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REGGEIZATION OF INTERNAL SYMMETRIES - II

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ABSTRACT

These notes (which will not be published in the present form) are concerned with Reggeization of U(6,6) symmetry. Our major conclusion is that if physical criteria could be formulated, the Reggeization procedure could, in principle, determine why physical particles correspond to parts of specified non-compact towers and not to others. The paper is concerned with the details of construction of different types of trajectories, though unresolved is the specification of the criteria to select out, for example, a Regge generalized trajectory with the content of, say, a Feynman tower. It is suggested that Majorana-like infinite-dimensional equations provide a concrete realization of Regge trajectories. These equations give rise to propagators for Regge trajectories. Using these one can develop a calculus in J-plane similar to the Feynman calculus which would represent multiple exchange of Regge trajectories.

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INTRODUCTION

The nature of internal symmetries of particle physics is mysterious. The exact U(1) symmetries which manifest themselves as charge and baryon-number conservation are presumably kinematical in character, outside of space-time structure. The broken symmetries like SU(2) and SU(3) may resemble the U(1) symmetries mentioned above or alternatively may be dynamical in character like shell model symmetries in nuclear physics, manifesting themselves for some class of phenomena and not for others, for some energy ranges and not for others. Proposals of this variety have been mentioned from time to time, among others by Yamaguchi [2)] who has argued that SU(2) and SU(3) are really no deeper than S_2 and S_3 (permutation group) symmetries, with a superimposition of a particular type of dynamics; by Lipkin [3)] who, starting with just three quarks, has claimed to obtain a number of SU(3) results <u>without</u> SU(3) and by Chew from whose point of view the symmetries can possibly be no deeper than a manifestation of bootstrap accidents. The successes of SU(2) or SU(3) algebra of currents do not gainsay this type of reasoning in any fundamental sense. If basic doublet or triplet fields (quarks) exist, the commutation relations for the algebra follow from the quark commutation relations. The successes of the algebra then only verify once again that within a certain dynamical domain a (broken) symmetry exists among the physical particle spectrum.

The hypothesis that SU(2) and SU(3) are dynamical symmetries would, in an obtuse manner, perhaps be welcome, since it would place these symmetries on a par with the other dynamical symmetries SU(4) and SU(6). One cannot say this of SU(2) in the presently explored energy domain but certainly so far as SU(3) is concerned it is equally as broken as SU(6). [4)]

In the first paper of this series, referred to as I, a beginning was made to take seriously the point of view of internal symmetries being possibly dynamical in character for predicting the physical particle spectrum. Given a rest symmetry (like SU(6)) to which known particles belong, one could assume that the dynamical interactions of these particles also exhibit SU(6) symmetry. One could then expand the physical S-matrix in partial waves, using for the harmonic analysis the group SU(6) rather than the conventional group $SU(2)_J$. A reggeization of the Casimir-invariants of SU(6) rather than the Casimir J (angular momentum) of $SU(2)_J$ would self-consistently predict a particle spectrum which would include, of course, the particles we started from. Also one may exploit the well-known connection of Regge pole theory with asymptotic scattering limits and hope that this may provide results in better accord with experiment than the predictions of higher-symmetry theories in the low-energy domain. It was found that, in general, the particles appearing on a generalized Regge trajectory correspond to the content of appropriate non-compact towers. The reggeization procedure therefore provided a link between Coleman's 57 varieties of SU(6).[5]

Although in this introduction we have so far stressed the possibility that internal symmetries may be dynamical in character, there is nothing in the formalism of Part I of this investigation to make this absolutely essential to the development. In that paper, attention was specifically devoted to the higher rest symmetry SO(n). These groups - the prototype of which is SO(6) combining J and I-spin - proved (for somewhat accidental reasons, explained below) particularly simple to handle. The rest symmetries like SU(6) or $SU(6) \otimes SU(6)$ are more complicated and, in a sense, much more interesting to reggeize. Unlike the case of SO(n) where there appeared just one Casimir which could be reggeized, there now appear more than one, leading to distinct types of trajectories. The distinct types can be placed in a one-to-one correspondence with some of the rungs of non-compact towers. It is the principal aim of this paper to explain where new complications make their appearance. Not resolved is the problem of defining physical criteria which should select one type of trajectory rather than another.

What makes the difference among the various cases? The simplest way to see it is to make a partial wave expansion based [6] not on the little groups for $p^2 > 0$, i.e. the rest symmetry groups O(n), SU(6) or $SU(6) \otimes SU(6)$, but based on the little groups for $p^2 < 0$, i.e. $O(n-1,1)$, $U(3,3)$ or $SL(6,C)$. (In enumerating these $p^2 < 0$ little groups we have assumed that the overall Lorentz group containing the relativistic structure we are dealing with is $O(n,1)$, $SL(6,C)$ or $U(6,6)$.) The non-compact groups $O(n-1,1)$, $U(3,3)$ or $SL(6,C)$ possess a number of unitary irreducible representations with Casimir invariants which take (for principal series) continuously as well as discretely infinite set of values. It is only the former, the continuously infinite Casimirs, which can be reggeized. The difference between $O(n-1,1)$ and $SL(6,C)$ lies in the simple circumstance that $O(n-1,1)$ possesses just one continuous Casimir, while $SL(6,C)$ can have as many as five.

In Gel'fand and Naimark's classification [7], $SL(6,C)$ possesses one non-degenerate and nine [8] degenerate types of series. The partial wave expansion, presumably, in general calls for inclusion of all the ten types of series. Meromorphy assumptions for the amplitude would give ten distinct types of Regge poles. The most degenerate series, for example, leads to the reggeization of just the quark number. For other series, not only the quark number but also other Casimir's are reggeized.

Is there any physical criterion (besides simplicity) by which one could assert either that only one type of series contributes or only one Casimir shows meromorphy? One may conjecture numerous types of criteria; to take one, it may be hoped that the asymptotic behaviour of the expansion functions associated with each type of series may be different; from the known asymptotic behaviour one may thus find that one can limit oneself to just one series in the expansion. Unfortunately we have been unable to find anything in the literature on the high-energy behaviour of the expansion functions. We feel that if this problem could be solved one may eventually be able to determine why certain classes of representations occur in the expansion giving a clue to what type of particles might be physically realized. The plan of these notes is as follows. In Sec.I we formulate the problem of embedding a rest symmetry in a relativistic structure and show some of the arbitrariness which may arise. A particularly crucial role in all subsequent development is played by the W-spin subgroup of the rest symmetry - this is the subgroup of the rest-symmetry which commutes with a Lorentz generator J_{03}. The W-spin subgroup equals the helicity J_{12} when the relativistic group is O(3,1); it may with reason have been called "generalized helicity" for other cases. Given a two particle rest state with total W-spin content w, the Lorentz boost $e^{iJ_{03}\zeta}$ in the centre-of-mass frame generates from it all states of rest-symmetry (or in the brick-wall frame, all states of the corresponding non-compact little group) whose W-spin content equals w. This problem is solved in Sec. 2 for $U(\nu)\otimes U(\nu)$ groups. In Sec. 3, we turn to the problem of construction of the rotation functions for the most degenerate class of representations of $U(\nu)\otimes U(\nu)$. The result, in spite of the complicated methods of derivation, turns out to be remarkably simple; for example $d^{N}_{00}(\theta) = C^{3}_{N}(\cos\theta)$ for U(6,6) case. Here C is the Gegenbauer function. A reggeization of N (the quark number) shows that the high-energy behaviour is proportional to $(\cos\theta)^{N}$. In the last section we turn to a different topic. This topic concerns the possible use of Majorana-like infinite-dimensional equations for providing a representation of Regge trajectories. *

1.1 Relativistic symmetry groups

For systems at rest we have, to begin with, the group of spatial rotations. This symmetry group is generated by the ordinary angular momentum $\underline{J}$. In addition to this it is usual to include space reflection as a symmetry. Beyond these we can have the various strictly internal symmetries such as $SU(2)_I$ or $SU(3)_F$. These groups are independent of the spatial rotations and reflections, that is to say: the generators I_i, F_i, ... etc. are positive parity rotational invariants. All of these symmetries operate on rest states; they leave invariant the manifold of states with $\underline{P} = 0$. This manifold will be finite dimensional if the symmetries are compact ones, otherwise it must be infinite dimensional.

It may be that we can reasonably extend the rest symmetry to include transformations which are independent of neither the internal symmetries nor of the ordinary rotations. The generators

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of such mixed transformations will be neither $O(3)_J$ nor $SU(2)_I$ singlets but must belong to some non-trivial representation of $O(3)_J \times SU(2)_I$ (for example). Again this overall symmetry group may or may not be compact.

Amongst the physical observables we have the generators, J_{0i} of pure Lorentz transformations. Evidently these do not commute with the generators of the rest symmetry, in particular, with $\underline{J}$. Thus, if we wish to keep an algebraic structure which includes the Lorentz transformations then it is necessary to envision an enlargement of the rest symmetry. If the rest symmetry is compact then this enlargement is essential. The relativistic algebra so obtained may or may not be finite dimensional. It must at least contain the algebra of the homogeneous Lorentz group, $O(3,1)$, and this means that all the members of the algebra may be grouped into multiplets of $O(3,1)$. In particular, all the generators of the rest symmetry - which of course fall into $O(3)$ multiplets - must be associated with (possibly broken) $O(3,1)$ multiplets.

A given rest symmetry may be embedded in more than one relativistic group. For example, an operator which belongs to a triplet of $O(3)_J$ may belong to an axial vector of $O(3,1)$ or to an antisymmetric tensor. The physical consequences must vary accordingly. The simplest example of this sort of thing occurs for the rest symmetry $SU(4)$. The generators of this symmetry take, in the quark representation, the form

$$\tfrac{1}{2}\sigma_i\ ,\ \tfrac{1}{2}\tau_j\ ,\ \tfrac{1}{2}\sigma_i\,\tau_j\ ,\ i,j=1,\ 2,\ 3$$

where σ_i and τ_j denote 2 x 2 Pauli matrices. One way to extend this group is to regard these matrices as a subset of

$$\tfrac{1}{2}\sigma_{\mu\nu}\ ,\ \tfrac{1}{2}\tau_j\ ,\ \tfrac{1}{2}\sigma_{\mu\nu}\tau_j\ ,\ \tfrac{1}{2}\gamma_5\ \tau_j$$

which is a representation of $SL(4,C)$. Another relativistic extension is provided by the set

$$\tfrac{1}{2}\sigma_{\mu\nu}\ ,\ \tfrac{1}{2}\tau_j\ ,\ \tfrac{i}{2}\gamma_\mu\gamma_5\,\tau_j$$

which is a representation of $O(6,1)$ - a smaller algebra than $SL(4,C)$.

That the physical consequences springing from two such distinct embeddings will differ between the two models is not surprising. The distinctions are made clear at an early stage if we proceed to the W-subgroups. The W-group is defined as that part of the rest-symmetry which is left invariant by a particular Lorentz transformation, J_{03}. This group does not include the spin but rather only one component of it, J_{12}. Since the Lorentz transformation $e^{-i\alpha J_{03}}$ can be used to boost a rest state into motion along the 3-axis we see that W-symmetry leaves the 3-component of momentum invariant. Moreover, the 2-particle product states with vanishing total momentum and with relative momentum directed along the 3-axis can be classified into W-representations. They constitute a manifold which is invariant under the W-group.

Usually it happens that an irreducible representation of the rest symmetry will contain some W-representations more than once. This leads to labelling problems. It will be necessary to introduce extraneous operators into the system in order to obtain a complete set of quantum numbers with which to label the states. These operators generally will not commute with J_{03} - otherwise they would belong to the W-algebra - and so, for example, will not be conserved in forward scattering where W-spin is conserved. For the case considered above, where SU(4) was embedded in either SL(4,C) or O(6,1) the W-groups are respectively,

$$SU(2) \otimes SU(2) \otimes U(1) \quad \text{and} \quad O(5),$$

the one containing 5 commuting operators and the other 6. Thus, the constraints on forward scattering amplitudes are stronger in the second symmetry than in the first.

