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ABSTRACT

The quasipotential type equation for the relativistic scattering amplitude is obtained with the help of a special kind of perturbation theory.

QUASIPOTENTIAL TYPE EQUATION FOR THE RELATIVISTIC SCATTERING AMPLITUDE

1. INTRODUCTION

In non-relativistic quantum mechanics the two-particle scattering amplitude $T(\vec{p},\vec{q}) = T(\vec{p}^2,\vec{p}\vec{q},\vec{q}^2)$ off the energy shell $\vec{p}^2 = \vec{q}^2 = 2mE$ *) is known - under definite conditions - to satisfy the Lippman-Schwinger equation [1]

$$T(\vec{p},\vec{q}) = -\frac{m}{4\pi} V(\vec{p},\vec{q}) + \left(\frac{m}{2\pi}\right)^3 \int \frac{d\vec{k}}{\vec{q}^2 - \vec{k}^2 + i\varepsilon} \quad V(\vec{p},\vec{k})\, T(\vec{k},\vec{q}) \quad , \tag{1.1}$$

where $V(\vec{p},\vec{q})$ is the Fourier transform of the potential (in the case of local spherical symmetrical field $V(\vec{p},\vec{q}) = V\left[(\vec{p}-\vec{q})^2\right]$).

Here the function T is assumed to be normalized, as usual, to the differential cross-section of the elastic scattering

$$|T|^2 = \frac{d\sigma}{d\Omega} \quad , E_p = E_q .$$

In the following sections when we consider Lorentz-invariant amplitudes we shall use another (more convenient for the relativistic case) normalization:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \quad \frac{1}{s} \; |T|^2 .$$

(s is the square of the total energy in CMS). The corresponding integral equations will then involve factors which do not tend to unity in the non-relativistic limit. This circumstance must be taken into account when comparing relativistic and non-relativistic approaches. We shall not decide upon normalization since only the essentials of the problem are of interest to us.

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*) All variables are related to the centre-of-mass system. The masses of the scattered particles are considered to be identical and equal to m.

Further, it will be convenient to have eq.(1.1) also written in terms of the energy and the scattering angle variables. After introducing the notations

$$E_p = \frac{\vec{p}^2}{2m} \quad , \quad E_q = \frac{\vec{q}^2}{2m} \quad , \quad E_k = \frac{\vec{k}^2}{2m} \quad ,$$

$$\cos\vartheta = \frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|} \quad , \quad \cos\theta = \frac{\vec{k}\cdot\vec{q}}{|\vec{k}||\vec{q}|} \quad ,$$

$$\cos\psi = \frac{\vec{p}\cdot\vec{k}}{|\vec{p}||\vec{k}|} = \cos\vartheta\cos\theta + \sin\vartheta\sin\theta\cos\varphi \quad ,$$

$$d^3k = k^2\,dk\,d\Omega = k^2\,dk\,d\cos\theta\,d\varphi \quad ,$$

$$\tag{1.2}$$

one gets

$$T(E_p, \cos\vartheta; E_q) = -\frac{m}{4\pi} V(E_p, \cos\vartheta, E_q) +$$

$$+ \frac{m^2}{2(2\pi)^3} \int \sqrt{\frac{2E_k}{m}}\, dE_k\, d\Omega\, \frac{V(E_p, \cos\psi, E_k)}{E_q - E_k + i\varepsilon}\, \dot{T}(E_k, \cos\theta, E_q). \tag{1.3}$$

In quantum field theory the system of two interacting particles may be described in the framework of the Bethe-Salpeter formalism [2]. Then the invariant scattering amplitude satisfies the equation (we write it in operator form):

$$T = I + I\ G_0\ T \tag{1.4}$$

where I is the interaction operator given by the sum of all irreducible (in B.-S. sense) Feynman diagrams with four ends and G_0 is the "free" two-particle propagator equal to the product of two full one-particle Green functions.

In eq.(1.4), contrary to (1.1), the amplitude T is considered off the mass shell, and the energy as well as the momentum are conserved. This fact provides the relativistic invariance of (1.4).

However, another way of relativization of (1.1) is logically admissible. That is, the amplitude T may be retained on the mass shell but now simultaneous conservation of <u>all the four components</u> of the energy-momentum vector should be dropped. Then, evidently, the four-dimensional symmetry of (1.1) will be kept.

It is clear that in such an approach the usual non-relativistic perturbation theory has to be suitably changed so that the non-conservation of the 4-momentum also holds in the intermediate states.

The corresponding covariant form of the old-fashioned perturbation theory is developed in [3,4,5]. We shall outline below the results of these papers, which will be necessary for us in what follows.

2. COVARIANT FORMULATION OF THE OLD-FASHIONED PERTURBATION THEORY

Let $S = 1 + iR$ be the relativistic scattering amplitude and $\tilde{H}(p)$ the Fourier transform of the Hamiltonian density $H(x)$ *):

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*) All operators are considered in the interaction representation.

$$\tilde{H}(p) = \int e^{-ipx} H(x) \, d^4x \ . \tag{2.1}$$

Then

$$R = R(\lambda \varkappa)\Big|_{\varkappa=0} \tag{2.2}$$

where $\varkappa$ is an invariant parameter, λ is an arbitrary four-dimensional vector having the properties of a 4-velocity

$$\lambda^2 = \lambda_0^2 - \vec{\lambda}^2 = 1, \quad \lambda_0 > 0 \tag{2.3}$$

and the operator $R(\lambda \varkappa)$ is determined by the equation **)

$$R(\lambda\varkappa) = -\tilde{H}(\lambda\varkappa) - \frac{1}{2\pi}\int \tilde{H}(\lambda\varkappa - \lambda\varkappa_1) \, \frac{d\varkappa_1}{\varkappa_1 - i\varepsilon} \, R(\lambda\varkappa_1) \ . \tag{2.4}$$

It is easy to see that (2.4) is equivalent to the Tomonaga-Schwinger equation for the scattering "half"-matrix $S(\sigma, -\infty)$ defined on the space-like plane $\lambda x = \sigma$

**) Further on we shall also need the equation of a more general form

$$R(\lambda\varkappa, \lambda\varkappa') =$$

$$= -\tilde{H}(\lambda\varkappa - \lambda\varkappa') - \frac{1}{2\pi}\int \tilde{H}(\lambda\varkappa - \lambda\varkappa_1) \, \frac{d\varkappa_1}{\varkappa_1 - i\varepsilon} \, R(\lambda\varkappa_1, \lambda\varkappa')$$

which reduces to (2.4) when $\varkappa' = 0$. (2.5)

$$i \frac{\partial S(\sigma,-\infty)}{\partial \sigma} = \left(\int H(x)\, \delta(\sigma - \lambda x)\, d^4x \right) S(\sigma,-\infty). \qquad (2.6)$$

The connection between $S(\sigma,-\infty)$ and $R(\lambda\varkappa)$ is given by

$$S(\sigma,-\infty) = 1 + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\lambda\varkappa) \frac{e^{i\varkappa\sigma}}{\varkappa - i\varepsilon}\, d\varkappa \qquad (2.7)$$

Let us now turn to eq.(2.4) for the operator $R(\lambda\varkappa)$. The surface $\varkappa = 0$ will be called the energy-momentum shell, since for $\varkappa \neq 0$ the 4-momentum of the system is conserved only up to the quantity $\lambda\varkappa$. It is important to stress that the scattering matrix does not depend on components of λ on the energy-momentum shell, i.e., it is a completely relativistic invariant quantity *). Therefore, for $\varkappa \neq 0$ the vector λ may be chosen to be collinear to any time-like vector occurring in a concrete problem. Each such choice will correspond to a completely definite way of going off the energy-momentum shell. It is, however, clear that the most suitable and symmetrical one is based on the assumption that

$$\lambda_n \sim \mathcal{P}_n \qquad (2.8)$$

where $\mathcal{P}_n$ is the total 4-momentum of the system.

In this case, in virtue of the translational invariance, the 4-velocity vector of the system is a conserved quantity outside the shell $\varkappa = 0$ as well. The invariant "mass" $\sqrt{\mathcal{P}_n^2} = \sqrt{s}$ alone is not conserved.

