FADDEEV EQUATIONS AND COULOMB EFFECTS IN $^3$He

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ERRATA AND ADDENDUM

P. 2, line 7 Instead of "with separable two-body matrices" please read "with separable two-body T-matrices".

P. 4, line 9 Instead of "they can be immediately used" please read "our equations can be immediately used".

P. 6, lines 10 and 11 Instead of "the T-matrix is not diagonal in $I$" please read "the T-matrix is not only diagonal in $I$".

P. 12, lines 12 and 13 Replace "However, we found that, in the case in which there is a non-separable interaction..." by "However, AGS also point out that, in the case in which there is a non-separable interaction..."

P. 14, lines 23 and 24 Delete the comment "but the potentials must be multiplied by a factor $(-2)^1$".

P. 16 Eq. (3.18) should be replaced by

$$F_{jj'}(q,q',E) = \frac{8}{3\sqrt{3}} \frac{e^2}{\hbar^2} \int d\rho^2 \frac{1}{(p-K)^2 + p_d^2} \frac{1}{(p-K)^2 + q^2 + q'E} \frac{1}{p^2} +$$

\[+ \frac{1}{(p-K)^2 + q^2 - E} \frac{1}{(p-K)^2 + p_d^2} \]
\[ + \frac{4}{3\sqrt{3}} \frac{e^2}{\pi^2} \frac{1}{c^2} \int dp^2 \frac{1}{(\vec{p} - \vec{q})^2 + p^2_c} \frac{1}{(\vec{p} + \vec{q}_1)^2 + q_1^2 - \varepsilon} \]

P. 16, line 4

Instead of "where \( \vec{k}_1 = \frac{1}{\sqrt{3}} \vec{q} + \frac{2}{\sqrt{3}} \vec{q}' \); \( \vec{k}_2 = \frac{2}{\sqrt{3}} \vec{q} + \frac{1}{\sqrt{3}} \vec{q}' \)"
please read "where \( \vec{k}_1 = \frac{1}{\sqrt{3}} \vec{q} + \frac{2}{\sqrt{3}} \vec{q}' \); \( \vec{k}_2 = \frac{2}{\sqrt{3}} \vec{q} + \frac{1}{\sqrt{3}} \vec{q}' \)"
and \( \vec{q} = \frac{2}{\sqrt{3}} (\vec{q}' - \vec{q}) \)

P. 16, line 6

Instead of "The integral is computed in Appendix I"
please read "The method of computation of the integrals is discussed in Appendix I".

P. 27, line 5
Replace all this line by: "vectors of the kernel of the Lippmann-Schwinger equations can be adapted without difficulty to the case we are considering, namely, coupled Lippmann-Schwinger equations with complex".

P. 31, lines 2 and 3
Replace these lines by "Here we discuss briefly the computation of one of the integrals involved in the definition (3.18); the other is much simpler and can be computed in a similar way. Consider: "...

P. 31
Replace \( F_{ij}(\vec{q}', \vec{q}; E) \) by \( J_{ij}(\vec{q}', \vec{q}, E) \) in Eqs. (AI.1) and (AI.3); \( \vec{q}_1 \) by \( \vec{q} \) and \( \vec{q}_2 \) by \( \vec{q}' \) in Eq. (AI.2); \( \frac{d}{\sqrt{\delta^2 - \alpha \gamma}} \)
by \( \frac{n^2}{\sqrt{\delta^2 - \alpha \gamma}} \) in Eq. (AI.5) and \( n^3 \) by \( n^2 \) in Eq. (AI.8).

P. 42
Replace Fig. II by new Fig. II attached.
ADDENDUM

We have omitted a non-separable contribution in our calculation, i.e., the one represented by the third term of the right-hand side of Fig. II. Its inclusion leads to correcting the function $F_{ij}(\tilde{q}', \tilde{q}, E)$ in Eq. (3.18) in the way indicated in this Errata. When this contribution is taken into account the numerical results are practically unchanged. For example, we obtain $\Delta_c = 0.65 \pm 0.05$ MeV for the S.K. potentials and $\Delta_c = 0.81 \pm 0.05$ MeV for the $(C_N + S_N)$ potentials. In order to perform the calculation one has to introduce a cut-off because the new contribution diverges as the Coulomb potential in the forward direction and the cut-off used in momentum space is the same as that used to handle the Coulomb-corrected form factors.

The estimates for the probabilities are also practically unchanged and all the conclusions of the paper are still valid.

We are grateful to Drs. P. Grassberger and W. Sandhas for pointing out to us this omission in our calculation.
FADDEEV EQUATIONS AND COULOMB EFFECTS IN $^3$He

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ABSTRACT

When two particles interact via a potential $V$ which is the sum of a separable potential $V_g$ and a Coulomb potential $V_c$, the two-body T-matrix can be obtained exactly and can be split in the usual way into a pure Coulomb T-matrix and a "nuclear" T-matrix which also contains Coulomb effects. Using the fact that the "nuclear" T-matrix is still separable in the off-shell variables, the formalism of Alt, Grassberger and Sandhas is applied to the calculation of Coulomb effects in the three-nucleon system, with the contribution of the non-separable Coulomb T-matrix fully taken into account. The Coulomb energy $\Delta_c$ of $^3\text{He}$ and the probability $P_{3/2}$ of finding the $^3\text{He}$ in a $I = 3/2$ state are calculated, using s-wave spin-dependent potentials of the Yamaguchi type to describe the two-nucleon interaction. This model yields values for $\Delta_c$ which are in reasonable agreement with the binding energy difference of $^3\text{H}$ and $^3\text{He}$, but predicts a negligible admixture of the $I = 3/2$ state in the $^3\text{He}$ wave function. The effect of the non-separable part of the p-p T-matrix on the binding energy and wave function of $^3\text{He}$ is discussed. Finally, the possible relevance of hard-core effects is pointed out.
I. INTRODUCTION

It is a well-known fact that a non-local separable potential yields a two-body T-matrix which is also separable in the off-shell variables. In the last few years, these potentials have been used extensively in three-body calculations; the reason being that the FADDEEV equations with separable two-body matrices become Fredholm integral equations in only one variable after angular momentum decomposition.

The application of the separable approximation to the three-nucleon system has had a reasonable success. The two-nucleon system contains, at low energies, the deuteron bound state with quantum numbers \( J = 1, I = 0 \), and a singlet virtual state (denoted by \( s \)) with quantum numbers \( J = 0, I = 1 \). The simplest assumption, as discussed by Lovelace, is that the two-body T-matrix is dominated by central forces in these two states. Tensor forces have also been taken into account — either phenomenologically or in an explicit fashion — in calculations of the \(^3\)H bound state and n-d scattering lengths. Perhaps the most complete calculation has been carried out by Schrenk and Mitra. These authors include tensor forces in the triplet state and hard-core effects in the singlet state using a variety of separable potentials that fit the two-body data, and conclude that the separable approximation provides a reasonable description of bound-state and scattering properties of the three-nucleon system at low energies.

We are concerned in this paper with a non-perturbative calculation of Coulomb effects in the three-nucleon system. The problem of two particles interacting via a potential \( V \) which is the sum of a separable potential \( V_s \) and a Coulomb potential \( V_c \) has been discussed by Harrington. This author showed that the problem can be solved exactly and that the resulting T-matrix can be split in the usual way in a pure Coulomb and a "nuclear" T-matrix. Needless to say, the "nuclear" T-matrix also contains Coulomb effects.
However, the crucial point is that it is still separable in the off-shell variables. In Sec. II we briefly review the results by Harrington which are relevant for our purposes and discuss the two-body interactions used as an input in our model.

In writing the equations for p-d scattering, one has to face the problem of the non-separable Coulomb part of the p-p T-matrix. This is accomplished in our calculation by using an elegant formalism devised to take into account non-separable contributions to the Lovelace equations, due to ALT, GRASBERGER and SANDHAS. In Sec. III we briefly review their formalism and derive the Lovelace equations for p-d scattering.