Insofar as it is sensible to assume any algebraic structure at all for the set of operators obtained from the generators of the rest symmetry by applying Lorentz transformations to them we must require this also for the displacement operators. Of course we can always generate new operators from those of the rest symmetry by commuting them with J_{03} and then commuting these new ones with each other, ... , etc. In general one has no reason to expect the system so generated to be a finite one - unless one is dealing with a system whose degrees of freedom are finite in number, such as the 3-dimensional harmonic oscillator - and so one need not find any algebraic structure. The assumption that one does find such a structure, however complicated, is a very restrictive one indeed. Having assumed the existence of a finite-rank homogeneous relativistic group, one is faced with the problem of what to do about the displacements. Again by applying homogeneous transformations to the Poincaré displacements P_μ, we will generate new operators, and if the set produced in this way is finite, they must comprise a multiplet of the homogeneous group. The simplest assumption we can make about these operators is that they all commute with one another. This of course is a highly restrictive dynamical assumption. We have no analogies from systems of finite degree such as the harmonic oscillator or the hydrogen atom on which to base it.

There is one rather special model (Fronsdal) [9] which avoids this multi-momentum situation. If one starts with a <u>non-compact rest symmetry</u> - one which contains the Lorentz group - then it is possible to construct a relativistic group which has the form of the direct product of O(3,1) with the rest symmetry, and the Poincaré 4-vector P_μ <u>in itself</u> may be looked upon as a multiplet of this group. With such an interpretation there is no need to incorporate additional momenta when completing the algebra. Of course, with such a non-compact group as rest symmetry, the manifold of rest states must always be infinite dimensional and, moreover, this infinity of states will be degenerate in mass.

If we must incorporate the momentum in a higher dimensional multiplet - as we shall have to if we wish to avoid the infinite degeneracy of Fronsdal's model - then there are two essential requirements to be met. Firstly, this multiplet must contain a 4-vector when decomposed relative to the physical Lorentz group and, secondly, the timelike component of this 4-vector must be a <u>singlet</u> of the rest symmetry group. The 4-vector can then be interpreted as the physical 4-momentum.

The next step, of course, is to associate particles with the irreducible representations of the inhomogeneous relativistic group. If such a representation is decomposed relative to the Poincaré group then, in general, a number of Poincaré representations will appear. There will be a continuum of masses and a (possibly infinite) range of spins. If we confine ourselves to the manifold which is generated from the rest states by application of Lorentz transformations then the mass continuum disappears and the range of spins becomes finite. These are the only parts we shall use. That is, we shall proceed with incomplete multiplets of the group thereby regarding it as a broken symmetry. The problem of how much symmetry to assign to the scattering operator is the central one of the whole exercise.

1.2 Decomposition of direct products and the partial wave expansion

To make a partial wave expansion we have to be able to decompose the direct product of two irreducible representations. This can be done largely by analogy with the Poincaré group. It amounts to discovering what representations of the rest symmetry can be projected out of the manifold of product states with fixed total momentum. We are not interested in the complete decomposition but rather only in the parts receiving contributions from our physically truncated states - those which are connected to the rest states by pure Lorentz transformations.

The 1-particle states with completely general momenta can presumably be generated from the rest states by applying sufficiently general boosts. However, we shall need only states of the form

$$|p\,\xi\rangle = e^{-i\theta J_{31}} \; e^{-i\alpha J_{03}} |\hat{p}\,\xi\rangle \tag{1}$$

or

$$|p\,\xi\rangle = e^{i(\pi-\theta)J_{31}} \; 3^{-i\alpha J_{03}} |\hat{p}\,\xi\rangle \tag{2}$$

where $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \alpha < \infty$ and the states $|\hat{p}\,\xi\rangle$ correspond to particles at rest - all components of p vanishing except for $p_0 = m$. These rest states span a unitary irreducible representation of the rest symmetry.

From the space of product states, $|p_1\xi_1, p_2\xi_2\rangle$ let us pick out those for which

$$(p_1 + p_2) = (E, 0, 0, 0)$$

These can be written in the form

$$
\begin{aligned}
|p_1\xi_1, p_2\xi_2\rangle &= e^{-i\theta J_{31}^{(1)}} e^{-i\alpha_1 J_{03}^{(1)}} e^{i(\pi-\theta)J_{31}^{(2)}} e^{-i\alpha_2 J_{03}^{(2)}} |\hat{p}_1\xi_1, \hat{p}_2\xi_2\rangle \\
&= e^{-i\theta J_{31}} e^{-i(\alpha_1 J_{03}^{(1)} - \alpha_2 J_{03}^{(2)})} e^{i\pi J_{31}^{(2)}} |\hat{p}_1\xi_1, \hat{p}_2\xi_2\rangle \\
&= e^{-i\theta J_{31}} e^{-i(\alpha_1 J_{03}^{(1)} - \alpha_2 J_{03}^{(2)})} \sum_{\xi_2'} |\hat{p}_1\xi_1, \hat{p}_2\xi_2'\rangle\langle\xi_2'| e^{i\pi J_{31}^{(2)}} |\xi_2\rangle
\end{aligned}
\tag{3}
$$

where $J_{31} = J_{31}^{(1)} + J_{31}^{(2)}$ and

$$E = m_1 \operatorname{ch}\alpha_1 + m_2 \operatorname{ch}\alpha_2, \quad 0 = m_1 \operatorname{sh}\alpha_1 - m_2 \operatorname{sh}\alpha_2$$

The states $|\hat{p}_1\xi_1, \hat{p}_2\xi_2\rangle$ belong to the direct product of two irreducible representations of the rest symmetry. This product could be decomposed into irreducible parts by the usual Clebsch-Gordan method if that were needed. However, the operator $\alpha_1 J_{03}^{(1)} - \alpha_2 J_{03}^{(2)}$ does not belong to the rest symmetry algebra and, moreover, it is not a scalar of that algebra. But it is a scalar of the W-group. Therefore, the operator $\exp \frac{1}{i}(\alpha_1 J_{03}^{(1)} - \alpha_2 J_{03}^{(2)})$, while being an infinite sum of different representations of the rest symmetry, is a scalar of the W-symmetry. We can expect an expansion of the form

$$e^{-i(\alpha_1 J_{03}^{(1)} - \alpha_2 J_{03}^{(2)})} |\hat{p}_1\xi_1, \hat{p}_2\xi_2\rangle = \sum_{n\xi} |n, \hat{p}\xi\rangle\langle n\xi|\xi_1\xi_2\rangle \tag{4}$$

where $\hat{p} = (\mathcal{E}, 0, 0, 0)$ and the coefficients $\langle n\, \xi \mid \xi_1\, \xi_2 \rangle$ must be proportional to coupling coefficients of the W-group. The label, n, is generally needed to separate the multiplicity of rest symmetry representations which appear. Its form can in principal be worked out explicitly in any given case.

The "rest" states $|n, \hat{p}\, \xi\rangle$ arrived at in this way belong to irreducible representations of the full relativistic group. Putting these things together we arrive at the desired decomposition. It has the form

$$|p_1 \xi_1, p_2 \xi_2\rangle = \sum_{n \xi \xi_2' \xi'} |n, \hat{p}\, \xi\rangle\langle \xi | e^{-i\theta J_{31}} | \xi'\rangle\langle n, \xi' | \xi_1 \xi_2'\rangle\langle \xi_2' | e^{i\pi J_{31}} | \xi_2\rangle \tag{5}$$

In order to apply this formula to a given symmetry there are essentially two things that must be computed:

(1) the functions $\langle \xi | e^{-i\theta J_{31}} | \xi'\rangle$

(2) the coefficients $\langle n \xi | \xi_1, \xi_2\rangle$

At least one of these jobs is difficult: which one depends upon our choice of basis vectors. If the labels ξ, are chosen so as to diagonalize the Casimirs of $O(3)_J$ then (1) is trivial while (2) is very complicated. On the other hand, if ξ diagonalizes the Casimirs of the W-group - which does not contain J_{31} - then (1) is complicated while (2) is relatively simple. The ideal solution, it would appear, should be the construction of the transformation matrices which connect these two bases. Actually these will be needed if we are to make any use at all of the W-labels since the physical input will always involve $O(3)_J$.

The matrix elements of an invariant operator, S, will be expressible in the form

$$S|n, p\, \xi\rangle = \sum_m |m, p\, \xi\rangle\langle m | S(p^2, \xi) | n\rangle \tag{6}$$

the amplitudes $\langle m | S(p^2, \xi) | n\rangle - \delta_{mn} = i\langle m | T(p^2, \xi) | n\rangle$ being the analogues of helicity amplitudes in the higher symmetry model, the helicity, λ, being here generalized to include the Casimirs of the W-group. If the operator S is unitary then, of course, so are the matrices $\langle m|S|n\rangle$,

$$\sum_{m} \langle m|S|n'\rangle \langle m|S|n\rangle^{*} = \delta_{nn'} \tag{7}$$

1.3 <u>Various models</u>

The rest symmetry contained in the Poincaré group is just O(3). We shall consider a number of extensions of this,

$$O(4) = SU(2) \otimes SU(2), \quad O(6) = SU(4), \quad SU(6), \ U(6) \otimes U(6)$$

Each of these groups contains O(3) and so their irreducible representations generally will include a range of spin values. With the exception of the first example which is too small, these groups contain also a rotationally invariant "internal" symmetry group. This means that their irreducible representations contain <u>multiplets</u> of the internal symmetry each associated with a definite spin.

We consider firstly the relativistic extensions of these groups. The first two examples have alternative extensions of comparable merit. The last two are fairly unambiguous. In order to treat these as uniformly as possible we shall exhibit the various generators in a representation by Dirac γ-matrices and $3 \otimes 3$ λ^j matrices. They are

(i) O(4) : $\sigma_{ab}, \ i\gamma_a\gamma_5 \quad , \quad a,b = 1,2,3$

(ii) O(6) : $\sigma_{ab}, \tau^i, \ \sigma_{ab}\tau^i \quad , \quad a,b,i = 1,2,3$

(iii) SU(6) : $\sigma_{ab}, \lambda^j, \ \sigma_{ab}\lambda^j \quad , \quad j = 1,2,..,8$

(iv) $SU(6) \otimes SU(6)$: $\sigma_{ab}, \ i\gamma_a\gamma_5, \ \lambda^j, \ \sigma_{ab}\lambda^j \ i\gamma_a\gamma_5\lambda^j, \ \gamma_0\lambda^j$

For all of these examples the Lorentz transformation J_{03} is represented by σ_{03}. Hence commuting the operators with σ_{03} and with each other will produce the relativistic extensions. We get

(i) $O(4) \rightarrow O(4,1)$ generated by $\sigma_{\mu\nu}$, $i\gamma_\mu \gamma_5$

(ii) $O(6) \rightarrow SL(4,C)$ " " $\sigma_{\mu\nu}, \tau^i, \sigma_{\mu\nu}\tau^i, \gamma_5 \tau^i$

(iii) $SU(6) \rightarrow SL(6,C)$ " " $\sigma_{\mu\nu}, \lambda^j, \sigma_{\mu\nu}\lambda^j, \gamma_5 \lambda^j$

(iv) $SU(6) \otimes SU(6) \rightarrow SU(6,6)$ generated by
$$\lambda^j, \gamma_\mu, \gamma_\mu \lambda^j, \sigma_{\mu\nu}, \sigma_{\mu\nu}\lambda^j, i\gamma_\mu\gamma_5, i\gamma_\mu\gamma_5\lambda^j, \gamma_5, \gamma_5\lambda_j$$

From the example (iv) we see that in the course of extending $SU(6) \otimes SU(6)$ by applying Lorentz transformations we have been forced to extend the rest symmetry itself:

$$SU(6) \otimes SU(6) \longrightarrow SU(6) \otimes SU(6) \otimes U(1)$$

where $U(1)$ is represented by γ_0. This did not happen in example (i) because $SU(2) \otimes SU(2)$ happens to be equivalent to an O-group. However, if we arbitrarily extend the rest symmetry by including γ_0, i.e.,

$$SU(2) \otimes SU(2) \longrightarrow SU(2) \otimes SU(2) \otimes U(1), \text{ or}$$

$$O(4) \rightarrow O(4) \otimes O(2)$$

then the relativistic extension coincides with the entire Dirac algebra,

$$SU(2) \otimes SU(2) \otimes U(1) \longrightarrow SU(2,2)$$

or $$O(4) \otimes O(2) \rightarrow O(4,2)$$

which is to be compared with $O(4) \rightarrow O(4,1)$, an entirely different model.

