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A LAGRANGIAN FORMULATION OF THE JOOS-WEINBERG WAVE EQUATIONS FOR SPIN-S PARTICLES

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ABSTRACT

The Lorentz covariant, spin-s, Joos-Weinberg equations are derived from a Lagrangian. This Lagrangian is written entirely in terms of an auxiliary field $\phi(x)$ and the Joos-Weinberg wave function $\psi(x)$ is defined in terms of $\phi(x)$. The function $\psi(x)$ describes a free, massive, spin-s particle, while $\phi(x)$ describes a free, massive, spin-s particle "decorated" with massless, spin-s neutrinos. The question of whether $\psi(x)$ or $\phi(x)$ is the wave function corresponding to reality is discussed. The interpretation of $\psi(x)$ as the actual wave function is favoured here.

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I. INTRODUCTION

Equivalent, Lorentz covariant, spinor formulations of the theory of a free mass-m, spin-s particle have been given by DIRAC[1], FIERZ and PAULI[2], RARITA and SCHWINGER[3], BARGMANN and WIGNER[4], JOOS[5], WEINBERG[6] and WEAVER, HAMMER and GOOD[7]. In fact it has been shown by PURSEY[8] that an infinite number of covariant formulations exist. These covariant formulations have the property that, for $s > 1$, the wave function must satisfy a wave equation plus one or more auxiliary conditions which are required to guarantee a physical mass and/or the correct spin.*)

Since the spin-s wave function must satisfy a set of auxiliary conditions in addition to the wave equation, a simple Lagrangian approach to these formulations does not work. Lagrangian approaches to the Dirac-Fierz-Pauli equations, Rarita-Schwinger and tensor equations for half-integral and integral spin, respectively, and the Bargmann-Wigner equations have been given by FIERZ and PAULI[2], CHANG[11] and GURALNIK and KIBBLE[12], respectively. These approaches all make use of one or more auxiliary fields which appear in the Lagrangian, along with the wave function, in order to get the wave equations and auxiliary conditions. The auxiliary fields vanish in the free-particle case but do not in the presence of an electromagnetic field.

Here a Lagrangian approach to the Joos-Weinberg equations is developed. The Lagrangian is written entirely in terms of an auxiliary field $\phi(x)$, and the wave function $\psi(x)$ is defined in terms of $\phi(x)$. While $\psi(x)$ is taken to be the actual wave function here, this question is open to

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*) While the Hamiltonian form of the Weaver-Hammer-Good formulation involves a unique[9] Hamiltonian without auxiliary conditions, the manifestly covariant form[10] does involve a wave equation plus an auxiliary equation, in the form of the Klein-Gordon equation, to guarantee a physical mass.

debate. HAMMER, McDONALD and PURSEY[13] derive essentially the same equations for $\phi(x)$ by a different method and interpret $\phi(x)$ as the wave function. The difference is this: For $s > 1$, $\psi(x)$ describes a free, mass-m, spin-s particle; on the other hand, $\phi(x)$ describes a free, mass-m, spin-s particle with spin-s, massless particles mixed in; that is, the massive particle is "decorated" with neutrinos. For $s = 1/2, 1$, $\psi(x)$ and $\phi(x)$ are identical and describe a simple massive particle without neutrinos. The $\phi(x)$ used here is slightly different from the Hammer-McDonald-Pursey $\phi(x)$, but this does not affect the discussion for the arbitrary spin case.

In the following, two a priori restrictions are made. The equations are not to involve operators of the form $(\partial_\mu \partial_\mu)^{\frac{1}{2}}$ or $i\partial/\partial t/\sqrt{-\nabla^2 + m^2}$, the energy sign operator. The former restriction necessitates slightly different treatments for integral and half-integral spin. The latter prevents the application of this method to the Weaver-Hammer-Good covariant formalism for integral spin. The Joos-Weinberg and Weaver-Hammer-Good equations are identical for half-integral spin and differ only by the presence of the energy sign operator in the Weaver-Hammer-Good integral spin equations. These restrictions are made because, while both operators are well defined in the free-particle case, they may not be so when electromagnetic interactions are considered.