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*) This is guaranteed by the local character of the interaction Hamiltonian [3]:

$$[H(x)\ ,\ H(y)] = 0$$

for

$$(x-y)^2 = (x_0 - y_0)^2 - (\vec{x} - \vec{y})^2 < 0.$$

Let us now describe the diagram technique in our formalism. For simplicity we shall choose the interaction Hamiltonian in the form:

$$H(x) = g:\psi^*(x)\psi(x)\varphi(x): + g:\chi^*(x)\chi(x)\varphi(x): \quad , \qquad (2.9)$$

where $\varphi(x)$ is the field operator of neutral scalar particles with mass μ , and $\psi(x)$ and $\chi(x)$ are non-hermitian fields, corresponding to two types of charged scalar particles with mass m. Let us introduce the Fourier-decompositions in the standard way

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \varphi(k) e^{ikx} d^4k =$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2k_0}} \left[\alpha^+(\vec{k}) e^{ikx} + \alpha(\vec{k}) e^{-ikx}\right] ,$$

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{iqx} \psi(q) d^4q =$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{q}}{\sqrt{2q_0}} \left[e^{-iqx} a(\vec{q}) + e^{iqx} b^+(\vec{q})\right] ,$$

$$\psi^*(x) = \frac{1}{(2\pi)^{3/2}} \int e^{ipx} \psi^*(p) d^4p =$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{2p_0}} \left[e^{ipx} a^+(\vec{q}) + e^{-ipx} b(\vec{p})\right] ,$$

$$\chi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{iqx} \chi(q)\, d^4q =$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{q}}{\sqrt{2q_0}} \left[e^{iqx} d^+(q) + e^{-iqx} c(\vec{q}) \right],$$

$$\chi^*(x) = \frac{1}{(2\pi)^{3/2}} \int e^{ipx} \chi^*(p)\, d^4p =$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{2p_0}} \left[e^{ipx} c^+(\vec{p}) + e^{-iqx} d(\vec{p}) \right] \tag{2.10}$$

Here α^+, α, a^+, a, ..., d are the creation and annihilation operators of particles and antiparticles described by the corresponding fields.

With the help of (2.1), (2.9) and (2.10) we find

$$\tilde{H}(\lambda\varkappa - \lambda\varkappa') = \int e^{-i\lambda\varkappa x} H(x)\, e^{i\lambda\varkappa' x}\, d^4x =$$

$$= \frac{g}{\sqrt{2\pi}} \int \delta(-\lambda\varkappa + \lambda\varkappa' + p + q + k) : \psi^*(p)\psi(q)\varphi(k) : d^4p\, d^4q\, d^4k +$$

$$+ \frac{g}{\sqrt{2\pi}} \int \delta(-\lambda\varkappa + \lambda\varkappa' + p + q + k) : \chi^*(p)\chi(q)\varphi(k) : d^4p\, d^4q\, d^4k.$$

(2.11)

The operator $\tilde{H}(\lambda\varkappa - \lambda\varkappa')$ is represented graphically in the following manner

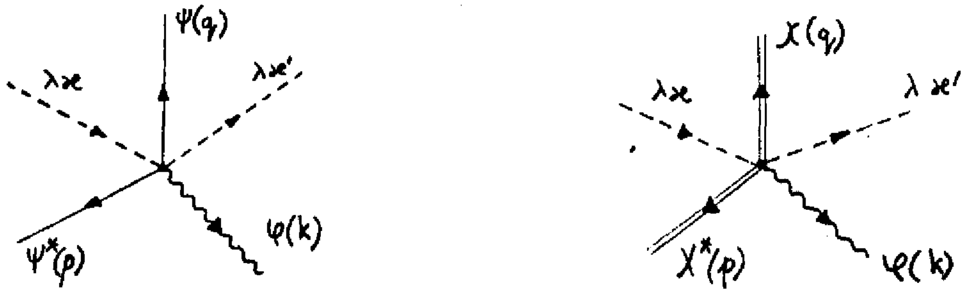

Fig. 1

A dotted line which carries the 4-momenta $\lambda\varkappa$ and $\lambda\varkappa'$, and corresponds in this case to the plane waves $e^{-i\lambda\varkappa x}$ and $e^{i\lambda\varkappa' x}$, will be called, in the following, a <u>quasiparticle</u>. In higher orders of the perturbation series this line can have "internal" parts, i.e., can go out from one vertex and come in another. To such a <u>virtual</u> quasiparticle we put in correspondence a propagator

$$G_0(x) = \frac{1}{2\pi}\frac{1}{\varkappa - i\varepsilon} \tag{2.12}$$

and a 4-momentum $\lambda\varkappa$. To the usual particles in intermediate states we assign the functions $\mathcal{D}^{(+)}(p) = \theta(p^0)\,\delta(p^2 - m^2)$ and $\Delta^{(+)}(p) = \theta(p^0)\,\delta(p^2 - M^2)$, since when the iterations of eqs.(2.4) and (2.5) are reduced to the normal form it is necessary to apply the Wick theorem for the usual product of normal products (see for instance [6]) and to use the following pairings:

$$\begin{aligned}
\underbrace{\psi(q)\,\psi^*}(p) &= \delta(q+p)\,D^{(+)}(p)\\
\underbrace{\psi^*(p)\,\psi}(q) &= \delta(p+q)\,D^{(+)}(q)\\
\underbrace{\chi(q)\,\chi^*}(p) &= \delta(q+p)\,D^{(+)}(p)\\
\underbrace{\chi^*(p)\,\chi}(q) &= \delta(p+q)\,D^{(+)}(q)\\
\underbrace{\varphi(k_1)\,\varphi}(k_2) &= \delta(k_1+k_2)\,\Delta^{(+)}(k_2)
\end{aligned} \tag{2.13}$$

Let us now suppose that we have solved eq.(2.5) and we have written the operator $R(\lambda\varkappa, \lambda\varkappa')$ in the normal form

$$R(\lambda\varkappa, \lambda\varkappa') = \sum_{n=m=\mu=0} \int F_{n,m,\mu}(\lambda\varkappa, \lambda\varkappa'; p_1 \dots p_n; p_1' \dots p_m'; q_1, \dots, q_n; q_1' \dots q_m'; k_1 \dots k_\mu)$$

$$\times :\psi^*(p_1)\dots\psi^*(p_n)\chi^*(p_1')\dots\chi^*(p_m')\,\psi(q_1)\dots\psi(q_n)\chi(q_1')\dots\chi(q_m')\,\varphi(k_1)\dots\varphi(k_\mu):$$

$$dp_1 \dots dp_n\, dp_1' \dots dp_m'\, dq_1 \dots dq_n\, dq_1' \dots dq_m'\, dk_1 \dots dk_\mu \,. \tag{2.14}$$

The coefficient functions F, appearing in front of the normal products in (2.15), determine at $\varkappa = \varkappa' = 0$ the probability amplitudes for different physical processes. They can be constructed in terms of a series in the coupling constant by means of a diagram technique. The corresponding rules are formulated in the following manner:

a) Draw the Feynman graph corresponding to the given process in the usual approach. Arbitrarily number its vertices and orient each internal line from the vertex with the larger number to the vertex with the smaller number assigning to it some 4-momentum p.

b) Connect with dotted lines the first vertex with the second, the second with the third, the third with the fourth, etc. Orient them in the direction of the increasing numbers and assign to each of them a 4-momentum $\lambda\varkappa_s$, where $s = 1, 2, \ldots, n-1$ is the number of the vertex which the given dotted line leaves. In addition, attach to the first vertex an incoming external dotted line with a 4-momentum $\lambda\varkappa$ and to the last vertex (with number n) an outgoing external dotted line with a 4-momentum $\lambda\varkappa'$.

c) To each internal dotted line with a 4-momentum $\lambda\varkappa_s$ put in correspondence a function $G_0(\varkappa_s) = \frac{1}{2\pi}\,\frac{1}{\varkappa_s - i\epsilon}$ and to each solid internal line with 4-momentum p a function $D^{(+)}(p) = \theta(p^0)\,\delta(p^2 - m^2)$ and $\Delta^{(+)}(p) = \theta(p^0)\delta(p^2 - \mu^2)$ (depending upon the kind of particle).

d) To each vertex of the diagram put in correspondence a factor $(-g/\sqrt{2\pi})$ and a four-dimensional δ-function, which takes into account the conservation law of the total 4-momentum of the incoming and outgoing particles and quasiparticles in the given vertex.

e) Integrate between infinite limits over all the variables $\varkappa_s$ and over all the independent momenta among the vectors p.

f) Repeat the operations called for in items a) ... e) for all n! numberings of the vertices of the given diagram and sum the resulting coefficient functions*). Multiply the result by

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*) Practically only the coefficient functions corresponding to topologically non-equivalent diagrams (dotted lines being taken into account) occur to be essentially different.

the factor $1/h$, where h is the number of permutations of the external vertices, appearing in the diagram in a symmetrical way. The performance of operations a) ... f) leads to the desired coefficient function.

Let us illustrate this procedure by concrete examples.

i) The self-energy of the φ-particle in the second order of the perturbation theory (for simplicity we shall not take into account the interaction with the χ-field).