Example (ii) provides an even sharper illustration of the arbitrariness of this programme since here we have only to change the <u>interpretation</u> of the rest symmetry - without increasing it - in order to change the relativistic group. Thus, if we make the replacement

$$\sigma_{ab}\,\tau^i \longrightarrow i\,\gamma_0\,\gamma_5\,\tau^i$$

which leaves the rest algebra unmodified, then the relativistic algebra becomes

$$\sigma_{\mu\nu}\,,\ \tau^i\,,\ i\,\gamma_\mu\,\gamma_5\,\tau^i$$

corresponding to the relativistic group O(6,1). Thus we have the alternatives

$$O(6) \begin{array}{l} \nearrow SL(4,C) \\ \searrow O(6,1) \end{array}$$

which are physically quite distinct.

Let us tabulate the groups together with their generators. We list in turn: the relativistic group, the rest symmetry from which it arose and the W-group.

(i)	O(4,1)	:	$J_{\mu\nu}$, $J_{\mu 5} = -J_{5\mu}$	$\mu,\nu = 0,1,2,3$
	O(4)	:	J_{ab} , J_{a5}	$a,b = 1,2,3$
	O(3)	:	J_{12} , J_{25} , J_{51}	
(i')	SU(2,2) = O(4,2)	:	$J_{\mu\nu}$, $J_{\mu 5}$, J_μ , J_5	
	O(4) ⊗ O(2)	:	J_{ab} , J_{a5} , J_0	
	O(3)	:	J_{12} , J_{25} , J_{51}	
(ii)	O(6,1)	:	$J_{\mu\nu}$, J^i , $J^i_{\mu 5}$	$i = 1,2,3$
	O(6)	:	J_{ab} , J^i , J^i_{a5}	
	O(5)	:	J_{12} , J^i , J^i_{15} , J^i_{25}	

(ii') SL(4,C) : $J_{\mu\nu}$, J^i , $J^i_{\mu\nu}$, J^i_5 $\quad i = 1,2,3$

SU(4) = O(6) : J_{ab} , J^i , J^i_{ab}

SU(2) ⊗ SU(2) ⊗ U(1) : J_{12} , J^i , J^i_{12}

(iii) SL(6,C) : $J_{\mu\nu}$, J^i , $J^i_{\mu\nu}$, J^i_5 $\quad i = 1,2,\ldots,8$

SU(6) : J_{ab} , J^i , J^i_{ab}

SU(3) ⊗ SU(3) ⊗ U(1) : J_{12} , J^i , J^i_{12}

(iv) SU(6,6) : $J_{\mu\nu}, J^i_{\mu\nu}, J_{\mu 5}, J^i_{\mu 5}, J_5, J^i_5, J_\mu, J^i_\mu, J^i$

SU(6) ⊗ SU(6) ⊗ U(1) : $J_{ab}, J^i_{ab}, J_{a5}, J^i_{a5}, J_0, J^i_0, J^i$

SU(6) : $J_{12}, J^i_{12}, J_{15}, J_{25}, J^i_{15}, J^i_{25}, J^i$

Let us consider next the labelling problem. Generally there are for each model two chains of subgroups whose Casimirs are useful for labelling states. Both chains will include the purely internal symmetry group but one chain will include the W-group while the other will include the spin group $O(3)_J$. Most of these chains have the awkward feature of providing incomplete sets of labels which means that their Casimirs must be supplemented by other operators constructed from the generators of the rest symmetry.

Only for the cases (i) and (i'), O(4,1) and O(4,2) are both chains completely sufficient. This case can be analysed completely. For case (ii), O(6,1), the W-chain is sufficient but the J-chain is not. For the other cases, (ii), (iii), (iv), neither chain is adequate.

Taking the various models in turn we list the operators, known or hypothetical, which will be needed to label the basis vectors in an irreducible representation of the rest symmetry.

(i) $O(4,1)$

We need 4 labels. These are given by the eigenvalues of the following operators:

$$\tfrac{1}{2} J_{ab} J_{ab} + J_{a5} J_{a5}$$

$$\tfrac{1}{2} \epsilon_{abc} J_{ab} J_{c5}$$

$$\begin{cases} J_{12}^2 + J_{25}^2 + J_{51}^2 & \text{(W-chain)} \\ J_{12}^2 + J_{23}^2 + J_{31}^2 & \text{(J-chain)} \end{cases}$$

$$J_{12}$$

The connection between the basis vectors of the two chains is provided by the unitary transformation $\exp\left(i \frac{\pi}{2} J_{35}\right)$.

(i') $O(4,2) = SU(2,2)$

This model is labelled by the same chains as in (i) with addition of the single operator, J_0, a Casimir of $O(4) \otimes O(2)$. There is therefore a total of 5 labels. The connection between the two chains is again provided by $\exp\left(i \frac{\pi}{2} J_{35}\right)$.

(ii) $O(6,1)$

Here we need 9 labels. The operators of the W-chain are complete. They are best given in terms of a straightforward set of $O(6)$ generators, J_{AB} ($A,B = 1,2,3,4,5,6$), defined in terms of the old set by

$$J_{12}, J_{23}, J_{31} = J_{12}, J_{23}, J_{31}$$

$$J_{45}, J_{56}, J_{64} = J^1, \; J^2, \; J^3$$

$$J_{16}, J_{14}, J_{15} = J^1_{15}, J^2_{15}, J^3_{15}$$

$$J_{26}, J_{24}, J_{25} = J^1_{25}, J^2_{25}, J^3_{25}$$

$$J_{36}, J_{34}, J_{35} = J^1_{35}, J^2_{35}, J^3_{35}$$

The required 9 operators are, for the W-chain,

$$\left.\begin{array}{l} \frac{1}{2} J_{AB} J_{AB} \\ \frac{1}{48} \epsilon_{ABCDEF} J_{AB} J_{CD} J_{EF} \\ J_{AB} J_{BC} J_{CD} J_{DA} \end{array}\right\} \text{Casimirs of } O(6)$$

$$\left.\begin{array}{l} \frac{1}{2} J_{\alpha\beta} J_{\alpha\beta} \\ J_{\alpha\beta} J_{\beta\gamma} J_{\gamma\delta} J_{\delta\alpha} \end{array}\right\} \begin{array}{l}\text{Casimirs of } O(5), \\ \alpha, \ldots, \delta = 1,2,4,5,6\end{array}$$

$$\left.\begin{array}{l} \frac{1}{2} J_{\mu\nu} J_{\mu\nu} \\ \frac{1}{8} \epsilon_{\mu\nu\lambda\rho} J_{\mu\nu} J_{\lambda\rho} \end{array}\right\} \begin{array}{l}\text{Casimirs of } O(4), \\ \mu, \nu = 1,4,5,6\end{array}$$

$\frac{1}{2}J_{ab} \ J_{ab}$ Casimir of $O(3)$, $a,b = 4,5,6$

J_{64} Casimir of $O(2)$

On the other hand, the operators of the J-chain must include, in addition to the 3 Casimirs of $O(6)$, the 4 Casimirs from $O(3)_J \otimes O(3)_I$ and its subgroup $O(2) \otimes O(2)$:

$$J_{12}^2 + J_{23}^2 + J_{31}^2$$

$$J_{45}^2 + J_{56}^2 + J_{64}^2$$

$$J_{12}$$

$$J_{64}$$

which make a total of 7. Thus it is generally necessary to construct two other operators, F_1 and F_2 say, which commute with each other and with these. We have no idea at this stage what would make the most suitable candidates.

Also, the problem of transforming from one basis to the other would have to be solved.

(ii') <u>SL(4,C)</u>

Here again we need 9 labels. Neither chain is sufficient in this case. For the W-chain we have, in addition to the three Casimirs of $O(6) = SU(4)$, a set of 5 from the W-group and its subgroups. They are

$$\left.\begin{array}{l} (J^i + J^i_{12})\,(J^i + J^i_{12}) \\ \\ (J^i - J^i_{12})\,(J^i - J^i_{12}) \\ \\ J_{12} \end{array}\right\} \quad \text{Casimirs of } SU(2) \otimes SU(2) \otimes U(1)$$

J^3

J^3_{12}

which makes a total of 8. Therefore we must construct one more operator G to complete the set.

For the J-chain we can adopt the same operators as were used for case (2). Of course the transformation between W- and J-bases must be recalculated.

(iii) SL(6,C)

The rest symmetry SU(6) generally requires 20 labels. Of these, 5 are provided by the Casimirs of SU(6) itself. In the W-chain another 11 are provided by $SU(3) \otimes SU(3) \otimes U(1)$ and its subgroups. It is necessary then to supplement these with 4 constructs, $H_1, \ldots, H_4$ in order to fill out the W-chain

$$20 = 5 + 11 + 4 \ .$$

The J-chain requires 8 new constructs

$$20 = 5 + 7 + 8 \ .$$

(iv) SU(6,6)

The rest symmetry $SU(6) \otimes SU(6) \otimes U(1)$ requires 41 labels. The Casimirs give $5 + 5 + 1 = 11$ of these. The W-chain yields another 20, leaving 10 to be made up:

$$41 = 11 + 20 + 10 \ .$$

The J-chain is even more hopeless.

The momentum operators must be assigned to appropriate representations of the homogeneous parts. The two fundamental requirements that: (1) the representation contain a Lorentz 4-vector and (2) that the timelike component of this vector be a singlet of the rest symmetry can presumably be satisfied by many different representations. We list here the simplest choice for each case.

(i)	O(4,1)	:	P = 5
(i')	O(4,2)	:	P = 15
(ii)	O(6,1)	:	P = 7
(ii')	SL(4,C)	:	P = $(4,\bar{4})$
(iii)	SL(6,C)	:	P = $(6,\bar{6})$
(iv)	SU(6,6)	:	P = 143

Evidently the 4-vector γ_μ is always included and, moreover, the set of matrices which commute with γ_0 can in each case be seen to be the rest symmetry. In all of these examples the rest symmetry coincides with the maximal compact subgroup.

1.4 The coefficients $\langle n\,\xi\,|\xi_1, \xi_2\rangle$

We shall assume now that a complete set of labels has been worked out for our relativistic group according to the W-chain. We shall assume in addition that the various Clebsch-Gordan coefficients which arise can be computed. As a preliminary let us make some conjectures concerning the Clebsch-Gordan problem.

Let us suppose that the states of an irreducible representation of the rest symmetry group can be written

$$|C\,F\,W\rangle \quad = \quad |C\,F\,\omega\,\lambda\rangle \tag{8}$$

where C denotes the set of Casimir invariants of the rest group, W denotes the set necessary to label a representation of the W-group and F denotes the set of supplementary labels which are necessary when a given W-group representation occurs more than once. It is also useful on occasion to separate the set of labels W into two parts, ω and λ, where ω denotes the Casimir invariants of the W-group and λ the rest.

The direct product of two representations presumably decomposes into the form

$$|C_1F_1W_1\ ,\ C_2F_2W_2\rangle = \sum_{\zeta CFW} |C_1C_2\,\zeta\,CFW\rangle\langle\zeta CFW|C_1F_1W_1,\ C_2F_2W_2\rangle \tag{9}$$

where ζ denotes a set of multiplicity parameters. How many of these there are can be conjectured on the basis of a simple counting argument: the number of parameters in the set $C_1\ C_2\,\zeta$ must equal the number in CFW if none is to be lost in the decomposition. Thus, for SU(n) where

number of C's $= n-1$

" " CFW $= (1 + 2 + 3 + \dots + n) - 1 = \frac{n(n+1)}{2} - 1$

" " ζ $= N_\zeta$

$$\therefore \qquad 2(n-1) + N_\zeta = \tfrac{1}{2}\, n(n+1) - 1 \tag{10}$$

i.e. $$N_\zeta = \tfrac{1}{2}(n-1)(n-2)$$

so that for SU(2): we have $N_\zeta = 0$, for SU(3): $N_\zeta = 1$, ... etc.