Since the Coulomb potential violates isospin symmetry in a very definite way, we discuss to some extent the SU(2) symmetry-breaking in the three-nucleon system and write the equations in a basis in Hilbert space in which the breaking is more transparent. The bound-state scattering amplitudes are no longer diagonal in I, the total isospin, and for a given value of \( I_Z \) we obtain a set of coupled equations in I. When \( I_Z = -\frac{3}{2} \), they reduce to the well-known equations for n-d scattering, but for \( I_Z = +\frac{3}{2} \) they show explicitly the Coulomb effects.

In Sec. IV we discuss the bound-state properties of the three-nucleon system and use a procedure suggested by AMADO to obtain the \(^3\)He wave function from the Lovelace equations for the p-d scattering amplitude. Sec. V contains the numerical results. The binding energy difference of \(^3\)He and \(^3\)H is calculated by computing the shift of the pole of the p-d scattering amplitude with respect to the pole of the n-d scattering amplitude. It should be emphasized that, although our calculation of the \(^3\)He Coulomb energy is based on the exact solution of a model, we are not saying that perturbation theory calculations of such a quantity are unreliable. Our interest in this model has been raised because it allows an exact computation of both \(^3\)He binding energy and its wave function. The isospin symmetry breaking due to the Coulomb interaction produces
not only the splitting of isospin multiplets but also configuration mixing, and therefore the $^3\text{He}$ wave function acquires a small $I = \frac{3}{2}$ component. We also discuss the calculation of the probabilities of finding the $^3\text{He}$ in the $I = \frac{3}{2}$ state, and in the $S'$ state of mixed spacial symmetry with $I = \frac{3}{2}$.

Finally, Sec. V also contains a discussion of our results.

It should be emphasized that, although our calculations are restricted to bound-state properties of the three-nucleon system, they can be immediately used as they stand to perform calculations of p-d scattering with the Coulomb corrections fully taken into account.
II. TWO-BODY INTERACTIONS

The simplest model used previously to describe the two-nucleon interaction assumes a spin-dependent separable potential of the YAMAGUCHI\textsuperscript{25}-26(type, defined by

$$V(t', \tilde{p}) = \lambda_s q_s(t') q_s(t) P_s + \lambda_d q_d(t') q_d(t) P_d$$ \hspace{1cm} (2.1)

where

$$q_i(t) = \frac{1}{\tilde{p}^2 + \rho_i^2} \hspace{1cm} i = s, d \hspace{1cm} (2.2)$$

and $P_s$, $P_d$ are the appropriate spin-isospin projection operators. The two-nucleon T-matrix is given in this case by

$$T(t', t, \tilde{p}, \tilde{e}) = q_s(t') t_s(t) P_s + q_d(t') t_d(t) P_d \hspace{1cm} (2.3)$$

where

$$t_i^{-1}(\tilde{e}) = \frac{1}{\lambda_i} + 4\pi \int_0^\infty dp \frac{t_i^2 q_i(t)}{p^2 - \tilde{e}} = \frac{1}{\lambda_i} + \frac{\pi^2}{\rho_i^2 (\rho_i^2 - \sqrt{\rho_i^2 - \tilde{e}})} \hspace{1cm} (2.4)$$
The singlet and triplet parameters in the potentials are determined by fitting the two-body data in a way described in Sec. V. In order to take into account Coulomb effects, one has to add to the potentials given by Eq. (2.1) a term of the form $V_c P_{1,1}$, where $V_c$ is the Fourier transform of the Coulomb potential and $P_{1,1}$ is the projection operator

$$P_{1,1} = \langle I=1, I_z=+1 | I=1, I_z=+1 \rangle$$  \hspace{1cm} (2.5)

Since the Coulomb potential acts only on $I = 1$ states, it modifies only the singlet part of the two-nucleon T-matrix. It is a well-known fact that if isospin is an exact symmetry the T-matrix is not diagonal in $I, I_z$ but also independent of $I_z$. The Coulomb potential breaks the symmetry in such a way that the singlet T-matrix is still diagonal, but acquires an $I_z$ dependence. Consequently, the two-nucleon T-matrix is given by

$$T(p', p; \varepsilon) = g_d(p') t_d(\varepsilon) g_d(p') P_d + g_c(p') t_c(\varepsilon) g_c(p') \left[ P_{1,0} + P_{1,-1} \right] +$$

$$+ \left[ T_c(p', p) + g_c(p') t_c(\varepsilon) g_c(p') \right] P_{1,1}$$  \hspace{1cm} (2.6)

In this equation $T_c(p', p)$ represents the pure Coulomb T-matrix, and the remaining coefficient of $P_{1,1}$ the "nuclear" T-matrix for p-p scattering. The modified form factors $g_c(p)$ have been calculated by HARRINGTON, with the result: 

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\[ q_c(F) = q_s(F) C_o(\eta) \exp\left[2\eta t_0^{-1}(p s_0^{-1})\right] \]  

(2.7)

where \( \eta = \frac{m e^2}{p} \), \( m \) is the reduced mass of the two-particle system, and

\[ C_o(\eta) = \left\{ 2\pi \eta \left[\exp\left(2\pi \eta\right) - 1\right]^{-1}\right\}^{1/2} \]  

(2.8)

The function \( t_c(E) \) is given by an expression analogous to eq. (2.4), except that it is the modified form factor which appears in the integrand. Therefore, one is faced with the problem of evaluating the integral

\[ I(E) = 4\pi \int_0^{\infty} \frac{dp}{(p + \tilde{\lambda})^2 (\tilde{\lambda}^2 - E)} \left[ p^2 C_o(\eta) \exp\left[4\eta t_0^{-1}(p s_0^{-1})\right]\right] \]  

(2.9)

The value of \( s_0 \) necessary to fit the two-body data is such that \( (m e^2/\beta_0) \approx 0.06 \). Consequently, this quantity can be used as an expansion parameter in the calculation of \( I(E) \), and terms of order \( (m e^2/\beta_0)^2 \) can safely be neglected. We refer to Harrington's paper for details.

As a result of the calculation one can get relations between p-p and n-n low-energy scattering parameters which are indeed very reasonable. If the parameters \( \beta_0 \) and \( \lambda_0 \) are fixed so as to yield \( a_{pp} = -7.81 \, F \) and \( r_{pp} = 2.80 \, F \), one gets for the corresponding parameters with the Coulomb effects removed the values \( a_{nn} = -18.0 \, F \) and \( r_{nn} = 2.93 \, F \). Although these values for \( a_{nn} \) and \( r_{nn} \) are a
little large \(^27\), this result indicates that the Coulomb inter-
action is rather well described by this formalism at low energies.
Anyway, this point is not of extreme importance to us since we are
not taking the attitude of introducing as an input for the \(^3\)He
calculation the p-n scattering data with Coulomb corrections. Our
inputs are the "nuclear" two-body T-matrix plus the pure Coulomb
T-matrix. The parameters will be determined as indicated in Sec. V
by fitting the singlet effective range and scattering length. Without
Coulomb corrections, this gives a reasonable model for the \(^3\)H. Our
attitude is to test the stability of the three-body model against
electromagnetic corrections to the strong interactions.

The calculations we carry out in this paper are restricted
to the bound-state region of the three-nucleon system. Since the two-
particle propagators \(t_i(E)\) appear in the Lovelace equations in the
form \(t_i(E - q^2)\), \(E\) being the total energy, it is clear that we need
the function \(t_0(E)\) below its cut on the positive real axis. For
\(E < 0\), the integral (2.9) can be evaluated\(^20\), and we obtain, setting
\(k = i\sqrt{-E}\)

\[
t_c^{-1}(k) = \frac{1}{\lambda_s} + \frac{\pi^2}{\beta_s (\beta_s + k)^2} + \\
\quad \frac{4\pi^2 \beta_s^2}{(\beta_s^2 - k^2)^2} \left(1 - \frac{k_0}{mE + k} + \ln \left[\frac{4k_0^2}{(\beta_s (\beta_s + k))^2}\right] + \frac{\beta_s^2 - k^2}{\beta_s^2}\right)
\]

(2.10)

It is important to notice that the singularity at \(k = \beta_s\) of the last
term of the right-hand side is only apparent. The factor in brackets
behaves like \((\beta_s^2 - k^2)^2\) when \(k \to \beta_s\).

