KEPLER'S EQUATION, FOCK VARIABLES, BACRY'S GENERATORS AND DIRAC BRACKETS

G. GYÖRGYI

1967
PIAZZA OBERDAN
TRIESTE
KEPLER'S EQUATION, FOCK VARIABLES, BACRY'S GENERATORS
AND DIRAC BRACKETS *

G. Györgyi **

TRieste
June 1967

* To be submitted to Nuovo Cimento. A preliminary
version of Sec. 2 of this work formed the content of

** On leave of absence from Central Research Institute for
Physics of the Hungarian Academy of Sciences, Budapest,
Hungary.
ABSTRACT

A formulation of the Kepler problem, manifestly invariant with respect to the SO(4) and SO(3,1) groups, respectively, is given in terms of the Fock variables and their canonical conjugates; one is led to introduce a new time parameter, proportional to the eccentric anomaly. A transformation of the dynamical variables performed in order to get back the standard time $t$ leads in a natural way to Bacry's generators. A manifestly SO(4,2) invariant formulation of the problem is given. The concept of the Dirac bracket is used to establish a connection with the usual three-dimensional description.
KEPLER'S EQUATION, FOCK VARIABLES, BAGRY'S GENERATORS AND DIRAC BRACKETS

"Finally I have brought to light and verified beyond all my hopes and expectations that the whole Nature of Harmonies permeates to the fullest extent, and in all its details, the motion of the heavenly bodies; not, it is true, in the manner in which I had earlier thought, but in a totally different, altogether complete way."

Johannes Kepler, 1619

Although the Kepler problem has been one of the central themes in analytical dynamics for several centuries and treated in innumerable textbooks, in recent years it has received renewed attention because of its "internal" mathematical symmetries. The SO(4) invariance dynamical group of the problem (for bound states) has been identified by Fock, who investigated the problem of accidental degeneracy of the H atom by considering the mathematical properties of its Schrödinger equation in momentum space (see however C. Klein's remark in Ref. 2). A link between Fock's result and Pauli's masterly derivation of the H spectrum was subsequently established by Bargmann. (For the more-dimensional generalization of the problem see Refs. 5,6). Nearly another thirty years have elapsed, however, before these results have found full appreciation by the community of physicists (for notable exceptions see, however, Refs. 7-20), and started to influence theoretical thinking. This, as well as the subsequent recognition of the role of the non-invariance group SO(4,1) by Barut, Budini and Fronsdal, (see also Refs. 22-29), as well as that of SO(4,2) by Malkin and Man'ko, Barut and Kleinert, and Fronsdal,
was closely related to attempts and successes of the group-theoretical interpretation of the properties of hadrons. A number of theoreticians seem to share the hope that the problem of Keplerian motion (motion in the Coulomb field) which played an outstandingly fruitful role throughout the history of physics in the context of the discovery of the Newtonian laws of dynamics and gravitation, of the existence of the atomic nucleus and of the fundamental rules of quantum theory, and in offering tests of relativity theory and quantum mechanics, will also help to create a dynamics of hadrons.

In spite of the thorough treatment of the Kepler problem mentioned above, there are still a few questions to be raised which, in our opinion, deserve attention. It is sometimes pointed out that the Kepler problem is "less trivial" than the harmonic oscillator. In fact, while dynamical symmetries of the harmonic oscillator can easily be exhibited in terms of primitive dynamical variables, independently of the special choice of representation, for the Kepler problem (H atom) FOCK has exploited special properties of the Schrödinger equation in momentum space. Fock's transformation has the interesting property that in general it does not preserve probabilities. The question may be raised whether Fock's variables can be used as canonical variables, and what their canonical conjugates are. In fact it seems desirable to have a "manifestly SO(4) (or SO(3,1)) invariant formulation of the Kepler problem. The SO(4,1) non-invariance generators for the (classical) Kepler problem have been obtained for the first time in BAGCHY's pioneering work. In view of the relatively complicated form of his generators, as opposed to the simplicity of the bracket relations to be satisfied, one may have the feeling that there must be something behind his results. One may also mention that for positive energies BANDER and ITZKSON use SO(4,1) as non-invariance group, while in the work by MUKUNDA, O'RAIFERTAIGH and SUDARSHAN the group SO(3,2) is mentioned to play this role. Finally, in connection with recent discussion of the SO(4,2) group and the "purely group-theoretical" description of the dynamics of the H atom, one may also desire to have a "manifestly SO(4,2) invariant" formulation of the Kepler problem.
In the present paper we limit ourselves to the classical Kepler problem. In Sec. 1 the intimate relation between the $\text{SO}(4)$ invariance and the derivation of Kepler's equation of time is pointed out. In Sec. 2 a manifestly $\text{SO}(4)$ (or $\text{SO}(3,1)$) invariant formulation of the problem is given in terms of the Fock variables and their canonical conjugates; one is led to introduce a new time parameter, proportional to the eccentric anomaly. Non-invariance ($\text{SO}(4,1)$, $\text{SO}(3,2)$ and $\text{SO}(4,2)$) generators are also constructed in terms of the Fock variables. In Sec. 3 a transformation is performed in order to get back the standard time $t$. BACRY's generators\(^{22}\) are obtained in a natural way and a manifestly $\text{SO}(4,2)$ invariant (purely group-theoretical) formulation of the problem is given. In Sec. 4 the concept of the Dirac bracket is used to establish a connection between the description in terms of the four-dimensional variables simply related to the BACRY generators ('Bacry variables') and that in terms of the standard three-dimensional variables $x$, $p$. Although the treatment presented here is purely classical, it is very tempting to assume that this approach might prove interesting also in the quantum mechanics.

