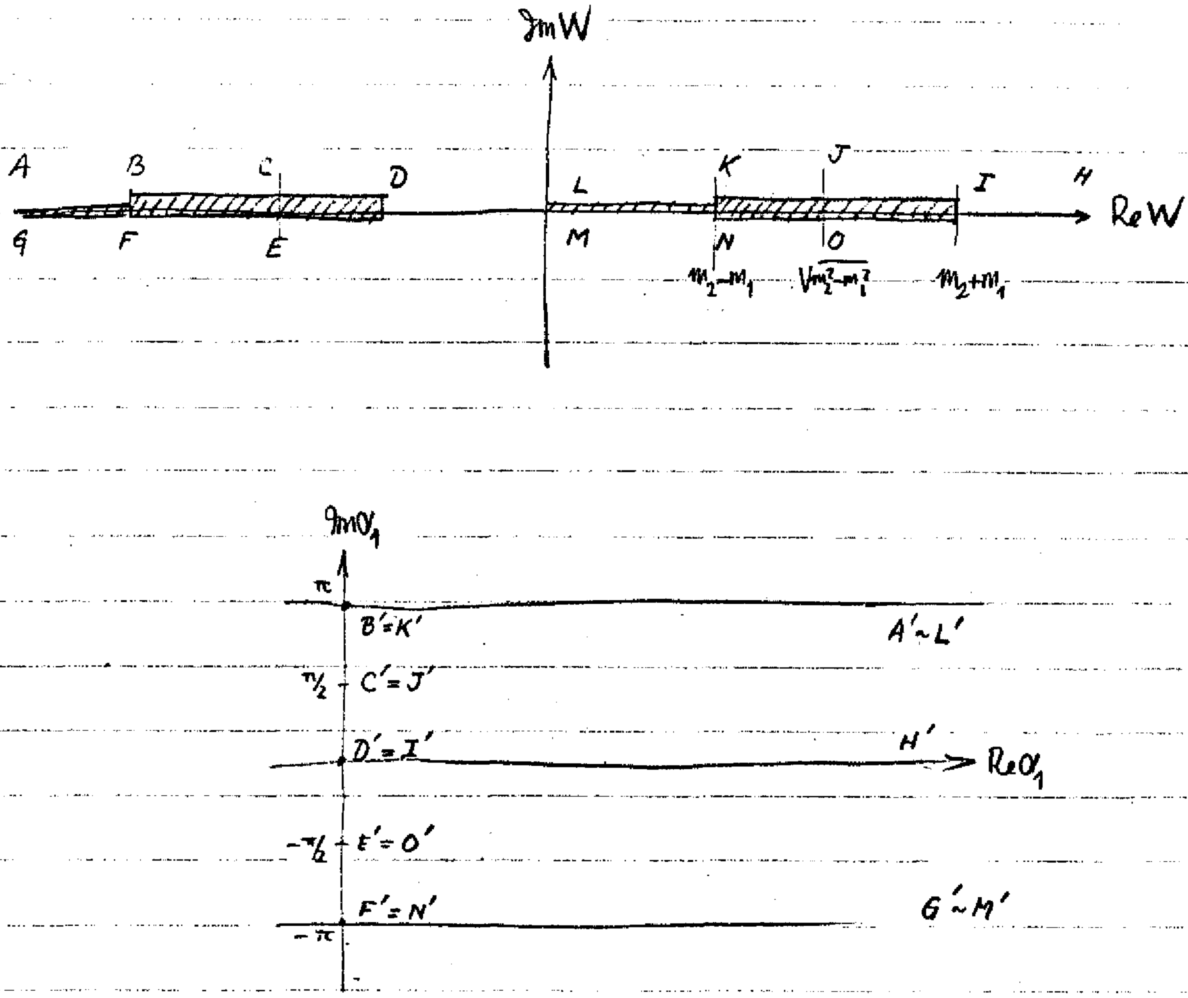

Fig. 2

The mapping $W \to \alpha_1(W)$, $m_1 > m_2$.

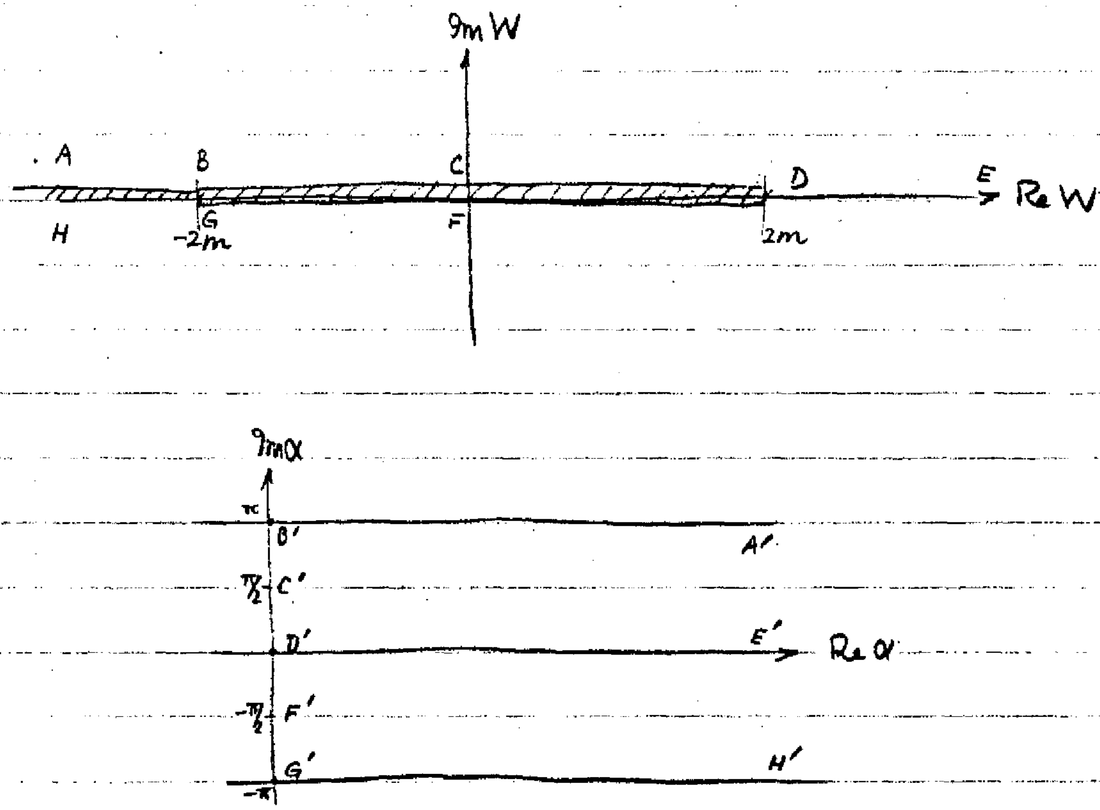

Fig. 3

The mapping $W \rightarrow \alpha_1(W) = \alpha_2(W)$, $m_1 = m_2 = m$.

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