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# THE SPACE-TIME CODE* †

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## ABSTRACT

It is known that the entire geometrical theory of a relativistic space-time can be summed up in two concepts, a space-time measure $\mu$ and a space-time causal or chronological order relation C ; in brief, a causal measure space. On grounds of finiteness, unity and symmetry, we argue that the observed macroscopic space-time may be the classical-geometrical limit of a causal quantum space. The necessary conceptual framework is provided. Mathematical individuals that naturally form causal spaces are symbol sets ordered by inclusion. The natural extension of this purely logical concept to quantum symbols is formulated. The problem is posed to give finite quantum rules for the generation of quantum symbol sets such that the order of generation becomes, in the classical limit, the causal order of space-time; as it were, to break the space-time code. The causal quantum spaces of three simple codes are generated for comparison with reality. The unary code (repetitions of one digit) gives a linearly ordered external world of one time dimension and a circular internal space. The binary code gives the future null-cone of special relativity and a circular internal space. The causal quantum space of "words" (sets of characters) in the binary code gives the solid light cone $t > (x^2 + y^2 + z^2)^{1/2}$ of special relativity and an internal space U(2, C) suitable for the description of charge and isospin. There is full translational and proper Lorentz invariance except at the boundary of the light cone, where the classical-geometrical limit fails. Plausible consequences of this model for cosmology and elementary particles are discussed. There is a quantum of time $\tau$ on the order of $\hbar/m_\pi c^2$, and the space-time complementarity relation $\Delta t\, \Delta x\, \Delta y\, \Delta z \gtrsim \tau^4$.

THE SPACE-TIME CODE

Until we find a satisfactory theory of space-time structure we shall be beset by the dilemma of the discrete versus the continuous, the dilemma already posed by Riemann in much the following terms:

i) A discretum has finite properties where a continuum does not. Natural quantities are to be finite.

ii) A discretum possesses natural internal structure. A continuum must have it imposed from without. Natural law is to be unified.

iii) A continuum has continuous symmetries where a discretum does not. Nature possesses continuous symmetries.

The third argument is especially serious for rotational symmetry which is much more difficult to counterfeit than translational. Subgroups can be found as dense as desired in the translation group that are not everywhere dense, but I do not think they exist for the rotation or Lorentz groups.

Since Riemann a new approach to this dilemma has become available. The same question about matter - is it finite or is it discrete - having been asked for two millenia, has in this century at last been answered: no. Matter is made neither of discrete nor wave-like objects but of quanta. Most fundamentally put, a quantum is a system whose mechanical properties form neither a discrete not a continuous Boolean algebra, but an algebra which is not even Boolean, being non-distributive. This non-distributive algebra is the algebra of subspaces of a separable Hilbert space and is naturally imbedded in an algebra of non-commutative quantities, the operators on that Hilbert space, which I call for brevity a quantum space. A quantum space is a third possibility for space-time too. This possibility would pass us cleanly between the horns of Riemann's dilemma:

i) A quantum space, like a discretum, has better convergence than a continuum - remember Planck and the Black Body.

ii) A quantum space, like a discretum, is born with internal structure and is even more unified, being coherent.

iii) A quantum space, like a continuum, possesses continuous symmetry groups.

The intrinsic structure Riemann meant for a discretum must have been like a checkerboard or a bathroom floor; a tesselation or graph in which the germs of a topology and a metric are present in the concepts of incidence and number. The world he faced was one three-dimensional continuum, space, changing in another one-dimensional continuum, time. Here too we have a better point of departure than Riemann. Since Einstein we have been confronted by just one four-dimensional continuum. More important, the structure of this continuum is not that of a changing metric space but that of a space with an order relation between its points, causal or chronological precedence $C$ . Alas, most of classical geometry ran off on the wrong road after dimension 1, from the point of view of relativity. The number line has been many things to many people: a metric geometry, a number field, an ordered space and so forth. Any of these exist in higher dimension. The development of higher-dimensional metric geometry flourished in the recognition that the world is a kind of higher-dimensional line and in the mistaken belief that the essential surviving property of the line is its metric structure. The important thing about the line for us is its order. The world is like a line, but in respect to the order structure, not the metric structure. For example the topology of space-time must be based on intervals $a \, C \, x \, C \, b$ , not balls $d(x, a) < r$ . Points at 0 interval can be as far apart as the stars. Evidently the propositional function of two points, $p \, C \, p'$ , expressing that $p$ is causally prior to $p'$ , is a simpler thing than an indefinite numerical distance function $d(p, p')$. We are unlikely to find an indefinite metric by counting squares on a space-time checkerboard and we are much better off hunting for structures that are born with order. The causal order $C$ determines the conformal structure of space-time, or nine of the ten components of the metric. The measure on space-time fixes the tenth component.