1. **FOUR-DIMENSIONAL SYMMETRY AND KEPLER'S EQUATION**

Given a sphere

$$\xi_{1}^{2} + \xi_{2}^{2} + \xi_{3}^{2} + \xi_{4}^{2} = a^{2} \quad (1.1)$$

in four-dimensional Euclidean space, a main circle on it may be represented in the parametric form

$$\xi_{\lambda}(\omega) = \alpha_{\lambda} \cos \omega + \beta_{\lambda} \sin \omega \quad (0 \leq \omega < 2\pi) \quad (1.2)$$

the four-vectors $\alpha_{\lambda}$, $\beta_{\lambda}$ are of the length $a$ and perpendicular to each other *:

* Greek (lower case) indices assume the values 1, 2, 3, 4. Summation over repeated indices is understood.
Instead of $\alpha_\lambda$, $\beta_\lambda$, this main circle may also be characterized by the second rank antisymmetric tensor

$$
\Phi_{\alpha\lambda} = \alpha_\alpha \beta_\lambda - \alpha_\lambda \beta_\alpha.
$$

The following relations are valid:

$$
\varepsilon_{\eta\zeta\sigma\tau} \Phi_{\eta\zeta} \Phi_{\sigma\tau} = 0,
$$

$$
\varepsilon_{\lambda\zeta\sigma\tau} \Phi_{\zeta\sigma} \Phi_{\tau} = 0.
$$

The triplets $\Phi_{23}$, $\Phi_{31}$, $\Phi_{12}$ and $\Phi_{14}$, $\Phi_{24}$, $\Phi_{34}$ behave under rotations in the three-dimensional subspaces of the $\xi^\alpha (\xi_1, \xi_2, \xi_3)$ as three-vectors. For these the notations $\Lambda = \frac{1}{2} \varepsilon_{ijk} \Phi_{jk}$, $\Omega = \Phi_{ij}$, will be used*. In this three-dimensional notation eq. (1.5) assumes the form

$$
\Lambda \Phi = 0;
$$

Eq. (1.6) yields the relations

$$
\Omega \times \xi - \Lambda \sqrt{\alpha^2 - \xi^2} = 0,
$$

$$
\Lambda \xi = 0
$$

* Latin (lower case) indices assume the values 1, 2, 3.
Eq. (1.9) expresses the fact that the curve obtained by projecting the circle (1.2) on the subspace of the vectors \( \overrightarrow{\xi} \) lies in a plane perpendicular to \( \overrightarrow{\Lambda} \) in this three-space. The equation of this curve is given by (1.8); choosing conveniently the axes of the co-ordinate system in this space parallel to the (mutually perpendicular) vectors \( \overrightarrow{Q} \), \( \overrightarrow{Q} \times \overrightarrow{Q} \) and \( \overrightarrow{\Lambda} \), respectively, the only non-zero component of (1.8) yields this equation in the form

\[
\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1
\]
(1.10)

(here \( s^2 = a^2/(1+\xi^2/a^2) \); \( \xi \) and \( \eta \) denote the first two components of \( \overrightarrow{\xi} \)). Any four-dimensional rotation, if applied to the circle (1.2) or the antisymmetric tensor (1.4), transforms (1.10) into another ellipse with the same semi-major axis \( a \); and any ellipse, having \( a \) as semi-major axis and the origin of the \( \overrightarrow{\xi} \)-space as centre, can be obtained by projecting an appropriately chosen main circle of the four-dimensional sphere (1.1) on the \( \overrightarrow{\xi} \)-space. This is just the family of the Kepler orbits corresponding to a given negative value of energy. One is thus led to the idea of establishing a correspondence between a Kepler motion on an elliptical orbit and a circular motion (on the above fictitious four-dimensional sphere). Essentially this correspondence was used by Kepler in his derivation of the celebrated Kepler equation (see, e.g., Ref. 36).

