FEYNMAN RULES FOR REGGEONS

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ABSTRACT

Rules are formulated for the evaluation of Feynman diagrams in which the virtual lines represent infinite multiplets with discrete and/or continuous spectra. In the simplest case this is a theory of Reggeon Feynman diagrams. The fact that infinite multiplets can represent multi-particle states with continuous mass spectra is emphasized, and a special case of "Compton" scattering via two-particle intermediary states is studied in detail. The kinematical structure of the amplitudes is fixed by the vertices, and is physically reasonable in all channels, particularly near $u = 0$ in unequal mass scattering. This allows a particularly convenient empirical representation of the dynamics. It should be emphasized that the mass spectrum of the infinite multiplets are perfectly general, and that no higher symmetry is implied.
I. INTRODUCTION

It is possible to conceive of physical states as members of infinite multiplets. However, there is no reason to expect this to be profitable if the states have to be mutilated in order to fit into a framework that is too narrow.

Early work on infinite multiplets idealized the states to the point where they became an infinite set of one-particle states with equal masses. In this case it is possible to invoke invariance under various non-compact groups to obtain predictions for certain form factors. The convenient notation of group representations can also be employed to construct explicit solutions of current algebra and superconvergence relations.

The next step is to consider one-particle states with unequal masses. One principal purpose of this report is to calculate amplitudes related to the exchange of such objects. It is found that both Regge pole contributions and Lorentz pole contributions appear. The behaviour of an exchange amplitude is studied in detail in Sec. IV. It turns out that the vertex functions suggested by infinite component local field theory (see below) gives a reasonable kinematical structure; in particular, this is true near the infamous point \( u = 0 \).

The physical states that occur in strong interaction scattering processes may be approximated by single-particle states — but poorly, even if the masses are unequal. Another main topic of this paper is a consideration of multiplets that include multi-particle states. It turns out that the inclusion of multiparticle states does not create essential complications.

Our approach is semi-empirical. All our scattering amplitudes are related to simple Feynman diagrams, in which the external lines are one-particle states, while each internal line represents an infinite multiplet. To define an amplitude it is necessary to specify (i) the vertex functions and (ii) the propagators. A completely empirical theory would leave both arbitrary a priori, and attempt the calculation of vertex functions and propagators from general physical requirements and additional assumptions like current algebra or superconvergence. An example of the opposite extreme is infinite-component local field theory, in which everything is deduced...
from the Lagrangian or from a field equation. The advantage of the empirical approach is that a wide range of physical requirements can be introduced as input. Infinite-component field theory, on the other hand, is a soluble model, and exact solubility is often very useful. We shall therefore steer a middle course, which may be described as searching for the physical world among the widest possible class of soluble models. Briefly, the idea is to leave propagators arbitrary, to be related directly to experiment, but to accept the vertex functions suggested by infinite component field theory.

Let the infinite multiplet be associated with an infinite set of fields, \( \psi_\sigma(x) \), \( \sigma = 1, 2, \ldots \). The states are expected to possess spin; therefore the generators of Lorentz transformations must include a spin part,

\[
\mathcal{L}_{\mu
u} \psi_\sigma(x) = i \left( \gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu \right) \psi_\sigma(x) + (S_{\mu\nu})_\sigma^{\tau} \psi_\tau(x)
\]

It is necessary to specify the spin matrices in some detail, because the physical eigenstates of mass must be defined in such a way that the mass is a Poincaré invariant. The "solubility" of our models is ensured by constructing a unitary irreducible representation of some group \( G \) that includes the spin group \( SO(3,1) \), acting on the index \( \sigma \) only. The simplest possibility is to take \( G = SO(3,1) \); this is the case considered in Sections II.1 and II.2. A larger group is required if multiparticle states are to be included; in Section II.3 we take \( G = SO(4,1) \).

The vertices are assumed to be defined by an interaction density that is local and non-derivative,

\[
\sum_{\sigma,\lambda,\tau} C_{\sigma\lambda\tau} \psi_\sigma(x) \phi_\lambda(x) \chi_\tau(x)
\]

and invariant under \( G \). If \( G = SO(3,1) \), then invariance is automatic for non-derivative local interactions. This type of vertex, when expressed in terms of eigenstates of mass and spin, contains very physical form factors, which is the main reason why we are
reluctant to give it up. The relationship between the above invariant density and vertex functions is, briefly, as follows. A physical state with definite mass and spin is associated with a particular set of entries in the infinite column formed by the components of \( \psi_\sigma^p(\mathbf{p}) \); \( \psi_\sigma^p(\mathbf{p}) \rightarrow \psi_\sigma^{(1)}(\mathbf{p}) \) for the state \(|1\rangle\), say. Supposing that the functions \( \psi_\sigma^{(1)}(\mathbf{p}) \), \( \psi_\sigma^{(2)}(\mathbf{p}), \cdots \) are known, the vertex function for the elementary interaction between three states \(|1\rangle, |2\rangle, |3\rangle\) is given by

\[
V_{123}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \sum_{\sigma\lambda} C_{\sigma\lambda} \psi_\sigma^{(1)}(\mathbf{q}_1) \psi_\lambda^{(2)}(\mathbf{q}_2) \psi_\lambda^{(3)}(\mathbf{q}_3)
\]

The simplest case is obtained by supposing that one of the fields is an ordinary scalar one-component field. Then the vertex function reduces to

\[
V_{123}(\mathbf{t}) = \sum_{\sigma\lambda} C_{\sigma\lambda} \psi_\sigma^{(1)}(\mathbf{q}_1) \psi_\lambda^{(2)}(\mathbf{q}_2)
\]

with \( \mathbf{t} = (\mathbf{p}_1 - \mathbf{p}_2)^2 \). Suppose that the states \(|1\rangle\) and \(|2\rangle\) are identical except that \( \mathbf{p}_1 \neq \mathbf{p}_2 \), then \( V_{123} \) is the scalar form factor \( K_1(\mathbf{t}) \) of this state. Clearly \( K_1(0) = 1 \) and \( K_1(\mathbf{t}) \rightarrow 0 \) as \( \mathbf{t} \rightarrow -\infty \), because the overlap between the two states is perfect when \( \mathbf{p}_1 = \mathbf{p}_2 \) and vanishes in the limit of infinite momentum transfer. Notice that the vertex functions have been defined by the above for physical mass-shell states only. Off-mass-shell vertex functions will be defined in terms of the Feynman amplitudes to which they contribute.

The propagators are denoted \( L^{-1} \) and are assumed to be invariant under Poincaré transformations. Acting on the fields \( \psi_\sigma \) in momentum space they are some complicated matrices:

\[
L^{-1} \psi_\sigma(\mathbf{p}) = \sum_{\tau} L^{-1}_{\sigma\tau}(\mathbf{p}) \psi_\tau(\mathbf{p})
\]

However, these matrices can be diagonalized. Let \( n \) be a set of Poincaré-invariant quantum numbers. For example, if \( G = SO(3,1) \), \( n \) is just the spin \( \ell \), defined invariantly by

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in terms of the Poincaré generators. The physical states are eigenstates of these quantum numbers. We suppose that they form a complete set, so that the physical states may be uniquely labelled by \( \ell \) and \( n \). Then \( L^{-1} \) is diagonal in this basis,

\[
L^{-1}_{\ell n}( \psi ) \rightarrow L^{-1}_{\ell n}( \psi ) \delta_{\ell n}.
\]

The empirical input into our models is represented by the arbitrary function \( L_n^{-1}(p) \). The singularities of this function determine the mass spectrum of the physical states of the model, as well as the Regge trajectories. If \( G = \text{SO}(3,1) \), then there can be only a discrete set of states, one for each value of the spin. But for larger groups the mass spectrum can have both a discrete and a continuous part. An example is discussed in Section II.3. It is known that the continuous part of the spectrum is capable of representing multiparticle states exactly in the nonrelativistic limit.