All the mathematical objects I can think of that are born with order and measure are sets of one kind or another ordered by inclusion and measured by counting. This leads to the idea that each point of space-time is some kind of assembly of some kind of thing and a point that is later is a point that is greater, regarded as such an assembly. The things have to be quanta if they are to be finite in kind, yet possess continuous symmetries and we could just as well call thèm chronons since their creation is to be the passage of time. However, they should not be ascribed mass or other mechanical properties, which are to emerge in a higher order of things, and it is safest to call them "digits". This reminds us of their abstract quality, forestalls meaningless questions and implies that their disjoint kinds are finite, like the binary or decimal digits. The basic object of the ordered quantum space, a quantum set of quantum digits, is then called a quantum "character".

This approach seems upside down from the point of view of general relativity; and general relativity seems upside down from here, seems too complicated a theory of too simple a thing to be fundamental rather than phenomenological. What is too complicated is the wedding-cake of laws that would have had to be legislated on the first day of creation: set theory, topology, differential manifolds and pseudometric geometry, with a sticky topping of quantization. What is too simple is the space-time point. It looks as if a point might be an enormously complicated thing. Each point, as Feynman pointed out, has to remember with precision the values of indefinitely many fields describing indefinitely many elementary particles; has to have data inputs and outputs connected to neighbouring points; has to have a little arithmetic element to satisfy the field equations; and all in all might just as well be a complete computer. Maxwell made his computer out of gears and idlers, Feynman is inclined towards digital rather than analogue components and I attempt to squirm between with quantum elements. But surely the laws of these complex structures should be simple.

So we are set the problem of breaking the space-time code: finding finite quantum rules for symbol generation such that the order of generation gives the causal order of space-time and thus the entire geometrical structure of space-time, in the classical limit.

The code seems unbelievably simple. If it were any simpler, there would be no space-time, just a one-dimensional time continuum. We always knew that the Lorentz group was as simple as could be - had Cartan's $A_1$ rating, so to speak; and presumably we were puzzled by the less compelling nature of the space-time signature +1-3, which is supposed to be theoretically prior. Perhaps here is the real reason for the dimension and signature of space-time. Working out the causal order of the three simplest codes, the first (characters in the unary code, repetitions of a single digit) gives the linear space-time, the second (characters in the binary code) gives the future null-cone $t = +(x^2 + y^2 + z^2)^{1/2}$ ; and the third (words in the binary code, admittedly chosen with the first two examples as a guide) gives the solid light-cone $t \geq +(x^2 + y^2 + z^2)^{1/2}$, with the Poincaré group as symmetry group as long as we stay away from the bounding cone where the classical limit breaks down and normal conceptions of space-time fail.

## ACKNOWLEDGMENTS

David Bohm said the world is a pattern of elementary causations at lunch in 1958. Richard P. Feynman said space-time points were like digital computers over dinner in 1961. Peter G. Bergmann said the signature of space-time must come from 2 x 2 hermitian matrices in class in 1964. Roger Penrose said all along that everything must be made of spinors. I came to believe them. It is not their fault.

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## I. THE ORDER OF SPACE-TIME

The classical space-times of special and general relativity may each be described completely by a measure space M and a partial ordering $p \, C \, p'$ (p "causes" $p'$, i.e., causally precedes $p'$) of the points $p, p'$ of M . The measure space gives the set theory and measure theory of space-time and finally determines $\sqrt{(-g)}$ . The causal relation $p \, C \, p'$ means that $p'$ is in the closed future light cone emanating from p and gives the topology, differential manifold structure and conformal geometry of space-time, finally determining $g_{\mu\nu}$ . Following Roger Penrose, we call a space with a causal order a causal space.

Here we consider new space-times likewise endowed with measure and order, but quantum rather than classical. By a quantum space we mean the $*$-algebra of operators on Hilbert space, thought of as containing the algebra of properties of an individual object. We suppose that the locational properties of a point in space-time form the same kind of structure as the mechanical properties of an elementary quantum mechanical system. Briefly,

$$\underline{\text{space-time is a quantum space } Q} \, . \tag{Q}$$

The quantum sets of a quantum space are the hermitian idempotents $\sigma$ of the $*$-algebra:

$$\sigma^2 = \sigma^* = \sigma \, .$$

The inclusion $\sigma \subset \sigma'$ of quantum sets is expressed by $\sigma'\sigma = \sigma$.

Two quantum sets $\sigma$ and $\sigma'$ are compatible, $\sigma \leftrightarrow \sigma'$, means the idempotents $\sigma, \sigma'$ commute: $\sigma\sigma' = \sigma'\sigma$ . The complement of a quantum set $\sigma$ is $1 - \sigma$ . The conjunction $\sigma \cap \sigma'$ is that idempotent included in both $\sigma$ and $\sigma'$ which is greatest in the sense of $\subset$ . The adjunction $\sigma \cup \sigma'$ is that idempotent including both $\sigma$ and $\sigma'$ which is smallest. The measure of a set $\sigma$ is $\operatorname{tr} \sigma$ and is normalized so that the two lowest values it assumes are 0 and 1. Then $\operatorname{tr} \sigma$ counts the maximal number of disjoint unit sets $s, s', \ldots$ in $\sigma$,

$$\operatorname{tr} s = 1 \; ; \; s \subset \sim s' \; ; \; s, s' \subset \sigma \, .$$

A frame means a maximal set of disjoint unit sets. The minimal $*$-algebra containing symbols $g$ obeying relations $\rho$ is called the $*$-algebra generated by $g$ and $\rho$ , $* \mathrm{alg}(g; \rho)$ . The things we call $*$-algebras are required to contain the number 1.