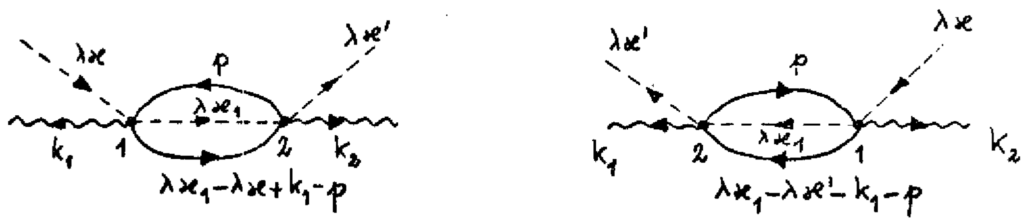

Fig. 2

$$F_{0,0,2} = \frac{1}{2!}\frac{g^2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\varkappa_1}{\varkappa_1 - i\varepsilon}\left\{\int d^4p\, D^{(+)}(p)\left[D^{(+)}(\lambda\varkappa_1 - \lambda\varkappa + k_1 - p) + \right.\right.$$

$$\left.\left. + D^{(+)}(\lambda\varkappa_1 - \lambda\varkappa' + k_2 - p)\right]\right\}\delta(\lambda\varkappa - \lambda\varkappa' - k_1 - k_2) \equiv$$

$$\equiv \delta^4(\lambda\varkappa - \lambda\varkappa' - k_1 - k_2)\sum(\lambda\varkappa, \lambda\varkappa', k_1, k_2) \qquad (2.15)$$

Without loss of generality here we may put $\varkappa' = 0$. If, in addition, we take into account the conservation law of the 4-momentum

$$\lambda\varkappa = k_1 + k_2 \qquad (2.16)$$

then after simple calculations we obtain the following result:

$$\Sigma = \Sigma(\varkappa) = \Sigma(0) + \varkappa \Sigma_{reg}(\varkappa^2) ,$$

where

$$\Sigma(0) = \frac{g^2}{2^5\pi} \int_{4m^2}^{\infty} \frac{dz}{z - \mu^2 - i\varepsilon} \sqrt{\frac{z - 4m^2}{z}} \qquad (2.17)$$

$$\Sigma_{reg}(\varkappa^2) = \frac{g^2}{2^6\pi} \Theta(\varkappa^2 - 4\mu^2) \int_{4m^2}^{\infty} \frac{dz}{z - \mu^2 - i\varepsilon} \frac{1}{\sqrt{z^2 - \mu^2 + \frac{\varkappa^2}{4}}} \sqrt{\frac{z - 4m^2}{z}} \qquad (2.18)$$

It is clear that the divergent part of (2.15) is concentrated in $\Sigma(0)$, to which a subtraction procedure*) must be applied. Evidently the form of $\Sigma(0)$ is exactly the same as in the usual formalism, that means, the mass counter-terms $\delta\mu^2$ are identical in both approaches.

ii) Elastic scattering of ψ- and χ-particles (second order).

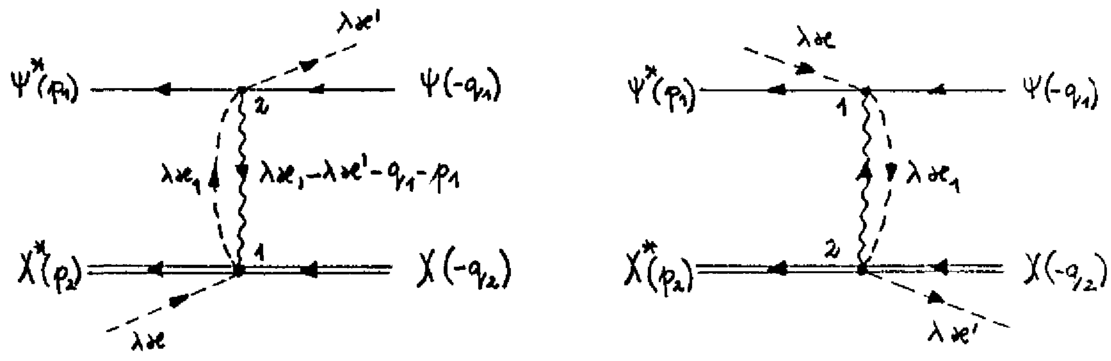

Fig. 3

*) Let us emphasize that the subtraction point here is $\varkappa = 0$. For this reason the relativistic invariance holds after removal of the divergences.

$$F_{1,1,0} = \frac{g^2}{(2\pi)^2}\,\delta^{(4)}(p_1+p_2-\lambda\varkappa-q_1-q_2+\lambda\varkappa')\times$$

$$\times\int_{-\infty}^{\infty}\frac{d\varkappa_1}{\varkappa_1-i\varepsilon}\left[\Delta^{(+)}(\lambda\varkappa_1-\lambda\varkappa'+q_1-p_1)+\Delta^{(+)}(\lambda\varkappa_1-\lambda\varkappa+p_1-q_1)\right]\equiv$$

$$\equiv\frac{1}{(2\pi)^2}\,\delta^{(4)}(p_1+p_2-\lambda\varkappa-q_1-q_2+\lambda\varkappa')\;M$$

(2.19)

Simple calculations give

$$M=\frac{g^2}{2\sqrt{\mu^2-t_{pq}+[(q_1-p_1)\lambda]^2}}\left\{\frac{1}{\varkappa'-\lambda(q_1-p_1)+\sqrt{\mu^2-t_{pq}-[(q_1-p_1)\lambda]^2}}\right.+$$

$$\left.+\frac{1}{\varkappa+\lambda(q_1-p_1)+\sqrt{\mu^2-t_{pq}+[(q_1-p_1)\lambda]^2}}\right\},\tag{2.20}$$

where $t_{pq}=(q_1-p_1)^2$.

If we choose the direction of λ in accordance with (2.8), putting*)

$$\lambda=\frac{q_1+q_2}{\sqrt{(q_1+q_2)^2}}=\frac{q_1+q_2}{\sqrt{s_q}},\tag{2.21}$$

---

*) It is clear that owing to the conservation of the 4-momentum in (2.19), the vector λ defined by (2.21) coincides with the vector $(p_1+p_2)/\sqrt{(p_1+p_2)^2}=(p_1+p_2)/\sqrt{s_p}$. Therefore, the 4-velocity does not vary in the interaction process.

the amplitude M becomes

$$M = \frac{g^2}{\sqrt{\mu^2 - t_{pq} + \frac{1}{4}(\varkappa - \varkappa')^2}\left(\frac{\varkappa + \varkappa'}{2} + \sqrt{\mu^2 - t_{pq} + \frac{1}{4}(\varkappa - \varkappa')^2 - i\varepsilon}\right)} \tag{2.22}$$

The quantities $\varkappa$, $\varkappa'$, $\sqrt{s_q}$, $\sqrt{s_p}$ are evidently connected by the following "conservation law":

$$\varkappa + \sqrt{s_q} = \varkappa' + \sqrt{s_p} . \tag{2.23}$$

iii) Some higher order diagrams.

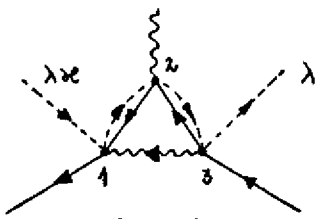

Fig.4

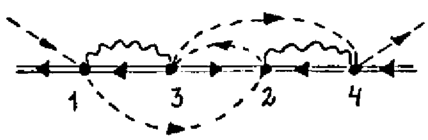

Fig. 5

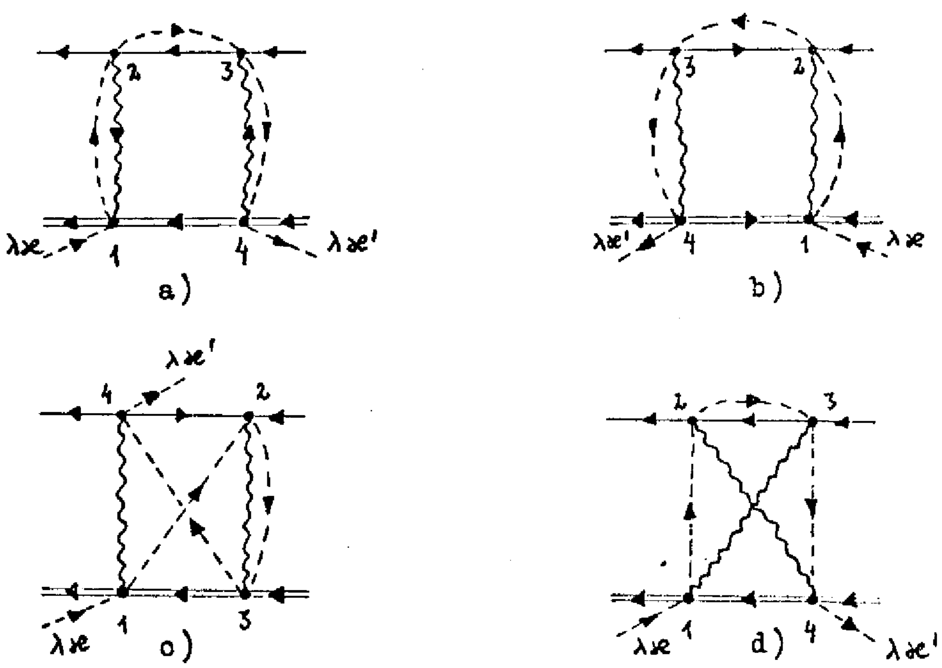

Fig. 6

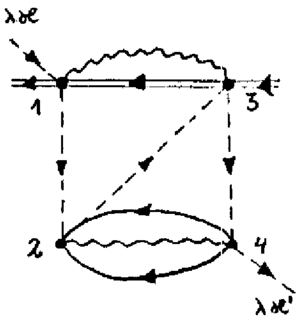

Fig. 7

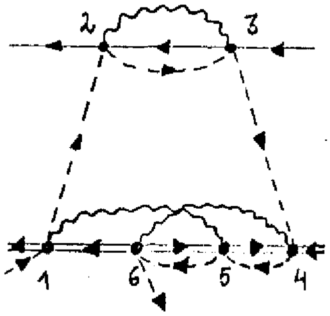

Fig. 8

One should stress that in the diagram technique considered the ordinary physical particles in intermediate states are real (and not virtual ones as it occurs in the Feynman technique). However, due to the presence of virtual quasiparticles in these states, the total 4-momentum of ordinary particles is no longer conserved. If there are external dotted lines in the diagram then, following the terminology adopted, the physical system in question is off the energy-momentum shell. In the case of emission or absorbtion of quasiparticles with zero 4-momenta ($\lambda\varkappa = \lambda\varkappa' = 0$) the system is on the energy-momentum shell. It follows that this diagram technique is general enough to describe physical processes on the shell as well as off the shell.*)