Going back to the modified form factor \(g_c(p)\), it follows
from eqs. (2.7) and (2.8) that it has an essential singularity at
\(p = 0\) due to the behaviour of \(C_0(\eta)\). This essential singularity
merely reflects the long-range nature of the Coulomb potential\(^{28}\). In order to avoid this difficulty, the Coulomb potential is cut off at a distance \(R\) much greater than the range \(\frac{1}{\beta}\) of the separable potential. In this condition, eq. (2.7) is not valid all the way down to \(p = 0\), and ceases to be valid for values of \(p\) such that

\[
\frac{R}{p} \approx \frac{m e^4}{p^2} \quad (2.11)
\]

In our calculations we have kept \(R\) as a free parameter and have used a form factor given by eq. (2.7) for \(p > p_o\), but defined for \(p \leq p_o\) by

\[
\exp\left[2\eta_o t g^{-1}\left(p_o (2\pi)^{1/2}\right)\right] \quad (2.12)
\]

where \(p_o^3 = \frac{m^2 e^4 R}{1}\), \(\eta_o = \eta(p_o)\).

All the results we quote in Sec. V correspond to a value of \(R = 20\). As a matter of fact, we have checked that they are quite insensitive to variations of this parameter up to \(50\). The range of the singlet potential that we use is of the order of 1.

III. THREE-BODY EQUATIONS

The problem of modifying the Lovelace equations in the case in which the two-body T-matrix contains non-separable terms such as, for example, the Coulomb T-matrix given by eq. (2.6), has been discussed by ALT, GRASSERGER and SANDNASS\(^{22}\). These authors introduce transition operators for bound-state elastic re-arrangement scattering, \(\mathcal{U}_{\alpha\alpha}(\mathcal{E})\), which are slightly different from those used by LOVELACE\(^{1}\). However, they yield the same scattering amplitude when their matrix elements are put on the energy-shell. In terms of the Lovelace transition operators \(\mathcal{U}_{\alpha\alpha}^{(\pm)}(\mathcal{E})\), the \(\mathcal{U}_{\alpha\alpha}(\mathcal{E})\) are defined by
where $H_\alpha = H_0 + V_\alpha$; $\alpha, \beta = 0, 1, 2, 3$, and $V_o \equiv 0$. These operators satisfy Faddeev-like equations,

$$U_{\rho \alpha} = -(1 - \delta_{\rho \alpha})(H_\alpha - E) + U_{\rho \alpha}^{(1)}$$

$$= -(1 - \delta_{\rho \alpha})(H_0 - E) + U_{\rho \alpha}^{(-)}$$

(3.1)

or

$$U_{\rho \alpha}(E) = -(1 - \delta_{\rho \alpha}) G^{-1}_0(E) - \sum_{\gamma \neq \rho} T_\gamma(E) G_\gamma(E) U_{\gamma \alpha}(E)$$

(3.2)

and $T_\gamma$ is the two-body T-matrix for the pair $\gamma$ in the three-body Hilbert space. The bound-state elastic or re-arrangement scattering operators are defined, as usual, by

$$\chi_{\alpha \beta \gamma}(E) = \langle \alpha | G_\alpha(E) U_{\beta \gamma}(E) G_\gamma(E) | \beta \gamma \rangle$$

(3.3)

where $G_0(E) = (H_0 - E)^{-1}$ and $T_\gamma$ is the two-body T-matrix for the pair $\gamma$ in the three-body Hilbert space. The bound-state elastic or re-arrangement scattering operators are defined, as usual, by

$$\chi_{\alpha \beta \gamma}(E) = \langle \alpha | G_\alpha(E) U_{\beta \gamma}(E) G_\gamma(E) | \beta \gamma \rangle$$

(3.4)

where $\langle p_\alpha | \alpha \rangle = \delta_{\alpha}^{\beta} (p_\alpha)$, and the corresponding scattering amplitudes are calculated by taking matrix elements of $\chi_{\alpha \beta \gamma}(E)$.
between plane waves representing the relative motion of the particle
\(\alpha\) (or \(\beta\)) and the \(n^{th}\) (or \(m^{th}\)) bound-state of the pair \((\beta \gamma)\) (or \((\alpha \gamma)\)).

In the case in which the two-body T-matrices are separable, LOVELACE has shown \(^1\) that eqs. (3.2) or (3.3) can be recast in the form of a set of coupled Lippmann-Schwinger equations for the operators \(X_{\alpha \eta}, B_m(E)\), with energy-dependent potentials which become complex above the three-particle threshold. Let us now assume that \(T_\alpha\) can be written in the form

\[
T_\alpha(E) = T_\alpha^S(E) + T_\alpha^r(E) \tag{3.5}
\]

where \(T_\alpha^r(E)\) represents the non-separable contribution. In order to handle this piece of the two-body T-matrix, AGS define a new set of operators \(U_{\alpha \eta}(E)\), as the solution of eq. (3.2) and (3.3) with only the non-separable part of \(T_\alpha(E)\):

\[
U_{\alpha \eta}(E) = -(1 - \delta_{\alpha \eta}) G_0^{-1}(E) - \sum_{\gamma \neq \alpha} T_{\alpha \gamma}(E) G_\gamma(E) U_{\gamma \eta}(E) \tag{3.6}
\]

and

\[
U_{\alpha \eta}(E) = -(1 - \delta_{\alpha \eta}) G_0^{-1}(E) - \sum_{\gamma \neq \alpha} U_{\alpha \gamma}(E) G_\gamma(E) T_{\gamma \eta}(E) \tag{3.7}
\]

After doing this, AGS proceed to show that eqs. (2.2) and (2.3) can again be reduced to a set of multi-channel Lippmann-Schwinger equations, namely
\[ X_{\omega n, \rho m}(E) = \mathcal{Z}_{\omega n, \rho m}(E) - \sum_{r, \delta + \alpha} \mathcal{Z}_{\omega n, r}(E) t_{\gamma r}(E) X_{r \gamma, \rho m}(E) \]  

(3.8)

where \( t_{\chi n}(E) \) are the Lovelace propagators, but the potentials are different and defined by

\[ Z_{\omega n, \rho m}(E) = \langle \alpha n \mid G_0(E) U_{\alpha \beta}(E) G_0(E) \mid \beta m \rangle \]  

(3.9)

It is clear from eqs. (3.6) and (3.7) that if \( T_{\chi} = 0 \), these potentials reduce to those defined by Lovelace. AGS suggest the use of perturbation theory to compute them. This means that one can solve eqs. (3.6) or (3.7) by iteration and replace the result in eq. (3.9) to obtain, to first order

\[ Z_{\omega n, \rho m}(E) = -(1 - \delta_{\omega \rho}) \langle \alpha n \mid G_0(E) \mid \beta m \rangle + \] 

\[ + \sum_{\gamma + \chi, \delta} \langle \alpha n \mid G_0(E) T_{\gamma}(E) G_0(E) \mid \beta m \rangle - \ldots \]  

(3.10)

However, we found that in the case in which there is a non-separable interaction only between one pair of particles, eqs. (3.6) and (3.7) can be trivially solved and the operators \( U_{\alpha \beta}(E) \) can be
computed exactly. Let us assume, for the sake of definiteness, that $T_1' \neq 0$, and $T_2' = T_3' = 0$. In this case, eqs. (3.6) and (3.7) imply that

$$U_{\alpha 1}'(E) = -(1 - \delta_{\alpha 1}) G_o^{-1}(E) \text{ for any } \alpha$$

$$U_{1\beta}'(E) = -(1 - \delta_{1\beta}) G_o^{-1}(E) \text{ for any } \beta$$

respectively. Therefore, we can summarize this result by stating that

$$U_{\alpha \beta}'(E) = -(1 - \delta_{\alpha \beta}) G_o^{-1}(E) \text{ if } \alpha = 1 \text{ or } \beta = 1 . \quad (3.13)$$