After fixing the manifold of singularities of the propagator so as to obtain the desired mass spectrum and Regge trajectories, we still have the residues of the singularities at our disposal. The residues form the absorptive part of the propagator and are related to the completeness relation,

\[
\sum_{\psi} \psi \cdot \text{Abs} \ L^{-1} \cdot \overline{\psi} = 1
\]

We believe that assumptions of current algebra and superconvergence can be formulated as conditions on the residue function, but this is not attempted in the present paper.
II. INFINITE MULTIPLETS AS INTERMEDIARY STATES

1. "Compton" scattering with single particle spectrum

We shall calculate the amplitude that corresponds to the Feynman diagram of Fig. 1, in which the straight lines represent Reggeons, i.e. particles belonging to an infinite multiplet, and the wavy lines are conventional scalar bosons, described by a field $A(x)$. In this first example the infinite multiplet is of the simplest possible kind: a single irreducible representation of the homogeneous Lorentz group.

Consider the irreducible representation $D(N)$ of $SO(3,1)$ that is realized on the generalized tensor field

$$\phi_{\mu_1 \cdots \mu_N}(x)$$

where the indices are four-vector indices. This representation is unitary if $(N+1)^2 < 1$; for definiteness we shall take $N$ real, with

$$-1 < N < 0$$

Then $D(N)$ belongs to the supplementary series of unitary representations.

The vertices are defined by the local invariant density

$$\bar{\phi} \phi A = \bar{\phi}^{\mu_1 \cdots \mu_N}(x) \phi_{\mu_1 \cdots \mu_N}(x) A(x)$$

The amplitude is

$$A_i = \sum \bar{\phi}(p_\perp) A(q_\perp) \cdot \phi L^{-1} \cdot \phi (p_\perp) A(q_\perp)$$

Here $L$ is the propagator for the virtual states, and the sum is over a complete set of intermediary states. It is necessary to choose definite states for the external lines; this will be done by replacing $\phi(p_\perp)$ and $\phi(p_\perp)$ by their respective spin-zero projections, e.g.
The next step is to expand the tensor $\mathcal{F}$ according to the irreducible representations of the little group $G$, where

$$Z^P = \phi_1 + \phi_2 = \phi_3 + \phi_4$$  \hspace{1cm} (II.6)

The expansion has the form

$$\mathcal{F}_{\mu_1...\mu_N} = \sum_{\ell=0}^{\infty} \sum_{\epsilon_1, \epsilon_2} \mathcal{F}_{\mu_1...\mu_2}^\epsilon \sum_{\ell = \ell_1, \ell_2, ...} a_{\epsilon, \epsilon}^{N}$$  \hspace{1cm} (II.7)

$$\times \Theta_{\ell_1, \mu_1} \cdots \Theta_{\ell_2, \mu_2} \lambda_{\ell_3, \mu_3} \cdots \lambda_{\ell_N, \mu_N}$$

where $\mathcal{F}_{\mu_1...\mu_2}$ describes the $2\ell + 1$ states with angular momentum $\ell$, and

$$\lambda_{\mu} = \mathcal{P}_{\mu} (P^*)^{-t}, \quad \Theta_{\mu} = \lambda_{\mu}^* \lambda_{\mu} - g_{\mu\nu}$$  \hspace{1cm} (II.8)

The coefficients $a_{\epsilon, \epsilon}^{N}$ are given in the appendix. Substitution of (II.5) and (II.7) in (II.4) gives

$$A_i = \sum_{\ell=0}^{\infty} \sum_{\epsilon} V_{\epsilon}^{(i)} (g_{12} g_{13}) \lambda_{3\mu_1} \cdots \lambda_{3\mu_N} \mathcal{F}_{\mu_1...\mu_N}^\epsilon$$  \hspace{1cm} (II.9)

$$\times L_1 L_2 \cdots L_N \mathcal{F}_{\nu_1...\nu_2}^{\epsilon^*} \lambda_{\nu_1} \cdots \lambda_{\nu_N} \epsilon \epsilon (g_{12} g_{13})$$

The vertex functions are calculated in the appendix and may be expressed in terms of 4-dimensional spherical functions.\(^{10}\),

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where \( V_i \) is the velocity of particle 1 in the centre-of-mass system,

\[
V_i = \sqrt{1 - \frac{\lambda^2}{(\lambda_1 \cdot \lambda)^2}}.
\]

In (II.9) it has been assumed that the propagator \( L^{-1} \) is diagonal in \( \ell \), which is required by Lorentz invariance. The sum over \( \ell \) is a sum over helicities; it may be carried out by means of the completeness relation (A-3), with the result

\[
A_i (s, \ell) = \sum_{\ell=0,1,...} (2\ell+1) P_{\ell} (\cos \theta_s) \tilde{d}^{(\ell)}_\ell (s) \tag{II.13}
\]

The partial waves are

\[
\tilde{d}^{(\ell)}_\ell (s) = \frac{1}{(\ell-N-1)! (\ell+N+1)!} \frac{1}{P_{\ell} (\lambda_1 \cdot \lambda)} L^{-1}_{\ell} (s) \frac{1}{P_{\ell} (\lambda_1 \cdot \lambda)} \tag{II.14}
\]

The variables \( s \) and \( \cos \theta_s \) are the conventional scattering parameters and (II.12) is the same as

\[
\mathcal{U}^2 = \left[ s - (m_1 + m_2)^2 \right] \left[ s - (m_1 - m_2)^2 \right] \left[ s + m_1^2 - m_2^2 \right]^{-2} \tag{II.15}
\]
The correct threshold factors \( v_1^l \) and \( v_3^l \) are hidden in the functions \( P_{N_l}^l \).

If the propagator \( L^{-1}(s) \) is independent of \( l \) — which implies that all the particles in the infinite multiplet have the same mass — then (II.13) reduces to an addition formula for four-dimensional spherical functions, and

\[
A_1(s, t) \rightarrow K_{oo}(t) \cdot L^{-1}(s) \tag{II.16}
\]

where \( K_{oo}(t) \) is the "form factor" of the spin-zero component of the infinite multiplet. \( \odot 11 \odot \) 6\)

\[
K_{oo}(t) \sim V_o^{(1)}(t) \sim P_{N_l}^l(\lambda_1, \lambda_3) \sim \frac{\sinh (N+1) t}{\sinh (N t)} \tag{II.17}
\]

\[
\cosh \phi = \lambda_1 \cdot \lambda_3
\]

2. "Strong" scattering with single particle spectrum

Next, consider the Feynman diagram of Fig. 2, in which all the lines represent particles belonging to infinite multiplets. In order that the formulae remain as simple as possible we shall assign all the external lines to the representation \( D(N) \) considered above, and the virtual states to \( D(2N) \). Then the vertices may be defined by the local invariant density

\[
\bar{\chi} \phi \phi = \bar{\chi}^{\mu_1 \cdots \mu_2N}(\chi) \phi_{\mu_1 \cdots \mu_2N}(\chi) \phi_{\mu_1 \cdots \mu_2N}(\chi) \tag{II.18}
\]