The four-vector \( \alpha^\lambda \) in (1.2) can be chosen for \( \beta > 0 \) so that \( \alpha^\lambda(a,0,0,0) \). By the application of an appropriate rotation in the 2-4 plane, \( \beta^\lambda \) can be transformed into \( \beta^\lambda(0,a,0,0) \). As a result of this the circle (1.2) is transformed into the circle

\[
\xi'(\omega) = a \cos \omega, \quad \eta'(\omega) = a \sin \omega.
\]
(1.11)

In Fig. 1 circle and ellipse are shown together; the correspondence
between individual points of the circle and the ellipse is established through the projection procedure described above: the image of P is Q. This is in fact the figure familiar from the textbooks, showing the relation between the true anomaly $\theta$ and the eccentric anomaly $\varpi$, and used to derive Kepler's equation. One simple way to establish this equation is the following. The Cartesian coordinates of the point Q on the ellipse (taking the axes of the ellipse as axes of coordinates) are $\xi = a \cos \varpi$, $\eta = b \sin \varpi$. In the system with the focus S as origin (and axes still parallel to the axes of the ellipse) one has

$$x = a(\cos \varpi - \varepsilon), \quad y = b \sin \varpi; \quad (1.12)$$

here $\varepsilon$ denotes the numerical eccentricity of the ellipse. The constant areal velocity about the focus S is

$$\frac{\pi a b}{T} = \frac{1}{2} \left( x \frac{dx}{dt} - y \frac{dy}{dt} \right) = \frac{a b}{2} (1 - \varepsilon \cos \varpi) \frac{d\varpi}{dt} \quad (1.13)$$

($T$ is the period of motion); thus

$$n = (1 - \varepsilon \cos \varpi) \frac{d\varpi}{dt}, \quad (1.14)$$

where

$$n = \frac{2\pi}{T}. \quad (1.15)$$

Kepler's equation, giving the relationship between the eccentric anomaly and the time (or the mean anomaly $n t$), is obtained herefrom by integration.
\[ n t = \omega - \varepsilon \sin \omega; \]  

(1.16)

It is supposed here that \( t = 0 \) for \( \omega = 0 \). It is worth noting that the distance \( r = (x^2 + y^2)^{\frac{3}{2}} \) is given by

\[ r = a(1 - \varepsilon \cos \omega); \]  

(1.17)

thus instead of (1.14) one may also write

\[ n = \frac{1}{a} \frac{d\omega}{dt}. \]  

(1.18)

2. FORMULATION OF THE KEPLER PROBLEM IN TERMS OF FOCK VARIABLES AND THEIR CANONICAL CONJUGATES

The principal constant of motion of the problem, the energy

\[ E = \frac{p^2}{2m} - \frac{G}{r} \]  

(2.1)

will be equivalently denoted also by

\[ E = \mp \frac{p_o^2}{2m} \mp \frac{G}{2a}, \]  

(2.2)

where \( p_o \) and \( a \) are non-negative numbers of dimension of momentum and length, respectively. The upper sign, here and also in the following formulae, refers to bound states, the lower sign to positive energy states. Further constants of motion, the existence of which is expressed by Kepler's laws, are the angular momentum vector

\[ \vec{L} = \vec{r} \times \vec{p} \]  

(2.3)
and the "vector of numerical eccentricity"

\[ \vec{e} = \frac{\vec{r}}{r} - \frac{\vec{p} \times \vec{L}}{m_0} \]  

(2.4)

also called Laplace integral or Lenz-Runge vector. For \( p_0 \neq 0 \) it is more useful to consider the vector

\[ \vec{A} = \frac{m g}{p_0} \left( \frac{\vec{r}}{r} - \frac{\vec{p} \times \vec{L}}{m g} \right). \]  

(2.5)

The two vectors \( \vec{L} \) and \( \vec{A} \) satisfy the condition

\[ \vec{L} \cdot \vec{A} = 0. \]  

(2.6)

The energy, if different from zero, may be expressed by them in the form

\[ E = \mp \frac{m g^2}{2} (A^2 + \vec{L}^2)^{-1}. \]  

(2.7)

It is convenient to arrange the components of \( \vec{L} \) and \( \vec{A} \) into a 4 x 4 scheme \( \left(F_{\alpha\beta}\right) \), where \( F_{ij} = \varepsilon_{ijk} l_k \) and \( (\pm)^3 F_{i4} = A_i \left[F_{\alpha\beta} = -F_{\beta\alpha} \right] \); \( (\pm)^3 \) expresses an extra \( i \) factor for positive energy. Using this notation one can write (2.6) in the form

\[ \varepsilon_{\sigma\tau} F_{\tau\sigma} F_{\sigma\tau} = 0; \]  

(2.8)

the energy can be written instead of (2.7) as

\[ E = -\frac{m g^2}{F_{\sigma\tau} F_{\sigma\tau}}. \]  

(2.9)
If $F_{\alpha\beta}$ is considered a second-rank tensor under four-dimensional orthogonal transformation ($a_{\alpha\beta}$), the latter preserve the form of the condition (2.8) and the value (2.9) of the energy. To secure the reality of the vectors $L$ and $A$, $a_{14}^{\pm}$, $a_{14}^{\pm}$, $a_{44}^{\pm}$ have to be chosen real ($0(4)$ for bound states, $0(3,1)$ for positive energy).