II. HALF-INTEGRAL SPIN

The Joos-Weinberg equations for a particle of mass m and half-integral spin s may be obtained from the following Lagrangian:

$$L = \tfrac{1}{2}\, \partial_{\mu_1} \cdots \partial_{\mu_{s-\frac{1}{2}}} \bar{\phi}(x)\, \gamma_{\mu_1 \ldots \mu_{2s}}\, \partial_{\mu_{s+\frac{1}{2}}} \cdots \partial_{\mu_{2s}} \phi(x) -$$

$$- \tfrac{1}{2}\, \partial_{\mu_1} \cdots \partial_{\mu_{s+\frac{1}{2}}} \bar{\phi}(x)\, \gamma_{\mu_1 \ldots \mu_{2s}}\, \partial_{\mu_{s+\frac{3}{2}}} \cdots \partial_{\mu_{2s}} \phi(x) +$$

$$+ m\, \partial_{\mu_1} \cdots \partial_{\mu_{s-\frac{1}{2}}} \bar{\phi}(x)\, \partial_{\mu_1} \cdots \partial_{\mu_{s-\frac{1}{2}}} \phi(x) \quad , \qquad (1)$$

where $x_\mu = (\vec{x}, it)$ and $\partial_\mu = \frac{\partial}{\partial x_\mu}$. The $2(2s+1) \times 2(2s+1)$ covariant matrices $\gamma_{\mu_1 \ldots \mu_{2s}}$ are symmetric with respect to the interchange of any two tensor indices and are the generalizations of the Dirac γ-matrices discussed by WEINBERG [6] and BARUT, MUZINICH and WILLIAMS [14]. The matrix $\phi(x)$ is a $2(2s+1)$ column matrix and

$$\bar{\phi}(x) = \phi^\dagger(x)\, \gamma_{4 \ldots 4} ,$$

where $\dagger$ denotes hermitian conjugation.

The function $\phi(x)$ is the massive, neutrino "decorated", spin-s particle wave function considered by HAMMER, McDONALD and PURSEY [13]. Its transformation properties are complicated because of the mixture of massive and massless particles. It transforms according to a generalized Lorentz transformation Λ given by HAMMER, McDONALD and PURSEY [13] and

$$\Lambda^{-1} \gamma_{\mu_1 \ldots \mu_{2s}} \Lambda = a_{\mu_1 \nu_1} \ldots a_{\mu_{2s} \nu_{2s}} \gamma_{\nu_1 \ldots \nu_{2s}} ,$$

where the $a_{\mu\nu}$ are the ordinary four-vector transformation coefficients. Essentially Λ is an ordinary Lorentz transformation operator in which the mass has been replaced by a mass operator, for example $(-p_\mu p_\mu)^{1/2}$. As a result then, the Lagrangian is a scalar and the derived wave equations are covariant.

The Lagrangian implies the following equation:

$$[\partial_{\mu_1} \ldots \partial_{\mu_{2s}} \gamma_{\mu_1 \ldots \mu_{2s}} + m(\partial_\mu \partial_\mu)^{s-\frac{1}{2}}]\, \phi(x) = 0 . \qquad (2)$$

The equation given by Hammer, McDonald and Pursey is their eq. (66),

$$[m\, \partial_{\mu_1} \ldots \partial_{\mu_{2s}} \gamma_{\mu_1 \ldots \mu_{2s}} + (\partial_\mu \partial_\mu)^{s+\frac{1}{2}}]\, \phi(x) = 0 .$$

This equation leads to spin-1/2 particles "decorated" with neutrinos while eq. (2) reduces to the ordinary Dirac equation. The discussion will be made in terms of eq. (2).