The amplitude is

\[
A_2 = \sum_k \bar{\phi}(k) \phi(k) \cdot \chi L^{-1} \bar{\chi} \phi(\lambda) \phi(\lambda_2) \tag{II.19}
\]
Again we consider the case when all external one-particle states are spinless, by making the substitution (II.5) for \( G(p_1), \ldots, G(p_4) \). For \( \chi_{\mu_1} \ldots \chi_{\mu_4} \) we write the obvious analogue of the expansion (II.7). Inserting these substitutions into (II.19) one obtains (details are given in the appendix):

\[
A_{2l} = \sum_{\ell=0}^{l} \sum_{\chi} V^{(2)}_{\ell}(p_1, p_4) Q_{\mu_1} \ldots Q_{\mu_4} \chi_{\mu_1} \cdots \chi_{\mu_4} \\
\times L_{\ell}^{(s)} \chi_{\nu_1} \ldots \chi_{\nu_4} Q^{(s)}_{\nu_1} \ldots Q^{(s)}_{\nu_4} V^{(2)}_{\ell}(s_1, s_2)
\]

(II.20)

where \( \hat{Q} = Q | Q^{-1} \), \( \hat{Q}' = Q' | Q'^{-1} \),

\[
2Q = p_1 - p_2 + s^{-1} (m_2^2 - m_3^2) (s_1 + s_2)
\]

(II.21)

\[
2Q' = p_4 - p_3 + s^{-1} (m_4^2 - m_5^2) (s_3 + s_4)
\]

and

\[
V^{(2)}_{\ell}(p_1, p_2) = (s^2 - \delta_{12}) \frac{iN}{4\pi} \sum_{t=1, l=2, \ldots} b_{\ell, t} \nu_{12}^{t-\ell}
\]

(II.22)

The coefficient is the polynomial

\[
b_{\ell, t} = \left[ (t-\ell)! (t+\ell+1)! \right]^{-1} P_{\ell-N} (\delta_{12} / s)
\]

(II.23)

with

\[
\delta_{12} = m_2^2 - m_3^2
\]
and the centre-of-mass "mean" velocity is given by

\[ \nu^2 = \frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2} / \left( s^2 - s \right) \]  \hspace{1cm} (II.24)

The coefficient \( b_{l,t} \) simplifies in the equal mass case:

\[ b_{l,t} \rightarrow \begin{cases} 0, & \text{if } t \text{ is odd} \\ \frac{(-)^l t (t-1)!! (2N-t)!! (t-l)!! (t+l+1)!!}{l!! (l-2l+1)!!}, & \text{if } t \text{ is even} \end{cases} \]  \hspace{1cm} (II.25)

in this case (II.22) is a simple hypergeometric series. \(^{124} \)

The sum over the intermediary states is evaluated just as before, and the final result is

\[ A_l = \sum_{l}(2l+1) P^l (\cos \theta) \frac{\psi^{(2)}_l (s)}{(s)^{l}} \]  \hspace{1cm} (II.26)

\[ \frac{\psi^{(2)}_l (s)}{(s)^{l}} = \frac{(l+2N+1)!!}{(l-2N-1)!!} \frac{V^{(2)}_l (\theta, \phi)}{V^{(2)}_l (\theta, \phi)} \]  \hspace{1cm} (II.27)

3. "Compton" scattering with multiparticle structure

Both examples studied so far suffer from the defect that the intermediary states are interpretable only as single-particle states, because every partial wave passes through a single intermediary state. It is possible to enlarge the set of intermediary states so that multiparticle intermediary states can be taken into account exactly.

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In Subsection II.1, let the group SO(3,1) be replaced by SO(4,1), and consider the irreducible representation \(D(N)\) of SO(4,1) that is realized on the generalized tensor field \(\varphi_{i_1\cdots i_N}\). The arguments \(x = x_0, x_1, x_2, x_3\) are the usual four-dimensional space-time coordinates, but the indices now take on the values 0, 1, 2, 3 and 4. The representation is unitary if \((N - \frac{3}{2})^2 < \left(\frac{3}{2}\right)^2\); we shall take \(N\) real with

\[-\frac{3}{2} < N < 0\]  

Then \(D(N)\) belongs to the supplementary series of unitary representations of SO(4,1).

The physical interpretation of the states of this SO(4,1) multiplet may be clarified most easily in terms of a basis that is adapted to the compact subgroup. Thus, let \(\lambda\) be a "timelike" unit 5-vector, for example \(\lambda = (1,0,0,0,0)\), and decompose the tensor \(\varphi_{i_1\cdots i_N}\) according to the subgroup \(G_\lambda\) — isomorphic to SO(4) — that leaves \(\lambda\) invariant. The decomposition is identical to (II.7),

\[
\varphi_{i_1\cdots i_N} = \mathcal{S} \sum_{n=0}^{\infty} \tilde{\varphi}_{i_1\cdots i_m} \sum_{t_2, t_3, \ldots} a^N_{m, t} \\
\times \Theta_{i_1 + t, \ldots} \Theta_{i_m, \ldots} \lambda_{i_1, \ldots} \lambda_{i_N}
\]  

except that the coefficients \(a^N_{n, t}\) are slightly different in this case (see appendix). Here \(\tilde{\varphi}_{i_1\cdots i_m}\) is symmetric, traceless and transverse to \(\lambda\):
For every \( n \), the tensor \( \mathcal{F}_{i_1 \ldots i_m} \) has \((n + 1)^2\) independent components that form the basis vectors for an irreducible representation of \( SO(4) \) — or \( G_\lambda \), to be more precise. The spins of these states are found by reducing \( \mathcal{F} \) according to the rotation subgroup of \( SO(4) \); for each \( n \) the values of \( l \) are \( 0, 1, \ldots, n \). The basis vectors are thus labelled by \( n, l, l_z \), with \( n = 0, 1, \ldots, l = 0, 1, \ldots, n \) and \( l_z = -l, -l + 1, \ldots, l \). This is the familiar structure of the bound states of the Schrödinger hydrogen atom. However, the basis vectors just introduced are not the eigenstates of the Hamiltonian. We now introduce a basis of physical states; then it turns out that both scattering and bound states can be accounted for.

The spin \( \lambda \) of a state with four-momentum \( \mathbf{p} \) is defined covariantly in terms of the little group \( G_\lambda \) — the subgroup of \( SO(3,1) \) that leaves \( \mathbf{p} \) invariant:

\[
\gamma^2 \ell(\ell + 1) = W_\mu W^\mu, \quad W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \gamma^\nu \gamma^\rho
\]

It is necessary to define the quantum number \( n \) covariantly, and this is done in a similar fashion. For every state of four-momentum \( \mathbf{p} \) choose a unit five-vector \( \lambda = \lambda(\mathbf{p}) \), \( \lambda^2 = \ell \), and let \( G_\lambda \) be the subgroup of \( SO(4,1) \) that leaves \( \lambda \) invariant. Then let \( n = n(\mathbf{p}) \) be defined by

\[
\lambda^2 m(m + 2) = \frac{1}{2} W_{ij} W^{ij}, \quad W_{ij} = \frac{1}{2} \epsilon_{ij\kappa\lambda} \lambda^\kappa\lambda^\lambda
\]