The direct product of two W-representations presumably decomposes in like manner:

$$|\omega_1\lambda_1,\ \omega_2\lambda_2\rangle = \sum_{\sigma\omega\lambda} |\omega_1\omega_2\sigma\omega\lambda\rangle\langle\sigma\omega\lambda|\omega_1\lambda_1,\ \omega_2\lambda_2\rangle \tag{11}$$

We shall sometimes write

$$\langle\sigma\omega\lambda\ \ |\omega_1\lambda_1\,,\omega_2\lambda_2\rangle = \langle\sigma W|W_1,\ W_2\rangle \tag{12}$$

Finally, we can define "W-scalar factors" in the rest symmetry Clebsch-Gordan coefficients by writing

$$\langle \zeta C F W \mid C_1 F_1 W_1, C_2 F_2 W_2 \rangle$$

$$= \sum_{\sigma} \langle \zeta C F \sigma w \| C_1 F_1 w_1, C_2 F_2 w_2 \rangle \langle \sigma W \mid W_1, W_2 \rangle \tag{13}$$

Using these notions for decomposing the product of two representations of the homogeneous group, we proceed as follows. Write the product state in the form

$$|p_1 C_1 F_1 W_1, p_2 C_2 F_2 W_2\rangle = U(G)\, e^{-i(\alpha_1 J_{03}^{(1)} - \alpha_2 J_{03}^{(2)})} |\hat{p}_1 C_1 F_1 W_1, \hat{p}_2 C_2 F_2 W_2\rangle \tag{14}$$

where G belongs to the rest group and the angles α_1, α_2 are chosen so as to have

$$(p_1 + p_2)_\mu = (E, 0, 0, 0) \equiv \hat{p}_\mu \tag{15}$$

where E denotes the total energy. This is not always the most general 2-particle centre-of-mass state but it is sufficient for our purposes.

Since J_{03} is a W-group invariant we can make the multipole decomposition

$$e^{-i(\alpha_1 J_{03}^{(1)} - \alpha_2 J_{03}^{(2)})} = \sum_{\hat{C}\hat{F}} N_{\hat{C}\hat{F}0}(E) \tag{16}$$

where the operator $N_{\hat{C}\hat{F}\hat{W}}(E)$ belongs to an irreducible representation of the rest symmetry. Now

$$N_{\hat{C}\hat{F}0}(E)\, |\hat{p}_1 C_1 F_1 W_1, \hat{p}_2 C_2 F_2 W_2\rangle$$

$$= \sum_{\bar{\zeta}\bar{C}\bar{F}W} N_{\hat{C}\hat{F}0}(E)\, |\hat{p}_1 C_1, \hat{p}_2 C_2, \bar{\zeta}\bar{C}\bar{F}W\rangle \langle \bar{\zeta}\bar{C}\bar{F}W \mid C_1 F_1 W_1, C_2 F_2 W_2\rangle$$

$$= \sum_{\substack{\bar{\zeta}\bar{C}\bar{F}W \\ \zeta C F}} |E, C_1 C_2, \bar{\zeta}\bar{C}\hat{C}, \zeta C F W\rangle \langle \zeta C F W \mid \hat{C}\hat{F}0, \bar{C}\bar{F}W\rangle \times \langle \bar{\zeta}\bar{C}\bar{F}W \mid C_1 F_1 W_1, C_2 F_2 W_2\rangle \tag{17}$$

Therefore,

$$|\, p_1 C_1 F_1 W_1, \; p_2 C_2 F_2 W_2 \rangle$$

$$= U(G) \sum_{\hat C \hat F} \sum_{\bar\xi \bar C \bar F} \sum_{\zeta C F W} |\epsilon, C_1 C_2, \bar\xi \bar C \hat C, \zeta C F W\rangle \times$$

$$\times \sum_{\sigma} \langle \zeta C F \omega \| \hat C \hat F o, \bar C \bar F \omega\rangle \langle \bar\xi \bar C \bar F \sigma \omega \| C_1 F_1 \omega_1, C_2 F_2 \omega_2\rangle \langle \sigma W | W_1 W_2\rangle$$

$$= \sum_{\substack{CFW,\\ \sigma F'W'}} \sum_{\substack{\zeta \bar\xi \bar C \hat C\\ \bar F \hat F}} |\epsilon, C_1 C_2, \bar\xi \bar C \hat C, \zeta C F' W'\rangle \langle \zeta C F \omega \| \hat C \hat F o, \bar C \bar F \omega\rangle \times$$

$$\times \langle \bar\xi \bar C \bar F \sigma \omega \| C_1 F_1 \omega_1, C_2 F_2 \omega_2\rangle \, D^{C}_{F'W', FW}(G) \langle \sigma W | W_1, W_2\rangle$$

$$= \sum_{C\sigma} \sum_{FW, F'W'} |\hat p, C_1 F_1 \omega_1, C_2 F_2 \omega_2, \sigma F \omega, C F' W'\rangle D^{C}_{F'W', FW}(G) \langle \sigma W | W_1, W_2\rangle \tag{18}$$

We have thereby constructed the expansion

$$|p_1 \xi_1, p_2 \xi_2\rangle = \sum_{n \xi \xi'} |n \hat p \; \xi'\rangle D_{\xi' \xi}(G) \langle n \xi | \xi_1 \xi_2\rangle \tag{19}$$

where

$$D_{\xi' \xi}(G) = \delta_{C' C} \, D^{C}_{F' W, FW}(G) \tag{20}$$

$$\langle n \xi \,|\, \xi_1 \xi_2\rangle = \langle \sigma W | W_1, W_2\rangle \tag{21}$$

and

$$n = \left\{ C_1 F_1 \omega_1, \; C_2 F_2 \omega_2, \; \sigma F \omega \right\} \tag{22}$$

The coefficient $\langle \sigma W | W_1, W_2 \rangle$ implies a finite limit to the summations over σ and W. The restricted range of W implies further restrictions on the summations over C and F. To find out what these are one must learn something of the structure of the representations, D^C.

The scattering amplitudes $\langle n' | T(p^2, \xi) | n \rangle$ corresponding to a strictly invariant S-matrix are now seen to carry the following labels:

$$\langle C_3 F_3 \omega_3, \; C_4 F_4 \omega_4, \; \sigma' F' \omega' | T^C(s) | C_1 F_1 \omega_1, \; C_2 F_2 \omega_2, \; \sigma F \omega \rangle \; . \tag{23}$$

They will appear together with the functions

$$\langle W_3, W_4 | \sigma' W' \rangle \; \langle CF' W' | e^{-i\theta J_{31}} | CFW \rangle \; \langle \sigma W | W_1, W_2 \rangle \tag{24}$$

and assorted Clebsch-Gordan coefficients.

2.1 <u>Enumeration of generalized partial wave amplitudes</u>

The representation functions

$$\mathcal{D}^C_{F'W', FW}(G) = \langle CF'W' | U(G) | CFW \rangle \tag{25}$$

which appear in the partial wave expansion remain to be computed in general. But before one can even begin to calculate them there is the fundamental problem of enumerating all C labels which contain the W and W' sublabels appearing above, i.e. <u>one has to determine and classify all rest representations which include the particular W spin representations</u>. Depending on the character of the rest symmetry, the problem alters its complexion, so we are forced to proceed from the general to the particular. In this section we determine all the representations; in the next section we turn to the classification problem.

Since the solution has already been given for all $O(\nu)$ representations by Salam and Strathdee in I, we shall restrict our attention to the case of the relativistic $SU(\nu, \nu)$ group where the rest symmetry is $SU(\nu) \otimes SU(\nu) \otimes U(1)$ and the W-group is $SU(\nu)$. Let us label the rest representations by $(N_1, N_2; \Gamma)$ where N_1, N_2 and Γ serve as labels for the $SU(\nu)$, $SU(\nu)$ and $U(1)$ groups; and let us label the $SU(\nu)_W$ representation by N. Thus N, N_1 and N_2 could be regarded as the dimensionalities of the representations. Our problem consists in seeing what $(N_1\ N_2; \Gamma)$ contain a given N. In fact Γ is totally irrelevant to this question since its generator commutes with the $SU(\nu)_W$ generators (but of course not with the Lorentz boost) and we can assert that all the spectrum of Γ values are allowed. Thus we shall ignore it henceforth. For the rest we must have

$$N_1 \otimes N_2 = N \oplus \cdots \tag{26}$$

The constraint on N_1 and N_2 are readily obtained if we invert the relation to read

$$\overline{N}_1 \otimes N = \sum N_2 \tag{27}$$

for we have only to vary N_1 in order to discover with what N_2 it must be associated. Thus simple multiplication solves the enumeration problem.

To exemplify the procedure take the case $\nu = 6$. One knows then that the commonly occurring baryons and mesons can be accommodated into the $(6, \overline{6})$ and $(56, 1)$ multiplets (with $\Gamma = 0, 3$). Two-particle states require the compositions

$1 \otimes 35, \quad 1 \otimes 56, \quad 35 \otimes 35, \quad 56 \otimes \overline{56}, \quad 35 \otimes 56$

of W-spin representations and in the reduction one meets the W-multiplets

$N = 1,\ 35,\ 189,\ 405, \ldots\ ;\ 56,\ 70,\ 700, \ldots$

These, then, have to be contained in (N_1,N_2) levels of $SU(6) \otimes SU(6)$, which can be found directly from the rule

$$\bar{N}_1 \otimes N = N_2$$

<u>Thus for $N = 1$</u> (W-singlet) we have all the rest states of the variety $C = (N,\bar{N})$.

<u>For $N = 35$</u> one obtains the sequence

$$(1,35),\ (35,1),\ (6,\bar{6}),\ (\bar{6},6),\ (6,\overline{84}),\ (\bar{6},84),$$
$$(6,\overline{180}),\ (\bar{6},180),\ (35,35),\ (189,1),\ (1,189),$$
$$(405,1),\ (1,405),\ \cdots,$$

and so on.

It is important however to remark that in general this multiplication problem leads to a ν-fold infinity which is connected to the single degree of freedom in Γ and the $\nu-1$ parameters which characterize a given N_1 representation (N_2 is thereby limited in its range).

We have not discussed the model where the relativistic group is $SL(2\nu, C)$ and the rest symmetry $SU(2\nu)$. The enumeration problem is rather more difficult here in that the W-spin symmetry $SU(\nu) \otimes SU(\nu) \otimes U(1)$ does not immediately lend itself to the above technique. Presumably, though, some variation of the method is possible.[9)]

2.2 Classification of the amplitudes

We now turn to the problem of ordering the rest-symmetry representations into various well-defined categories. This is altogether a more difficult problem to solve than that of enumeration as we must first define what precisely distinguishes one class from the next; and our guide can most simply come from the theory of unitary representations of non-compact groups. The reason why we use non-compact groups is that their unitary infinite-dimensional representations can be grouped into various sets characterized by differing degrees of "degeneracy", to each degree of which is associated a special content of the representations of the maximal compact subgroup. The clue for the choice of non-compact group is obtained from the "crossed channel" analysis where the rest symmetry is "continued" (rather like $O(3) \to O(2,1)$ in ordinary partial wave analysis). Thus for our models the continuations are to

Relativistic symmetry	Rest symmetry	W symmetry	Crossed channel symmetry
$SO(\nu,1)$	$SO(\nu)$	$SO(\nu-1)$	$SO(\nu-1,1)$
$SL(2\nu,C)$	$SU(2\nu)$	$SU(\nu)\otimes SU(\nu)\otimes U(1)$	$SU(\nu,\nu)$
$SU(\nu,\nu)$	$SU(\nu)\otimes SU(\nu)\otimes U(1)$	$SU(\nu)$	$SL(\nu,C)\otimes O(1,1)$

Notice that, in general, the W group is the maximal compact subgroup of crossed channel symmetry.

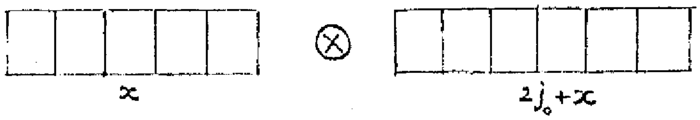
Hereafter we shall confine our consideration to the last of these models.

The unitary representations of $SL(\nu, C)$ will be distinguished by their $SU(\nu)$ content and with these distinctions we shall be able to arrange the corresponding $SU(\nu)\otimes SU(\nu)$ representations into different classes. It may be that this classification of $SU(\nu)\otimes SU(\nu)$ sequences can also be achieved through the use of $SU(\nu, \nu)$ classes of unitary representations. This is a matter for speculation as at the present time one does not have available the full series of $SU(\nu, \nu)$ representations.