3. EQUATION FOR THE SCATTERING AMPLITUDE OFF THE ENERGY-MOMENTUM SHELL

In this section we proceed to the solution of our main problem - the construction of an equation for the scattering amplitude T off the energy-momentum shell (see Sec. 1). We shall consider the elastic scattering of ψ- and χ-particles. By definition

$$\langle \vec{p}_1, \vec{p}_2 | R^c(\lambda\varkappa) | \vec{q}_1, \vec{q}_2 \rangle = (2\pi)^6 \langle 0 | a(\vec{p}_1) c(\vec{p}_2) | R^c(\lambda\varkappa) | a^+(\vec{q}_1) c^+(\vec{q}_2) | 0 \rangle =$$

$$= (2\pi)^4 \delta(p_1 + p_2 - \lambda\varkappa - q_1 - q_2) \frac{T}{\sqrt{(2p_1^0)(2p_2^0)(2q_1^0)(2q_2^0)}} , \qquad (3.1)$$

*) Similarly, the Feynman diagram technique is also suitable for calculations of physical quantities off the mass-shell.

where the index "c" symbolically indicates that in the perturbation theory decomposition of (3.1), there are only diagrams connected with respect to the solid lines. Under the condition (2.21) the amplitude T is a function of three independent variables*)

$$T = T(\varkappa, t_{pq}, s_q) = T(s_p, t_{pq}, s_q) \tag{3.2}$$

where, as before,

$$s_q = (q_1 + q_2)^2 \quad , \quad s_p = (p_1 + p_2)^2 \quad , \quad t_{pq} = (p_1 - q_1)^2 \qquad (3.3.)$$

$$\sqrt{s_p} = \sqrt{s_q} + \varkappa$$

By passing to the centre-of-mass system $\vec{q}_1 + \vec{q}_2 = \vec{p}_1 + \vec{p}_2 = 0$ and putting (compare with Sec. 1)

$$\begin{aligned} \vec{p}_1 &= -\vec{p}_2 = \vec{p}, \\ \vec{q}_1 &= -\vec{q}_2 = \vec{q}, \\ \cos\vartheta &= \frac{\vec{p}\,\vec{q}}{|\vec{p}|\,|\vec{q}|} \end{aligned} \tag{3.6}$$

---

*) Eq.(3.2) can also be written in the form

$$T = T(s_p, t_{pq}, u_{pq}, s_q)$$

where $u_{pq} = (p_1 - q_2)^2$ is related to s_p, s_q, t_{pq} by

$$\sqrt{s_p s_q} + t_{pq} + u_{pq} = 4m^2 \tag{3.4}$$

It is evident that on the shell $\varkappa = \sqrt{s_p} - \sqrt{s_q} = 0$, (3.4) is identical with the well-known equality

$$s_p + t_{pq} + u_{pq} = 4m^2 \ . \tag{3.5}$$

we have

$$
\begin{aligned}
s_p &= 4E_p^2 = 4(m^2 + \vec{p}^2) , \\
s_q &= 4E_q^2 = 4 \ (m^2 + \vec{q}^2) , \\
t_{pq} &= 2(m^2 - E_p \ E_q + \sqrt{E_p^2 - m^2} \ \sqrt{E_q^2 - m^2} \quad \cos\vartheta) \\
u_{pq} &= 2(m^2 - E_p \ E_q - \sqrt{E_p^2 - m^2} \ \sqrt{E_q^2 - m^2} \quad \cos\vartheta)
\end{aligned}
\tag{3.7}
$$

Finally

$$
\begin{aligned}
T &= T(s_p, t_{pq}, s_q) = T(\vec{p}, \vec{q}) = \\
&= T(E_p, \cos\vartheta, E_q)
\end{aligned}
\tag{3.8}
$$

From (3.8) it can be seen that the relativistic amplitude T which we consider is a direct generalization of the non-relativistic Lippmann-Schwinger amplitude.

The equation for T can be obtained in two different ways. The first one is connected with the use of the basic operator equation (2.4) (or (2.5)). The second one resembles the procedure applied in the derivation of the B-S equation [2].

In both cases it is convenient to go over to matrix notation in the $\varkappa$ - space [3, 5] by putting formally

$$
\begin{aligned}
R(\lambda\varkappa, \lambda\varkappa') &= \langle\varkappa | R | \varkappa'\rangle , \\
\tilde{H}(\lambda\varkappa - \lambda\varkappa') &= \langle\varkappa | H | \varkappa'\rangle , \\
\frac{1}{2\pi} \ \delta(\varkappa - \varkappa') \quad \frac{1}{\varkappa - i\varepsilon} &= \langle\varkappa | G_0 | \varkappa'\rangle .
\end{aligned}
\tag{3.9}
$$

In new notations the basic operator equation becomes

$$
R = - H - H \ G_0 \ R. \tag{3.10}
$$

It is useful to introduce the "full" propagator G of the quasiparticle. We define it in the following way:

$$G = G_0 - G_0 \; R \; G_0 \; . \tag{3.11}$$

From (3.11) and (3.10) it is easy to find an equation for G.

$$G = G_0 - G_0 \; H \; G \tag{3.12}$$

from where

$$G = \frac{1}{G_0^{-1} - H} = \frac{1}{2\pi} \frac{1}{\varkappa + \frac{1}{2\pi} H - i\varepsilon} \; . \tag{3.13}$$

Let us recall that here H is an operator in the space of the physical particle states as well as in the space of the quasiparticle "states". A concrete realization of this operator is given in [5]

$$H = \int a(\varkappa) \widetilde{H}(-\lambda\varkappa) \, d\varkappa \tag{3.14}$$

where the quasiparticle's wave function $a(\varkappa)$ is defined by

$$\langle \varkappa_1 | \, a(\varkappa) \, | \varkappa_2 \rangle = \delta(\varkappa_1 + \varkappa - \varkappa_2) \tag{3.15}$$

To derive the equation for the scattering amplitude from eq.(3.10), let us make in (3.10) one iteration

$$R = - H + HG_0H + HG_0HG_0R \tag{3.16}$$

and make the substitution

$$R = R' - \frac{1}{1 - HG_0HG_0} \; H \; . \tag{3.17}$$

The result is

$$R' = HG_0H + HG_0HG_0R' \tag{3.18}$$

or

$$R' = K_2 + K_2 G_0 R' \qquad (K_2 \equiv H G_0 H) \ . \tag{3.19}$$

In virtue of (2.11) and (3.17) one may conclude that the matrix elements of the operators R and R' are indentical with one another for the transitions between states which do not involve the φ-particles.*) Since we are interested in the elastic scattering of the ψ- and χ-particles which represents the transition of this kind, the "prime" in (3.18) and (3.19) will be omitted.

Let us now take from both sides of (3.19) matrix elements of the form (3.1)

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_2 | R | \vec{q}_1, \vec{q}_2 \rangle &= \langle \vec{p}_1, \vec{p}_2 | K_2 | \vec{q}_1, \vec{q}_2 \rangle + \\ &+ \int \langle \vec{p}_1, \vec{p}_2 | K_2 | \vec{k}_1, \vec{k}_2 \rangle \frac{d\vec{k}_1 d\vec{k}_2}{(2\pi)^6} G_0 \langle \vec{k}_1, \vec{k}_2 | R | \vec{q}_1, \vec{q}_2 \rangle + \\ &+ \sum_{n' \neq |k_1, k_2\rangle} \langle \vec{p}_1, \vec{p}_2 | K_2 | n' \rangle G_0 \langle n' | R | \vec{q}_1, \vec{q}_2 \rangle \end{aligned} \tag{3.20}$$

The summation in the right-hand side of (3.20) is carried over a complete set of "bare" states of the considered fields, excluding the two-particle state $|2\rangle \equiv |\vec{k}_1, \vec{k}_2\rangle = (2\pi)^3 a^+(\vec{k}_1) c^+(\vec{k}_2) | 0 \rangle$ whose contribution we have separated explicitly.