Consider the vector product $A \times \vec{r}$; using (2.1-2), (2.5) and (2.7), the following relation can be obtained:

$$A \times (\vec{r} \mp \vec{e}) - \frac{L}{P} \sqrt{a^2 - (\vec{r} \mp \vec{e})^2} = 0 \quad (2.10)$$

where

$$\vec{e} = \frac{1}{P_0} \vec{A} \quad (2.11)$$

may be called the "vector of linear eccentricity". One has further

$$L \cdot (\vec{r} \mp \vec{e}) = 0 \quad (2.12)$$

Eq. (2.12) confines the motion to a plane perpendicular to $L$, while (2.10) gives the equation of the Kepler orbit (cf. (1.8-9)). Analogous relations can be obtained for the momentum vector:

$$A \times \vec{p} + L \frac{\vec{p} \times \vec{p}^2}{2P_0} = 0 \quad (2.13)$$

$$L \vec{p} = 0 \quad (2.14)$$

Eq. (2.14) expresses the fact that $\vec{p}$ moves in a plane perpendicular to $L$; (2.13) yields the equation of the curve described by $\vec{p}$ (or $\vec{v} = \vec{y}/m$; this latter curve is the hodograph).
It is natural to introduce the four-component quantities \( \mathcal{Q}_\alpha \) and \( \mathcal{T}_\alpha \):

\[
\mathcal{Q}_\sigma = \mathbf{r} \mp \mathbf{e}, \quad \mathcal{Q}_4 = \frac{1}{(\pm)^{1/2}} \sqrt{\pm \left[ a^2 - (\mathbf{x} \mp \mathbf{e})^2 \right]},
\]

\[
\mathcal{T}_\sigma = \frac{2p_\sigma^2}{p_0^2 \pm p_0^2} \mathbf{p}, \quad \mathcal{T}_4 = \frac{(\pm)^{1/2}}{p_0^2 \pm p_0^2} \frac{p_0^2 \mp p_0^2}{p_0^2 \pm p_0^2} p_0
\]

normalized according to

\[
\mathcal{Q}_\sigma \mathcal{Q}_\sigma = a^2,
\]

\[
\mathcal{T}_\sigma \mathcal{T}_\sigma = \pm p_0^2.
\]

In (2.15) and (2.10) the sign of the square-root has to agree with that of \(-\mathbf{F} \cdot \mathbf{P}\). The four-component equations

\[
\varepsilon_{\alpha} \mathcal{Q}_{\sigma \tau} F_{\sigma \tau} \mathcal{Q}_{\tau} = 0,
\]

\[
\varepsilon_{\alpha} \mathcal{Q}_{\sigma \tau} F_{\sigma \tau} \mathcal{T}_{\tau} = 0
\]

are equivalent to (2.10-12) and (2.13-14), respectively. If \( \mathcal{Q}_\alpha \) and \( \mathcal{T}_\alpha \) are regarded as four-vectors, and, along with the skew-symmetrical second-rank tensor \( F_{\alpha \beta} \), are subjected to the four-dimensional orthogonal transformations described above, then the transformed four-vectors \( \mathcal{Q}'_\alpha \), \( \mathcal{T}'_\alpha \) also correspond to a possible Kepler motion with the same energy. The four-vector nature of (2.16) has provided the basis of Fock's work, who has exhibited the hidden four-dimensional symmetry of the Kepler problem (H atom) by considering the momentum space (for bound states) as the stereographic projection of a four-dimensional sphere. On the other hand (2.15) makes clear why this symmetry remains hidden in the usual position representation. The vector \( \mathbf{F} \) is not a covariant quantity possessing simple transformation

* The variables used in Fock's work are actually \( u_\alpha = \frac{1}{p_0} \mathcal{T}_\alpha \) (normalized according to \( u_\alpha u_\alpha = \pm 1 \); cf. p. 13). For the sake of brevity we shall refer to the \( \mathcal{T}_\alpha \) (and sometimes to the \( \mathcal{Q}_\alpha \) and \( \mathcal{Q}_\alpha \)) simply as Fock variables.
properties in four dimensions; it is the vector \( \vec{x} = \vec{r} + \vec{e} \), defining the position with respect to the geometric centre of the orbit rather than with respect to the centre of attraction, which forms part of the four-vector (2.15).

Furthermore, the validity of the following relations may be established:

\[
\frac{\pi_{\sigma} \pi_{\sigma}}{m} + \frac{\phi}{(\phi_{\tau} \phi_{\tau})^{\frac{1}{2}}} = 0,
\]

(2.21)

\[
\pi_{\sigma} \phi_{\sigma} = 0
\]

(2.22)

and

\[
F_{\alpha\beta} = \phi_{\alpha} \pi_{\beta} - \phi_{\beta} \pi_{\alpha},
\]

(2.23)

whence

\[
\phi_{\alpha} = \frac{F_{\alpha\sigma} \pi_{\sigma}}{\pi_{\sigma} \pi_{\sigma}},
\]

(2.24)

\[
\pi_{\alpha} = -\frac{F_{\alpha\sigma} \phi_{\sigma}}{\phi_{\sigma} \phi_{\sigma}}.
\]

(2.25)

These formulae suggest that, e.g. for bound states, \( \phi_{\alpha} \) and \( \pi_{\alpha} \) may be considered as position and momentum vectors, respectively, of a mass point moving along a main circle on a four-dimensional sphere with energy-dependent radius (cf. (2.21)). The Kepler orbit and the curve described by \( \vec{y} \) (hodograph) are, according to (2.15-16), parallel and stereographic projections of this main circle on the three-dimensional configuration and momentum space, respectively. Similar relationship may be anticipated between the positive energy Keplerian motion and the inertial motion of a mass point on a four-dimensional hyperboloid.