The covariant γ-matrix has the property that

$$(\partial_{\mu_1}\ldots\partial_{\mu_{2s}}\gamma_{\mu_1\ldots\mu_{2s}})^2 = (\partial_\mu\partial_\mu)^{2s} \quad , \tag{3}$$

for arbitrary $s = 0, 1/2, 1, 3/2, \ldots$; so squaring up eq. (2) gives

$$(\partial_\mu\partial_\mu - m^2)\,(\partial_\mu\partial_\mu)^{2s-1}\,\phi(x) = 0 \quad . \tag{4}$$

This implies that

$$(\partial_\mu\partial_\mu - m^2)\,\psi(x) = 0 \quad , \tag{5}$$

where

$$\psi(x) = (\partial_\mu\partial_\mu)^{2s-1}\,\phi(x) \, , \tag{6}$$

Operating on eq. (2) with $(\partial_\mu\partial_\mu)^{2s-1}$ and using eq. (4) gives

$$(\partial_{\mu_1}\ldots\partial_{\mu_{2s}}\gamma_{\mu_1\ldots\mu_{2s}} + m^{2s})\,\psi(x) = 0 \quad . \tag{7}$$

Eqs. (5) and (7) are the Weinberg equations.

A close look at eq. (4) shows that, in addition to the solutions $\phi_m(x)$ and $\phi_0(x)$, where

$$(\partial_\mu\partial_\mu - m^2)\,\phi_m(x) = 0$$

and

$$(\partial_\mu\partial_\mu)\,\phi_0(x) = 0 \quad ,$$

there are solutions such that

$$(\partial_\mu\partial_\mu)^n\,\phi(x) \neq 0$$

but

$$(\partial_\mu\partial_\mu)^{n+1}\,\phi(x) = 0$$

for $1 \leqslant n \leqslant 2s - 2$. So eq. (2) has massive and massless solutions plus other solutions which cannot be so easily interpreted. The function $\psi(x)$ is a simple mass m solution.

For the special case $s = 1/2$, $\psi(x)$ and $\phi(x)$ are identical and eq. (2) or (7) is the usual Dirac equation. The Hammer-McDonald-

Pursey equation does not reduce to the usual Dirac equation but retains massless solutions in the $s = 1/2$ case.

III. INTEGRAL SPIN

For the integral spin-s case the Joos-Weinberg equations may be derived from the following Lagrangian:

$$L = \partial_{\mu_1} \ldots \partial_{\mu_s} \bar{\phi}(x) \gamma_{\mu_1 \ldots \mu_{2s}} \partial_{\mu_{s+1}} \ldots \partial_{\mu_{2s}} \phi(x) -$$

$$- \partial_{\mu_1} \ldots \partial_{\mu_s} \bar{\phi}(x) \partial_{\mu_1} \ldots \partial_{\mu_s} \phi(x) -$$

$$- 2m^2 \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \bar{\phi}(x) \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \phi(x) . \qquad (8)$$

The wave equation is

$$[\partial_{\mu_1} \ldots \partial_{\mu_{2s}} \gamma_{\mu_1 \ldots \mu_{2s}} + (\partial_\mu \partial_\mu)^{s-1} (-\partial_\mu \partial_\mu + 2m^2)] \phi(x) = 0 . \qquad (9)$$

This is eq. (101) of Hammer, McDonald and Pursey.

This equation may be treated in a slightly different way. Eq. (9) is just a matrix eigenvalue equation. Since the components of ∂_μ commute, eq. (9) may be rewritten as

$$\partial_{\mu_1} \ldots \partial_{\mu_{2s}} \gamma_{\mu_1 \ldots \mu_{2s}} \phi_\pm(x) = \pm (\partial_\mu \partial_\mu)^s \phi_\pm(x) =$$

$$= (\partial_\mu \partial_\mu)^{s-1} (\partial_\mu \partial_\mu - 2 m^2) \phi_\pm(x) \quad . \qquad (10)$$

The eigenvalues of the operator matrix $\partial_{\mu_1} \ldots \partial_{\mu_{2s}} \gamma_{\mu_1 \ldots \mu_{2s}}$ are the operators $\pm (\partial_\mu \partial_\mu)^s$. This was not done for half-integral spin because of the a priori restriction concerning $(\partial_\mu \partial_\mu)^{1/2}$.