Using again this information in eqs. (3.6) and (3.7) we find

$$U_{\alpha \beta}'(E) = -(1 - \delta_{\alpha \beta}) G_o^{-1}(E) - T_1'(E) \text{ if } \alpha, \beta \neq 1 . \quad (3.14)$$

Therefore, only the potentials $Z_{\alpha \mu, \rho \nu}(E)$ such that $\alpha, \beta \neq 1$ are modified and the final exact result is

$$Z_{\alpha \mu, \rho \nu}(E) = -(1 - \delta_{\alpha \mu}) \langle \alpha \mu | G_o(E) | \rho \nu \rangle +$$

$$+ (1 - \delta_{\alpha \mu})(1 - \delta_{\rho \nu}) \langle \alpha \mu | G_o(E) T_1'(E) G_o(E) | \rho \nu \rangle$$

$$(3.15)$$
It is obvious that these results can be immediately applied to our problem, $T'_1(E)$ being the pure Coulomb T-matrix. When we treat the three particles as identical by introducing the isospin formalism, the fact that the series solution of AGS reduces to just the first and second term is guaranteed by the isospin transformation properties of the Coulomb T-matrix as given, for example, by eq. (2.6).

In what follows we shall keep in the potentials only terms of order $e^2$, and therefore we replace $T'_1$ by $V'_1$, the Coulomb potential. This is consistent with our previous approximation for the Coulomb-corrected form factor $g_c(P)$, where we have also retained the terms of order $e^2$. Therefore, all the Coulomb corrections to the potentials $Z_{\alpha\mu,\beta\nu}(E)$ will be of first order in $e^2$, because it can be checked immediately that the condition $\alpha, \beta \neq 1$ in eq. (3.15) implies that the matrix element of $G \cdot V_c G$ never has to be calculated between states that contain Coulomb-corrected form factors. Moreover, this matrix element can be computed analytically if the initial and final states contain form factors of the YAMAGUCHI type$^{25}$ discussed in Sec. II. Such a calculation is described in Appendix I.

Having established where we stand, we proceed with our calculation along the usual patterns. In the first place, the identity of the particles can be used to reduce the number of equations. As a result, they no longer depend on the index $\alpha$ that labels pairs of particles, but the potentials must be multiplied by a factor $(-2)^{-1}$. Since we are including neither tensor nor L-S forces, the orbital and spin angular momentum decouple and we can discuss the equations in the L-S representation. The $^3$H and $^3$He are, in this model, bound states in the $L = 0, S = \frac{1}{2}$ partial wave. The spin dependence of the two-body forces that we use can be taken into account in the three-body equations by means of recoupling coefficients$^1$.

As far as the treatment of isospin is concerned, instead of working in a basis in which the individual isospins of the bound state and the remaining particle are diagonal, we choose to work in a basis in which the total isospin $I$ is diagonal. Since the Coulomb interaction violates $I$ but not $I_2$, the bound-state scattering operators will still be diagonal in $I_2$, but will have non-vanishing matrix elements between states of different $I$. Therefore, for a given eigenvalue of $I_2$, let us say $M$, we must expect to obtain a set of coupled
equations in $I$. Setting $N = \frac{1}{2}$ or $(-\frac{1}{2})$, we shall obtain the equations for $p-d$ or $n-d$ scattering respectively.

The eigenstates of $I$ available for the three-nucleon system are $I = \frac{1}{2}$ and $I = 3/2$. If a pair of nucleons are in a $\frac{1}{2}$ state, the total isospin can only be $I = \frac{1}{2}$. However, if the pair of nucleons are in a $\frac{3}{2}$ state, the total isospin can take both values $\frac{1}{2}$ or $3/2$. Therefore it is clear that we are dealing here with a three-channel system.

Our formulation of the problem, as well as the kinematics we use, is the same as in Ref. 1). We shall label $q'$ the relative momentum of the pair $(q_1, q_2)$ and the particle $\alpha$ in the total centre-of-mass system, and $p'$ the relative momentum of the particles $\beta$ and $\gamma$ in their own centre-of-mass system. There are factors absorbed in the definition of $q'$ and $p'$ in such a way that the kinetic energy of three free particles is $E = \frac{1}{2}p'^2 + \frac{1}{2}q'^2$. The equations for our problem are of the form

$$\chi_{n'N'}^{M} (q', q', E) = -2 \chi_{n'N'}^{M} (q', q', E) -$$

$$-2 \sum_{n'N'} \left\{ d_{n}^{n'} \begin{pmatrix} K_{n}^{M} (q', q', E) \chi_{n'N'}^{M} (q', q', E) \right\}$$

$$\chi_{n'N'}^{M} (q', q', E)$$

(3.16)

where the channel indices $(n, I)$ can take the values $(d, \frac{1}{2})$, $(s, \frac{1}{2})$, and $(s, 3/2)$. It only remains to specify the potentials and the kernel of eq. (3.16). Let us define, for the sake of simplicity, the functions

$$\chi_{n}^{(d)} (q', q', E) = \left( \frac{d}{s} \right)^{\frac{3}{2}} \frac{q^{(d)} q^{(s)}}{p^2 + q^2 - E}$$

(3.17)
and

\[
F_{ij}(q', q, \xi - \beta) = \frac{8}{5\pi} \frac{q'^2}{\pi^2} \int \frac{1}{(p - k_1)^2 + q^2 - E} \cdot \frac{1}{(p - k_2)^2 + q^2 - E} \cdot \frac{1}{p^2}.
\]

(3.18)

where \( k_1 = \frac{1}{\sqrt{3}} q + \frac{2}{\sqrt{3}} q', \quad k_2 = \frac{2}{\sqrt{3}} q + \frac{1}{\sqrt{3}} q' \). In eq. (3.17) \( \xi_i(\vec{p}) \) are the form factors defined in Sec. II, and the indices \( i, j \) can take the values \( d, s \) or \( c \). In eq. (3.18) the indices \( i, j \) can only take the values \( d \) or \( s \). The integral is computed in Appendix I, where a more explicit formula for \( F_{ij}(q', q, \xi - \beta) \) is given. With these definitions, it is shown in Appendix II that the potentials are given by

\[
L_{d_\frac{1}{2}, d_\frac{1}{2}} (q', q, \xi + \beta) = -\frac{1}{2} \left[ f_{dd} - F_{dd} \right]
\]

(3.19)

\[
L_{d_\frac{1}{2}, s_\frac{1}{2}} (q', q, \xi + \beta) = -\frac{1}{\sqrt{3}} f_{dc} - \frac{i}{2\sqrt{3}} \left[ f_{ds} - F_{ds} \right]
\]

(3.20)

\[
L_{d_\frac{1}{2}, s_\frac{1}{2}} (q', q, \xi + \beta) = -\frac{1}{\sqrt{6}} f_{dc} + \frac{i}{\sqrt{6}} \left[ f_{ds} - F_{ds} \right]
\]

(3.21)
These potentials must still be multiplied by the spin recoupling coefficients. Those not listed here can be obtained without difficulty by recalling that they are symmetric under exchange of all the variables (including channel indices) in the initial and final states.

We can use Feynman-like diagrams to represent these particle-bound-state potentials. The function \( f_{c,i} \) can be represented by the diagram of Fig. I, and the function \( (f_{ij} - F_{ij}) (i,j \neq c) \) by that of Fig. II. These diagrams will be helpful when we compare our method with perturbation theory.