The time evolution of the motion in four dimensions is governed, by virtue of the Newtonian equations of motion.
by the following equations:

\[
\frac{d\hat{r}}{dt} = \frac{\hat{p}}{m}, \quad \frac{d\hat{p}}{dt} = -\frac{q}{\tau} \frac{\hat{F}}{\tau},
\]  

(2.26)

where

\[
\frac{1}{(q_\rho q_\rho)^{\frac{1}{n}}} \frac{d\varpi_\rho}{dt} = \frac{1}{m} \varpi_\rho, \quad \frac{1}{(q_\rho q_\rho)^{\frac{1}{n}}} \frac{d\varpi_\rho}{dt} = -mn^2 \varpi_\rho,
\]  

(2.27)

for elliptic orbits \( n \) is related to the period \( T \) through (1.15). The time element \( dt \) is thus not invariant under the orthogonal transformations of the four-vectors (2.15-16), described above. It is natural to introduce an invariant "time" parameter \( \tau \) obeying the relation

\[
d\tau = \pm \frac{(q_\rho q_\rho)^{\frac{1}{n}}}{T} dt.
\]  

(2.29)

The equations of motion

\[
\frac{d\varpi_\rho}{d\tau} = \frac{1}{m} \varpi_\rho, \quad \frac{d\varpi_\rho}{d\tau} = \mp mn^2 \varpi_\rho
\]  

(2.30)

written in this new parametrization are Hamilton's canonical equations (with \( \tau \) as time) corresponding to

\[
H = -\frac{\text{mg}^2}{F_{\sigma_\tau}^2 \sigma_\tau}
\]  

(2.31)

(cf. eq. (2.9)). When deriving eqs. (2.30) from (2.31), the
constraints (2.21-22) must be properly taken into account.

From (1.18), (2.17) and (2.29) the equality

$$\text{d} n \text{ d} \tau = \text{d} \omega \text{,} \quad (2.32)$$

establishing a simple relationship between the invariant time parameter and the eccentric anomaly, can be obtained for elliptic orbits.

While eqs. (2.30) describe (for bound states) uniform motion along a main circle on the four-dimensional sphere with \( \tau \) as time, eqs. (2.27) show that if \( t \) is considered as time, this uniform character is lost: the (angular) velocity of the particle becomes proportional to \((Q_\sigma Q_\sigma)^{3/2} = a/r \). With \( \tau \) as time, the distribution of momenta is uniform along the circle described by \( \Pi_\alpha \). The distribution corresponding to \( t \) is obtained from this uniform distribution by multiplying it by \( t/\alpha \); by virtue of eqs. (2.1-2) this factor can also be written in the form \( 2P_\alpha^2 \). In the FOCK theory\(^1\) of the H atom this is reflected by the fact that the probability \( |\Phi(\vec{r})|^2 \text{ d}^3 p \) in the \( \vec{p} \) space is obtained by multiplying the probability \( |\phi(\vec{u}_\alpha)|^2 \text{ d}^4 Q_\alpha \), corresponding to the force-free motion on a four-dimensional sphere, just by the factor

(The notation of Ref. 26 is used here; \(\vec{u}_\alpha \) denotes the unit vector parallel to Fock's four-momentum \( \Pi_\alpha \)).

One may define the Poisson bracket

$$\{X, Y\}_F = \frac{\partial X}{\partial Q_\alpha} \frac{\partial Y}{\partial P_\alpha} - \frac{\partial X}{\partial P_\alpha} \frac{\partial Y}{\partial Q_\alpha} \quad (2.33)$$

in terms of the Fock variables \( \Pi_\alpha \), \( Q_\alpha \). The question naturally arises, how this bracket is related to the standard Poisson bracket

$$\{X, Y\}_N = \frac{\partial X}{\partial x_5} \frac{\partial Y}{\partial p_5} - \frac{\partial X}{\partial p_5} \frac{\partial Y}{\partial x_5} \quad (2.34)$$
in the variables $\vec{x}(x_1, x_2, x_3)$ and $\vec{p}(p_1, p_2, p_3)$ used in the Newtonian description of the problem.