A unit set, or equivalently a vector in the Hilbert space on which the $*$-algebra $Q$ acts, represents a maximally precise determination of location in space-time.

## Quantum relations

Examples of quantum relations between two quantum objects from elementary quantum mechanics are

$e\, R_1\, e'$ : Electrons $e, e'$ have the same $L_z$ .

$b\, R_2\, b'$ : Baryons $b, b'$ are bound into a triplet-s internal state.

$p\, R_3\, p'$ : Particle $p$ has a greater kinetic energy than particle $p'$.

Classically, and we suppose quantally as well, a relation $R$ between two things $p, p'$ is a set $R$ in the space of pairs $(p, p')$ and $p\, R\, p'$ means the same as $(p, p') \in R$ . When $p$ is described by a quantum space $Q$ , the pairs $(p, p')$ are described by the tensor product $Q \times Q$ of the $*$-algebra $Q$ with itself, the tensor square, which is generated by two commuting replicas of $Q$ . If $\sigma$ is a set in $Q$ , we shall show which of the two points $p(1)$ , $p(2)$ of $Q \times Q$ is intended by a symbolic argument, like $\sigma(1)$. Thus if $\{s, s', \ldots\}$ is a frame for $Q$ , $\{s(1)\, s'(2)\}$ is a frame for $Q \times Q$ .

For a quantum space-time $Q$, then, any quantum relation between two points $p(1)$ , $p(2)$ is a hermitian idempotent expressible as a linear combination of products like $a(1)\, b(2)$ , where $a, b$ are quantities in $Q$. In particular, we suppose the causal relation $C$ is such.

The quantum theory of relations is more complicated (to me) than the classical theory because of the possibility of coherent superposition and the impossibility of perfect identity.

Quantum identity

We shall wish to speak, for example, of reflexive relations. Classically, that R is reflexive means

$$p = p' \ . \subset . \ p R p' \ .$$

This could be transcribed to quantum theory if we knew what = meant. Classically = is described by the diagonal set in the cartesian product

$$"=" = \bigcup_x \{x(1)\ x(2)\} \ .$$

Since we have no underlying set of x's to sum over, this definition can only be suggestive for the quantum case. For example, if s ranges over all unit sets

$$\bigcup_s s(1) \cap s(2) = 1 \ ,$$

the unit operator in $Q \times Q$, which is trivial,

We shall have to settle for identity with respect to some complete set of commuting quantities. By <u>an</u> identity relation I shall mean one of the form

$$I = \bigcup_s s(1)\ s(2)$$

where $\{s\}$ is a frame for Q .

By a reflexive quantum relation R, I mean one that follows from some quantum identity I

$$I \subset R \ .$$

This behaves well in the classical limit. Symmetry of a relation S has obvious quantum meaning: $p S p' = p' S p \equiv p S^T p'$ . Antisymmetry is a stronger form of reflexivity

$$p A p' \cap p' A p \subset p I p'$$

where I is an identity relation.

Transitivity is a concept of three-variable logic and the tensor cube $M(1) \times M(2) \times M(3)$

$$p(1)\ T\ p(2)\ \cap\ p(2)\ T\ p(3) \subset p(1)\ T\ p(3) \ .$$

We can define a partial ordering, at last, as a reflexive, antisymmetric, transitive quantum relation. It is a pity, but this concept does not behave well under conjunction. The conjunction of two reflexive relations which are not compatible is not reflexive, but can even be null. We shall not assume the causal order relation is a partial ordering.

<u>Strong relations</u>

I call a quantum relation R strong when it is incompatible with any separate identity relation for either member.

$$p\ R\ p' \leftrightarrow p\ I\ p'' \ , \ p''\ I\ p' \ .$$

The true quantum relations are strong. For two quanta to be bound into the ground state of the hydrogen atom is a strong relation between them. The causal relation between space-time points proposed later is a strong relation.

<u>Internal co-ordinates</u>

While classically the causal relation C is a partial ordering, we have seen that this concept does not generalize well to quantum theory and I shall soften the requirements on the quantum relation C .