The calculation of the propagators is also discussed in Appendix II. Since two nucleons in a \( \pi \) state and a third one can be in two total isospin states, the propagator \( t_c(E) \) now becomes a matrix in total isospin space:
The first and second row— or column— correspond to $I = \frac{1}{2}$ or $3/2$, respectively. The deuteron propagator remains unchanged, and is given by

$$t_{\alpha\beta}^{1/2} = \begin{pmatrix} \frac{2}{3} t_c(E) + \frac{1}{3} t_s(E) & \sqrt{\frac{2}{3}} [t_c(E) - t_s(E)] \\ \sqrt{\frac{2}{3}} [t_c(E) - t_s(E)] & \frac{1}{3} t_c(E) + \frac{2}{3} t_s(E) \end{pmatrix}$$

Finally, the kernel of eq. (3.16) is:

$$K_{\alpha\beta}^{M} (\vec{q}, \vec{q}', E) = \sum_{\eta} Z_{\alpha\beta}^{M} \tilde{Z}_{\alpha\beta}^{M} (\vec{q}, \vec{q}', E) \langle \eta | t_{\alpha\beta}^{M} | \eta \rangle (E - \vec{q}^2)$$

We have written our equation for the case $M = \frac{1}{2}$; that is to say, they are valid to discuss p-d scattering or the $^3$He bound state. It is not necessary to write explicitly the equations for $M = -\frac{1}{2}$ because they can be obtained from the previous ones by switching off the Coulomb interaction. If this is the case, the modified form factors $g_c(\vec{p})$ coincide with the singlet form factors $g_s(\vec{p})$, and the same thing happens with the propagators $t_c(E)$ and $t_s(E)$. Therefore, the indices $c$ and $s$ in all our previous equations.
became indistinguishable, and it is evident that:

i) Both the potentials and propagators are now diagonal in $I$. As a matter of fact, $t^2_{s,\bar{s},s,\bar{s},1}(E)$ reduces to the singlet propagator times the unit matrix. Consequently, the two-channel equations for $I = \frac{1}{2}$ are decoupled from the one-channel equation for $I = 3/2$.

ii) The numerical coefficients that multiply the functions $f_{dd}$, $f_{ds}$ and $f_{ss}$ are precisely the isospin recoupling coefficients quoted by Lovelace$^1$.

This is a consistency check on our results.

IV. $^3$He WAVE FUNCTION

We are interested in calculating not only the Coulomb energy of $^3$He, but also its wave function. There are several ways of doing this, and we choose to follow a procedure suggested by Amado$^{23}$ to extract the wave function from the Lippmann-Schwinger equations for the T-matrix. This method was used by Amado to calculate the $^3$H wave function$^{23}$.

It is a well-known fact that the off-shell matrix elements have a pole in the parameter $E$ at the binding energy $E_B$ of the two-particle system, and that the residues factorize in the off-shell variables$^1$, namely

$$
\langle \tilde{\psi}' | T(E) | \tilde{\psi} \rangle \sim \frac{\varphi(\vec{p}') \varphi(\vec{p})}{E + E_B} + \text{terms regular at } E = E_B
$$

(4.1)

The bound-state wave function is given by the product of the bound-state form factor and the free Green function, that is,
The same is true for the three-body problem: the bound-state scattering amplitudes \( \langle \tilde{q}' | \, X_{\alpha n, \gamma m}(E) | \tilde{q} \rangle \) have a pole at the binding energy of the three-body bound state, and their residues factorize in the off-shell variables. However, if we calculate the residue of these amplitudes and generalize the preceding argument, we do not obtain the full three-body wave function but rather the vertices for finding the bound state as a system composed of a nucleon and a \( \pi \) or \( \rho \) state. It is rather easy to convince oneself that, in order to obtain the three-body wave function, one has to calculate the residue at the pole in a break-up amplitude, namely, an amplitude in which there are three free particles in the final state.

The break-up amplitude for the process \( 1 + (2,3) \rightarrow 1 + 2 + 3 \) can be obtained in the AGS formalism in terms of the bound-state scattering amplitudes given by the integral equation (3.8). Such a break-up amplitude is the matrix element \( \langle \tilde{q}' | X_{\gamma n}, P_{\gamma}^i | q \rangle \) of an operator defined by:

\[
X_{\gamma n, P_{\gamma}^i}(E) = -Z_{\gamma n, P_{\gamma}^i}(E) - \sum_{\alpha, m} X_{\gamma n, \alpha m}(E) T_{\alpha m}(E) Z_{\gamma m, P_{\gamma}^i}(E) \tag{4.3}
\]

Notice that this is not an integral equation. In the case in which there are non-separable interactions only between one pair of particles, let us say \( \gamma' \), the operators \( Z_{\gamma n, P_{\gamma}^i}(E) \) are given exactly by

\[
\psi(q) = \frac{1}{i^2 + E_{\gamma}} \phi(q)
\]

(4.2)
The meaning of eq. (4.3) is exhibited in terms of Feynman-type diagrams in Fig. III. Since we are interested in obtaining the residue of the matrix element of $X_{lnp_1}(E)$ at $E = E_B$ (binding energy of the three-body bound state), we can drop the first term of the right-hand side of eq. (4.3) because only the bound-state scattering amplitudes are singular at $E = E_B$.

The effect of the non-separable interactions on the residue of the break-up amplitude is taken into account by: (a) the bound-state scattering amplitudes, because they are calculated with potentials that include the non-separable contributions, and (b) the second term of the right-hand side of eq. (4.4), in which the non-separable terms produce a "final-state interaction", as is obvious from Fig. III. This term complicates the analysis of the three-body wave function. We have carried out the analysis as far as possible by replacing $T_r$ by the Coulomb potential in the final-state interaction term, but in spite of the fact that some of the integrals involved can be computed analytically, we are left with quintuple integrals to be computed numerically in the final formulas for the probabilities we want to calculate. Therefore, we have neglected the contribution of such "final-state interaction" terms, since we cannot estimate them in a reliable way. They are of the same order as other contributions we have taken into account, and therefore our final numerical results should be considered only as estimates of the order of magnitude of the different probabilities we calculate.

Taking into account the identity of the particles, neglecting the "final-state" interaction terms, and introducing spin and isospin, we compute the right-hand residue of $\langle \tilde{q}_1 | X_{lnp_1} | \tilde{q}_1 \rangle$, which we
The wave function is obtained by multiplying the residue by the Green function \((\hat{p}_1^2 + \hat{d}_1^2 - \hat{E}_B)^{-1}\). However, before giving our results it is convenient to recall some properties of the spin and isospin wave functions of three nucleons. We shall use here the same notation as Schiff and write the doublet spin states of three nucleons in the form:

\[
\chi_1 = \frac{i}{\sqrt{6}} \left[ \alpha_1 \alpha_2 \beta_3 + \alpha_1 \beta_2 \alpha_3 - 2 \beta_1 \alpha_2 \alpha_3 \right]
\]

(4.5)

\[
\chi_2 = \frac{i}{\sqrt{2}} \left[ \alpha_1 \alpha_2 \beta_3 - \alpha_1 \beta_2 \alpha_3 \right]
\]

(4.6)

These spin functions correspond to \(S = \frac{1}{2}, S_z = \frac{1}{2}\). Since spin is conserved in our problem, the results are independent of \(S_z\). The spin functions \(\chi_1\) and \(\chi_2\) correspond to coupling first the pair \((2,3)\) in a state with \(S^{23} = 1\) or \(S^{23} = 0\), respectively. These two functions are the basis of the two-dimensional representation of the permutation group of three objects and transform under permutations into linear combinations of themselves, as discussed in Ref. 31).

We also need the isospin wave function. There are two functions corresponding to \(I = \frac{1}{2}, I_z = \frac{1}{2}\), which we call \(\eta^I\) and \(\eta^{I^*}\). They have the same structure and transformation properties under permutations as \(\chi_1, \chi_2\). There is also the isospin wave function corresponding to \(I = \frac{3}{2}, I_z = \frac{3}{2}\), which we call \(\eta^{I=3/2}\). Its explicit form is not needed; we must only remember that it is a totally symmetric function.