Eq. (10) implies that

$$m^2(\partial_\mu \partial_\mu)^{s-1} \phi_+(x) = 0$$

and

$$(\partial_\mu\partial_\mu - m^2)(\partial_\mu\partial_\mu)^{s-1} \phi_-(x) = 0 ,$$

and therefore that

$$(\partial_\mu\partial_\mu - m^2)(\partial_\mu\partial_\mu)^{s-1} \phi(x) = 0 . \qquad (11)$$

The wave function is defined by

$$\psi(x) = (\partial_\mu\partial_\mu)^{s-1} \phi(x) , \qquad (12)$$

and so eq. (11) implies

$$(\partial_\mu\partial_\mu - m^2) \psi(x) = 0 . \qquad (13)$$

Operating on eq. (9) with $(\partial_\mu\partial_\mu)^{s-1}$ and using eqs. (12) and (13) we obtain

$$(\partial_{\mu_1}\ldots\partial_{\mu_{2s}} \gamma_{\mu_1\ldots\mu_{2s}} \pm m^{2s}) \psi(x) = 0 . \qquad (14)$$

Here also, eq. (9) has solutions such that

$$(\partial_\mu\partial_\mu)^n \phi(x) \neq 0$$

but

$$(\partial_\mu\partial_\mu)^{n+1} \phi(x) = 0$$

for $1 \leqslant n \leqslant s-2$. So eq. (9) has solutions which are not simple massless solutions.

For $s = 1$ the $\psi(x)$ and $\phi(x)$ are identical and eq. (9) reduces to the spin-1 equation given in SHAY and GOOD[15] which was shown to be equivalent to the two spin-1 Weinberg equations. This spin-1 equation has no massless solutions.

IV. MOMENTUM SPACE WAVE FUNCTIONS

It is instructive to take the Fourier transforms of eqs. (2) and (8) and to look closely at the momentum space wave functions $\phi(p)$. For half-integral and integral spin the momentum space wave functions satisfy the following equations:

$$[p_{\mu_1} \dots p_{\mu_{2s}} \gamma_{\mu_1 \dots \mu_{2s}} - i\, m (p_\mu p_\mu)^{s-\frac{1}{2}}]\ \phi(p) = 0 \qquad (15)$$

and

$$[p_{\mu_1} \dots p_{\mu_{2s}} \gamma_{\mu_1 \dots \mu_{2s}} - (p_\mu p_\mu)^{s-1} (p_\mu p_\mu + 2m^2)]\ \phi(p) = 0\ , \qquad (16)$$

respectively. These are simple matrix equations.

The eigenvalues of $p_{\mu_1} \dots p_{\mu_{2s}} \gamma_{\mu_1 \dots \mu_{2s}}$ are $\pm (p_\mu p_\mu)^s$ for any s . Here $(p_\mu p_\mu)^{1/2}$ is simply a number and, except for an arbitrariness in phase, is well defined. Since $\phi(p)$ is an eigenmatrix of $p_{\mu_1} \dots p_{\mu_{2s}} \gamma_{\mu_1 \dots \mu_{2s}}$, eqs. (15) and (16) become

$$[\pm (p_\mu p_\mu)^s - i\, m\, (p_\mu p_\mu)^{s-\frac{1}{2}}]\ \phi_\pm(p) = 0 \qquad (17)$$

and

$$[\pm (p_\mu p_\mu)^s - (p_\mu p_\mu)^s - 2\, m^2\, (p_\mu p_\mu)^{s-1}]\ \phi_\pm(p) = 0\ . \qquad (18)$$

The massive solutions of eq. (17) satisfy

$$[\pm (p_\mu p_\mu)^{\frac{1}{2}} - i\, m]\ \phi_\pm(p) = 0\ .$$

In the rest frame, with a suitable definition of the phase of $(p_\mu p_\mu)^{\frac{1}{2}}$, one can relate the $\pm$ signs in eq. (17) to the sign of the energy of the particle. No such identification is possible in eq. (18).