The constants of motion (2.23) satisfy the Poisson bracket relation

$$\{F_{\alpha\beta}, F_{\gamma\delta}\}_F = \delta_{\alpha\gamma} F_{\beta\delta} - \delta_{\alpha\delta} F_{\beta\gamma} + \delta_{\beta\gamma} F_{\alpha\delta} - \delta_{\beta\delta} F_{\alpha\gamma}$$

identical in form with the $(\ , \ )_N$ bracket relations obeyed by the $F_{\alpha\beta}$. The contact transformations of the canonical variables $Q_\alpha, \pi_\alpha$, generated by the $F_{\alpha\beta}$ via the bracket $(\ , \ )_P$, form the $SO(4)$ and $SO(3,1)$ invariance dynamical symmetry groups of the problem for negative and positive energies, respectively. The next step would be to construct generators of non-invariance transformations. The four-vectors

$$P_{\alpha} = (Q_\sigma, \pi_\sigma)^{\frac{\lambda}{\phi}} \pi_\alpha$$

and

$$R_{\alpha} = (\pm \pi_\sigma \pi_\sigma)^{\frac{\lambda}{\phi}} Q_\alpha$$

satisfy the following Poisson bracket relations:

$$\begin{align*}
\{F_{\alpha\beta}, P_{\gamma}\}_F &= \delta_{\alpha\gamma} P_{\beta} - \delta_{\beta\gamma} P_{\alpha} \\
\{P_{\alpha}, P_{\beta}\}_F &= -F_{\alpha\beta}, \\
\{F_{\alpha\beta}, R_{\gamma}\}_F &= \delta_{\alpha\gamma} R_{\beta} - \delta_{\beta\gamma} R_{\alpha} \\
\{R_{\alpha}, R_{\beta}\}_F &= \mp F_{\alpha\beta}.
\end{align*}$$
The relations (2.35), (2.38) define the Lie algebra of SO(4,1) and SO(3,2) for negative and positive energies, respectively. On the other hand, (2.35), (2.39) give SO(4,1) for both cases. One has, further,

\[ (p_{\alpha}^{\ast}, R_{\beta}) = -\delta_{\alpha\beta} M, \]  

\[ (p_{\alpha}^{\ast}, m) = 0, \]  

\[ (p_{\alpha}^{\ast}, m) = -R_{\alpha}, \]  

\[ (R_{\alpha}^{\ast}, m) = \pm p_{\alpha}, \]

where

\[ M = (\tilde{\sigma}^{\sigma} \tilde{\sigma}^{\sigma})^{1/2}(\pm \tau_{\sigma} \tau_{\sigma})^{1/2}. \]

All the Poisson bracket relations (2.35), (2.38-40) taken together define the SO(4,2) Lie algebra, again for both positive and negative energies. The contact transformations generated by M via the bracket \(( , )_p\) describe the time evolution of the motion (with \(\tau\) as time). In fact (2.40 c-d) are equivalent to the equations of motion (2.30), the quantity M being a simple function of the Hamiltonian (2.30): \(M = \sqrt{1 + m^2 / 2H}\). The non-invariance contact transformations generated by the quantities \(p_{\alpha}\) and \(R_{\alpha}\) via the bracket \(( , )_p\) violate the constraint (2.21). One notes further that neither \(p_{\alpha}\) nor \(R_{\alpha}\) agrees (in the case of negative energy) with the SO(4,1) generators obtained by BACHY \(^{22}\) as solutions of the differential equations entailed by the corresponding \(( , )_N\) Poisson bracket relations. It is interesting to observe, however, that Bacry's generators can be expressed (for the value zero of the parameter \(\alpha\) occurring in Baery's formula) by \(p_{\alpha}\) and \(R_{\alpha}\) in the following simple form:
\[ \mathbf{A}_\alpha = P_\alpha \cos \psi + R_\alpha \sin \psi, \quad (2.42) \]

where

\[ \psi = \frac{p_0 \mathbf{B}(E)}{mg} + \Theta(E), \quad (2.43) \]

Bacry's \( \mathbf{B} \) and \( S \) are obtained from (2.42) for \( \alpha = 1, 2, 3 \) and 4, respectively. It may be remarked further that similar relationship exists between our \( R_\alpha \) and the generators \( \mathbf{B}, S \) given by Han \(^{27}\).

The existence of the relations (2.21–22) connecting the positions and momenta might, of course, suggest the use of the Dirac bracket \( (, ) \) instead of the Poisson bracket \( (, ) \). It is better, however, to defer the introduction of the Dirac bracket to the last section. In the next section we perform first a transformation of the dynamical variables to get back the standard time parameter \( t \).