As further rationalization, I point out that there is abundant indication that the physical relation C is not antisymmetric at all. Call a unitary transformation U of M an <u>internal</u> symmetry if U leaves C unaffected in the sense that

$$U(1)\ (1C2)\ U(1)^{*} = U(2)\ (1C2)\ U(2)^{*} = 1C2 \ .$$

This is not to be confused with the milder concept of a symmetry transformation, which leaves C invariant in the sense that

$$U(1)\ U(2)\ (1C2)\ U(2)^{*}\ U(1)^{*} = 1C2 \ .$$

There are many transformations in elementary quantum mechanics that are internal in this sense. For example, isospin rotation and charge conjugation seem to have no effect upon causal dependency, do not affect the metrical relations between objects in space-time. This is why they are called internal. Dropping the antisymmetry of C makes room for such transformations in the geometrical foundations of physics. Such quantum orderings correspond approximately to higher-dimensional geometries in classical theories of space-time. I posit, in brief, that:

<u>the causal relation is a transitive quantum relation C .</u> (C)

<u>Quantum co-ordinates</u>

By a co-ordinate in a classical or quantum space-time we mean a real quantity on the measure space. In quantum measure spaces, $*$-algebras, this concept coincides with that of an arbitrary self-adjoint q-number in the $*$-algebra itself. If f is such a co-ordinate and $\sigma$ is a set, classical or quantum, f takes on a definite value on $\sigma$ if and only if

$$f \sigma = \lambda \sigma$$

and then $\lambda$ is the value. In general a co-ordinate f has an expectation value on each set $\sigma$, given by

$$< f > = \operatorname{tr} \sigma f / \operatorname{tr} \sigma \ .$$

An external co-ordinate system means a set of co-ordinates $\{x\}$ that determine the causal relation in the sense that $p(1)\, C\, p(2)$ can be expressed in terms of the $\{x(1), x(2)\}$. (More precisely, everything that commutes with the $x(1)$, $x(2)$ commutes with C .)

An internal co-ordinate y is one compatible with the causal relation, or, equivalently, one that generates an internal symmetry.

These definitions should be taken with care. An internal co-ordinate can be part of an external co-ordinate system. An external co-ordinate as such is not defined.

Every co-ordinate x determines a binary relation $L_x$ : $x(1) < x(2)$ ("less in x") represented by the same element of $Q(1) \times Q(2)$ as the function $\theta(x(2) - x(1))$ where $\theta$ is the step function

$$\theta(x) = 1 \ , \quad x \geq 0 \ ,$$
$$= 0 \ , \quad x < 0 \ .$$

Evidently $L_x$ is a transitive relation and is compatible both with $x(1)$ and $x(2)$ .

Let us call a co-ordinate $t$ a time co-ordinate if it is greater for points which are later; more exactly, if $C \subset L_t$ .

The time co-ordinates make up a convex subset of the $*$-algebra $Q$ . I shall call the extreme points of this convex set <u>pure</u> times (following the old terminology for density matrices, which likewise form a convex set in their $*$-algebra).

In particular the time co-ordinates admitted by the causal ordering of special relativity likewise form a convex set and the "pure" times are the null co-ordinates, like $t - x$ .

## II. THE ORDER OF CHARACTERS

The most restrictive possible model for $C$ is the order relation of a code. A code is a set of primitive symbols that I shall call "digits", together with a rule for assembling these. This rule defines an order relation for the assemblies, which I call "characters"; namely, the order of generation. One character $X(1)$ is put before another $X(2)$ in this ordering if $X(2)$ can be produced from $X(1)$, which requires that $X(1)$ be a part of $X(2)$ . Let us formulate a quantum code.

By a quantum code I mean the causal quantum space $X$ of an object $X$ called a quantum character which is a quantum set of an unspecified number of quantum individuals called digits. The order relation in $X$ , which I call $P$ , means that one character $X(1)$ is "part of" another character $X(2)$.

Let $D$ be the $*$-algebra of one quantum digit and let the quantum numbers of this digit be represented as operators on a finite-dimensional

Hilbert space $\{\delta\}$. A vector $\delta$ represents a maximally precise specification of the digit. To form X, the $*$-algebra of sets of such digits, we apply what is usually called Bose-Einstein quantization. The vectors $\delta$, which already have the repertory of operations that define a Hilbert space, are given in addition the operations of multiplication and adjoint of a $*$-algebra, $\delta_1 \delta_2$ and $\delta^*$. X is the $*$-algebra generated by D and the commutation relations

$$\begin{aligned} \delta_1^* \delta_2 - \delta_2 \delta_1^* &= \tau (\delta_1 | \delta_2) , \\ \delta_1 \delta_2 - \delta_2 \delta_1 &= 0 , \end{aligned} \qquad (\rho_c)$$

using the inner product $(|)$ of D. A vector $\delta$ given such algebraic structure is also called a creator of a digit in the quantum set $|\delta)(\delta|$ of D. $\tau$ is a factor put in to help define the classical limit $\tau \rightarrow 0$. It appears later as a quantum of time.