Going back to our problem, it is clear that when we solve the homogeneous integral equations with the same kernel as eq. (3.16) at the bound-state energy \(E = E_B\), we obtain the vertices for finding the \(^3\)He as a \(d-N\) bound state with \(I = \frac{3}{2}\) or as a \(S-N\) bound state with
I = \frac{1}{2} or 3/2. We call this function \( \Gamma_d(\frac{1}{2}, q) \), \( \Gamma_s(\frac{1}{2}, q) \) and \( \Gamma_s(3/2, q) \) respectively. Of course, it is only the overall normalization of these three functions which is arbitrary, since they satisfy coupled integral equations.

A straightforward calculation gives the wave function as

\[
\Psi(p_1, p_2) = \frac{1}{2} \left( P_{12} + P_{21} + P_{13} \right) \left[ F_1(p_1, p_2) \chi_1 \chi_2 + \\
+ F_2(p_1, p_2) \chi_2 \chi_1 + G(p_1, p_2) \chi_1 \chi_2 \right]
\]  

where \( P_{ij} \) are permutation operators that interchange the coordinates (including spin and isospin) of particles \( i \) and \( j \), and the functions \( F_1 \), \( F_2 \) and \( G \) are given by

\[
(p^2 + q^2 - E_0) F_1(q, p) = \Gamma_d(\frac{1}{2}, q) \left[ t_d(E_0 - q^2) q_d(p) \right]
\]  

\[
3(p^2 + q^2 - E_0) F_2(q, p) = \Gamma_s(\frac{1}{2}, q) \left[ 2 t_c(E_0 - q^2) q_c(p) + t_s(E_0 - q^2) q_s(p) \right] + \\
+ \frac{1}{2} \Gamma_s(\frac{3}{2}, q) \left[ t_c(E_0 - q^2) q_c(p) - t_s(E_0 - q^2) q_s(p) \right]
\]  

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The complete antisymmetry of the wave function given by eq. (4.7) is guaranteed by the properties of the spin and isospin wave functions and the fact that the functions $F_1, F_2, G$ depend only on the modulus of $p$ and $q$. Here again we can check very easily that, if the Coulomb interaction is switched off, these results coincide with the expression written by Amado for the $^3\text{H}$ wave function. Carrying out the symmetrization required by eq. (4.11) we obtain

\[ \langle P_1, q_1 \rangle = \frac{1}{\sqrt{2}} \Phi_0 + \sqrt{\frac{3}{8}} \left( W_1 \Phi_1 - W_2 \Phi_2 \right) + \]
\[ + \frac{13}{2} \left( \gamma_{1/2} \gamma_{1/2} - \gamma_{1} \gamma_{1} \right) \eta^{1/2} \]

\[ \text{(4.11)} \]

where the spin-isospin wave functions $\Phi_0, \Phi_1, \text{and} \Phi_2$ are defined in Ref. 31) as

\[ \Phi_0 = \frac{1}{\sqrt{2}} \left( \eta_{z} \gamma_{z} - \gamma_{z} \eta_{z} \right) \]
\[ \text{(4.12)} \]

\[ \Phi_1 = \frac{1}{\sqrt{2}} \left( \eta_{z} \gamma_{z} - \gamma_{z} \eta_{z} \right) \]
\[ \text{(4.13)} \]
\[
\Phi_2 = \frac{1}{\sqrt{2}} \left( \chi_z \eta_1 + \chi_1 \eta_z \right)
\]  
(4.14)

and the functions \( u, w_1, w_2, v_1 \) and \( v_2 \) can be calculated in terms of \( F_1, F_2 \) and \( G \). Let us define

\[
\Phi_{\pm}(q, p) = F_1(q, p) \pm F_2(q, p)
\]  
(4.15)

Then those functions are given by the following formulas:

\[
\begin{align*}
\psi &= \frac{1}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} (q_1, p_2) + \frac{1}{\sqrt{2}} (q_2, p_3) + \frac{1}{\sqrt{2}} (q_3, p_1) \right) \\
\psi_4 &= \frac{1}{\sqrt{15}} \left[ \frac{1}{\sqrt{2}} (q_1, p_2) + \frac{1}{\sqrt{2}} (q_2, p_3) - 2 \frac{1}{\sqrt{2}} (q_3, p_1) \right] \\
\psi_5 &= \frac{1}{\sqrt{2}} (q_1, p_2) - \frac{1}{\sqrt{2}} (q_2, p_3)
\end{align*}
\]  
(4.16) (4.17) (4.18)

The functions \( v_1 \) and \( v_2 \) depend on \( G(q, p) \) in the same way as \( w_1 \), \( w_2 \) depend on \( f_+(q, p) \).

The first term in the right-hand side of eq. (4.11) represents the dominant \( S \) state of the three-nucleon system, the function \( u \) being totally symmetric. The second term represents the \( S' \) state of mixed spatial symmetry, and the third term the \( I = 3/2 \) component of the wave function which is present here due to the isospin symmetry-breaking induced by the Coulomb interaction.
V. NUMERICAL RESULTS AND CONCLUSIONS

In order to calculate the Coulomb energy $\Delta_c$ of $^3$He, we consider the Faddeev-Lovelace equations (3.16) and project the $l = 0$ partial wave. The $^3$H and $^3$He binding energies are computed by searching for the zeros of the Fredholm determinants of the equations corresponding to $M = -1/2$ and $M = 1/2$, respectively. Their difference yields the desired Coulomb energy $\Delta_c$.

We have carried out two different calculations of $\Delta_c$ by choosing two different sets of parameters $\lambda_s$, $\beta_s$, $\lambda_d$, $\beta_d$ to describe the two-body interactions. In the first case we have used the same parameters as SITENKO and KHARCHENKO$^5$ (SK) in their calculation of the triton binding energy. These parameters are obtained by fitting the deuteron binding energy and the scattering length in the $^3S_1$ state, and the scattering length and effective range in the $^1S_0$ state. This model overestimates the $^3$H binding energy, which comes out to be $-11.96$ MeV. In the second case we have used the parameters corresponding to the NAQV1$^{32}$ potentials defined as $(C_N + S_N)$ in Ref. 12). They are obtained by fitting the two-body data with a central plus a tensor separable interaction in the $^1S_0$ state. Only the central part of the separable potential is retained in the three-body calculation. The $^3$H binding energy is underestimated by these potentials, which yield the value of $-6.95$ MeV for that quantity. In both cases the singlet scattering length takes the value $a_s = -23.7$ F.

We have used both sets of separable potentials with the same purpose in mind as GUPTA and MITRA$^{33}$ in a recent perturbative calculation of $\Delta_c$. These authors calculate $\Delta_c$ by computing the expectation value of the Coulomb potential, using exact $^3$H wave functions given by this kind of model. As they point out, the difference of the results for the above two cases is expected to give an indication of the explicit contribution of the tensor forces to this calculation.

As far as the $^3$He wave function is concerned, it is calculated by using the formulas developed in Sec. 4 and by solving numerically the homogeneous equations corresponding to eq. (3.16) to compute.
\( \gamma_{q}(\frac{1}{2}, q), \gamma_{q}(\frac{3}{2}, q) \) and \( \gamma_{q}(3/2, q) \). This is equivalent to obtaining an eigenfunction of the kernel of eq. (3.16) corresponding to an eigenvalue equal to one. Most of the analysis carried out by Weinberg\(^{34} \) concerning the properties of the eigenvalues and eigenvectors of the kernel of the Lippmann-Schwinger equations with complex potentials. We exploit the general properties of the eigenvalues discussed by Weinberg and solve numerically the homogeneous equations on a computer by a rapidly convergent iterative method\(^{35} \).

The probabilities of the states \( S, S' \) and \( I = 3/2 \), which we call \( P_{S} \), \( P_{S'} \), and \( P_{3/2} \), respectively, can be calculated at once from eq. (4.11) by recalling that the spin and isospin wave functions are defined in such a way that they are normalized to one\(^{31} \). The integrals involved in the calculation of the probabilities can be reduced to a triple integral: one integral over angles and two integrals over the modulus of two independent momenta. Gaussian quadratures are used to evaluate them, with a mesh of seven points for the angular integral and twenty points for the momentum integrals. Due to our rather crude momentum mesh, the errors in the final results for the probabilities have been estimated in the order of 30%. These big errors, together with the fact that we have neglected what we called "final-state interaction" terms in the previous section, indicate that our results should be considered as estimates of the probabilities.