Thus \( n(n + 2) \) is the Casimir operator of the stability group \( G_\lambda \) of \( \lambda(\mathbf{p}) \). If \( \lambda(\mathbf{p}) \) is "timelike", \( \lambda^2 = 1 \), then \( G_\lambda \) is isomorphic to \( SO(4) \) and \( n \) takes the values \( 0, 1, \ldots \); but if \( \lambda(\mathbf{p}) \) is spacelike, \( \lambda^2 = -1 \), then \( G_\lambda \) is isomorphic to \( SO(3,1) \) and the spectrum of \( n \) is continuous except for at most one discrete point.
It is clear that the little group $G_\phi$ must be a subgroup of $G_\lambda(\mathfrak{g})$, therefore $\lambda(\mathfrak{g})$ must have the form

$$
\lambda(\mathfrak{g}) = (c, \xi, \zeta) \quad , \quad c^2 \xi^2 - \zeta^2 = 1 . \quad (I I . 3 1)
$$

where $c$ and $\xi$ are real numbers that may depend on the mass $p^2$. The spectrum of $n$ depends on $p^2$:

$$
n = 0, 1, \ldots, \text{if } c^2 \xi^2 - \zeta^2 = +1 \quad (I I . 3 2)
$$

$$
n = \text{continuous, if } c^2 \xi^2 - \zeta^2 = -1
$$

but except for this restriction the quantum numbers $p_\mu$, $n$ and $\ell$ are independent. Since $n$ and $\ell$ is a complete set of quantum numbers for the states of momentum $p_\mu$, each physical state must have a mass

$$
m = m(n, \ell) \quad \text{that is determined by } n \text{ and } \ell . \quad \text{That is, the mass shell is defined by a relation of the form}
$$

$$
L(p^2, n, \ell) = 0 . \quad (I I . 3 3)
$$

On the other hand, the physical states are defined by the singularities of the propagator $L^{-1}$. Since this is Poincaré invariant it can be diagonalized by taking $n$ and $\ell$ diagonal, so that $L^{-1}$ reduces to $L^{-1}_{n,\ell}(p^2)$. Clearly

$$
L^{-1}_{n,\ell}(p^2) = [L(p^2, n, \ell)]^{-1} . \quad (I I . 3 4)
$$

The physical states with real mass are thus defined by the propagator $L^{-1}$ in the following way: For every positive value of $p^2$ one calculates $\lambda(p)$ and determines the spectrum of $n$. Then one searches for solutions of Eq. (II.33), keeping $p^2$ fixed and allowing $n$ to vary over this spectrum. With an appropriate choice of the functions $c$ and $\xi$ in (II.31) it is possible to obtain a physical mass spectrum that is partly continuous. In particular, it is possible
to obtain a spectrum that coincides with that of the Schrödinger hydrogen states, including the electron-proton scattering states, by taking

$$c = \left[ \frac{\alpha}{\hbar} \left( \psi^2 - M^2 \right) \right]^{-\frac{1}{2}}, \quad \lambda = \left[ \psi^2 (1 + \frac{\alpha^2}{\hbar^2}) - M^2 \right]^{-\frac{1}{2}}$$

(II.35)

and

$$\mathcal{L} \psi, m, \xi = \left( m + 1 \right) \sqrt{M^2 - \xi^2} - e^2 \sqrt{M^2}$$

(II.36)

where $\mu$ and $M - \mu$ are the electron and proton masses.

Now let us calculate the scattering amplitude that corresponds to the Feynman diagram of Fig. 1, when the straight lines represent the states of the $SO(4,1)$ irreducible multiplet, and the wavy lines are conventional scalar bosons, described by the field $A(x)$. The vertices are defined by the local invariant density

$$\overline{\psi} \psi A = \overline{\psi} \psi^\nu(x) \psi_{\nu}^\mu(x) A(x)$$

(II.37)

and the amplitude is

$$A_1 = \sum \overline{\psi} (\gamma_3) \psi (\gamma_3) \cdot \frac{\overline{\psi}}{\psi} \cdot \psi (\gamma_3) A (\gamma_3)$$

(II.38)

We shall assume that the physical states include at least one discrete state, with $n = \lambda = 0$, and we shall calculate the elastic "Compton" scattering amplitude from this state. Thus

$$\psi_{i_1 \cdots i_\nu} (\gamma_1) \rightarrow \lambda_{i_1 \cdots i_\nu}$$

(II.39)

$$\psi_{i_1 \cdots i_\nu} (\gamma_3) \rightarrow \lambda_{i_1 \cdots i_\nu}$$

where, for example,
\[ \lambda_i = (\mathcal{C}_i \delta_i, \mathcal{N}_i) \]  
\[ \mathcal{C}_i = \mathcal{C}(\mathcal{N}_i), \quad \mathcal{N}_i = \mathcal{N}(\mathcal{N}_i) \]  

Let \( \mathcal{Z} \) and \( \mathcal{Z}_2 \) be defined by \( \mathcal{Z}_2 = \mathcal{Z}_1 + \mathcal{Z}_2 = \mathcal{Z}_3 + \mathcal{Z}_4 \), and \( \mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2) \). Then

\[ \lambda = (\mathcal{C}, \mathcal{N}), \quad \mathcal{C}^2 - \mu^2 = 1 \]  

\[ \mathcal{C} = \mathcal{C}(\mathcal{s}), \quad \mu = \mathcal{N}(\mathcal{s}), \quad \mathcal{s} = \mathcal{P}^2. \]

To begin with, consider values of \( \mathcal{s} \) such that \( \lambda_i \) is "timelike", \( \lambda_i^2 = +1 \), and \( \mathcal{n} = \mathcal{n}(\mathcal{Z}) \) has the discrete spectrum \( 0, 1, \ldots \). Then (II.29) is an expansion of \( \mathcal{Y} \) into a complete set of off-mass-shell states. Substituting (II.29) and (II.39) into (II.38), and assuming hydrogen-like degeneracy

\[ L_{n, \lambda}^{-1} = L_{n}^{-1}, \quad \mathcal{l} = 0, 1, \ldots, \mathcal{n} \]  

one may repeat the calculation from (II.9) to (II.15). There is only a slight change of the coefficients \( a_{n, \mathcal{t}}^{\mathcal{H}} \), due to the enlargement of the group \( \text{SO}(3,1) \) to \( \text{SO}(4,1) \). The appendix gives formulae valid for \( \text{SO}(\mathcal{f}, 1) \). The result is

\[ A_{n}(s, t) = \sum_{n = 0, 1, \ldots} (2n + 2) P_{n, 4}(\mathcal{s}) f_{n}(t) \]  

(II.43)

The "partial" amplitude is

\[ f_{n}^{(\mathcal{l})} = (\mathcal{l} - \mathcal{N} - 1)!(\mathcal{l} + \mathcal{N} + 2)! P_{n, \mathcal{S}}^{-\mathcal{m}}(\lambda_{2}, \lambda_{1}) L_{n}^{-\mathcal{l}}(s) P_{n, \mathcal{S}}^{-\mathcal{m}}(\lambda_{1}, \lambda_{2}) \]  

(II.44)
and the variables $Z$ and $\omega_i$ are

$$Z = \left[ (\lambda_1 \cdot \lambda_2 - \lambda_3^2 (\lambda_1 \cdot \lambda_3)) \right] [\lambda_1 \cdot \lambda_2 - \lambda_3^2]^{\frac{1}{2}} [\lambda_3^2 - \lambda_3^2]^{\frac{1}{2}}$$

(II.45)

$$\omega_i^2 = 1 - \lambda_i^2 / (\lambda_1 \cdot \lambda_2)^2$$

(II.46)

If $L_n^{-1}$ is independent of $n$, then the sum in (II.43) is an addition formula for 5-dimensional spherical functions, and

$$A_{14}(s, t) \rightarrow K_{00}(t) \cdot L_n^{-1}(s)$$

(II.47)

where $K_{00}(t)$ is the form factor for the ground state $n = 0$. When $\lambda^2 = +1$, then $|Z| < 1$ and the series (II.43) converges inside an ellipse in the complex $Z$-plane, provided the function $L_n^{-1}(s)$ does not increase too fast as $n \rightarrow \infty$. When $\lambda^2 = -1$, then $Z > 1$, and the series does not converge unless $L_n^{-1}(s)$ decreases unreasonably fast as $n \rightarrow \infty$. Hence (II.43) is not applicable to the case of spacelike $\lambda$.