The null word $\phi$ (more correctly, the property of being null) is the unit quantum set of X defined by $\delta^* \phi = 0$ for all $\delta$. The explicit power series solution of this equation gives

$$\phi = \sin(2\pi t/\tau)/(2\pi t/\tau)$$

where

$$t = \Sigma \, \delta \, \delta^* ,$$

summed over an orthonormal basis $\{\delta\}$ of D.

As in any quantum space, the quantum sets of characters, the hermitian idempotents of X, are ordered by inclusion:

$$\sigma \subset \sigma' \quad \text{means} \quad \sigma\sigma' = \sigma .$$

This has absolutely nothing to do with the relation P we seek, which is not of the quantum sets of X but of individual characters. The individual described by X happens itself to be an assembly of other individuals and this is therefore a problem in a higher-order quantum logic than the simple propositional calculus of $\subset$, $\cup$, $\cap$, $\sim$. The classical limit of the relation P is quite clear: the individual is still a kind of set and the partial ordering of such sets defines a partial ordering P of such individuals. For example, there is no doubt about when one set of classical co-ordinates and momentum values

$$(x_1 , p_1 ; x_2 , p_2 ; \dots ; x_m , p_m)$$

is part of another such set

$$(x_1' , p_1' ; x_2' , p_2' ; \dots ; x_n' , p_n') .$$

It is when the individual cannot be represented by a point in a classical set that any problem arises.

Each creator $\delta$ creates a certain digit connected with that creator and each $\delta\delta^*$ is a quantum co-ordinate in X related to how many of the $\delta$ kind of digit appears in the character. Each $L_{\delta\delta^*}$ is a well-defined ordering of X , expressing the relation of having less of the $\delta$ kind of digit. We define the quantum ordering by digit count, $\chi(1)\ P\ \chi(2)$ (character 1 is "part" of character 2) to mean

$$P \equiv \bigcap_{\delta} L_{\delta\delta^*} .$$

The conjunction is taken over all $\delta$ . The null character $\phi$ , the "digit vacuum", is part of every character in the sense that

$$\chi(1) \in \emptyset . \subset . \chi(1)\ P\ \chi(2)$$

and no other character is part of the null character:

$$\chi(1)\ P\ \chi(2) \cap \chi(2) \in \emptyset : \subset : \chi(1) \in \emptyset .$$

Unary code

We first compute the weak ordering of characters in one digit $\delta$ . It is a review of the harmonic oscillator.

The $*$-algebra D of the digit is now the complex plane K , the idempotents of K being 0 , "$\delta$ does not exist" and 1, "$\delta$ exists". X, the

$\ast$-algebra of characters, is generated by one creator $\delta_0$ and the relation

$$\delta_0^* \delta_0 - \delta_0 \delta_0^* = \tau .$$

The digit count (times $\tau$) is

$$t = \delta_0 \delta_0^*$$

and the weak ordering P that provides the chronological order of this little world is

$$P = L_t = \theta(t(2) - t(1)).$$

Then the null character $\phi$ , the element of X which obeys

$$\phi^2 = \phi^* = \phi$$

and

$$\delta_0^* \phi = 0 ,$$

is uniquely defined by these conditions as the power series

$$\phi = \frac{\sin 2\pi (t/\tau)}{2\pi (t/\tau)} .$$

This element of the $\ast$-algebra X indeed annihilates the unit sets $t = \tau, 2\tau, \ldots$ and preserves the unit set $t = 0$ . (It helps to recall that $(\sin 2\pi n)/(2\pi n)$ is the projector on the ground state of the harmonic oscillator, where $n = a^* a$.) A frame for X is $\{\delta^{*n} \phi\}$ and the chronological ordering of these unit sets is shown in Fig. 1.

This is evidently a trivial kind of partially ordered space for our purposes with too simple a structure. It is not quite as simple as Fig. 1 would suggest. Fig. 1 shows a line and this space has to be called two-dimensional, I think. I have not yet been able to formulate the concept of dimension for causal quantum spaces, but in the classical limit the algebra becomes that of complex functions on K and the underlying set and measure theory is that of the complex plane. The ordering P is by radius and a neighbourhood of a point is an annulus centered about the origin. Therefore points at the same distance from the origin cannot be separated causally. The radius r is the external co-ordinate. It is a Newtonian sort of world in that there is an absolute time and in that causal effects can propagate with infinite speed in one of the co-ordinates, the "internal co-ordinate", which may be taken to be a polar angle $\theta$ . The transformations $\delta_0 \to e^{i\alpha}\delta_0$ are internal transformations and they are generated by the time $t$ .