The equality (3.20) is only one of the equations amongst the infinite system of linked integral equations **), which is equivalent to the operator relation (3.19). Successively excluding the matrix elements $\langle n' | R | \vec{q}_1 \vec{q}_2 \rangle$ from (3.20) with the help of other equations of the given system, we can obtain, for the quantity $\langle \vec{p}_1, \vec{p}_2 | R | \vec{q}_1, \vec{q}_2 \rangle$ the following closed integral equation:

---

*) This statement is also valid in the case when we add to the Hamiltonian (2.11), counter-terms quadratic in the field φ.

**) Such kind of linked integral equations connecting amplitudes of different processes are the subject of investigation in the Tamm-Dancoff method and also in Ref. [7].

$$\langle \vec{p}_1,\vec{p}_2 | R | \vec{q}_1,\vec{q}_2 \rangle = \langle \vec{p}_1,\vec{p}_2 | K | \vec{q}_1,\vec{q}_2 \rangle +$$

$$+\frac{1}{(2\pi)^6}\int \langle \vec{p}_1,\vec{p}_2 | K | \vec{k}_1,\vec{k}_2 \rangle G_0 \langle \vec{k}_1,\vec{k}_2 | R | \vec{q}_1,\vec{q}_2 \rangle d\vec{k}_1 d\vec{k}_2 , \tag{3.21}$$

where we have introduced the notation

$$K \equiv K_2 \frac{1}{1-G_0(1-\Pi_2)K_2} = K_2 + K_2 G_0 (1-\Pi_2) K_2 + \cdots \tag{3.22}$$

(Π_2 is the projection operator onto two particle states $|2\rangle = |\vec{K}_1,\vec{K}_2\rangle$).

Eq. (3.21) is an operator equation in $\varkappa$-space. If we write it in matrix form, we shall have, taking into account (3.9),*)

$$\langle \vec{p}_1,\vec{p}_2 | R(\lambda\varkappa,\lambda\varkappa') | \vec{q}_1,\vec{q}_2 \rangle = \langle \vec{p}_1,\vec{p}_2 | K(\lambda\varkappa,\lambda\varkappa') | \vec{q}_1,\vec{q}_2 \rangle +$$

$$+\frac{1}{(2\pi)^7}\int \langle \vec{p}_1,\vec{p}_2 | K(\lambda\varkappa,\lambda\varkappa_1) | \vec{k}_1,\vec{k}_2 \rangle \frac{d\vec{k}_1 d\vec{k}_2 d\varkappa_1}{\varkappa_1 - i\varepsilon} \langle \vec{k}_1,\vec{k}_2 | R(\lambda\varkappa,\lambda\varkappa') | \vec{q}_1,\vec{q}_2 \rangle , \tag{3.23}$$

or for $\varkappa' = 0$,

$$\langle \vec{p}_1,\vec{p}_2 | R(\lambda\varkappa) | \vec{q}_1,\vec{q}_2 \rangle = \langle \vec{p}_1,\vec{p}_2 | K(\lambda\varkappa,0) | \vec{q}_1,\vec{q}_2 \rangle +$$

$$+\frac{1}{(2\pi)^7}\int \langle \vec{p}_1,\vec{p}_2 | K(\lambda\varkappa,\lambda\varkappa_1) | \vec{k}_1,\vec{k}_2 \rangle \frac{d\vec{k}_1 d\vec{k}_2 d\varkappa_1}{\varkappa_1 - i\varepsilon} \langle \vec{k}_1,\vec{k}_2 | R(\lambda\varkappa_1) | \vec{q}_1,\vec{q}_2 \rangle . \tag{3.24}$$

---

*) Here we introduce a supplementary definition: $K(\lambda\varkappa,\lambda\varkappa') \equiv \langle\varkappa | K | \varkappa'\rangle$. Its uniqueness follows from (3.22) and the equality $K_2 = HG_0H$.

Eq.(3.21) cannot yet be considered as the one required for the scattering amplitude since, according to (3.1), the latter must be expressed in terms of connected diagrams whereas the expression for $\langle \vec{p}_1 \vec{p}_2 | R(\lambda \varkappa) | \vec{q}_1 \vec{q}_2 \rangle$ involves contributions from the vacuum loops and unconnected graphs of the form shown in Figs. 7 and 8 as well. On the other hand, however, (3.24) possesses a number of properties which will be characteristic for the final equation too. Therefore, it is reasonable to study (3.24) more thoroughly.

Let us introduce an "unconnected" amplitude M corresponding to the matrix element $\langle \vec{p}_1 \vec{p}_2 | R(\lambda \varkappa) | \vec{q}_1 \vec{q}_2 \rangle$ by defining it in complete analogy with the "connected" amplitude T (see (3.1))

$$\langle \vec{p}_1 \vec{p}_2 | R(\lambda \varkappa) | \vec{q}_1 \vec{q}_2 \rangle =$$

$$= (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2 - \lambda \varkappa) \times$$

$$\times \frac{1}{\sqrt{(2p_1^0)(2p_2^0)(2q_1^0)(2q_2^0)}} M(\lambda \varkappa; p_1, p_2, q_1, q_2). \tag{3.25}$$

As in (3.25) let us explicitly take into account the conservation of the 4-momentum in the kernel and in the free term of eq.(3.24)

$$\langle \vec{p}_1, \vec{p}_2 | K(\lambda\varkappa, \lambda\varkappa_1) | \vec{k}_1, \vec{k}_2 \rangle = (2\pi)^4 \delta^{(4)}(\lambda\varkappa_1 + p_1 + p_2 - \lambda\varkappa - k_1 - k_2) \times$$

$$\times \frac{1}{\sqrt{(2p_1^0)(2p_2^0)(2k_1^0)(2k_2^0)}} V(\lambda\varkappa, p_1, p_2; \lambda\varkappa_1, k_1, k_2); \tag{3.26}$$

$$\langle \vec{p}_1, \vec{p}_2 | K(\lambda\varkappa, 0) | \vec{q}_1, \vec{q}_2 \rangle = (2\pi)^4 \frac{\delta^{(4)}(p_1 + p_2 - \lambda\varkappa - q_1 - q_2)}{\sqrt{(2p_1^0)(2p_2^0)(2q_1^0)(2q_2^0)}} V(\lambda\varkappa, p_1, p_2; 0, q_1, q_2)$$

Substituting (3.25) and (3.26) in (3.24) and cancelling the δ-function common for both the parts, we find

$$M(\lambda\varkappa; p_1, q_1+q_2+\lambda\varkappa-p_1, q_1, q_2) = V(\lambda\varkappa, p_1, p_2; 0, q_1, p_1+p_2-\lambda\varkappa-q_1) +$$

$$+\frac{1}{(2\pi)^3}\int V(\lambda\varkappa, p_1, p_2; \lambda\varkappa_1, k_1, p_1+p_2-\lambda\varkappa+\lambda\varkappa_1-k_1)\frac{d\varkappa_1}{\varkappa_1-i\varepsilon}\, d^4k_1\, D^{(+)}(k_1)\times$$

$$\times D^{(+)}(q_1+q_2+\lambda\varkappa_1-k_1)\, M(\lambda\varkappa_1; k_1, q_1+q_2+\lambda\varkappa_1-k_1, q_1, q_2),$$

(3.27)

The graphical interpretation of (3.27) is as follows:

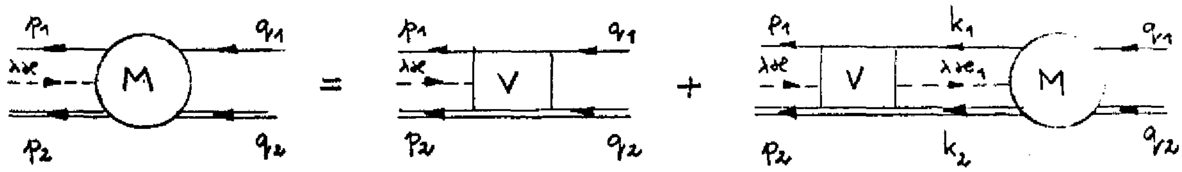

Fig. 9

On imposing the conditions (2.21), eq.(3.27) can be written in the form

$$M(s_p, t_{pq}, s_q) = V(s_p, t_{pq}, s_q\, ; s_q) +$$

$$+\frac{1}{(2\pi)^3}\int V(s_p, t_{pk}, s_k; s_q)\, d^4k_1\, D^{(+)}(k_1)\,\frac{M(s_k, t_{kq}, s_q)}{s_k+t_{kq}+u_{kq}-4m^2-i\varepsilon}\;,$$