To obtain an expression for the amplitude that is valid for values of $P$ for which $\lambda$ is spacelike, one may perform a Sommerfeld-Watson transformation with respect to the variable $n$. When $L_n^{-1}$ is independent of $n$ this can be rigorously justified: The expansion (II.43) expresses a spherical function for the
representation $D(N)$ of $SO(4,1)$ in terms of the spherical functions of the unitary representations $D(n)$ of the subgroup $G\lambda$, which is isomorphic to $SO(4)$ when $\lambda$ is timelike. There exists another expansion of the same function, in terms of the spherical functions of the unitary representations of $SO(3,1)$ or, more generally, the subgroup $G\lambda$ with spacelike $\lambda$. The unitary representations of $SO(3,1)$ that occur correspond to the ranges

$$n = -1 + i\rho, \quad \rho = \text{real, and } -1 < n < 0$$ \hspace{1cm} (II.49)'

and the expansion can be written as an integral over these ranges of $n$. Furthermore, this integral converges for $\lambda^2 = \pm 1$, and may be related to the sum (II.43) by a Sommerfeld-Watson transformation when $\lambda^2 = -1$.\hspace{1cm} (10) We now postulate that the properties of $L_n^{-1}(s)$ are such as not to upset the possibility of performing the transformation, except that poles of $L_n^{-1}(s)$ may lie in the path of the deformation of the contour, and have to be taken into account.

The substitution of the sum in (II.43) by an integral over a hairpin contour, and the subsequent deformation of the contour are completely straightforward. For definiteness let us suppose that $L_n^{-1}(s)$ has the form

$$L_n^{-1}(s) = g_n(s) / [n - \alpha(s)]$$ \hspace{1cm} (II.50)'

where $g_n(s)$ is analytic and sufficiently well behaved at infinity when $\text{Re} n > 1$. The function $g_n(s)$, together with most of the s-dependent or constant factors in (II.44), may be lumped together in the function

$$\beta_m(s) \equiv \frac{n+1}{\lambda m n m} \frac{(n+N+2)!(m-N)!}{(m-N)!} \times g_m(s) \mathcal{P}_{N_1}^{-m}(\lambda_1, \lambda) \mathcal{P}_{N_2}^{-m}(\lambda_2, \lambda)$$ \hspace{1cm} (II.51)'

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Then we obtain

\[ A_1 = (2\pi i)^{-1} \int_{-1-\infty}^{-1+i\infty} \frac{\beta_n(s)}{(s-n)[s-\alpha(s)]} P_m \phi(-2) \]

\[ + \frac{\beta_n(s)}{\alpha(s)-N} P_{\alpha(s),N}(-2) \]

\[ + \frac{\beta_n(s)}{N-\alpha(s)} P_N(-2) \quad (11.52) \]

The second term is due to the pole of the propagator at \( n = \alpha(s) \), and occurs if \( \text{Re} \, \alpha(s) > -1 \) only. The third term is due to the pole of \( \mathcal{C}_n^{(1)} \) at \( n = N \), and occurs if \(-1 < N < 0 \) only. \(^{15}\)

Finally, a formula that is valid for \( \lambda^2 = -1 \) is obtained by eliminating \( \lambda \) in favour of \( s \), and continuing the right-hand side of (11.52) to values of \( s \) such that \( \lambda^2 = -1 \). To follow the analytic continuation in detail, let us assume that the spectrum consists of an infinite set of one-particle states (bound states), with masses that increase as \( \alpha(s) \) take the values 0, 1, 2, ..., and reach an accumulation point at \( s_0 \) as \( \alpha(s) \to \infty \), plus a continuum of two-particle states (ionized states) from \( s = s_0 \) to infinity. Then, as \( s \to s_0 \) from below, \( \alpha(s) \to \infty \) along the real axis, while for \( s > s_0 \), \( \alpha(s) \) is on the line \( \text{Re} \, \alpha(s) = -1 \). Thus \( \alpha(s) \) has a branch-point at the threshold \( s = s_0 \). The physical sheet, cut from \( s = s_0 \) to \( +\infty \) corresponds to \( \text{Re} \, \alpha(s) > -1 \); the upper (lower) side of the cut corresponds to \( \text{Im} \, \alpha(s) > 0 \) (\( < 0 \)). Write \( n = -1 + i\rho \) for \( \text{Im} \, n > 0 \) and \( n = -1 - i\rho \) if \( \text{Im} \, n < 0 \), then the first term in (11.52) takes the form

\[ \int_0^\infty d\rho \frac{\mathcal{I}(\rho, s, t^2)}{\rho^{\alpha + \left( \alpha(s) \right)^2}} \]

This is very similar to a dispersion integral. The mapping \( s \to \left[ 1 + \alpha(s) \right]^2 \) maps the physical sheet of complex \( s \) on the...
complex plane cut from 0 to \(-\infty\). The bound state poles are contained in the second term in (II.52), in the factor \(\csc \alpha(s)\) in \(\beta \alpha(s)\). Other singularities, among them the anomalous thresholds, are contained in the \(s\)-dependence of \(\beta_n(s)\). If the propagator is that given by (II.36), and if \(\xi\) and \(\kappa\) are given by (II.35), then in the nonrelativistic limit \(\kappa \to \infty\) the above amplitude satisfies exact 2-particle unitarity.

III. INFINITE MULTIPLET EXCHANGE

1. "Compton" scattering with \(u > 0\)

Here we shall calculate the amplitude that corresponds to the Feynman diagram of Fig. 3. When the momentum transfer \(p_1 - p_2\) is timelike — that is, when

\[
(m_1 - m_2)^2 > u > 0
\]  

(III.1)

then this can be done by repeating the calculation of Section II.1.

In the expansion (II.7) we now have

\[
\lambda = (p_1 - p_2)u^{\frac{2}{3}}
\]  

(III.2)

instead of (II.8). In the power series (II.10) \(V_1\) is still defined by (II.12), but in (II.16) \(s\) is replaced by \(u\):

\[
V_1^2 = \left[ u - (m_1 + m_2)^2 \right] \left[ u - (m_1 - m_2)^2 \right] \left[ u + m_1^2 - m_2^2 \right]^{-2}
\]  

(III.3)

When \(u\) is in (III.1),

\[
0 < V_1^2 < 1;
\]  

(III.4)

Hence the power series (II.10) is convergent in this domain, and the results (II.13), (II.14) are valid after \(s\) is replaced by \(u\):

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This is not a partial wave expansion of the amplitude $A_3$. It looks like the partial wave expansion of the u-channel amplitude, continued analytically from $u > (m_1 + m_2)^2$ to the range (III.1), and in fact the amplitudes $A_1$ and $A_3$ are related by conventional s-u crossing, provided the propagator is sufficiently analytic. This is because the particles that are interchanged in s-u crossing are described by conventional local fields. Below, it will be shown that this normal analytic behaviour extends to negative $u$.