3. GETTING BACK THE TIME \( t \), BACRY'S GENERATORS

Define the quantities

\[ \begin{align*}
\mathbf{a}_\alpha &= \Pi_\alpha \cos \eta (\tau - t) + \eta \eta \mathbf{a}_\alpha \sin \eta (\tau - t), \\
\mathbf{b}_\alpha &= \mathbf{a}_\alpha \cos \eta (\tau - t) - \eta \eta \mathbf{a}_\alpha \sin \eta (\tau - t) \\
\mathbf{c}_\alpha &= \mathbf{a}_\alpha \cos \eta (\tau - t) - \eta \eta \mathbf{a}_\alpha \sin \eta (\tau - t) \\
\mathbf{d}_\alpha &= \mathbf{a}_\alpha \cos \eta (\tau - t) - \eta \eta \mathbf{a}_\alpha \sin \eta (\tau - t)
\end{align*} \quad (3.1) \]

and

\[ \begin{align*}
\mathbf{a}_\alpha &= \Pi_\alpha \sinh \eta (\tau - t) - \eta \eta \mathbf{a}_\alpha \sinh \eta (\tau - t), \\
\mathbf{b}_\alpha &= \Pi_\alpha \sinh \eta (\tau - t) - \eta \eta \mathbf{a}_\alpha \sinh \eta (\tau - t)
\end{align*} \quad (3.2) \]

-16-
for negative and positive energies, respectively. Using (2.30) the following equations of motion may be established:

\[ \frac{d}{dt} c^\alpha = \frac{i}{m} b^\alpha; \quad \frac{d}{dt} b^\alpha = \mp m \hbar^2 c^\alpha. \]  

(3.3)

The four-component dynamical quantities \( b^\alpha, c^\alpha \) obey, in virtue of (2.17-18), (2.22) and (2.28), the normalization conditions

\[ b^\sigma \bar{b}^\sigma = \hbar \sigma \bar{\sigma} = \pm \hbar^2, \]  

(3.4)

\[ c^\sigma \bar{c}^\sigma = \bar{c}^\sigma c^\sigma = \sigma^2. \]  

(3.5)

Combining (3.4-5) with (2.2) one obtains the constraint

\[ \frac{\gamma_\sigma b^\sigma}{m} + \frac{\gamma_\bar{\sigma}}{c^\sigma \bar{c}^\sigma} = 0. \]  

(3.6)

Furthermore the validity of the following relations may be established:

\[ b^\sigma c^\sigma = 0; \]  

(3.7)

\[ F_{\alpha \beta} = c^\alpha b^\beta - c^\beta b^\alpha; \]  

(3.8)

\[ \varepsilon_{\alpha \beta \gamma \tau} F_{\gamma \tau} b^\beta = 0; \]  

(3.9)

\[ \varepsilon_{\alpha \beta \gamma \tau} F_{\gamma \tau} c^\beta = 0; \]  

(3.10)

\[ \eta^\alpha = -\frac{F_{\alpha \sigma} c^\sigma}{c^\sigma c^\sigma}; \]  

(3.11)

\[ c^\alpha = \frac{F_{\alpha \sigma} b^\sigma}{b^\sigma b^\sigma}. \]  

(3.12)
The eqs. of motion (3.3) are Hamilton's canonical equations (with \( t \) as time) corresponding to (2.31). When deriving (3.3) from (2.31), the form (3.8) of \( F_{\alpha \beta} \) has to be used, and the constraints (3.6-7) must be properly taken into account.

The new canonical variables \( b_{\alpha}, c_{\alpha} \) may also be used to define a Poisson bracket:

\[
(X, Y)_B = \frac{\partial X}{\partial c_{\sigma}} \frac{\partial Y}{\partial b_{\sigma}} - \frac{\partial X}{\partial b_{\sigma}} \frac{\partial Y}{\partial c_{\sigma}}. \tag{3.13}
\]

For \( F_{\alpha \beta} \) the \( (\ , \ )_B \) Poisson bracket relations have exactly the same form as the \( (\ , \ )_P \) and \( (\ , \ )_N \) Poisson bracket relations (cf. (50)). The quantities

\[
b_{\alpha} = (c_{\sigma} c_{\sigma})^{\frac{1}{2}} b_{\alpha} \tag{3.14}
\]

and

\[
c_{\alpha} = (\pm b_{\sigma} b_{\sigma})^{\frac{1}{2}} c_{\alpha} \tag{3.15}
\]

satisfy the relations:

\[
\begin{align*}
(F_{\alpha \beta}, B_{\sigma})_B &= \delta_{\alpha \sigma} b_{\beta} - \delta_{\beta \sigma} b_{\alpha}, \\
(B_{\alpha}, B_{\beta})_B &= -F_{\alpha \beta}, \\
(F_{\alpha \beta}, C_{\gamma})_B &= \delta_{\alpha \gamma} c_{\beta} - \delta_{\beta \gamma} c_{\alpha}, \\
(C_{\alpha}, C_{\beta})_B &= -F_{\alpha \beta}.
\end{align*} \tag{3.16}
\]

One has further

\[
(B_{\alpha}, C_{\beta})_B = -\delta_{\alpha \beta} M, \tag{3.18a}
\]

-18-
\[(\mathbf{F}_{\alpha}, M)_B = 0, \quad (3.18b)\]
\[(\mathbf{B}_{\alpha}, M)_B = -c_{\alpha}, \quad (3.18c)\]
\[(\mathbf{C}_{\alpha}, M)_B = \pm B_{\alpha}, \quad (3.18d)\]

where

\[M = \left( Q_{\alpha} Q_{\alpha'} \right)^{3/2} \left( \mp \Pi_{\sigma} \Pi_{\sigma'} \right)^{3/2} = \left( C_{\sigma} C_{\sigma'} \right)^{3/2} \left( \pm b_{\sigma} b_{\sigma'} \right)^{3/2}, \quad (3.19)\]

in complete analogy with (2.36-41).