(3.28)

where s_p, t_{pq}, s_q are the invariant variables defined in (3.3) and s_k, t_{kq} and u_{kq} quantities defined by similar equalities

$$
\begin{aligned}
s_k &= (k_1 + k_2)^2 \\
t_{kq} &= (k_1 - q_1)^2 \\
u_{kq} &= (k_1 - q_2)^2
\end{aligned}
\tag{3.29}
$$

and (cf. (3.4))

$$
\sqrt{s_k s_q} + t_{kq} + u_{kq} = 4m^2 . \tag{3.30}
$$

Further, it is natural to consider (3.28) in CMS. Introducing, in addition to (3.6) – (3.8), the notations*)

$$
\vec{k}_1 = -\vec{k}_2 = \vec{k} \quad ; \quad 4E_k^2 = s_k \quad ; \quad \cos\theta = \frac{\vec{k}\cdot\vec{q}}{|\vec{k}||\vec{q}|} \; ;
$$

$$
d^3k = k^2 dk \sin\theta \, d\theta \, d\varphi = k^2 \, dk \, d\Omega \; ;
$$

$$
\cos\psi = \frac{\vec{p}\cdot\vec{k}}{|\vec{p}||\vec{k}|} = \cos\vartheta\cos\theta + \sin\vartheta\sin\theta\cos\varphi \tag{3.31}
$$

$$
V(s_p, t_{pq}, s_q; s_q) = V(\vec{p}, \vec{q}; E_q) = V(E_p, \cos\theta, E_q; E_q),
$$

$$
V(s_p, t_{pk}, s_k; s_q) = V(\vec{p}, \vec{k}; E_q) = V(E_p, \cos\psi, E_k; E_q), \tag{3.32}
$$

---

*) From (3.29) and (3.31), it evidently follows (compare with (3.7)) that

$$
t_{pk} = 2(m^2 - E_p E_k + \sqrt{E_p^2 - m^2}\,\sqrt{E_k^2 - m^2}\,\cos\psi)
$$

$$
t_{kq} = 2(m^2 - E_k E_q + \sqrt{E_k^2 - m^2}\,\sqrt{E_q^2 - m^2}\,\cos\theta)
$$

we have

$$M(\vec{p},\vec{q}) = V(\vec{p},\vec{q};E_q) + \frac{1}{(4\pi)^3}\int V(\vec{p},\vec{k};E_q)\frac{d^3\vec{k}}{\sqrt{\vec{k}^2+m^2}}\,\frac{M(\vec{k},\vec{q})}{E_k(E_k-E_q-i\varepsilon)} \quad , \tag{3.33}$$

or

$$M(E_p,\cos\vartheta,E_q) = V(E_p,\cos\vartheta,E_q;E_q) +$$

$$+\frac{1}{(4\pi)^3}\int V(E_p,\cos\psi,E_k;E_q)\sqrt{\frac{E_k^2-m^2}{E_k^2}}\,\frac{dE_k\,d\Omega}{E_k-E_q-i\varepsilon}\,M(E_k,\cos\theta,E_q). \tag{3.34}$$

If, as before, we abstain ourselves from questions of normalization and connectedness of the matrix element (3.25), we may say that (3.33) is the relativistic analogue of the Lippmann-Schwinger eq.(1.1). This analogy is clearly seen in the spherical co-ordinate system (eqs.(1.3) and (3.34)), since the factors $\sqrt{2E_k/m}$ and $\sqrt{(E_k^2 - m^2)/E_k^2}$ in front of the integrals are the modulae of the particle's velocities in the non-relativistic and relativistic cases, correspondingly.

On the other hand, the equation we have obtained is very close in form to the equation for the scattering amplitude in the quasi-potential approach proposed several years ago by Logunov and Tavkhelidze [8]*)

$$T(\vec{p},\vec{q}) = V(\vec{p},\vec{q};E_q) + \frac{1}{4(2\pi)^3}\int V(\vec{p},\vec{k};E_q)\frac{d\vec{k}}{\sqrt{\vec{k}^2+m^2}}\,\frac{T(\vec{k},\vec{q})}{E_k^2-(E_q+i\varepsilon)^2} \tag{3.35}$$

*) At present there is much literature devoted to the analysis and applications of the quasipotential approach (see for instance [9-20]).

The difference between (3.35) and (3.33) is that the denominators in these equations do not have the same dependence on the energy. We have to stress that the quantity V , playing the role of a potential in both (3.33) and (3.35), is in general a complex function of the energy E_q. This fact is the main feature of the equations considered and for this reason we shall call, after Logunov and Tavkhelidze, V the quasipotential, and eq.(3.33) the quasipotential equation.

In [9] it has been proved that, for a real quasipotential, eq.(3.35) leads to the relativistic two-particle unitarity condition

$$\operatorname{Im} T(\vec{p},\vec{q}) = \frac{1}{(8\pi)^2}\sqrt{\frac{E^2-m^2}{E^2}}\int d\Omega_k\, T^*(\vec{p},\vec{k})\,T(\vec{k},\vec{q}) \tag{3.36}$$

$$\left(E^2=\vec{p}^{\,2}+m^2=\vec{q}^{\,2}+m^2=\vec{k}^{\,2}+m^2\right).$$

Let us show now that, under the same assumptions, from eq.(3.33) for the amplitude M, also follows the condition (3.36). To do this let us introduce matrix notations

$$M(\vec{p},\vec{q}) = \langle\vec{p}|M|\vec{q}\rangle\ ;\ V(\vec{p},\vec{k};E_q) = \langle\vec{p}|V(E_q)|\vec{k}\rangle\ ;$$

$$\langle\vec{p}|g(E_q)|\vec{k}\rangle = \delta(\vec{p}-\vec{k})\frac{1}{E_k^2(E_k-E_q-i\varepsilon)} \equiv$$

$$\equiv \frac{\langle\vec{p}\,|\,\hat{I}\,|\vec{k}\rangle}{E_k^2(E_k-E_q-i\varepsilon)} \tag{3.37}$$

and rewrite (3.33) in the form

$$M = V(E_q) + \frac{i}{(4\pi)^3}V(E_q)(\operatorname{Im} g)M + \frac{1}{(4\pi)^3}V(E_q)(\operatorname{Re} g)M, \tag{3.38}$$

where

$$\operatorname{Im} g = \hat{I}\,\frac{\pi}{E_k^2}\,\delta(E_k-E_q)$$

$$\operatorname{Re} g = \hat{I}\,\frac{1}{E_k^2}\,P\,\frac{1}{E_k-E_q}\,. \tag{3.39}$$

From (3.38) it follows that the operator

$$N = M\left[1 + \frac{i}{(4\pi)^3}(\mathrm{Im}\, g)M\right]^{-1} \tag{3.40}$$

satisfies the equation

$$N = V(E_q) + \frac{1}{(4\pi)^3} V(E_q)(\mathrm{Re}\, g)N \tag{3.41}$$

Hence, the quantity N is real when V is real and due to this, from (3.40), we have

$$M^*\left[1 - \frac{i}{(4\pi)^3}(\mathrm{Im}\, g)M^*\right]^{-1} = M\left[1 + \frac{i}{(4\pi)^3}(\mathrm{Im}\, g)M\right]^{-1} \tag{3.42}$$

In the spinless case under consideration one has, owing to the T-invariance,

$$\langle \vec{p}|M|\vec{q}\rangle = \langle \vec{q}|M|\vec{p}\rangle$$

Taking this relation into account, we find from (3.42),

$$M^*\left(1 + \frac{i}{(4\pi)^3}(\mathrm{Im}\, g)M\right) = \left(1 - \frac{i}{(4\pi)^3}(\mathrm{Im}\, g)M^*\right)M \quad ,$$

or

$$M - M^* = \frac{2i}{(4\pi)^3} M^*(\mathrm{Im}\, g)M \tag{3.43}$$

Returning, in the last equality, to the old notations, taking into account (3.39) and going to the energy-shell*) $E_p = E_q = E_k = E$, it is easy to check that the condition (3.36) for the amplitude M is valid.

So far we have investigated only the kinematical structure of eq.(3.24) for the matrix element $\langle \vec{p}_1 \vec{p}_2 | R(\lambda \varkappa) | \vec{q}_1 \vec{q}_2 \rangle$. Doing this, we completely ignored the existence of "unconnected" parts in this quantity. Now we shall partially make up this deficiency, postponing the detailed analysis until the next section, where an equation for the scattering amplitude of the type (3.33) will be derived without the help of eq.(3.10).

According to (3.22) in order to construct the quasipotential it is necessary to sum infinite series of terms of increasing powers in g^2. Then, evidently, each term of the series, owing to the presence of the operator $(1 - \Pi_2)$, multiplied by the quasiparticle's propogator, has the property that it cannot be split into two parts connected with each other by two $\mathcal{D}^{(+)}$-functions and the function $G_0(\varkappa)$ (irreducibility condition).

Direct calculation shows that unconnected parts in (3.22) already appear in g^2-order. Let us demonstrate how eq.(3.33) must be rebuilt in order that unconnected parts in the scattering amplitude do not appear. Here again it is reasonable to use the matrix notations (3.37) in the space of the functions, but now instead of the "matrix" $\langle \vec{p} | g | \vec{k} \rangle$ it is more convenient to consider the matrix

$$\langle \vec{p} | g^{(0)} | \vec{k} \rangle = \frac{1}{(2\pi)^3} \frac{\delta(\vec{p} - \vec{k})}{4E_p(E_p - E_q - i\varepsilon)} = \frac{1}{(2\pi)^3} \frac{\delta(\vec{p} - \vec{k})}{s_p + t_{pq} + u_{pq} - 4m^2 - i\varepsilon} \qquad (3.44)$$

keeping as a whole the invariant form $d^3\vec{k}/2\sqrt{k^2 + m^2}$ of the three-dimensional volume element in momentum space. As a consequence, eq.(3.33) becomes

$$M = V + Vg^{(0)}M \qquad (3.45)$$

*) In fact (3.43) is a continuation of the unitarity condition off the energy shell.

