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HARMONIC ANALYSIS IN TERMS OF THE HOMOGENEOUS LORENTZ GROUP

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HARMONIC ANALYSIS

IN TERMS OF THE HOMOGENEOUS LORENTZ GROUP

Following Toller's realization that the elastic forward scattering amplitude admits of an O(3, 1) little group invariance on account of vanishing momentum transfer four-vector, considerable interest has lately arisen in partial wave analyses using unitary representations of the homogeneous Lorentz group.

The physical consequences of using this type of expansion are two-fold; first the resolution of $t = 0$ singularity problems which had made their appearance in Regge analyses based on the standard angular momentum O(3) expansion (evasions versus conspiracies); second, the appearance of "daughter" poles associated with a (Toller) pole in the complex O(3, 1) plane.

This note is concerned with two important generalizations:

(i) Expansion of the flipless elastic amplitude for all values of momentum transfer, and

(ii) Extension of the expansion to the spin-flip amplitudes.

Our methods only use group theory to provide the boundary conditions on the suggested expansions; otherwise they are simply based upon the fact that any function $T(\zeta)$ square integrable* in the sense that

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* The (near) forward scattering amplitudes do not satisfy the square-integrability criterion. However, one may assume that for large negative t, there exists a region of s where the criterion is satisfied. The expansion can then be continued in t to small negative t and eventually to $t > 0$. In this continuation process one will pick up poles in the complex σ-plane which pierce the background integral. It is also possible that cut singularities could be associated with the supplementary series of the Lorentz group. If $T(\zeta)$ is not square-integrable, the expansion (i) may need also a supplementary series integral ranging from $0 \leq \sigma \leq 1$, it appears tempting to postulate that the existence of such an integral can be associated with the cut phenomena in the complex σ-plane.

$$\int_0^\infty d\zeta \, sh^2\zeta \, |T(\zeta)|^2 < \infty$$

can be expanded in terms of the principal series of functions of the homogeneous Lorentz group. Specifically,

$$T_{s\lambda s'}(\zeta) = \sum_{|j_0| \leq \min(j,j')} \int_0^{i\infty} d\sigma \, (j_0^2 - \sigma^2) \; T_{ss'}(j_0, \sigma) \; d^{j_0\sigma}_{s\lambda s'}(\zeta) \tag{1}$$

Here σ and j_0 are the two Casimir operators of O(3, 1) ; $0 \leqslant \sigma \leqslant i\infty$ and j_0 is discrete. Thus consider the elastic scattering amplitude in the centre-of-mass frame and define a reduced amplitude $T_{s'\lambda' s\lambda}$ by

$$\langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle =$$

$$= \sum_{s\lambda, s'\lambda'} \langle s_4 \lambda_4 | s_2 \lambda_2 , s'\lambda' \rangle \; T_{s'\lambda', s\lambda}(s,t) \; \langle s_3 \lambda_3 , s\lambda | s_1 \lambda_1 \rangle \tag{2}$$

Here the amplitude on the left is the conventional helicity amplitude with λ's as the eigenvalues of J_{12} in the respective rest frames.

(1) In the forward direction, $T_{s'\lambda' s\lambda}$ (s , t = 0) = $\delta_{\lambda\lambda'} \, T_{s'\lambda s}$ (s) , has the <u>group theoretic</u> expansion (1) (with $ch\,\zeta = \frac{s-u}{4m_1 m_2}$), and $\lambda = \lambda'$. <u>For $t \neq 0$</u> ($\lambda = \lambda'$) , we use (1) to write

$$T_{s'\lambda, s\lambda}(s,t) = \sum_{j_0} \int d\sigma \, (j_0^2 - \sigma^2) \; T_{s',s}(j_0, \sigma, t) \; d^{j_0\sigma}_{s'\lambda s}(\zeta_t) \tag{3}$$

with,

$$\operatorname{ch}\zeta_t = (\hat{p_1+p_3})(\hat{p_2+p_4}) = (s-u)\left[(4m^2-t)(4m'^2-t)\right]^{-1/2} \tag{4}$$

Clearly (3) satisfies all group-theoretic boundary conditions at $t = 0$.