The decomposition of $SL(\nu,C)$ unitary irreducible representations with respect to the maximal compact subgroup $SU(\nu)$ is a classic problem which has been completely solved by Gel'fand and Naimark. They have given necessary and sufficient conditions for the occurrence of an $SU(\nu)$ representation in an $SL(\nu,C)$ one. Before we state them in Naimark's language, let us give rules of thumb for working out the content of irreducible $SL(\nu,C)$ representation of the principal degenerate and principal non-degenerate series.[10] We start with the well-known case of $SL(2,C)$ and observe that the content of the principal series representation (j_0, σ)

$$j_0 = 0, \tfrac{1}{2}, 1, 3/2, \cdots$$
$$-\infty \leqslant \sigma \leqslant \infty$$

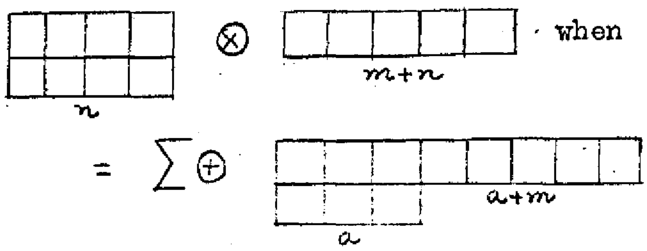
relative to the SU(2) subgroup is the same as results from the multiplication of two irreducible SU(2) representations



as $x \to \infty$. Above, x and $2j_0 + x$ denote the number of boxes in in the Young tableau. This special case is highly suggestive and we show below how we can generalize it to SL(3,C) and then to SL(n,C).

<u>SL(3,C)</u>. There are two types of principal series – the principal non-degenerate and principal degenerate.

i) Principal degenerate series are characterized by two numbers $(m\rho)$ $m \geqslant 0$ integral, $-\infty \leqslant \rho \leqslant \infty$, the content of the representation being given by the product



when $n \to \infty$

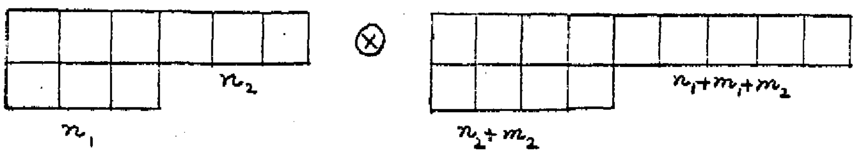
These SL(3,C) representations are the so-called Feynman towers and in their decomposition each SU(3) representation occurs just once. Note that, as for SL(2,C), the content does not depend upon ρ, a <u>common</u> feature of ∞-dimensional representations.

<u>Example:</u>

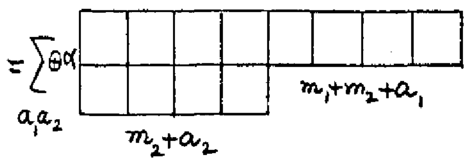
$$(0,\rho) \equiv 1 \oplus 8 \oplus 27 \oplus \cdots$$

$$(3,\rho) \equiv 10 \oplus 35 \oplus 81 \oplus \cdots$$

ii) Principal non-degenerate series are characterized by 4 numbers $(m_1 m_2, \rho_1 \rho_2)$, $m_1 m_2$ non-negative integers and $-\infty \leqslant \rho_1 \rho_2 \leqslant \infty$. The content of the above representation is obtained from the product



with $n_1, n_2 \to \infty$.



$\left(\begin{array}{l} a_1,\ m_2 + a_2 \text{ are non-negative integers} \\ \text{such that } 0 \leqslant a_1 + 2a_2 \equiv 0 \pmod 3 \end{array} \right)$

where α is the multiplicity which can be different from 1. Note the appearance of the double ∞, a characteristic which immediately distinguishes this case from the principal degenerate series representations.

SL(ν,C). Here the degeneracy types are defined by the partition of ν of the form

$$\nu = \nu_1 + \nu_2 + \cdots + \nu_\tau \qquad 2 \leqslant \tau \leqslant \nu, \ \nu_i \geqslant 1 \tag{28}$$

and the representations are characterized by $2(\tau-1)$ numbers

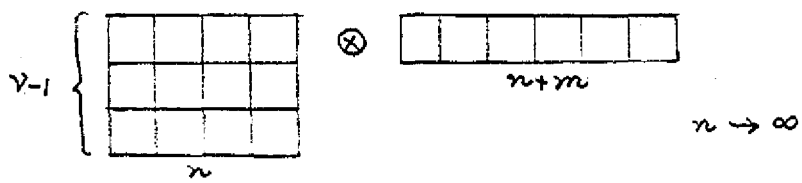
$$\begin{pmatrix} m_1 & m_2, & \cdots & m_{\tau-1} \\ \rho_1 & \rho_2, & \cdots & \rho_{\tau-1} \end{pmatrix} \qquad \begin{array}{c} m_i \text{ integers} \\ -\infty \leqslant \rho_i \leqslant \infty \end{array} \tag{29}$$

To find the content we again draw the diagrams:

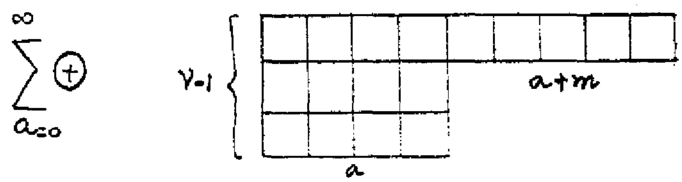
i) Most degenerate type: This is given by

$$\nu_1 = \nu - 1 \quad , \ \nu_2 = 1$$

If a representation of this type is characterized by $(m\,\rho)$, then its content is determined from



As for SL(2,C) and SL(3,C) cases, this can be calculated explicitly to give

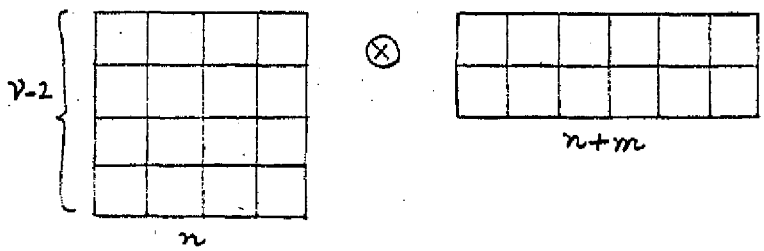


each representation occurring just once.

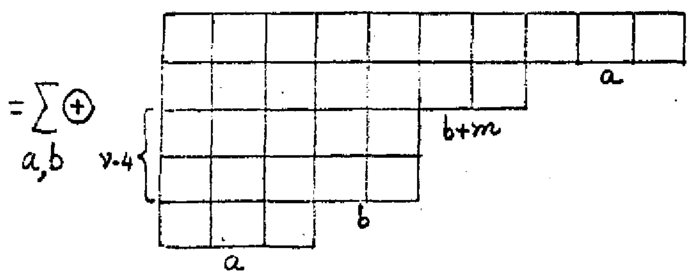
<u>Next most degenerate types:</u> There are many different series here given by different partitions of ν into <u>two</u> parts. The first is given by

$$\nu_1 = \nu - 2 \quad , \quad \nu_2 = 2$$

A representation of this type <u>still requires 2 independent Casimir operators</u>. Denoting a particular one of this type by (m, ρ) its content is depicted in



$n \to \infty$



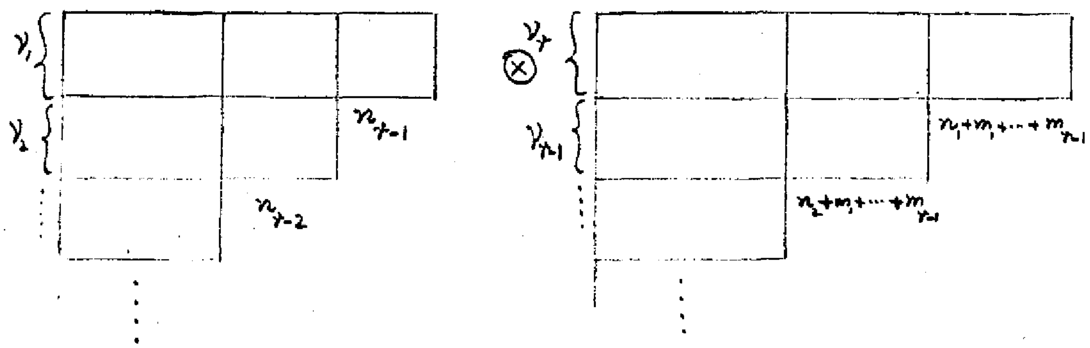
Each representation is again occurring once. The classes given here all have this characteristic.

This process can be continued step by step. To see what the general structure looks like, we describe below the case of the less degenerate representations.

Indeed the procedure is to draw two "large" diagrams which are conjugate to each other [11] and add the diagram

$$(m_1+m_2+\cdots+m_{r-1},\ m_2+m_3+\cdots+m_{r-1},\ \cdots,\ m_{r-1})$$

to the second one and multiply symbolically



with $n_1, n_2, \ldots \to \infty$.

The distinction between the different degrees of degeneracy is simply in the possible equality of the rows of the two SU(ν) diagrams separately which also necessitates equality of the corresponding m's. The partition

$$\nu = \nu_1 + \nu_2 + \cdots + \nu_r$$

which fixes the degeneracy type, restricts the second diagram such that all n_i are zero except for

$$n_{\nu_r},\ n_{\nu_r+\nu_{r-1}},\ \cdots,\ n_{\nu_r+\nu_{r-1}+\cdots+\nu_1}$$

In other words, for this type, the first ν_r rows of the second diagram are equal, then the next ν_{r-1} are equal, etc. The ν-th row is always taken to have zero length.

Finally we state Gel'fand and Naimark's theorem on the content.

To obtain the content of the $SL(\nu,C)$ representation characterized by the partition

$$\nu = \nu_1 + \nu_2 + \cdots + \nu_r \qquad 2 \leq r \leq \nu \,, \; \nu_i \geq 1$$

and a set of $2(r-1)$ numbers

$$\begin{pmatrix} m_1, & m_2, & \cdots & , m_{r-1} \\ \rho_1, & \rho_2, & \cdots & , \rho_{r-1} \end{pmatrix}$$

$\left(m_i \text{ integers}, \; -\infty \leqslant \rho_i \leqslant \infty \right)$

(all the $2(\nu-1)$ Casimir operators are functions of just these $2(r-1)$ numbers), search for the weight

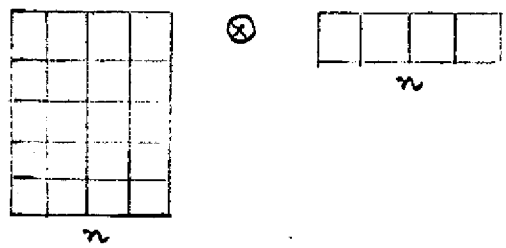
$$\left(\underbrace{m_1, m_1, \cdots, m_1}_{\nu_1}, \; \underbrace{m_2, m_2, \cdots, m_2}_{\nu_2}, \cdots \right)$$

in representations of $SU(\nu)$. The number of times this weight appears in a representation of $SU(\nu)$ as a singlet of the subgroup $SU(\nu_1) \times SU(\nu_2) \otimes \ldots \otimes SU(\nu_{r-1})$ is the multiplicity of occurrence of the $SU(\nu)$ representation in the $SL(\nu,C)$ representation.

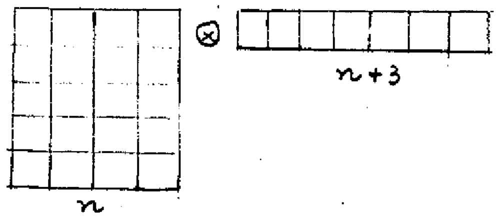
The trick that we have used to connect a weight with a Young pattern is the following. If an $SU(\nu)$ representation includes a particular weight, it also includes all its Weyl transforms. One of these is the highest weight of some irreducible representation and hence is associated with a Young diagram.

The proof of the rules that we have given for the general cases has not yet been rigorously established. Essentially it lies in identifying the multiplicity of a weight in an $SU(\nu)$ representation with the multiplicity of its occurrence in a particular decomposition. However, for the most degenerate and the next most degenerate cases that we have described (in all these cases the multiplicities were just one), the rules are undoubtedly exact. These rules are very plausible and we believe they are generally correct.