The annihilation amplitude $A_4$ associated with the Feynman diagram of Fig. 4 can be calculated in exactly the same way when $u$ is positive, and the answer is given by precisely the same formulae (III.5) and (III.6). Nevertheless, $A_4$ is not related to $A_3$ by the usual analytic continuation in the invariant kinematical variables. The reason for this is that $\lambda_3$ is not an analytic function of $\hat{q}_3$.

Let us take $L^{-1}_I(u)$ independent of $I$, then the sums that define $A_3$ and $A_4$ can be summed exactly to (compare (II.17)):

$$A_3 = K_{00}(t) L^{-1}_I(u), \quad A_4 = K_{00}(s) L^{-1}_I(u)$$

where

$$K_{00}(t) \sim \mathcal{P}_{\lambda_1, \lambda_2} (\lambda_1, \lambda_2), \quad \lambda_1 \cdot \lambda_2 = \frac{m_1^2 + m_2^2 - t}{2m_1 m_2}$$

$$K_{00}(s) \sim \mathcal{P}_{\lambda_1, \lambda_2} (\lambda_1, \lambda_2), \quad \lambda_1 \cdot \lambda_2 = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}$$

-20-
The two amplitudes are identical when expressed in terms of \( \lambda_1 \) and \( \lambda_3 \), but these vectors, being parallel to the physical momentum vectors, do not change sign under s–t crossing, and \( \lambda_1 \cdot \lambda_3 \) is not crossing invariant. This unconventional crossing behaviour of form factors has been discussed in more detail previously.\(^{16}\)

2. **"Strong" scattering with \( u > 0 \)**

We now turn to the evaluation of the amplitude \( A_5 \) for the Feynman diagram of Fig. 5. The amplitudes \( A_2, A_5 \), unlike \( A_1, A_3 \), are not related by conventional s–u crossing. In the range (III.1) the calculation may be carried out precisely as in Section II.2; one notices that \( \nu_{12}^2 \), defined by (II.24), varies from 0 to -1, so that the series (II.22) converges. The result is given by (II.26), (II.27) after \( s \) is replaced by \( u \) throughout. The breakdown of conventional crossing symmetry occurs in the same way as in the preceding discussion of the annihilation amplitude \( A_4 \).

3. **"Compton" scattering with \( u < 0 \)**

The expansions (II.13) and (III.5) are representations of the amplitudes \( A_1 \) and \( A_3 \) as sums over the contributions of irreducible representations of the little group \( G_\lambda \); that is, over the spin of the virtual state. In both cases \( \lambda \) was timelike, \( G_\lambda \) was compact and the reduction (II.7) gave a discrete sum over spins. Turning now to the case of spacelike momentum transfer, we meet with the difficulty that (II.7) should be replaced by an integral over \( \lambda \), since the range of \( \lambda \) is now continuous. It is doubtful that the tensor method can be pushed as far as to give a useful formula of this type. An alternative is to begin by expanding the tensor \( \gamma_{\mu_1 \cdots \mu_n} \) according to a discrete set of non-unitary representations of \( G_\lambda \), with \( n = N, N-1, \ldots \). However, since we already have an expansion for the amplitude \( A_3 \) for \( u > 0 \), the simplest
procedure is to continue $A$ analytically to negative $u$ by means of a Sommerfeld-Watson transformation. This is the same procedure that was followed in Section II.3. As in that case, we note that the transformation is certainly possible, barring complications due to unreasonable properties of the propagator.

For definiteness, let us suppose that $L^{-1}_A(u)$ has the form

$$L^{-1}_A(u) = G(u) \sqrt{[\lambda - \alpha(u)]}$$  \hspace{1cm} (III.9)

where $G(u)$ is analytic in $\text{Re} \lambda > -\frac{1}{2}$ and sufficiently well behaved as $\lambda \to \infty$. The physical mass spectrum is given by the solution of

$$\alpha(m_{\lambda}^2) = \lambda, \quad \lambda = 0, 1, 2, ...$$

and $\alpha(u)$ for negative $u$ defines a Regge trajectory. Let us introduce the abbreviation

$$\beta_\Lambda(u) = \frac{2\ell + 1}{2\pi i \ell} (\ell + N + 1) ! (\ell - N) !$$

$$\times G_{\Lambda}(u) \frac{\mathcal{P}^{-\ell}_{N+1} (\lambda; \lambda)}{\mathcal{P}^{-\ell}_{N+1} (\lambda; \lambda)}$$  \hspace{1cm} (III.10)

Then the result of a completely straightforward Sommerfeld-Watson transformation of (III.5) is

$$A = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} d\ell \beta_\Lambda(u) \frac{1}{(\ell - N)[\ell - \alpha(u)]} \mathcal{P}_{\ell} (-\cos \theta_u)$$

$$+ \beta_\Lambda(u) \frac{1}{\alpha(u - N)} \mathcal{P}_{\alpha(u)} (-\cos \theta_u)$$

$$+ \beta_N(u) \frac{1}{N - \alpha(u)} \mathcal{P}_N (-\cos \theta_u)$$  \hspace{1cm} (III.11)
The third term is the contribution of the first of a string of fixed poles \(^{17}\) of the partial amplitude, at \(J = N, N-1, \ldots\). In the approximation \(L^{-1}_\lambda(s) = L^{-1}(s)\) the first and the third term add up to a four-dimensional spherical function and we recognize the Lorentz pole amplitude of Toller. Indeed it is not surprising that the amplitude for the exchange of an irreducible representation of the Lorentz group, in the equal mass case, should be the same as a Lorentz pole contribution. The second term in (III,11) is a dynamical Regge pole contribution. Notice that the Regge pole moves with \(u\), while the Lorentz pole is fixed. The only interdependence of the two is that their residues are equal in magnitude but of opposite sign in the case of coincidence, which occurs at the value of \(u\) for which \(\kappa(u) = N\). Thus for \(\kappa(u)\) near \(N\) the amplitude is dominated by a "Regge dipole" with finite moment. \(^{18}\)

IV. APPLICATIONS

1. Multiparticle states

With the inclusion of multiparticle states in infinite multiplets we hope that a principal obstacle to genuine physical applications has been removed. It is known that an efficient treatment of the non-relativistic Coulomb problem is feasible \(^{13}\); probably the relativistic pion-nucleon problem should be attempted next. BARUT and KLEINERT \(^{19}\) considered the \(I = \frac{1}{2}\) states, and found that a group larger than \(SO(3,1)\) seems to be needed, because there are several isobars with the same spin and parity. Here we have pointed out that the representation of multiparticle states also requires large multiplets. Perhaps \(SO(4,1)\) or \(SU(2,2)\) is applicable to this problem—it is not hard to invent a propagator that corresponds to a single discrete state (the nucleon) and a continuum (nucleon + several pions). It seems natural to expand the group in another direction as well, to include isospin and strangeness and perhaps even \(SU(6)\). This can be done by taking \(G = SL(6,C)\) or \(SU(6,6)\). One of the objections against these groups has been the absence of experimental evidence for resonances with high isospin, but if
continuous mass spectra are introduced then no resonances are needed. Multiparticle states with arbitrarily high isospin certainly occur.