(2) In order to see how to incorporate the spin-flip amplitudes, note from the standard O(3) partial wave expansion

$$T_{(\lambda)}(s,t) = \sum_j (2j+1)\, T^j_{(\lambda)}(s)\, d^j_{\lambda_1-\lambda_2,\lambda_3-\lambda_4}(\theta_s)$$

that for small t , we have the threshold behaviour $(\sin\frac{1}{2}\theta_s)^{|\Delta|} \sim t^{|\Delta|/2}$ where

$$\Delta = (\lambda_1-\lambda_2)-(\lambda_3-\lambda_4) = (\lambda_1-\lambda_3)-(\lambda_2-\lambda_4)\,. \tag{5}$$

We can isolate the spin-flip factors $t^{|\Delta|/2}$ from the amplitude and define new (invariant) amplitudes by the definition:

$$T_{s'\lambda',s\lambda}(s,t) = \sum_J \langle s'\lambda' \mid |\Delta|\Delta, J\lambda\rangle\, t^{|\Delta|/2}\, T^{(s')}_{J\lambda s}(s,t) \tag{6}$$

It is easy to check that the count* of the new amplitudes $T^{(S')}_{J\lambda S}$ is exactly the same as that of the amplitudes $T_{s'\lambda',s\lambda}$ (i.e., $(2S'+1)(2S+1)$). Also one can easily show that $\frac{\partial^{|\Delta|/2}}{\partial t^{|\Delta|/2}} T_{s'\lambda',s\lambda}$ has a group theoretic form $\sum_J \langle s'\lambda' \mid |\Delta|\,\Delta\,,\,J\lambda\rangle\, T_{J\lambda s}(s,0)$ at the boundary value $t = 0$. With

* (up to the imposition of discrete C , P , T symmetries)

(6) we are now able to make the desired expansion for the new reduced amplitudes,

$$T^{(S')}_{J\lambda S}(s,t) = \sum_{j_0} \int d\sigma\, (j_0^2-\sigma^2)\;\; T^{(S')}_{JS}(j_0,\sigma,t)\;\; d^{j_0\sigma}_{J\lambda S}(\zeta_t) \tag{8}$$

One may now approximate to these amplitudes by a set of Toller poles. The set of amplitudes $T^{(S')}_{J\lambda S}$ is not a unique set; in particular, one could construct in complete analogy with these an alternative set $T^{(S)}_{S'\lambda' J'}$ where the spin-flip factors are isolated from the "vertex" S' rather than S. The important point however, is that in a scheme of Tollerization, the high-energy behaviour of the amplitude which depends on an asymptotic expansion of the functions $d^{j_0\sigma}_{J\lambda S}(\zeta)$ does not depend on the quantum numbers J and S but only on σ, j_0 and λ ($d^{j_0\sigma} \sim (\mathrm{ch}\,\zeta)^{\sigma-1-|j_0-\lambda|}$.

The physical consequences of introducing Toller poles into (8) at $\sigma - 1 = a(j_0, t)$ is to provide a set of integrally spaced Regge daughters (n) at $\alpha_n(t) = a(j_0, t) - 1 - n$, with their residues related to the leading one simply through factors which enter the group-theoretic decomposition of O(3,1) into O(2,1) representation. This has been discussed in detail by Toller and Sciarrino and Freedman and Wang and one can take over their formalism without change.

In the unequal mass case the previous arguments require little modification. Thus, we propose the very same expansion (2) (6) (8) with the same definition of ζ_t except that we replace $t^{|\Delta|/2}$ by the threshold factor $|\phi(s,t)|^{\Delta}$ where $\phi(s,t) = 0$ is the curve which denotes the boundary of the physical region. The group-theoretic necessity for this change can be seen as follows. The formalism we have presented is based on a completeness relation for the functions $d^{j,\sigma}_{S\lambda S'}$ and the possibility of expansion of a square integrable function in terms of these. We had to guarantee at each stage that the group theoretic boundary ~~boundary~~ conditions for the equal mass case at $t = 0$ or more exactly at $\theta_s = 0$ are respected. For unequal masses $\phi(s,t) \sim (\sin \frac{\theta_s}{2})$. The new point that emerges is the occurrence of singular residues (corresponding to factors $t^{-1}(m_2^1 - m_3^2)$ and $t^{-1}(m_2^2 - m_4^2)$) if one makes a Regge daughter decomposition and passes from an expansion based on $d^{j,\sigma}$ (ζ_t) to the normal expansion d^J (θ_t).

The formalism presented in this note (formula 8) is no more complicated than the normal angular momentum decomposition. The functions $d^{j,\sigma}$ are tabulated in Toller's papers (see also "Partial wave analysis" Part I (IC/67/9)). We believe this decomposition based on the Lorentz group harmonics will soon completely replace the normal O(3) analyses so far as Reggeization is concerned. It would be useful to start afresh on a systematic analysis of all 2-body scattering data from this point of view.