Case $\nu = 6$. The most degenerate series follows from the multiplications



mesons



baryons

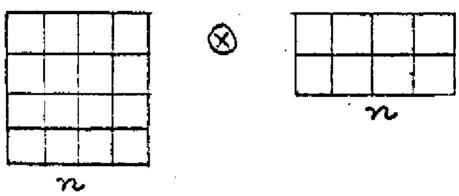
which correspond to the Feynman towers $\sum(N, \bar{N})$ and $\sum(N, \overline{N+3})$ respectively.

In detail,

$(1,1)$, $(\bar{6},6)$, $(\overline{21}, 21)$ ... mesons

$(1,56)$, $(\bar{6}, 126)$, ... baryons

A typical next most degenerate series would correspond to



and would yield

$(1,1)$, $(\bar{6},6)$, $(\overline{15},15)$ ... mesons

This particular variety of next most degenerate representations is obviously (for the baryons at least) not useful. And so on for the least degenerate series. Based upon the (1,56) and (1,1) fundamental diagrams we can generate the series

$$(N,\bar{N}) \qquad \text{mesons}$$

$$(1,56),\quad (\bar{6},126),\quad (\overline{15},504)\ \ldots \qquad \text{baryons}$$

3.1 Meson and baryon supermultiplets

The lowest-lying hadron states can be accommodated into the (56,1) and $(6,\bar{6})$ representations of SU(6) $\otimes$ SU(6). Upon that there is unanimous agreement; but what is far less settled is into which multiplets one should place the higher mesons and baryons, linked to which doubt is the possibility that the orbital excitation scheme SU(6) $\otimes$ SU(6) $\otimes$ O(3) with its smaller multiplet structures $(56,1,\ell)$ and $(6,\bar{6},\ell)$ may be preferred to the SU(6) $\otimes$ SU(6) scheme with its widely increasing multiplet dimensionalities. The ℓ-excitation scheme has as its basis the non-relativistic quark bound-state picture and gives rise to only octets and decuplets of SU(3) whereas the scheme SU(6) $\otimes$ SU(6) we are presenting will differ from the ℓ- excitation essentially in predicting the existence of SU(3) $\overline{10}$, 27 and 35's.

From the theoretical point of view there exists a sequence of meson and baryon representations, the Feynman towers, which is characterized by a single Casimir operator, e.g.,

$$(1,1),\quad (6,\bar{6}),\quad (21,\overline{21})\ \ldots \qquad \text{for the meson tower,}$$

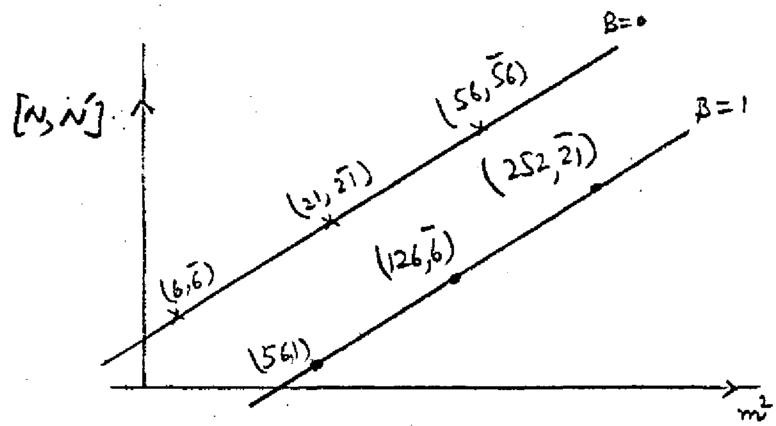
and

$$(56,1),\quad (126,\bar{6})\ \ldots \qquad \text{for the baryon tower.}$$

These towers belong to the most-degenerate series of unitary representations of SU(6,6); they are thus the most rudimentary of the classes and for that reason are theoretically amongst the most favoured. At first glance the series would seem to be experimentally discredited as it unquestionably predicts an alarming multitude of new particles in ever larger SU(3) representations (..., 27, 35, ...) which shows no signs of being established. But recently, fairly convincing arguments have been advanced which show that the W-spin selection rules inhibit the production of these higher states in simple 2 or 3-body channels;[12] the prevention of their observation then largely destroys the most direct evidence against the Feynman series.[13] There may of course be other objections why the degenerate series should not be seriously considered and we may be unable to meet them. For the present, however, let us pursue the study of the series if only on grounds of mathematical simplicity.

The important characteristic of these particular towers is that they are specified by a single label, the quark number N. If we neglect the mass differences within the supermultiplets, we may plot the mass^2 versus the integer N. The points when connected make the generalized Regge trajectories.

Possible Regge trajectories in the quark number plane



3.2 Reduction of the N-trajectory into J-trajectories

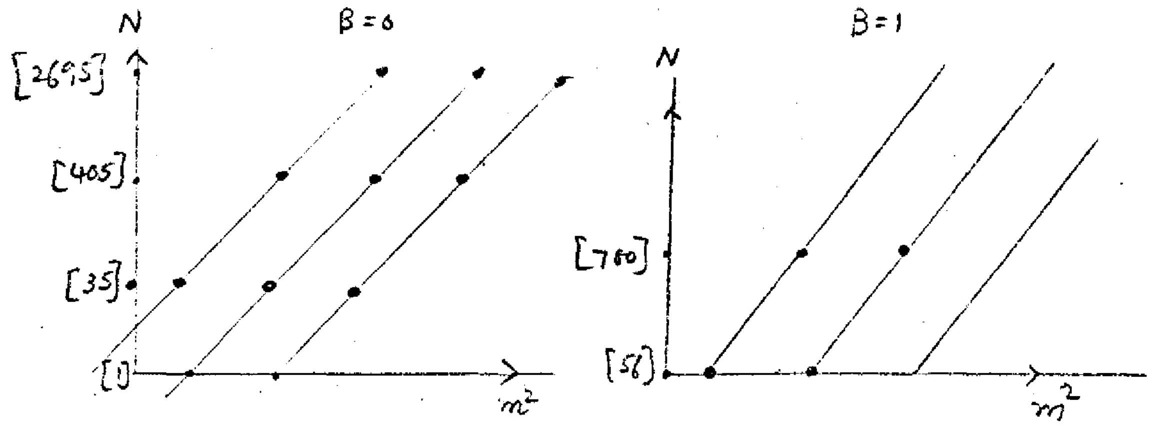
Here we shall study the implications of the N-trajectory hypothesis with regard to the properties of the resulting J-plane trajectories belonging to various SU(3) multiplets. The clue to obtaining the J-plane families is to examine the content of the integer N representations which correspond to the particle poles. We can follow the chain

$$SU(6) \otimes SU(6) \rightarrow SU(6) \rightarrow SU(2) \otimes SU(3)$$

to determine the spin which is associated with a particular SU(3) representation for any given $(N,\bar{N})$ state.

In the first stage of the chain it is easy to see that one obtains an infinity of SU(6) families.

Decomposition of N-trajectories into SU(6)-trajectories



by simply drawing trajectories through the resulting SU(6) particle poles. The residues associated with these trajectories must vanish for "negative integer" [N] simply because we only encounter positive dimensions in the reduction SU(6) ⓧ SU(6) → SU(6).

Examples: $(6,\bar{6}) = 1 \oplus 35$

$(21,\overline{21}) = 1 \oplus 35 \oplus 405$ etc.

Likewise for the baryons. Thus already at the level of the first chain we encounter an infinite family of trajectories; it is important to note that these satellites are a consequence of the internal symmetry group [14] and have nothing to do with the daughter trajectories of Toller, Freedman & Wang. [15]

Let us proceed to the next and final stage of the chain. Here one fixes on a particular SU(6) trajectory, say the leading trajectory I, and where it passes through "integer" N we perform the SU(6) reduction into SU(3) ⓧ SU(2) multiplets. For example, with the mesons, in an obvious notation

$$\begin{aligned} 1 &= (1,1) \\ 35 &= (1,3) + (8,\ 1+3) \\ 405 &= (1,5) + (8,\ 3+5) + (10+\overline{10},\ 3) + (27,\ 5+3+1) \\ &\quad + (1,1) + (8,\ 1+3) \end{aligned}$$

$$
\begin{aligned}
2695 = \; & (1,7) + (8,\, 5+7) + (10+\overline{10},\, 5) + (27,\, 7+5+3) \\
& + (1,3) + (8,\, 3+5) + (10+\overline{10},\, 3+1) + (27,\, 5+3+1) \\
& \qquad\quad + (8,\, 1+3) \qquad\qquad\qquad\qquad\quad + (27,\, 3) \\
& + (35+\overline{35},\, 5+3) + (64,\, 7+5+3+1)
\end{aligned}
$$

we can trace out the families of parallel trajectories for any particular SU(3) multiplet.

Four observations are in order about the properties of these J-trajectories:

1. All residues for negative J should vanish.

2. For sufficiently high SU(3) representations all residues occurring below a certain critical mass should vanish, e.g. with the <u>27</u> fold, no particles with masses $< \hat{m}_2$ can materialize.

3. All self adjoint SU(3) multiplets of the variety (λ,λ) have associated the leading trajectory I. The less self adjoint they become [i.e. in the notation (λ,μ) as $|\lambda-\mu|$ increases] the lower the leading trajectory. E.g., with the <u>10</u> fold, the first trajectory which gives rise to particle poles lies one unit below the leading <u>8</u> trajectory.

4. The number of members in succeeding generations of trajectories increases. Indeed it is convenient to enumerate the number of trajectories occurring in each generation:

   $\underline{1}$ has the sequence (1, 1, 2, 2, ...)
   $\underline{8}$ has the sequence (1, 3, 5, 7, ...)
   $\underline{27}$ has the sequence (1, 3, 6, 10, ..)
   $\underline{10}$ has the sequence (0, 1, 2, 3, ..)

   of the first, second, third, etc., generations.

Evidently a parallel set of remarks and method of enumeration can be applied to the baryon hypermultiplet.

3.3 Calculation of the d-functions for the most degenerate case

Super-singlet scattering is the basic process for providing the representation functions $d^{c}_{00}(\theta)$, the analogues of $P_J(\cos\theta)$ for O(3). Actually we shall treat the case of SU(ν) x SU(ν) rest symmetry as this is no more difficult to work out than SU(6) $\otimes$ SU(6); the meson tower of the most degenerate series may be tensorally represented by the sequence

$$\phi \;,\; \phi^{\hat{b}}_{a} \;,\; \phi^{(\hat{b}_1,\hat{b}_2)}_{(a_1,a_2)} \;,\; \ldots\ldots \; \phi^{(\hat{b}_1,\hat{b}_2,\ldots\hat{b}_N)}_{(a_1,a_2,\ldots a_N)},$$

and the baryon tower by

$$\psi_{(a_1 a_2 a_3)} \;,\; \psi^{\hat{b}_1}_{(a_1 a_2 a_3 a_4)} \;,\; \ldots \;,\; \psi^{(\hat{b}_1,\hat{b}_2,\ldots,\hat{b}_N)}_{(a_1 a_2 \ldots\ldots, a_{N+3})} \ldots$$

corresponding to particles at rest. The boosting to arbitrary momentum p may be effected in the standard manner by imbedding these tensors into finite $SU(\nu,\nu)$ representations subject to the subsidiary Bargmann-Wigner equations; thus we will obtain the relativistic set of states

$$\phi(p),\ \phi_A^B(p),\ \dots\ \dots\ \phi_{(A_1\dots A_N)}^{(B_1\dots B_N)}(p),\ \dots$$

$$\psi_{(A_1A_2A_3)}(p),\quad \psi_{(A_1A_2A_3A_4)}^{B_1},\ \dots\quad \psi_{(A_1\dots A_{N+3})}^{(B_1\dots B_N)}\ \dots$$

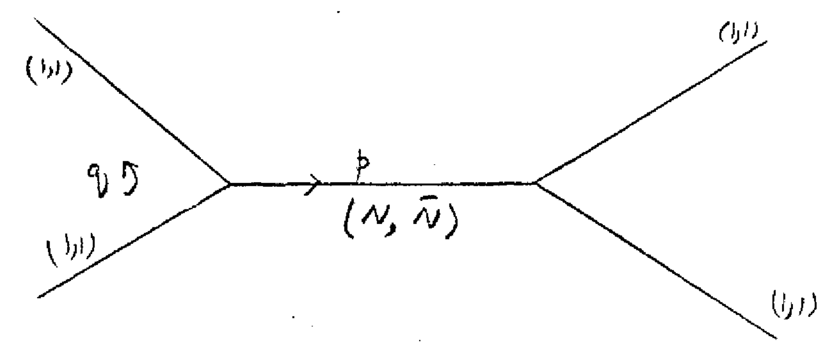
with $[SU(\nu)\otimes SU(\nu)]_p$ as the little group at arbitrary p.