2. **Exchange amplitudes, near \( u = 0 \) especially**

Recently the behaviour of scattering amplitudes near \( u = 0 \) has been studied by many authors \(^{20}\). In particular, it has been pointed out \(^{21}\) that the customary analytic continuation from one channel to another can give an inconvenient representation of a scattering amplitude, due to the wrong or incomplete separation of the kinematical factors. Thus, in the \( u \)-channel, in which the partial wave expansion is written down, kinematical factors appear which guarantee the correct threshold behaviour, and there is reason to hope that the partial wave amplitudes are smooth functions of \( u \). After analytic continuation to the \( s \)-channel, these factors appear in the individual Regge pole contributions to the amplitude. Applications to experimental data have shown that, in this case, the rapid variation of the kinematical factors has to be cancelled by rapidly varying Regge residues. The confusion that has reigned recently concerning the analytic structure of unequal mass scattering amplitudes near \( u = 0 \), is perhaps partly due to an inconvenient treatment of kinematical factors. The amplitudes calculated in this paper may be regarded as models of Regge pole exchange. As has been stressed repeatedly, the "vertex functions" must be regarded as being essentially kinematical. Now we suggest that they are of a form that combines the correct threshold behaviour in one channel with the correct behaviour near \( u = 0 \) in the other.

Consider the simplest case, the exchange of a minimal multiplet in "Compton" scattering. The scattering amplitude corresponding to the Feynman diagram of Fig. 3 was found to be given by the \( u \)-channel partial wave series

\[
A_1(s,t) = \sum_{\ell=0,1,2,\ldots} (2\ell+1) P_{\ell}(\cos \theta_u) f_\ell^{(1)}(\kappa) \quad (IV.1)
\]
The series converges in both the relevant regions, \( u > (m_1 + m_2)^2 \) and \( 0 < u < (m_1 - m_2)^2 \), but not uniformly near \( u = 0 \). The partial wave is

\[
\mathcal{f}_k^{(1)}(u) = (\lambda - N - 1)! (\lambda + N + 1)! L_k^{1}(u) \\
\times P_{N,\frac{1}{2}}^{-\ell} \left( \frac{\delta_{12} + u}{2m_1 \sqrt{u}} \right) P_{N,\frac{1}{2}}^{-\ell} \left( \frac{\delta_{12} - u}{2m_2 \sqrt{u}} \right)
\]

(IV.2)

In the \( u \)-channel, near the threshold \( u = (m_1 + m_2)^2 \),

\[
\frac{\delta_{12} + u}{2m_1 \sqrt{u}} \approx 1 + \frac{m_2}{2m_1} \left( \frac{u}{(m_1 + m_2)^2} - 1 \right)
\]

(IV.3)

Thus (IV.2) contains the correct threshold factor; however, in other regions of the variable \( u \), the spherical function behaves quite differently. Next, consider the positive neighbourhood of \( u = 0 \). Here, provided \( N > -1 \),

\[
P_{N,\frac{1}{2}}^{-\ell} \left( \frac{\delta_{12} + u}{2m_1 \sqrt{u}} \right) \approx \frac{1}{(N+\ell+1)!} \left( \frac{u}{(m_1 + m_2)^2} - 1 \right)^{\frac{1}{2} \ell}
\]

and

\[
\Lambda_1(s,t) \big|_{u=0} \approx u^{-N} \sum_{\ell} (2\ell + 1) \left( \frac{\ell-N-1}{(\ell+N+1)!} \right) L_\ell^1(u) P_{\ell}^{1}(\cos \theta_u)
\]

(IV.4)

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In the equal mass case, when $L^{-1}_1(u)$ is independent of $\lambda$, the sum is equal to

$$L^{-1}(u) (1 - \cos \theta_u)^N \approx L^{-1}(0) u^N (a+bs)^N \quad (IV.5)$$

where $a$ and $b$ are constants depending on the masses. The singularity of the factor $u^{-N}$ in (IV.4) is exactly cancelled by the factor $u^N$ in (IV.5). Thus it is seen that, in the case of trivial dynamics, the kinematical structure does not give a singularity at $u = 0$. In fact we knew this already since in the degenerate case the series (IV.1) was summed exactly to (II.16).

Finally, let us introduce non-trivial dynamics. If the function $L^{-1}_1(u)$ does not have a singularity at $u = 0$, then the analytic structure of the sum (IV.4) is given by the behaviour of the summand for large $\ell$. We may therefore approximate $L^{-1}_1(u)$ by

$$L^{-1}_1(u) \approx \frac{\ell}{\ell - \alpha(u)} , \text{ for large } \ell .$$

Since $\alpha(u)$ is of the order of 1 when $u \ll 0$, the sum behaves like

$$\sum_{\ell} (2\ell+1) \frac{(\ell-N-1+\frac{\sigma-1}{2})!}{(\ell+N+1+\frac{\sigma-1}{2})!} P_{\ell} (\cos \theta_u) \sim (1-\cos \theta_u)^{N+\frac{\ell}{2}-\frac{\sigma}{2}} \quad (IV.6)$$

Thus, near $u = 0$

$$A_2(s,t) \approx u^{\frac{1-\sigma}{2}} (a+bs)^{N+\frac{1}{2}-\frac{\sigma}{2}}$$

The analytic structure of $A_2(s,t)$ near $u = 0$ is determined by the analytic structure of the propagator function $L^{-1}_1(u)$ near $\ell = \infty$. Notice that this result implies a cancellation of singularities between the several terms in the Regge representation (III.11).
We are thus confirmed in the expectation that the vertex functions of local interaction between infinite multiplets gives rise to reasonable kinematical factors, not only near the normal threshold, but at other critical points as well. A more complete investigation of scattering involving particles with spin should be carried out.
The expansion of the generalized tensor $\mathcal{G}_{\mu_1...\mu_n}$ according to the compact subgroup $G_\lambda$ is given by (II.7), with the coefficient

$$a_{m,t}^n = \left[ (N-t)! (t-m)! (t+n+f-2)! \right]^{-1} \quad (A.1)$$

Here, and throughout the appendix, the formulae are derived for the group $SO(f,1)$ with arbitrary $f$, although $f = 3$ or 4 are the only cases of actual interest. The normalization is such that the invariant inner product is

$$(\mathcal{G}, \mathcal{G}') = \mathcal{G}^{\mu_1...\mu_n} \mathcal{G}'_{\mu_1...\mu_n}$$

$$= \sum_{m=0,1,...} \frac{m!}{(2m+f-2)! (m+N-1)!} \mathcal{G}^{\mu_1...\mu_n} \mathcal{G}'_{\mu_1...\mu_n} \quad (A.2)$$

which is the same as

$$\sum_{\tilde{P}} \tilde{G}^{\mu_1...\mu_n} \tilde{G}^{\nu_1...\nu_n} \tilde{t}_{\nu_1} ... \tilde{t}_{\nu_n}$$

$$= \delta_{m,n} (2m+f-2) \overline{P}_{m,f} (-\varrho \varrho') \frac{(m+N+f-2)!}{(m+N-1)!} \quad (A.3)$$

Here

$$\overline{P}_{m,f} = \sum_{s=m,N-2,...} (-)^{tm-ts} \frac{(m+s+f-4)!}{(m-s)! s!} \overline{Z}^s \quad (A.4)$$

are the $f$-dimensional Legendre polynomials. Eq. (A.3) holds for any pair $p,q$ of spacelike unit vectors that are orthogonal to $\lambda$. The $G_\lambda$ tensors $\tilde{G}$ are symmetric and traceless and transverse to $\lambda$, that is

$$\chi^{\mu} \tilde{G}_{\mu_1...\mu_n} = 0 \quad (A.5)$$
For arbitrary $f$, the vertex function (II.10) is

$$V_n^{(1)}(f_1 f_2) = (\lambda \cdot \lambda)^N \sum_{t \geq 0} a_x^t \nu_1^{t-m}$$

$$= (-1)^m \frac{(m-N-1)!}{(2m+f-2)!} (\lambda \cdot \lambda)^N \xi_1^m \left( \frac{m-N}{2}, \frac{m+N}{2}; i^{m+t}, \nu_1^{-2} \right)$$

where

$$\nu_1^2 = 1 - \lambda^2 / (\lambda \cdot \lambda)^2$$

The $f$-dimensional spherical functions are

$$P_{N+t+1} (\lambda \cdot \lambda) = \frac{(N+f-2)!}{N! (2m+f-2)!} (\lambda \cdot \lambda)^N \nu_1^m$$

$$\times \xi_1^m \left( \frac{m-N}{2}, \frac{m+N}{2}; i^{m+t}, \nu_1^{-2} \right)$$

$$= \frac{(N+f-2)!}{N!} \left[ (\lambda \cdot \lambda)^2 - 1 \right]^{t-f} P_{N+f+1} (\lambda \cdot \lambda)$$

When $f = 3$, then (A.6) and (A.8) yield the expression (II.11) for the vertex function $V_n^{(1)} (f_1 p_2)$.