Binary code

Let the digit now have two states. The creator $\delta$ is a two-component vector $\delta^{\alpha}$ , $\alpha = 0, 1$ . For any such $\delta$ there is an associated co-ordinate $\delta\delta^*$ and an order relation $L_{\delta\delta^*}$ . The ordering $P$ of $X$ is to be

$$\begin{aligned} P &= \bigcap_{\delta} L_{\delta\delta^*} \\ &= \bigcap_{\delta} \theta(\delta\delta^*(2) - \delta\delta^*(1)) \\ &= \bigcap_{\delta} \theta(\Delta\delta\delta^*) \; . \end{aligned}$$

Here and in the following, $\Delta f \equiv f(2) - f(1)$. Let $c$ be a generic complex two-vector $(c_\alpha)$:

$$P = \bigcap_{c} \theta(c_\alpha \Delta\delta^{\alpha}\delta^{\beta *} c^*_\beta) \; .$$

Thus is the order of two characters $X(1)$, $X(2)$ determined by the relative value of the respective quantities

$$\delta^{\alpha}\delta^{\beta *} = T^{\alpha\beta^*} \equiv (T) \; .$$

$T^{00*}$ counts 0's , $T^{11*}$ counts 1's and $T^{12*}$ tells about coherent superpositions of 0 and 1 . $X(1)\,P\,X(2)$ means that for all $c$ , $c\,\Delta T\,c^* \geq 0$ . In the classical limit, where the four quantities $T^{\alpha\beta *}$ commute, the conclusion is swift. The condition on $\Delta T$ is invariant under $\Delta T \to \Lambda\Delta T\Lambda^*$ , where $\Lambda$ is any matrix of $GL(2, C)$ , for $\Lambda$ simply shuffles the $c$'s . The condition is therefore a function of the invariants of $\Delta T$ under $GL(2, C)$, which reduce to the signs of the eigenvalues of $\Delta T$ . The eigenvalues must be non-negative. That is, $P$ means

$$\mathrm{tr}\,\Delta T \geq 0 \; , \quad \det \Delta T \geq 0 \; .$$

Since the $\Delta T^{\alpha\beta *}$ are not hermitian we introduce, for convenience, co-ordinates t x y z through

$$T = \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix}$$

The relation P becomes

$$\Delta t \geq 0 \quad , \quad \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \geq 0 \quad .$$

This is quite unexpected and pleasing. The first non-trivial code we try yields the order of the future null-cone of special relativity. (We have $\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0$ as an identity in the classical limit.) The $(t, x, y, z)$ are an external co-ordinate system. The fourth co-ordinate (loosely speaking, the one missing from the $(t, x, y, z)$) is totally irrelevant to the ordering P which is our candidate for the chronological ordering C , and is readily shown to be an angle invariant under $\Lambda$ .

The transformation $T \rightarrow \Lambda T \Lambda^*$ which leaves the ordering P invariant in the classical limit leads us to suspect the quantum space X might be Lorentz invariant. Indeed there is a unitary transformation of the entire X algebra that leaves P invariant and sends $T \rightarrow \Lambda T \Lambda^*$ . In fact we have here the Majorana representation of the Lorentz group in thin disguise. To expose it we define quantities

$$\psi^0 = (\delta^0 - i\,\delta^{1*})/\sqrt{2}$$

$$\psi^1 = (\delta^1 + i\,\delta^{0*})/\sqrt{2}$$

and their adjoints. The $\delta^\alpha$ were like creators of spin-1/2 bosons. The $\psi^\alpha$ obey the Majorana commutation relations $[\psi^0, \psi^1] = -i$ , $[\psi^\beta, \psi^{\alpha *}] = 0$. $(\rho_M)$ . The relations $\rho_M$ are invariant under the SL(2, C) transformation $\Psi \rightarrow \Lambda \Psi$ . Since the $\psi$ generate the $*$-algebra X, there exists a unitary transformation $U(\Lambda)$ accomplishing this transformation:

$$U \psi U^* = \Lambda \psi \; .$$

Moreover, the new order is equivalent to the old order:

$$L_{\psi\psi^*} = L_{\delta\delta^*} \ .$$

Covariant external co-ordinates are

$$x^{\alpha\beta^*} = \psi^{\alpha} \psi^{\beta^*} = x^{\mu} \sigma_{\mu}^{\ \alpha\beta^*} .$$

Because of the relations $\rho_M$ ,

$$x^{\mu} x_{\mu} = -\tau^2$$

which approaches $x^{\mu} x_{\mu} = 0$ when $\tau \to 0$ .

The point is that this Lorentz invariance of X guarantees that of the measure in the classical limit, which must therefore approach the relativistic measure on the light cone

$$d\mu = dx\,dy\,dz/t \ .$$

The space X might have been conformally flat and still have metrical curvature. It is flat.

I cannot refrain from pointing out that in the classical limit

$$X(1) \epsilon\, \emptyset . \subset . X(1)\ P\ X(2) \ ;$$

that is, there is a character in the beginning. The character is $\emptyset$ . For $\tau > 0$ , however, no one character occupies this role because of complementarity.

## III. THE ORDER OF WORDS

We now consider incoherent superpositions of characters. This means that we construct an assembly of characters, a "word", and regard the assembly averages as properties of a single character, a random character of the assembly. This entails disregarding all quantities involving more than one character, "interactions" between characters. It suffices first to consider an assembly of two characters.