Let us introduce now the Green function $\mathcal{G}$ for the given two-particle system putting by definition

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(0)} \, M \, \mathcal{G}^{(0)} \tag{3.46}$$

From (3.46) and (3.45) we find

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(0)} \, V \, \mathcal{G} \tag{3.47}$$

and taking into account (3.44) we have

$$\mathcal{G} = \frac{1}{[\mathcal{G}^{(0)}]^{-1} - V} =$$

$$= \frac{1}{(2\pi)^3} \; \frac{1}{s + t + u - 4m^2 - \frac{1}{(2\pi)^3} \; V - i\mathcal{E}} \tag{3.48}$$

Therefore, the combination $s + t + u = 4m^2$ of the Mandelstam variables s,t,u, in our scheme coincides with the inverse free Green function of the two-particle system (compare with the Klein-Gordon operator $p^2 - m^2$ for one particle).

Defining the wave function of the system by the relation

$$\Psi_q(\vec{p}) = 2(2\pi)^3 \sqrt{\vec{p}^{\,2} + m^2} \; \delta(\vec{p}-\vec{q}) +$$

$$+ \frac{M(\vec{p},\vec{q})}{s_p + t_{pq} + u_{pq} - 4m^2 - i\mathcal{E}} \; ,$$

we can obtain for $\psi_{\vec{q}}(\vec{p})$ an analogue of the Schrödinger equation in the p-representation (compare with [8])

$$(s_p + t_{pq} + u_{pq} - 4m^2) \; \psi_q(\vec{p}) =$$

$$= \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2\sqrt{\vec{k}^2 + m^2}} \; V(\vec{p},\vec{k};E_q) \; \psi_q(\vec{k}) \, . \tag{3.49}$$

Let us now suppose that we have separated all the unconnected parts $\tilde{V}$ in the quasipotential so that

$$V = \tilde{V} + V^{c} \tag{3.50}$$

Substituting (3.50) in (3.48), we have

$$G = \frac{1}{(g^{(0)})^{-1} - \tilde{V} - V^{c}} = \frac{1}{(\tilde{g}^{(0)})^{-1} - V^{c}} , \tag{3.51}$$

where the function $\tilde{g}^{(0)} = \frac{1}{(g^{(0)})^{-1} - \tilde{V}}$ obviously satisfies the equation*)

$$\tilde{g}^{(0)} = g^{(0)} + g^{(0)} \tilde{V} \tilde{g}^{(0)}$$

In virtue of (3.51) we can also write

$$g = \tilde{g}^{(0)} + \tilde{g}^{(0)} V^{c} g \tag{3.52}$$

We shall further define the connected amplitude M^{c}

$$g = \tilde{g}^{(0)} + \tilde{g}^{(0)} M^{c} \tilde{g}^{(0)} \tag{3.53}$$

From (3.52) and (3.53) it is easy to see that the equation for M^{c} is

$$M^{c} = V^{c} + V^{c} \tilde{g}^{(0)} M^{c} , \tag{3.54}$$

---

*) The operations we perform are usual procedure in the many-body scattering problem.

$$M^c(\vec{p},\vec{q}) = V^c(\vec{p},\vec{q};E_q) + \int V^c(\vec{p},\vec{\ell};E_q)\frac{d\vec{\ell}}{2\sqrt{\vec{\ell}^2+m^2}}\tilde{g}^{(0)}(\vec{\ell},\vec{k};E_q)\frac{d\vec{k}}{2\sqrt{\vec{k}^2+m^2}}M^c(\vec{k},\vec{q}) \tag{3.55}$$

We see, thus, that the rebuilding of eq.(3.33), whose aim was to separate the connected part of the scattering amplitude, has led to a change of the kernel and the free Green function ("the energy denominator") keeping the main feature of (3.33), i.e., a three-dimensional integration in the k-space with an invariant volume element.

4. DERIVATION OF THE EQUATION FOR THE SCATTERING AMPLITUDE ON THE BASIS OF THE DIAGRAM TECHNIQUE

As mentioned above, in this section we shall obtain the equation for the scattering amplitude without using eq.(3.10).

Let us suppose that we know the decomposition (2.15) and that we have separated from these infinite series the terms which do not contain vacuum loops. Let us denote these terms by

$$\hat{R}(\lambda\varkappa,\lambda\varkappa') = \sum_{n=m=\mu=0} \int \hat{F}_{n,m,\mu}(\lambda\varkappa,\lambda\varkappa';p_1\ldots p_n;p'_1\ldots p'_m;q_1\ldots q_n;q'_1\ldots q'_m,k_1,\ldots k_\mu)\times$$
$$\times :\Psi^*(p_1)\ldots\Psi^*(p_n)\chi^*(p'_1)\ldots\chi^*(p'_m)\Psi(q_1)\ldots\Psi(q_m)\chi(q'_1)\ldots\chi(q'_m)\varphi(k_1)\ldots\varphi(k_\mu): \tag{4.1}$$
$$dp_1\ldots dp_n\,dp'_1\ldots dp'_m\,dq_1\ldots dq_n\,dq'_1\ldots dq'_m\,dk_1\ldots dk_\mu .$$

with the condition that

$$\hat{F}_{0,0,0} = 0 \quad .$$

Putting

$$\hat{R}(\lambda\varkappa,\lambda\varkappa') = \langle\varkappa|\hat{R}|\varkappa'\rangle \tag{4.2}$$

we shall define the total Green function of the quasiparticle corresponding to $\hat{R}$ (cf. (3.11))

$$\hat{G} = G_0 + G_0 \hat{R} G_0 . \tag{4.3}$$

Further, it is convenient to consider the "full" normal pairings of the ψ- and χ-fields in the p-representation. Let us define them as

$$\mathcal{D}_\psi = \frac{1}{2\pi} \langle 0 | N \psi(1) \hat{G} \psi^*(2) | 0 \rangle$$

$$\mathcal{D}_\chi = \frac{1}{2\pi} \langle 0 | N \chi(1) \hat{G} \chi^*(2) | 0 \rangle \tag{4.4}$$

$$\mathcal{D}_{\psi\chi} = \frac{1}{(2\pi)^2} \langle 0 | N \psi(1) \chi(2) \hat{G} \psi^*(3) \chi^*(4) | 0 \rangle ,$$

etc,

where the 4-momenta on which the fields depend are denoted by number arguments, N is the symbol of the normal product and in the vacuum expectation value the vacuum of non-interacting fields is used.

Substituting (4.3) in (4.4) we obtain, taking into account (4.2) and (4.3),

$$2\pi \mathcal{D}_\psi(1,2) = G_0 \underbrace{\psi(1)\psi^*}(2) + $$
$$+ \int G_0 \underbrace{\psi(1)\psi^*}(p_1) \hat{F}_{1,0,0}(p_1, q_1) \underbrace{\psi(q_1)\psi^*}(2) G_0 \, dp_1 dq_1$$

$$2\pi \mathcal{D}_\chi(1,2) = G_0 \underbrace{\chi(1)\chi^*}(2) +$$
$$+ \int G_0 \underbrace{\chi(1)\chi^*}(p_1') \hat{F}_{0,1,0}(p_1', q_1') \underbrace{\chi(q_1')\chi^*}(2) G_0 \, dp_1' dq_1' \tag{4.5}$$

$$
\begin{aligned}
(2\pi)^2 \mathcal{D}_{\psi\chi}(1,2;34) = {} & G_0 \underbrace{\psi(1)\psi^*(3)}\,\underbrace{\chi(2)\chi^*(4)} + \\
+ {} & G_0 \int \underbrace{\psi(1)\psi^*(p_1)}\,\hat{F}_{1,0,0}(p_1,q_1)\,\underbrace{\psi(q_1)\psi^*(3)}\,dp_1\,dq_1\,\underbrace{\chi(2)\chi^*(4)}\,G_0 + \\
+ {} & G_0 \int \underbrace{\chi(2)\chi^*(p_1')}\,\hat{F}_{0,1,0}(p_1',q_1')\,\underbrace{\chi(q_1')\chi^*(4)}\,dp_1'\,dq_1'\,\underbrace{\psi(1)\psi^*(3)}\,G_0 + \\
+ {} & G_0 \int \underbrace{\psi(1)\psi^*(p_1)}\,\underbrace{\chi(2)\chi^*(p_1')}\,\hat{F}_{1,1,0}(p_1,p_1',q_1,q_1')\times \\
\times {} & \underbrace{\psi(q_1)\psi^*(3)}\,\underbrace{\chi(q_1')\chi^*(4)}\,dp_1\,dp_1'\,dq_1\,dq_1'\,G_0
\end{aligned}
\tag{4.6}
$$