We shall thereby frame all calculations in the M-function basis so that after contraction over the external particle wave functions we will obtain the $d_{00}(\theta)$ functions. As we shall show later, the more general $d_{WW'}(\theta)$ functions are derived from the $d_{00}(\theta)$ by appropriate differentiations. However there are a few conclusions which we should anticipate: 1. For $\theta \neq 0$ we can only expect $SU(3)\otimes SU(3)$ selection rules, i.e. equality of the $SU(3)\otimes SU(3)$ labels within W and W' in $d_{WW'}(\theta)$. Also, since J_{31} transforms as a <u>35</u> of the W-group we should expect that when W and W' differ to the extent that the first non-vanishing C.G. coefficient derives from $\underbrace{35\otimes 35\otimes\cdots\otimes 35}_{n}\otimes N_W = N_{W'}$ then $d_{WW'}(\theta)\sim\theta^n$ for small θ.

2. When $\theta\to 0$ the selection rules enlarge to $SU(6)_W$ and in fact we know that $d_{WW'}(0)\propto\delta_{WW'}$.

3.4 Evaluation of $d_{00}(\theta)$

To evaluate $d_{00}(\theta)$ we can adopt the usual rule of inhomogeneous $SU(\nu,\nu)$ theory which reduce the vertex symmetry to just $SU(\nu)_W$ through inclusion of the momentum factors [16] P_A^B. This answer turns out to be a very simple Gegenbauer function exhibited later in eq. (43). We will have two different $SU(\nu)_W$ groups operating at each end (they can be related to each other by the rotation $e^{-i\theta J_{31}}$ which become identical for collinear processes $(\theta = 0)$.



We make use of the vertex function

$$\langle p+\tfrac{q}{2}|\, j^{(A_1 A_2 \cdots A_N)}_{(B_1 B_2 \cdots B_N)}(0)\, |p-\tfrac{q}{2}\rangle = \Gamma^{(A_1 \cdots A_N)}_{(B_1 \cdots B_N)}(p,q) \tag{30}$$

and the propagator

$$\begin{aligned} &\langle 0|\, T\{\phi^{(B_1 B_2 \cdots B_N)}_{(A_1 \cdots A_N)}(p)\,\phi^{(A_1' \cdots A_N')}_{(B_1' \cdots B_N')}(-p)\}\, |0\rangle \\ &= \Delta^{(B_1 \cdots B_N)\,(A_1' \cdots A_N')}_{(A_1 \cdots A_N)\,(B_1' \cdots B_N')}(p) \end{aligned} \tag{31}$$

If we retain only the pole contribution to the amplitude we can safely make use of the subsidiary conditions on Δ. Then from symmetry and covariance requirements we get

$$\begin{aligned} &\Delta^{(B_1 \cdots B_N)\,(A_1' \cdots A_N')}_{(A_1 \cdots A_N)\,(B_1' \cdots B_N')} \\ &= \sum_{\substack{\text{permutations in} \\ A,\,B}} \frac{(p+m)^{A_1'}_{A_1} \cdots (p+m)^{A_N'}_{A_N}\,(p-m)^{B_1}_{B_1'} \cdots (p-m)^{B_N}_{B_N'}}{(N!)^2\,(2m)^{2N}\,(p^2-m^2)} \end{aligned} \tag{32}$$

and

$$\Gamma^{(A_1 \cdots A_N)}_{(B_1 \cdots B_N)}(p,q) = g_N \sum_{\substack{\text{permutations} \\ \text{in } A}} \frac{1}{N!}\, q^{A_1}_{B_1} \cdots q^{A_N}_{B_N} \tag{33}$$

whence we have the amplitude

$$\begin{aligned} T_N(p;q,q') = &\frac{g_N^2}{(N!)^2 (2m)^{2N} (p^2-m^2)} \Big[\sum_{\substack{\text{permutations} \\ \text{in } A}} q^{A_1}_{B_1} \cdots q^{A_N}_{B_N}\Big] \times \\ &\times (p+m)^{A_1'}_{A_1} \cdots (p+m)^{A_N'}_{A_N}\,(p-m)^{B_1}_{B_1'} \cdots (p-m)^{B_N}_{B_N'} \times \\ &\times \Big[\sum_{\substack{\text{permutations} \\ \text{in } A'}} q'^{B_1'}_{A_1'} \cdots q'^{B_N'}_{A_N'}\Big] \end{aligned} \tag{34}$$

The reduction of the numerator is a combinatorial problem involving the partitioning into the sets

$$[\mathrm{Tr}\,(K)]^{r_1}[\mathrm{Tr}\,(K^2)]^{r_2}\cdots\quad[\mathrm{Tr}\,(K^N)]^{r_N}$$

where

$$N = r_1 + 2r_2 + \cdots \quad + N r_N\,;\quad r_1, r_2, \cdots r_N \quad \text{non negative}$$

and K stands for the matrix $K = q(p+m)q'(p-m)$.

The number of times a particular partition occurs is

$$\frac{(N!)^2(0!)^{r_1}(1!)^{r_2}\cdots\quad((N-1)!)^{r_N}}{(1!)^{r_1}(2!)^{r_2}\cdots\quad(N!)^{r_N}\,r_1!\,r_2!\cdots r_N!}$$

$$= \frac{(N!)^2}{1^{r_1}\,2^{r_2}\cdots\quad N^{r_N}\,r_1!\,r_2!\cdots r_N!} \tag{35}$$

To see what this multiplies we need to evaluate

$$\mathrm{Tr}\,(K^n) = \mathrm{Tr}\,[\overbrace{q(p+m)q'(p-m)\cdots\quad(q(p+m)(q')(p-m)}^{n}] \tag{36}$$

This is more easily done in the rest frame of p since the expression reduces to

$$\mathrm{Tr}\,(K^n) = (2m)^{2n}\quad\tfrac{1}{2}\,\mathrm{Tr}\,[(\underline{q}\cdot\underline{\gamma}\quad \underline{q}'\cdot\underline{\gamma})^n]$$

$$= 2\,(-4m^2\,|q||q|')^n\,\cos n\theta \qquad ;\ \cos\theta = \hat{q}\cdot\hat{q}'. \tag{37}$$

Thus a given partition contributes

$$\frac{(N!)^2\,(-4m^2\,|q||q'|.)^N}{1^{r_1}\,2^{r_2}\cdots\quad N^{r_N}\,r_1!\,r_2!\cdots r_N!}\,(2\cos\theta)^{r_1}(2\cos 2\theta)^{r_2}\cdots\,(2\cos N\theta)^{r_N}$$

to the numerator and <u>in toto</u> we get

$$T = \frac{g_N^2 \left(-|q||q'| \right)^N}{p^2 - m^2(N)} \sum_{\text{partititions}} \frac{\left(\frac{\gamma \cos\theta}{1}\right)^{\gamma_1} \left(\frac{\gamma \cos 2\theta}{2}\right)^{\gamma_2} \cdots \left(\frac{\gamma \cos N\theta}{N}\right)^{\gamma_N}}{\gamma_1! \ \gamma_2! \ \cdots \ \gamma_N!} \tag{38}$$

Upon making the series expansion

$$\cos n\theta = \sum_{\gamma=0}^{\infty} (-1)^{\gamma} \, 2^{n-2\gamma-1} \cos^{n-2\gamma}(\theta) \, \frac{n \, \Gamma(n-\gamma)}{\Gamma(\gamma+1)\,\Gamma(n-2\gamma+1)} \tag{39}$$

the expression simplifies remarkably to the series

$$T = \frac{g_N^2 \left(-|q||q'|\right)^N}{p^2 - m^2} \frac{1}{\Gamma(\nu/2)} \sum_{\gamma=0}^{\infty} \frac{(-1)^{\gamma}\, \Gamma\left(\frac{1}{2}\nu + N - \gamma\right)}{\Gamma(\gamma+1)\,\Gamma(N-2\gamma+1)} (2\cos\theta)^{N-2\gamma} \tag{40}$$

which can in turn be recast in hypergeometric form:

$$T = \frac{-g_N^2 \left(|q||q'|\right)^N}{p^2 - m^2} \frac{\Gamma\left(N + \frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\nu\right)\Gamma(N+1)} (2\cos\theta)^N \times$$

$$\times \; {}_2F_1\left(-\tfrac{1}{2}N, \ -\tfrac{1}{2}N + \tfrac{1}{2} \ ; \ -N - \tfrac{1}{2}\nu + 1 \ ; \ \cos^{-2}\theta\right) \tag{41}$$

and is nothing else but a Gegenbauer function!

$$T = \tfrac{1}{2}\nu \, \frac{g_N^2 \left(|q|\,|q'|\right)^N}{p^2 - m^2(N)} \, C_N^{\frac{1}{2}\nu}\left(\hat{q}\cdot\hat{q}'\right) \tag{42}$$

Thus we have obtained

$$d_{00}^{N}(\theta) = C_N^{\frac{1}{2}\nu}(\cos\theta) . \tag{43}$$

We now presume that the analytical continuation to the "Regge amplitude" replaces T by

$$T = \tfrac{1}{2}\nu \, \beta(p^2) \, \frac{\left(|q|\,|q'|\right)^{\alpha(p^2)}}{\sin \pi\alpha(p^2)} \; C_{\alpha(p^2)}^{\frac{1}{2}\nu}\left(\hat{q}\cdot\hat{q}'\right) \tag{44}$$

This is the master formula we were aiming for.

Of particular interest is the ordinary J-plane trajectory reduction of the N-plane trajectory.[17] This may be obtained from the series

$$C_N^{\frac{1}{2}\nu}(\cos\theta) = \tfrac{1}{2} \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}\nu + r)\, \Gamma(N + \frac{1}{2}\nu - r)}{\Gamma(r+1)\, \Gamma(n - r + 1) \left(\Gamma(\frac{1}{2}\nu + 1)\right)^2} \cos(N - 2r)\theta \tag{45}$$

and

$$\cos n\theta = -\tfrac{1}{8} n \sum_{k=0}^{\infty} \frac{(2n - 4k + 1)\, \Gamma(k - \frac{1}{2})\, \Gamma(n-k)}{\Gamma(k+1)\, \Gamma(n - k + \frac{3}{2})} P_{n-2k}(\cos\theta) \tag{46}$$

Hence

$$C_N^{\frac{1}{2}\nu}(x) = \sum_{K=0}^{\infty} a_{NK\nu} \; P_{N-2K}(x)$$

where (47)

$$a_{NK\nu} = -\frac{(2N-4K+1)}{16\left[\Gamma(\frac{1}{2}\nu+1)\right]^2} \sum_{\gamma=0}^{K} \frac{\Gamma(\gamma+\frac{1}{2}\nu)\,\Gamma(N+\frac{1}{2}\nu-\gamma)\,(N-2\gamma)\,\Gamma(K-\gamma-\frac{1}{2})\,\Gamma(N-K-\gamma)}{\Gamma(\gamma+1)\;\Gamma(N-\gamma+1)\quad\Gamma(K-\gamma+1)\,\Gamma(N-K-\gamma+\frac{3}{2})}$$

(48)

is a finite series (for fixed K) in Γ functions. (One may directly show for $\nu = 1$ that $a_{NK1} = \delta_{K0}$).