The results we obtained for \( \Delta_{c} \) and \( P_{3/2} \) are shown in Table I. In order to study the effect of the non-separable T-matrix on the binding energy and wave function of \( ^{3}\text{He} \), we have recalculated \( \Delta_{c} \) and \( P_{3/2} \) by neglecting everywhere the contribution of the Coulomb T-matrix to the Faddeev-Lovelace equations. It turns out that its contribution is quite small; it only accounts for a few per cent of the predicted \( \Delta_{c} \) for both sets of parameters. The estimates of \( P_{3/2} \) also remain unchanged. Therefore, as far as \( \Delta_{c} \) and \( P_{3/2} \) are concerned, the main Coulomb effects are contained in the nuclear p-p T-matrix.

These results seem to contradict one's intuition, because it can be immediately argued that, had we chosen to include the contribution of T\(_{c}\) in a perturbative way, we should probably have got an effect of the order of \( \Delta E = (\frac{1}{\mu}^{1}v_{3}\text{He}^{1}, T_{c}, \frac{1}{\mu}^{1}v_{3}\text{He}^{1}) \), where
\( \Psi_{3_H} \) is the wave function calculated by neglecting \( T_c \). Then it is obvious that, since
\[
\Psi_{3_{He}} \sim \Psi_{3_H} \text{ and to first order in } e^2, \quad T_c \sim V_c,
\]
the contribution expected from \( T_c \) is precisely the perturbation theory result. Therefore, one is led to the conclusion that, as far as \( \Delta c \) is concerned, it should be accounted for mainly by the effects of the non-separable Coulomb part of the p-p T-matrix.

However, it is easy to convince oneself that in this case intuition fails and leads to double-counting. Let us first consider a perturbation expansion of the p-p transition matrix and make the rather trivial remark that perturbation diagrams containing only a certain number of Coulomb potentials (Fig. IV) contribute to \( T_c \), but diagrams containing interference terms between the Coulomb and the nuclear potential (Fig. V) contribute to the "nuclear" T-matrix, that is to say, to the Coulomb-corrected form factor. If we consider now all the possible perturbation theory diagrams for the three-nucleon system of first order in \( V_c \), it is clear that the contribution of the class of diagrams shown in Fig. VI is taken into account by \( T_c \), but the contribution of the class of diagrams shown in Fig. VII is included in the Coulomb-corrected form factors. This example shows clearly that the usual perturbation theory contributions are shared in our approach by the Coulomb-corrected form factors and the corrections introduced by \( T_c \), and that one cannot a priori decide which \( \Delta c \) if any will give the dominant contributions. This one is a model-dependent question; in our case it happens that the nuclear potentials are such that the corrections to the Lovelace potentials introduced by \( T_c \) are strongly depressed.

Our values for \( \Delta c \) are reasonably close to the experimental value of 0.764 MeV for the electromagnetic mass difference of \( 3_H \) and \( 3_{He} \). The first and second sets of parameters slightly underestimate and overestimate such a mass, respectively. These results do not agree with the work of GUPTA and MITRA\(^{33}\), in the sense that in their perturbative calculation, in addition to getting larger values for \( \Delta c \), they obtained that the first set of parameters predicts a larger value than the second. We have no comments to make on this point because a close comparison of our results with perturbation theory is certainly not meaningful. We determine the potentials that appear in the Faddeev-Lovelace equations to first
order in $e^2$, but afterwards we solve the equations numerically. Therefore, our results are not first-order results.

It has been suggested recently that the Coulomb energy of $^3\text{He}$ may be considerably smaller than the binding energy difference of $^3\text{H}$ and $^3\text{He}$\textsuperscript{36}). A recent calculation that includes in the $^3\text{H}$ wave function the effect of a soft repulsive core of the nuclear forces\textsuperscript{37}) indicates that this discrepancy may be of the order of 18%. Needless to say, if this discrepancy is a real effect, it implies the existence of a charge asymmetry of the nuclear forces, in particular that the n-n interaction is stronger than the p-p interaction\textsuperscript{37}). Moreover, it has been shown that in separable potential models the inclusion of hard-core effects - which we have neglected - tends to reduce the binding energy of the three nucleon system by approximately 10-15\%\textsuperscript{19}). It seems plausible to us that our results will tend to agree with those of Ref. 37) if hard-core effects in the nucleon-nucleon force are included.

We also find that for both sets of parameters $P_3 \sim 1^2$, in agreement with previous calculations\textsuperscript{11}), and $P_{3/2} \sim 0.01\textsuperscript{38})$. As far as $P_{3/2}$ is concerned, such a small value represents a negligible mixing of the $I = 3/2$ state in the $^3\text{He}$ wave function. These results are not very surprising; we know that wave functions are much more sensitive to the details of the input forces than binding energies or scattering cross-sections\textsuperscript{23}). The separable potentials of the Yamaguchi type used in this model are purely attractive, and that is certainly not the case for the nucleon-nucleon system. The probability $P_S$, has proved to be insensitive to the inclusion of tensor forces\textsuperscript{11}), but the inclusion of hard-core effects can raise $P_S$ up to a value of approximately $2\%\textsuperscript{18})$. This is roughly the amount of $S'$ state admixture compatible with inelastic electron scattering from $^3\text{H}$ and $^3\text{He}\textsuperscript{39})$, with the rate of slow neutron capture in deuterium\textsuperscript{40}) and with the nuclear binding energy\textsuperscript{41}). However, such an admixture of the $S'$ state cannot account for the striking difference between the charge form factors of both nuclei. GIBSON and SCHIFF\textsuperscript{42}) consider that the most promising candidate to explain the remaining discrepancy between both form factors is the $I = 3/2$ state.
The fact that our model predicts such a small admixture of the $I = 1/2$ state should not be taken as an argument against that possibility. It remains to be seen what happens to $P_{3/2}$ when more realistic two-body forces are considered. The next step is to include hard-core effects in the two-nucleon system. Work is in progress in this direction and we shall report on it in the near future. Finally, it must be emphasized that the methods we have discussed in this paper can be immediately applied to the study of p-d scattering, with the Coulomb corrections fully taken into account.

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Here we discuss briefly the computation of the integral involved in the definition (3.18); that is

\[
F_i((\xi', \eta'; E) = \frac{\alpha}{3\sqrt{5}} \frac{e^2}{\pi^2} \int \frac{d\xi'}{[\xi' - \xi_z]^2 + \mu_z^2} \cdot \int \frac{d\eta'}{[\eta' - \eta_z]^2 + \mu_z^2}.
\]

By making a decomposition in simple fractions, we can write (AI.1) in the form

\[
F_i((\xi', \eta'; E) = \frac{1}{\sqrt{3}} \frac{e^2}{\pi^2} \left[ I(\xi_z, \eta_z, \mu_z) - I(\xi_z, \eta_z, \mu_z) + I(\xi_z, \eta_z, \mu_z) \right].
\]

where

\[
\begin{align*}
\xi_1 &= \frac{1}{\sqrt{3}} \xi_1 + \frac{2}{\sqrt{3}} \xi_2 \\
\xi_2 &= \frac{2}{\sqrt{3}} \xi_1 + \frac{1}{\sqrt{3}} \xi_2 \\
\mu_1^2 &= \eta_1^2 - s \\
\mu_2^2 &= \eta_2^2 - s
\end{align*}
\]