Next, consider the "strong" vertex defined by (II.18).

The wave functions for the ground states ($n = 0$) of the $SO(f,1)$ tensors $\rho_{\mu}$... of the initial states are

$$\lambda_1 \mu_1 ... \lambda_1 \mu_N \quad \text{and} \quad \lambda_2 \mu_{n1} ... \lambda_2 \mu_{nN}$$

Let $\lambda$ and $\eta$ be two orthogonal vectors, with

$$\lambda^2 = 1, \quad \lambda \cdot \eta = 0$$
and define four numbers $a, b, c, d$ by

$$\lambda_1 = a\lambda + b\eta, \quad \lambda_2 = c\lambda + d\eta \quad (A.11)$$

This implies that $\lambda_1, \lambda_2$ and $\lambda$ are linearly dependent, which is guaranteed by momentum conservation if $f = 3$; for $f \neq 3$ it is a considerable loss of generality. Then

$$S \lambda_1 \lambda_2 = ac \lambda^2 + bd \eta^2 + (ad + bc)S \lambda \eta$$

$$= \alpha \lambda \eta + \beta \eta \lambda + \gamma S \lambda \eta$$

and thus

$$S [\lambda_1] \cdot [\lambda_2]^r = S \eta^2 \lambda_1 \eta^{r-2} \lambda_2$$

$$= S \sum_{r,s=0,1} (N)(N-r) \lambda^{n-r} \lambda^s \eta^{r} [\lambda]^{2n-r-2s} [\eta]^{2s} \quad (A.12)$$

This is to be multiplied by

$$\lambda_{m_1 \ldots m_r} = S \sum_{m=0,1} \ldots \lambda_{m_1 \ldots m_r} \sum_{\eta^{m_1 \ldots m_r}} a_{m_1 \ldots m_r}^{2n} \quad (A.13)$$

$$\times \theta_{m_1 \ldots m_r} \ldots \theta_{m_1 \ldots m_r} \lambda_{m_1 \ldots m_r} \eta^{m_1 \ldots m_r}$$

The symmetrization introduces a numerical factor, equal to the probability that the $2N-k \lambda$'s in $(A.12)$ hit the $2N-r-2s \lambda$'s in $(A.13)$. This factor is $(\frac{2N}{r})^{-1}$ if $r = r+2s$ and zero otherwise. The only terms that remain are those with $r = r+2s$.

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where

\[ V_{m}^{(2)}(\lambda_1, \lambda_2) = (\frac{-\alpha}{\beta})^{\lambda m} (Y^2 - 4\alpha\beta)^{\frac{1}{4}} \]

and

\[ b_{m,t} = \left( \frac{t}{\lambda m} \right)^{\lambda m} \left( \frac{t + \lambda m + \frac{1}{2} - 2}{1 + \frac{1}{2}} \right)^{-1} \right] \left( 1 - \frac{4\alpha\beta}{3} \right)^{-\frac{1}{4}} \]

In the case \( f = 3 \), which applies to Sec. II.2, we set

\[ \lambda_1 = \frac{p_1}{m_1}, \quad \lambda_2 = \frac{p_2}{m_2}, \quad \lambda = \frac{P}{(P^2)^{\frac{1}{2}}} \]

If we use the conservation law \( 2P = p_1 + p_2 \) and define \( s = (p_1 + p_2)^2 \), then

\[ \frac{\alpha}{\beta} = \frac{(s^2 - \Delta^2)}{4s}, \quad Y^2 = \frac{s^2}{m_1^2 m_2^2 s} \]

\[ b_{\alpha} = \frac{[s - (m_1 + m_2)]^2 [s - (m_1 - m_2)]^2}{(s^2 - \Delta^2)} \]

\[ \left[ 1 - 4\alpha\beta / s^2 \right]^{-\frac{1}{4}} = \frac{\delta}{s}, \quad \delta = m_1^2 - m_2^2 \]
If instead $2P = p_1 - p_2$, then these formulae remain valid if $s$ is replaced by $u = (p_1 - p_2)^2$. However, in this case (A.15) converges only if $0 < u < (m_1 - m_2)^2$. Substituting (A.18) into (A.15), (A.16) we obtain (II.22), (II.23) of the main text.

Finally, for the convenience of the reader, we write down the well known formula:

$$
\cos \Theta_3 = \frac{(s + m_1^2 - m_2^2)(s - m_1^2 + m_2^2) + 2s(m_2^2 + m_4^2) - t - u}{\left[ s - (m_1^2 + m_2^2) \right] \left[ s - (m_1^2 - m_2^2) \right] \left[ s - (m_2^2 + m_4^2) \right]^{1/2}}
$$
REFERENCES AND FOOTNOTES


4. Most earlier attempts in this direction have been based on unwarranted assumptions and are of limited applicability. The first paper was published by C. COCHO and H. AR-RASHID, Nuovo Cimento 47, 874 (1967). The methods of the present report have already been used in a model based on the harmonic oscillator, but the physical meaning of the results remained unclear because of the internal symmetry that was introduced there; see G. COCHO, C. FRONSDAL, I.T. GROESKY and R. WHITE, Phys. Rev., to appear.


7. We mean: it follows from Lorentz invariance.

8. A less trivial example is discussed in Section II.3.

9. Completeness of the physical states in the sense of physical unitarity depends on the mass spectrum. The sum in (II.4) is over a set of basis states that is complete with respect to the inner product of the unitarity group representation. The group invariant metric is $\sum_{\nu} \psi^\dagger \psi$, while the physical metric is $\sum_{\nu} \mathcal{S}(\text{Abs } L^{-1}) \psi^\dagger$. In the Dirac theory this latter quantity is the positive energy projection operator.
10. The precise definition of the hyperspherical functions used in this paper see Eq. (A.8) of the Appendix. A derivation of several relevant addition formulae may be found in C. Fronsdal, "On the supplementary series of representations of semi-simple non-compact groups", ICTP Internal Report 15/1967.


14. One may use (II.35) and interpret $K_0(t)$ as the form factor of the ground state of the hydrogen atom. The answer is not precisely the electric form factor, because our external field is a scalar, but the anomalous threshold occurs at the right place.

15. In the application to hydrogen the value is $N = -1$.


17. All the fixed poles disappear when $N = -1$, or if we use the principal, instead of the supplementary series of representations.

18. Regge multipoles have been introduced by R. Gatto, unpublished.


22. Convergence of the series (IV.6) requires that $N - \frac{3}{2} \sigma > -\frac{5}{4}$. 

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