We then continue our results to complex $N = \alpha$, still writing

$$C_\alpha^{\frac{1}{2}\nu}(x) = \sum_{K=0}^{\infty} a_{\alpha K\nu} \; P_{\alpha-2K}(x) \tag{49}$$

and keeping the same (finite) series definition of $a_{\alpha K\nu}$. This relation provides us with the J-Regge decomposition of the amplitude

$$T = \tfrac{1}{2}\nu \frac{(|q|\,|q'|)^{\alpha}}{\sin \pi\alpha} \,\beta \sum_{K=0}^{\infty} a_{\alpha K\nu} \; P_{\alpha-2K}(x) \tag{50}$$

corresponding to the exchange of an infinity of (equal signature) SU(3) singlet satellite trajectories spaced out at

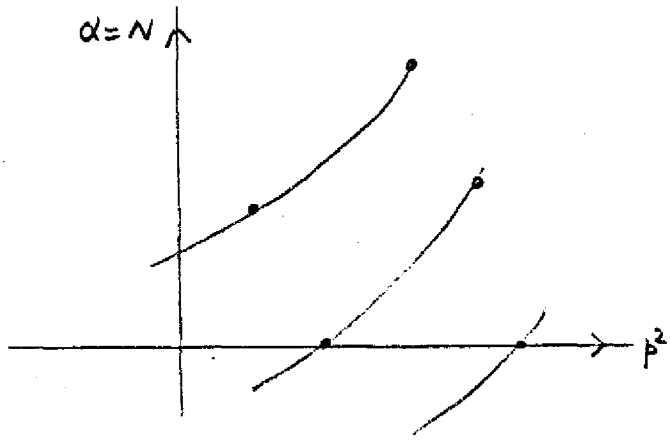
$$\alpha_K = \alpha - 2K$$

with residue ratios

$$\frac{\beta_K}{\beta_{K'}} = \frac{a_{\alpha K\nu}}{a_{\alpha K'\nu}} \tag{51}$$

etc.

Notice, however, that the K'th satellite has associated the threshold factor $(|q|/|q'|)^{\alpha}$ of the parent, not the expected centrifugal factor $(|q|/|q'|)^{\alpha-2K}$. This must be so since the ratio of residues (at the integers at least) are prescribed numbers which must be independent of p^2; hence the "kinematical" factors must cancel exactly.



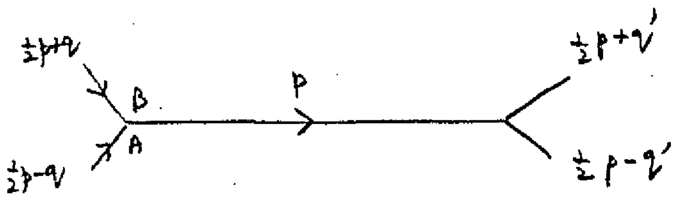
3.5 Scattering of higher representations

The simplest examples where the generalized representation functions make an appearance (generalized in the sense $P_J \rightarrow d^J_{\lambda\lambda'}$ for the O(3) case) are the processes

$$(1,1) + (6,\bar{6}) \longrightarrow (1,1) + (1,1)$$

or

$$(6,1) + (1,\bar{6}) \longrightarrow (1,1) + (1,1)$$



In the invariant amplitudes' language which we have been using, this is simply understood as the fact that the M-function acquires a pair of free indices T_A^B which remain to be contracted over the B.W wave functions representing the external particles. After contraction we obtain the generalized $d^{N}_{ow}(\theta)$ functions appropriate to SU(6) $\otimes$ SU(6).

There exists a simple technique for discovering the structure of T_A^B due to the exchange of the $(N,\bar{N})$ multiplet. It is a simple generalization of the usual method for J-reggeizing invariant amplitudes[18)] and consists in carrying out differentiations over the momentum 15-vector before projection onto the 4-dimension physical subspace. Thus, $T_A^B = \partial T/\partial q_B^{\;A}$ where T represents the scalar scattering amplitude. One might imagine that this procedure, involving the 15 momenta with consequent difficulties in multiplication, tracing operations, etc., would rapidly become prohibitive as the number of indices increases. Fortunately this is not the case if we follow the simple rule:

If

$$\frac{\partial T(q,q')}{\partial q_i} = -4m^2 q'_i F - 4m^2 q_i G \tag{52}$$

then

$$\nu \frac{\partial T}{\partial q^A_B} = \left[(p+m)\, q'\,(p-m)\right]_A^B F + \left[(p-m)\, q\,(p+m)\right]_A^B G. \tag{53}$$

The proof of this rule is established by noting that

$$\frac{1}{n}\frac{\partial}{\partial q^A_B} \mathrm{Tr}(K^n) = \frac{\left(-4m^2|q||q'|\right)^{n-1}}{\sin\theta}\begin{bmatrix}\left[(p+m)\, q'(p-m)\right]_A^B \sin n\theta \\ -\frac{|q'|}{|q|}\left[(p-m)\, q(p+m)\right]_A^B \sin(n-1)\theta\end{bmatrix} \tag{54}$$

in direct relation to the naive differentiation procedure

$$\frac{1}{n}\frac{\partial}{\partial q_i}\left(-4m^2 q q'\right)^n \cos n\theta$$

$$= \left(-4m^2 |q|\,|q'|\right)^{n-1}\left[-4m^2 \frac{|q'|}{|q|} q_i \cos n\theta + \frac{n \sin n\theta}{\sin\theta}\left(\frac{q'_i}{|q||q'|} - \frac{q_i}{|q|^2}\right)\right] \tag{55}$$

$$= \frac{\left(-4m^2|q||q'|\right)^{n-1}}{\sin\theta}\left[-4m^2 q'_i \sin n\theta + \frac{|q'|}{|q|}\left(-4m^2 q_i\right)\sin(n-1)\theta\right]$$

From the above rule applied to

$$T = \tfrac{1}{2}\nu \frac{\left(|q|\,|q'|\right)^N C_N^{\frac{1}{2}\nu}(\hat{q}\cdot\hat{q}') \quad g_N^2}{p^2-m^2} \tag{56}$$

we directly deduce

$$\frac{\partial T}{\partial q_i} = \tfrac{1}{2}\left(|q||q'|\right)^{N-1}\nu \left[\begin{array}{l} N q_i \frac{|q'|}{|q|} C_N^{\frac{1}{2}\nu} + q'_i C_N^{\frac{1}{2}\nu\,\prime} \\ \quad - q_i \frac{q\cdot q'}{|q|^2} C_N^{\frac{1}{2}\nu\,\prime} \end{array}\right] \frac{g_N^2}{p^2-m^2}$$

$$= \tfrac{1}{2}\nu g_N^2 \frac{\left(|q|\,|q'|\right)^{N-1}}{p^2-m^2}\left[q'_i C_N^{\frac{1}{2}\nu}(\hat{q}\cdot\hat{q}') - q_i \frac{|q'|}{|q|} C_{N-1}^{\frac{1}{2}\nu-1\,\prime}(\hat{q}\cdot\hat{q}')\right] \tag{57}$$

whence we immediately obtain

$$T_A{}^B = \frac{g^2}{8m^2}\,\frac{\left(|q|\,|q'|\right)^{N-1}}{p^2-m^2}\left\{\begin{array}{l}\left[(p-m)\,q\,(p+m)\right]_A^B \frac{|q'|}{|q|} C_{N-1}^{\frac{1}{2}\nu-1\,\prime}(\hat{q}\cdot\hat{q}') \\ -\left[(p+m)\,q'(p-m)\right]_A^B C_N^{\frac{1}{2}\nu\,\prime}(\hat{q}\cdot\hat{q}')\end{array}\right\} \tag{58}$$

Finally we have only to contract out indices over the external wave functions such as

$$\phi_A^B\left(\tfrac{1}{2}p+q\right)=\left[\left(\tfrac{1}{2}p+q+m\right)\left(\gamma_\mu\phi_\mu-\gamma_5\phi_5\right)\right]_A^\beta \tag{59}$$

to discover the $d^N_{WO}(\theta)$ functions. The answer for the W-singlet state is already known (it would be obtained by dropping the $\phi_{1,2,5}$ components above). Let us instead evaluate the function for the SU(3) ⊗ SU(3) singlet piece of the 35-dimensional representation by performing the trace.

$$\mathrm{Tr}\left[T\left(\tfrac{1}{2}p+q+m\right)\gamma_1\right]=-\frac{g^2m\left(|q||q'|\right)^{N-1}}{p^2-m^2}\,q'_1\,C_N^{\frac{1}{2}\nu'}(q\cdot q') \tag{60}$$

Thus we conclude that

$$d^N_{W'=35,\,W=1}(\theta)=\sin\theta\, C_N^{\frac{1}{2}\nu'}(\cos\theta) \tag{61}$$

in agreement with the small θ behaviour one should have expected on general grounds. By similar techniques it should be possible to derive more complicated d functions for SU(ν) ⊗ SU(ν).

REFERENCE AND FOOTNOTES

1. A. SALAM and J. STRATHDEE, ICTP, Trieste, preprint IC/67/39. In the text, this paper is referred to as I.

2. Y. YAMAGUCHI, Phys. Letters 9, 281 (1964).

3. J. LIPKIN, "Symmetry without symmetry in the quark model," Rehovoth preprint, (1966).

4. F.J. DYSON, "Symmetry groups in nuclear and particle physics", W.A. Benjamin, New York (1966).

5. S. COLEMAN, Univ. of California, Berkeley, California, preprint no: 12/65.

6. It is well known that if one makes partial wave analyses based not on the rest symmetry (little group for $p^2 > 0$) but based on the non-compact little group for $p^2 < 0$, utilizing unitary representations of these groups, one does not have to use any Sommerfeld-Watson transform; the transform is built into the expansion.

7. I.M. GEL'FAND and M.A. NAIMARK, Unitäre Darstellungen der klassichen Gruppen, Akademie-Verlag, Berlin (1957).

8. As will be explained in Section 2.3, the various series of principal series representations of SL(6,C) correspond to partitions of six into at least two parts. There are ten such partitions. The partition (1,1,1,1,1,1) corresponds to the principal non-degenerate series, while the others correspond to degenerate series of representations.

9. One procedure which suggests itself is to construct the Young tableaux for the W-spin representation (n_1, n_2, γ)

15. M. TOLLER, CERN preprint Th-780 (1967). This contains an extensive list of references and summarizes the author's work on the subject.

D.Z. FREEDMAN and J.M. WANG, "O(4) symmetry and Regge pole theory," Phys. Rev. (to be published).

16. Note that if one were to admit Lorentz but not ISU(ν, ν) covariant vertex factors of the variety $(C^{-1}P)^{AB}$ the symmetry would be broken down to the Poincaré group $\otimes$ SU(3) level.

17. We can distinguish between two generalized signature trajectories, those of even and odd N. From the discussion of the previous section we know that the even signature trajectory lies one unit above the odd signature trajectory (I versus II).

18. See, e.g.,

a) M.D. SCADRON, Imperial College, London, preprint ICTP/67/23.

b) H.F. JONES and M.D. SCADRON, Imperial College, London, preprint ICTP/67/28.

and to take their outer product. The resulting tableaux are the patterns belonging to SU(2ν), the different types being obtained as we vary λ through integer values 0,1,2,... Unfortunately this method does not provide a quick answer to the question of the multiplicity of W-spin representations contained in a given rest representation. Moreover (see next section) the classification of the multiplets is a difficult problem requiring a full knowledge of all types of representations of SU(ν, ν). In this connection see, e.g., M.L. WHIPPMANN, J. Math. Phys. 6, 1534 (1965).

10. We are grateful to Professor C. Fronsdal, Drs. J. Niederle and J. Fischer for directing our attention to these in Gel'fand and Naimark's book (Ref. 7) and to Professor Fronsdal also for explaining these techniques to us. In this connection see also C. Fronsdal, "The theory of representations of non-compact Lie algebras" in "High energy physics and elementary particles", (IAEA, Vienna 1965) page 585.

11. This is essentially due to the unitarity of the considered representations.

12. D. HORN, H.J. LIPKIN and S. MESHKOV, Phys. Rev. Letters 17, 1200 (1966).

13. Another way which would help to suppress their production is the possibility of mass splittings within the multiplets thus raising the levels like $(\overline{10}, 27, ..)$ above the (1, 8, 10) ones.

14. If one were to generalize the daughter phenomena of current interest associated with the O(4) symmetry, its counterpart here would be obtained from the U(12) group. Thus $(6,\overline{6})$ would be accompanied by its daughters $(6,\overline{6})$, (35,1), $(1,\overline{35})$ and (111) etc.