The calculation of the causal measure space following from the symmetric binary code immediately suggests a candidate for the causal measure space of special relativity. By the causal quantum space W of two character words in the symmetric binary code, we mean the quantum space X(1) x X(2) where X is the quantum space of characters in the symmetric binary code, with the order relation P based on the sums of the digit counts for X(1) and X(2). Since each X separately gives a future null vector in the classical limit, X(1) x X(2) gives the sum of two future null vectors for an external co-ordinate system and such a sum ranges over the solid cone

$$t > \sqrt{(x^2 + y^2 + z^2)} .$$

The causal order being Minkowskian, we have simply to calculate the measure to see if the space is metrically as well as conformally flat. By Lorentz covariance, the trace in the classical limit,

$$\mathrm{tr}\, f = \int (d\,\delta)\ f(\delta_j^\beta)$$

must have the form

$$\mathrm{tr}\, f = \int (dx)\ \rho(x^2)\, f(x^\mu).$$

Here $(d\delta)$ is the product of four elements of area, one from each of the complex $\delta_i^\alpha$ planes, and therefore is of degree 8 in the $\delta$, while (dx) is the Minkowski measure, of degree 4 in the $x^\mu$, which are of degree 2 in the $\delta$. Therefore $\rho$ is of degree 0. Since when $\tau \to 0$ there are no constant lengths left in the theory, we must have $\rho(x^2)$ = const. The measure is Minkowskian, the space is flat.

The internal space can be represented as the collection of complex numbers $\delta_i^\alpha$ that all map into one $x^\mu$. Without loss of generality the point $x^\mu = (1000)$ can be taken and a simple calculation shows the internal space has the structure of U(2, C). The word in the beginning, of course, is $\emptyset\emptyset$.

## IV. THE ORDER OF SPACE-TIME

The path from fundamental principles to observational conclusions has been so unexpectedly short and the conceptual economy so great, that I am obliged to regard the principles as deserving more intensive study. I must propose that the structures we ordinarily identify as single points of space-time are, or are reached by, approximately incoherent assemblies of sets of binary elements to which ordinary quantum principles apply and that a point we regard as later is an assembly of greater sets. The immediate consequences, we have seen, are the four-dimensionality of space-time, the signature +1-3 of its pseudometric structure, the existence of an origin and a bounding null-surface for the universe at which the classical approximation breaks down, the Poincaré invariance of special relativity away from the boundary region and the existence of a certain internal space. At the present epoch T the number of binary elements in the sets must be on the order of $T/\tau$, where $\tau$ is the quantum of time we have introduced.

## V. THE SIZE OF THE CHRONON

Principles seem to have led me on a strange trip again. They still tug, gently and mistakably, in a quarter on which, I suspect, the theory of gravity lies. It is scary going fast in new water; please let me stop and play awhile – in supernatural units, where $\hbar = c = 1 \sim 2\pi$.

There are already three conspicuous traces of the chronon size about us, leading to the estimate $\tau \sim 10^{-23}$ sec.

The mass spectrum

Whatever matter is, when it moves in the pinball machine of Fig. 2 it moves in a periodic system. Even near the classical limit this period shows up as a periodicity of $t$ of size $\tau$ and the four periodicities of the internal space $U(2, C)$. In the space of propagation vectors of matter there will therefore be bands of transmission and bands of rapid attenuation by Bragg scattering. Since the system is Lorentz invariant the band structure must be Lorentz invariant and therefore will be made up of mass shells. This presumably gives the mass spectrum of the elementary particles. The size of the first gaps will be determined by the period $\tau$. Taking the $\mu$ meson as typical, we have

$$\tau \sim 1/m_\mu$$

The size of the nucleus

The space-time co-ordinates obey an uncertainty relation. The canonical volume element $(d\delta)$, the product of 4 complex differentials $d\delta_i^\alpha$ and their complex conjugates, directly gives the number $dn$ of disjoint unit quantum sets in a region of classical space

$$dn = (d\delta)/(2\pi\tau)^4 .$$

The relation between $\delta$ and the co-ordinates $x, y$ is simplest said by thinking of $\delta$ as a $2 \times 2$ complex matrix in the classical limit. Then

$$x = \delta\delta^H$$

where $\delta^H$ is the transposed matrix of complex conjugate elements: $H = CT$. Let the unique polar factorization of $\delta$ be

$$\delta = \xi y ,$$

$\xi$ positive definite, $y$ unitary. Then $y$ makes a fine internal co-ordinate and the full co-ordinate transformation is

$$x = \delta\delta^H \quad , \quad y = \delta(\delta\delta^H)^{-1/2}$$

and conversely

$$\delta = x^{1/2} y \quad .$$

Thus integrals can be transformed according to

$$\int (d\delta) \sim \int (dx) \int (dy)$$

where $\sim$ conceals a pure numeric and $(dy)$ is an element of volume in $U(2, C)$. For functions $f(x)$ of the external co-ordinates alone,

$$\int (d\delta)\, f \sim \int (dx)\, f \quad .$$

Thus the number of disjoint unit quantum sets in a cell $\Delta t\, \Delta x\, \Delta y\, \Delta z$ is

$$\Delta n \sim \Delta t\, \Delta x\, \Delta y\, \Delta z \,/\, \tau^4 \quad .$$

The minimum space-time volume per unit set is thus $\Delta t\, \Delta x\, \Delta y\, \Delta z \sim \tau^4$.