By making a decomposition in simple fractions, we can write (AI.1) in the form

\[
F_i((\xi', \eta'; E) = \frac{1}{\sqrt{3}} \frac{e^2}{\pi^2} \left( \mu_1^2 - \eta_1^2 \right) \left( \mu_2^2 - \eta_2^2 \right) \left[ I(\xi_z, \eta_z, \mu_z) - I(\xi_z, \eta_z, \mu_z) + I(\xi_z, \eta_z, \mu_z) \right].
\]
where

\[
I(\xi, \eta, \kappa, \eta) = \int \frac{d^3p}{(p^2 - k^2)^2 + \eta^2} \cdot \frac{1}{p^2} \cdot \frac{1}{(p^2 - \lambda^2)^2 + \eta^2} \quad (AI.4)
\]

An integral of this form with $1/p^2$ replaced by $\lambda^2$ was computed by R.R. Lewis Jr. We have checked that there are no difficulties in setting $\lambda^2 = 0$ in his result, which in our case reads

\[
I(\xi, \eta, \kappa, \eta) = \frac{1}{\sqrt{\delta^2 - \alpha^2}} \ln \frac{\delta + (\delta^2 - \alpha^2)^{\frac{1}{2}}}{\delta - (\delta^2 - \alpha^2)^{\frac{1}{2}}} \quad (AI.5)
\]

where

\[
\alpha = \left[ (\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 \right] \left[ k_1^2 + \eta_1^2 \right] \left[ k_2^2 + \eta_2^2 \right] \quad (AI.6)
\]

\[
\delta = \eta_1 (k_1^2 + \eta_1^2) + \eta_2 (k_2^2 + \eta_2^2)
\]

From these relations we find

\[
\Delta^2 = \delta^2 - \alpha = - \left[ k_1^2 (\eta_1^2 + k_1^2) - k_1 (\eta_1^2 + k_1^2) \right] < 0 \quad (AI.7)
\]

Therefore, using (AI.7), eq. (AI.5) can be rewritten in the form

\[
I(\xi_1, \eta_1, \kappa_1, \eta_1) = \frac{\pi^3}{|\Delta|} \frac{t^3}{t^{\delta - 1}} \left( \frac{2 \delta |\Delta|}{\delta^2 - \Delta^2} \right) \quad (AI.8)
\]
APPENDIX II

We want to discuss here the isospin dependence of the potentials and propagators that appear in eqs. (3.16). We shall neglect spin indices because we are working with central forces and the spin dependence of the potentials can be taken into account in the way described in Ref. 1). From eq. (3.15) we can see that the potentials are basically matrix elements of the Green function \( G_0(E) \). If there were no isospin in the problem, the potentials called \( \mathcal{Z}_{n'n'}(q', q; E) \) would be given by matrix elements of \( G_0(E) \) between an initial and a final state in which there is a particle and a bound state called \( n \) (or \( n' \)) with relative momentum \( q \) (or \( q' \)) in the total centre-of-mass system, respectively. For equal-mass particles, such a potential is given explicitly by 1)

\[
\mathcal{Z}_{n'n'}(q', q; E) = \langle q', n' | G_0(E) | q, n \rangle = \left( \frac{4^3}{3} \right) \frac{g_n(p) g_{n'}(\tilde{p})}{p^2 + q^2 - E}
\]

where \( p = \frac{1}{\sqrt{3}} q - \frac{2}{\sqrt{3}} q' \), \( \tilde{p}' = \frac{2}{\sqrt{3}} q - \frac{1}{\sqrt{3}} q' \) and \( g_n(p) \) are the form factors of the bound states. The same is true in the case in which, instead of bound states, we have virtual or resonant states of two particles.

When isospin is introduced and the second term in the right-hand side of eq. (3.15) is taken into account − with \( V_f^c \), the Born approximation, instead of \( T_f^c \) − we use a basis in Hilbert space in which the three isospins of the individual particles, the isospin \( I_n \) of the bound state \( n \), the total isospin \( I \) and its \( z \) component \( I_z \) are diagonal. The eigenvalues of \( I_z \) are labelled \( M \). Therefore, the potentials we need are given by
These matrix elements can be calculated by recalling that $G_0(E)$ is the identity in isospin space. Using eq. (AII.1), everything reduces to Clebsch-Gordan algebra. However, one has to be careful because we have seen in Sec. II that in the case where there is isospin symmetry-breaking, the two-body $T$-matrix, and consequently the form factors $g^{ti(E)}$ depend on the $z$-component of the isospin $I_n$ of the two-body subsystem. We label $M$ the eigenvalues of $I_n^Z$, and denote the form factors by $g^{ti(E)_M}$. For the singlet state $I_n = 1$ and it is clear that

$$g^{ti(E)_1} = g^{ti(E)}$$

Therefore, the potentials are given by

$$Z^m_{n', I', n I} (q', q; E) = \sum\sum\sum \langle M, m; M, m | T | M, m; M, m \rangle.$$
where the functions $f_{M_{n'}}, M_n(q',q;E)$ and $F_{M_{n'}}, M_n(q',q'q;E)$ are given in eqs. (3.17) and (3.18), respectively.

It is interesting to discuss in general what happens when the Coulomb interaction is switched off. In this case, obviously, the second term of the right hand side of eq. (AII.4) vanishes, but also in this case the form factors no longer depend upon $M_n$, and all the functions of $q'$, $q$ can be factored outside the sum over Clebsch-Gordan coefficients. What remains is by definition an isospin recoupling coefficient $^1$), which of course vanishes unless $I - I'$. Therefore, eq. (AII.4) shows explicitly how the isospin violation in the three-body system is induced by the corresponding violation in the two-body subsystems.

As far as the propagators $t_n(E - q^2)$ are concerned, they described the propagation of a particle and a bound state labelled $n$. However, it is clear from our discussion in Sec. II that in the case in which there are Coulomb interactions they also depend on $M_n$. We denote the propagators by $t_n(E - q^2)$. When $I_n = 1$, it is clear that
\[ t_1(E) = t_c(E) \quad ; \quad t_0(E) = t_{-1}(E) = t_e(E) \quad \text{(AII.5)} \]

In the basis we are using, the propagators are

\[ t_{nI_{\frac{1}{2}}I'}^M(E) = \sum_{M_n} t_{M_n}(E). \]

\[ \cdot \langle I_{n, \frac{1}{2}}, I_{M, I} | I_{n, M_n, \frac{1}{2}, M-M_n} \rangle \langle I_{n, M_n, \frac{1}{2}, M-M_n} | I_{n, \frac{1}{2}, I', M} \rangle \quad \text{(AII.6)} \]

From this equation one can easily obtain eq. (3.25).
TABLE I

Results of the calculation of the Coulomb energy $\Delta_c$ of $^3$He and of the probability $P_{3/2}$ of the $I = 3/2$ state. The notation for the potentials is explained in Sec. V.

<table>
<thead>
<tr>
<th>Potential</th>
<th>$\Delta_c$ (MeV)</th>
<th>$P_{3/2}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S.K.</td>
<td>$0.68 \pm 0.05$</td>
<td>$\sim 0.01$</td>
</tr>
<tr>
<td>$G_N + S_N$</td>
<td>$0.84 \pm 0.05$</td>
<td>$\sim 0.01$</td>
</tr>
</tbody>
</table>
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FIGURE CAPTIONS

Fig. I : Graphical representation of particle-bound-state potentials that contain a Coulomb-corrected form factor.

Fig. II : Graphical representation of the non-separable corrections to the potentials that do not contain a Coulomb-corrected form factor.

Fig. III : Graphical representation of the form of the p-d break-up amplitude.

Fig. IV : Set of p-p perturbation diagrams that contribute to the pure Coulomb T-matrix. Wavy crossed lines represent the Coulomb potential.

Fig. V : Set of p-p perturbation diagrams that contribute to the "nuclear" T-matrix. Dotted crossed lines represent the nuclear potential.

Fig. VI : Set of three-body perturbation diagrams, to first order in $V_c$, whose effect is taken into account by the non-separable corrections to the bound-state potentials.

Fig. VII : Set of three-body perturbation diagrams, to first order in $V_c$, whose effect is taken into account by the Coulomb corrections to the two-body form factors.
Fig. I

Fig. II