It is evident that in the perturbation theory the functions (4.5) are represented by self-energy type diagrams (see, for instance, Fig.5), and the expression (4.6) corresponds to diagrams describing the scattering of ψ- and χ-particles (see, for instance, Figs. 5 and 8). The function $\hat{F}_{1,1,0}(p_1,p_1',q_1,q_1')$ is naturally split into two parts

$$
\hat{F}_{1,1,0} = \hat{F}^{(0)}_{1,1,0} + \hat{F}^{c}_{1,1,0}
\tag{4.7}
$$

where the first one corresppnds to unconnected graphs (for instance, Fig. 8) and the second one to connected graphs (Fig.6). Introducing the notation

$$
\begin{aligned}
(2\pi)^2 \mathcal{D}^{(0)}_{\psi\chi}(1,2;3,4) = {} & G_0 \underbrace{\psi(1)\psi^*(3)}\,\underbrace{\chi(2)\chi^*(4)} + \\
+ {} & G_0 \int \underbrace{\psi(1)\psi^*(p_1)}\,\hat{F}_{1,0,0}(p_1,q_1)\,\underbrace{\psi(q_1)\psi^*(3)}\,dp_1\,dq_1\,\underbrace{\chi(2)\chi^*(4)}\,G_0 + \\
+ {} & G_0 \int \underbrace{\chi(2)\chi^*(p_1')}\,\hat{F}_{0,1,0}(p_1',q_1')\,\underbrace{\chi(q_1')\chi^*(4)}\,dp_1'\,dq_1'\,\underbrace{\psi(1)\psi^*(3)}\,G_0 + \\
+ {} & G_0 \int \underbrace{\psi(1)\psi^*(p_1)}\,\underbrace{\chi(2)\chi^*(p_1')}\,\hat{F}^{(0)}_{1,1,0}(p_1,p_1';q_1,q_1')\,\underbrace{\psi(q_1)\psi^*(3)}\times \\
\times {} & \underbrace{\chi(q_1')\chi^*(4)}\,dp_1\,dp_1'\,dq_1\,dq_1'\,G_0
\end{aligned}
\tag{4.8}
$$

we shall have from (4.6)

$$\mathcal{D}_{\psi\chi}(1,2;3,4) = \mathcal{D}^{\circ}_{\psi\chi}(1,2;3,4) +$$

$$+\frac{1}{(2\pi)^2} G_0 \underbrace{\psi(1)\psi^*}(p_1)\underbrace{\chi(2)\chi^*}(p_1')\hat{F}^{c}_{1,1,0}(p_1,p_1';q_1,q_1')\underbrace{\psi(q_1)\psi^*}(3)\underbrace{\chi(q_1')\chi^*}(4)\,dp_1\ldots dq_1' . \tag{4.9}$$

If we carry out all the reasoning done in the usual formalism when one derives the Dyson equation for the one-particle propagators or the Bethe-Salpeter equation for the two-particle Green function, we can write the relation (4.9) in the form of an equation for the function $\mathcal{D}_{\psi\chi}$. Essentially we have only to introduce a convenient definition of irreducible diagrams. Let us consider all connected diagrams with four external lines of ψ and χ types and two dotted ends. Let us suppose that all lines are oriented in the same way as Figs. 3 and 6 and that the 4-momenta of the ψ and χ particles satisfy the condition $p^2 = m^2$, $p_0 > 0$. We shall call a diagram belonging to this class, irreducible, if it cannot be split into two connected parts which are linked by one dotted line, oriented from left to right and by a pair of ψ and χ lines, oriented from right to left. For instance the diagrams in Figs. 3, 6b, 6c, 6d are irreducible in this sense, and the diagram in Fig. 5a is reducible.

The set of all irreducible diagrams we shall denote by V . Then from (4.9) follows

$$\mathcal{D}_{\psi\chi}(1,2;3,4) = \mathcal{D}^{\circ}_{\psi\chi}(1,2;3,4) + \mathcal{D}^{\circ}_{\psi\chi}(1,2;5,6)\overset{(-)}{V}(5,6;7,8)\mathcal{D}_{\psi\chi}(7,8;3,4), \tag{4.10}$$

where, as before, all the quantities are operators in the $\varkappa$-space and an integration in the momenta space is carried over the repeated number arguments*). The sign (-) at V shows that in eq.(4.10) it is possible to omit the diagrams which have in the external ψ

---

* Due to the specific kind of the diagram technique this integration is in fact carried over the three-dimensional $\vec{k}$-space.

and χ lines, self-energy parts connected with the rest of the diagram by no more than one dotted line (these parts are taken into account in $\Delta^0_{\psi\chi}$). Examples of diagrams of this kind are given in Fig. 10.

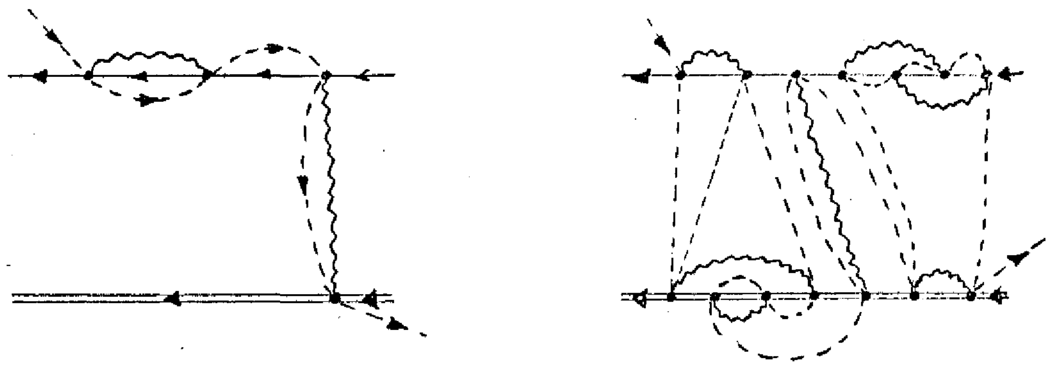

Fig. 10

Defining the scattering amplitude T by the relation

$$\mathcal{D}_{\psi\chi} = \mathcal{D}^{\circ}_{\psi\chi} + \mathcal{D}^{\circ}_{\psi\chi}\, T\, \mathcal{D}^{\circ}_{\psi\chi} \tag{4.11}$$

we shall have from (4.10)

$$T(1,2;3,4) = V^{(-)}(1,2;3,4) + V^{(-)}(1,2;5,6)\, \mathcal{D}^{\circ}_{\psi\chi}(5,6;7,8)\, T(7,8;3,4). \tag{4.12}$$

Although this equation looks like an analogue of the Lippmann-Schwinger eq. (1.1), it has a more complicated structure, because the function $\Delta^0_{\psi\chi}(5,6,7,8)$ is not diagonal in the momentum representation (compare with eq. (3.55)). However, it is clear that in (4.12) we can use the diagonal "free" function

$$\frac{1}{(2\pi)^2}\, G_0\, \underbrace{\psi(5)\psi^*(7)}\, \underbrace{\chi(6)\chi^*(8)} \tag{4.13}$$

instead of $\Delta^0_{\psi\chi}(5,6,7,8)$ if one simultaneously substitutes $V^{(-)}$ by V. Finally, we obtain

$$T(1,2;3,4) = V(1,2;3,4) + \frac{1}{(2\pi)^2} V(1,2;5,6)\, G_0\, \underbrace{\psi(5)\psi^*}(7) \cdot$$

$$\cdot \underbrace{\chi(6)\chi^*}(8)\, T(7,8;3\,4) \quad , \tag{4.14}$$

which is completely analogous to (1.1) and as is not difficult to verify, coincides, under condition (2.21), with (3.33)*).

The derivation of the quasipotential equation for the scattering amplitude given in the present Section shows that in our formalism this equation plays the same role as the Bethe-Salpeter eq.(1.4) in the usual approach. Correspondingly, the kernel of the equation obtained — the quasipotential — can also be built with the perturbation theory using specific irreducible diagrams.

To each choice of λ corresponds an invariant quasipotential and an invariant energy denominator (Green function). For this reason, the form of the quasipotential on the energy-momentum shell, contrary to the scattering amplitude, in general depends on the choice of λ . The only exception is the quasipotential V_2 in the Born approximation. For instance, in our case, V_2 is given by the expression (cf. (2.22))**)

$$V_2(\vec{p},\vec{k};E_q) = \frac{g^2}{\sqrt{\mu^2+(\vec{p}-\vec{k})^2}}\,\frac{1}{\left(E_p+E_k-2E_q-i\varepsilon+\sqrt{\mu^2+(\vec{p}-\vec{k})^2}\right)} \quad , \tag{4.15}$$

which at $E_p = E_k = E_q$ reduces to the invariant pole term

$$\frac{g^2}{\mu^2-(k_1-p_1)^2} \quad .$$

---

*) Let us recall that in (4.14), contrary to (3.33), only connected diagrams contribute to the scattering amplitude.

**) Eq.(3.33) with the kernel (4.15) has been studied in Ref. [22].

5. CONCLUSION

The formalism developed here, is similar to the quasipotential approach of Logunov and Tavkheldize and can be applied in scattering and bound state problems of relativistic particles for quantitative calculations as well as for purposes of phenomenological description. From our point of view it is very interesting to connect this approach with the recent investigations where decompositions of the scattering amplitude in terms of matrix elements of the Lorentz group are studied.

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