There seems no experimental obstacle to ascertaining that an event, say a photoproduction, took place in a nuclear volume $\sim r_0^{\,3}$ in the transit time $r_0$ of a photon. It follows that

$$r_0^{\,4} > \tau^4 \quad .$$

Even if $10^4$ events could be localized in that region it would lower the bound on $\tau$ only by 10. I take it therefore that

$$\tau \sim r_0 \quad .$$

<u>High-energy cross-sections</u>

Look at two billiard balls in their centre-of-mass frame, approaching each other with Einstein factors $\gamma >> 1$. If their transverse cross-section is $\sigma \sim a^2$, their longitudinal radius shrinks to $a/\gamma$ and the maximum time of contact is likewise $t \sim a/\gamma$. The maximum 4-volume of intersection, thought of as defining the event of head-on collision, is then $\sigma^2/\gamma^2$. For sufficiently large $\gamma$ this must eventually become less than $\tau^4$ and head-on collision becomes a logically impossible*) event. There should be a pronounced preference for small-angle scattering, then, when

$$\gamma > \sigma / \tau^2 \quad .$$

*) I mean the conjunction or intersection of the two world-tubes vanishes, $A \cap B = 0$, not that they are disjoint $A \perp B$. The event occurs with reduced amplitude, experimentally speaking. This kind of impossible we do every day, it just takes a little longer.

## REFERENCES


1. For the algebra of quantum sets or properties or propositions, see J. M. Jauch, Foundations of Quantum Mechanics, 1968. The first I know of to suggest space-time was a quantum space is Hartland Snyder, Quantized Space-Time, Phys. Rev. 71, 38 (1947). Commutation relations of Snyder's form

$$[x^\mu , x^\nu] = \text{const } J^{\mu\nu}$$

are valid for our $x^\mu$ , but there must be a difference in sign somewhere: we get a discrete time and a continuous space; he gets the reverse.

2. Spaces getting their topology from an order relation are in Garrett Birkhoff, Lattice Theory, 1937, 1948. The important unifying role of this concept for relativity has been stressed by Roger Penrose.

3. Odd statistics lead to a short-lived world; classical Maxwell-Boltzmann statistics lead to one with a growing number of spatial dimensions. Even statistics are just right.

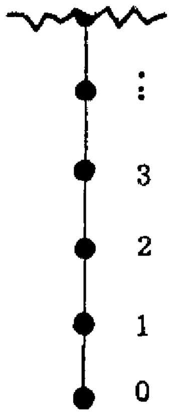


Fig. 1

The quantized time axis

This graph defines the causal order of a causal quantum space whose classical limit is the real time axis. The quantum space is isomorphic to the linear harmonic oscillator and is the space of a character in the quantum unary code. A dot • is an element of a frame in the Hilbert space. An upward line segment is a causal order relation. In the classical limit this space becomes the positive time-axis with the usual causal order and measure and an internal space in the form of a circle arising from the possibility of coherent superpositions of adjoint elements.

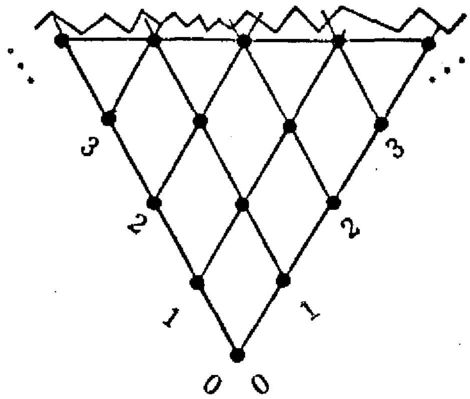


Fig. 2

The quantized null-cone

This graph indicates the causal order of a causal quantum space whose classical limit is the future null-cone of special relativity. The quantum space is isomorphic to the two-dimensional harmonic oscillator and is the space of a character in the binary symmetric quantum code. Dots • and line segments mean the same as in Fig. 1. In the classical limit the space becomes the three-dimensional future null-cone of special relativity with its usual causal order and measure and an internal space in the form of a circle. The apparent doubling of dimensions arises from the possibility of coherent superpositions of near elements.

While no element in the graph is Lorentz invariant, the Lorentz group can act on the space through unitary transformations that leave the causal order invariant, in fact by the Majorana representation of the proper Lorentz group. In this model, the exact isotropy of space-time is a quantum effect due to the existence of coherent superpositions of near elements in the diagram, while incoherent superpositions then provide points off the null-cone and account for the homogeneity of space-time.

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