It follows then from eqs. (17) and (18) that the momentum space wave functions have the general form

$$\phi(p) = \phi_m(p)\ \delta(p_\mu p_\mu + m^2) + \sum_{n=0}^{a} \frac{\phi_0^{(n)}(\vec{p})}{(p_\mu p_\mu)^n}\ \delta(p_\mu p_\mu)\ , \qquad (19)$$

where $a = s-2$ or $s-3/2$ for integral of half-integral spin, respectively. For the half-integral solutions of Hammer, McDonald and Pursey, one has $a = s-1/2$. Their integral solutions are the same as above.

It is clear that the $\dfrac{\phi_0^{(n)}(\vec{p})}{(p_\mu p_\mu)^n}\,\delta(p_\mu p_\mu)$ terms for $n > 0$ will cause complications in the space-time wave function

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4 p\, \phi(p)\, e^{i p_\mu x_\mu}$$

unless some restriction removes them or one extends the integrand into the complex plane in a suitable way. This makes it difficult to interpret $\phi(x)$ and to have any idea how these singularities act when electromagnetic fields are introduced. Note that the singularities are not present for $s = 1/2,\ 1,\ 3/2,\ 2.$

The wave function $\psi(x)$ as defined in eqs. (6) and (12), may be written as

$$\psi(x) = \frac{m^b}{(2\pi)^{3/2}} \int d^4 p\, \phi_m(\vec{p})\, e^{i p_\mu x_\mu}\, \delta(p_\mu p_\mu + m^2) \ ,$$

where $b = 4s-2$ or $2s-2$ for half-integral or integral spin, respectively. The function $\psi(x)$ corresponds exactly to the massive part of the Hammer, McDonald and Pursey wave function.

V. DISCUSSION

The Joos-Weinberg equations have been derived from a Lagrangian. This approach involves writing the Lagrangian entirely in terms of an auxiliary field $\phi(x)$ and then appropriately defining the wave function $\psi(x)$ in terms of $\phi(x)$. The function $\phi(x)$, which is considered here to be merely an auxiliary field without direct physical significance, is considered by Hammer, McDonald and Pursey to be the wave function itself. The two formulations where $\phi(x)$ or $\psi(x)$ is the wave function are inequivalent even in the free-particle case. The function $\psi(x)$ is obtained by operating on $\phi(x)$ with a projection operator which has no inverse. Further, $\psi(x)$ corresponds exactly to the massive solution of $\phi(x)$.

For the case where $\phi(x)$ is interpreted as the wave function one should notice the presence of some arbitrariness in the half-integral spin case. The $\phi(x)$ given here is the simple Dirac wave function for $s = 1/2$; zero mass solutions are present for $s \geq 3/2$; and the "singular" solutions appear for $s \geq 5/2$. The Hammer-McDonald-Pursey wave function has zero mass solutions for $s \geq 1/2$ and "singular" solutions for $s \geq 3/2$. Both functions $\phi(x)$ are the same for integral spin, a simple spin-1 particle, zero mass solutions for $s \geq 2$ and "singular" solutions for $s \geq 3$. While one can pick out the massive and massless solutions for free particles, when electromagnetic fields are inserted by replacing p by p-eA the "singular" solutions are mixed in also. The interpretation of these "singular" solutions and their behaviour when electromagnetic fields are introduced is a problem.

The interpretation of $\psi(x)$ as the wave function is simple. The extension of the equations to include the electromagnetic interaction is not simple, however. Certainly one can replace p by p-eA in the Lagrangian without difficulty but it is no longer clear what the definition of $\psi(x)$ is in terms of $\phi(x)$. This occurs because, since the components of π_μ do not commute and therefore the eigenvalue property of $\pi_{\mu_1} \ldots \pi_{\mu_{2s}} \gamma_{\mu_1 \ldots \mu_{2s}}$ cannot be used and squaring this matrix becomes complicated. The real test of the usefulness of this Lagrangian approach to the Joos-Weinberg equations is, however, the inclusion of the electromagnetic interaction.

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