The contact transformations of the canonical variables \(b_{\alpha}\), \(c_{\alpha}\) generated by the \(F_{\alpha B}\) via the \(\{ , \}_B\) bracket again form the \(SO(4)\) and \(SO(3,1)\) invariance groups for positive and negative energies, respectively. The contact transformations generated by \(M\) via the bracket \(\{ , \}_B\) again describe the time evolution of the motion (now with \(t\) as time). Eqs. (3.18 c-d) are equivalent to the equations of motion (3.3). The non-invariance transformations generated by \(B_{\alpha}\) and \(C_{\alpha}\) via the bracket \(\{ , \}_B\) again violate one of the constraints to be satisfied by the dynamical variables. The cure for this will be the Dirac bracket (see next section). Our first task is, however, to examine more closely the non-invariance generators \(B_{\alpha}\) and \(C_{\alpha}\).

By virtue of (3.1-2), (3.14-15) these quantities can be expressed in the form

\[
\begin{align*}
B_{\alpha} &= P_{\alpha} \cos n(\tau - t) + P_{\alpha} \sin n(\tau - t), \\
C_{\alpha} &= P_{\alpha} \cos n(\tau - t) - P_{\alpha} \sin n(\tau - t)
\end{align*}
\]

\[ (3.20) \]
and

\[
\begin{align*}
B_\alpha &= P_\alpha \coth n(\tau - t) - R_\alpha \sinh n(\tau - t), \\
C_\alpha &= R_\alpha \coth n(\tau - t) - P_\alpha \sinh n(\tau - t)
\end{align*}
\]

(3.21)

for negative and positive energies, respectively. Kepler's equation yields the argument in (3.20) in the following form:

\[
n(\tau - t) = \varepsilon \sin n \tau,
\]

(3.22)

To obtain this, eq. (2.32) has been combined with (1.16); it is supposed that \( \varepsilon > 0 \) for \( t = 0 \). The positive energy analogue of (3.22) is

\[
n(\tau - t) = \varepsilon \sinh n \tau.
\]

(3.23)

The right-hand sides of (3.22-23) may be expressed in the following form:

\[
\varepsilon \sin n \tau = \frac{\mathbf{p}_0 \cdot (\mathbf{p} \times \mathbf{r})}{mg},
\]

(3.24)

\[
\varepsilon \sinh n \tau = \frac{\mathbf{p}_0 \cdot (\mathbf{p} \times \mathbf{r})}{mg}.
\]

(3.25)

Using these, one finds that the quantities \( B_\alpha \), \( C_\alpha \) given by (3.20) are none other than the (2.42) Baarey generators specialized for \( \Theta = 0 \) and \( \Pi/2 \), respectively; the quantities under (3.21) represent the positive energy versions of the latter.

The validity of the following relations may be established:
\[
F_{\alpha \sigma} F_{\rho \tau} = -B_{\alpha B_{\rho \tau}} C_{\alpha} C_{\rho},
\]
\[
F_{\alpha \sigma} B_{\rho \tau} M C_{\alpha} = 0,
\]
\[
B_{\alpha} C_{\alpha} = 0,
\]
\[
C_{\alpha} C_{\alpha} = M^2.
\]

(3.26)

\[
\begin{align*}
\varepsilon_{\alpha \beta \gamma \tau} F_{\beta} & = 0, \\
\varepsilon_{\alpha \beta \gamma \tau} F_{\beta} C_{\tau} & = 0,
\end{align*}
\]

(3.27)

\[
M F_{\alpha \beta} = C_{\alpha} B_{\beta} - C_{\beta} B_{\alpha}.
\]

It is convenient to arrange the SO(4,2) generators \(F_{\alpha \beta}, B_\alpha, C_\alpha\) and \(M\) into the following antisymmetric 6 x 6 scheme *:

\[
G_{IJ} = \begin{pmatrix}
0 & F_{12} & F_{13} & F_{14} & iB_1 & (\mp) \frac{i}{2} C_1 \\
F_{21} & 0 & F_{23} & F_{24} & iB_2 & (\mp) \frac{i}{2} C_2 \\
F_{31} & F_{32} & 0 & F_{34} & iB_3 & (\mp) \frac{i}{2} C_3 \\
F_{41} & F_{42} & F_{43} & 0 & iB_4 & (\mp) \frac{i}{2} C_4 \\
iB_1 & iB_2 & iB_3 & iB_4 & 0 & \frac{1}{(\mp) i} M \\
-(\mp) \frac{i}{2} C_1 & -(\mp) \frac{i}{2} C_2 & -(\mp) \frac{i}{2} C_3 & -(\mp) \frac{i}{2} C_4 & -(\mp) \frac{i}{2} M & 0
\end{pmatrix}
\]

(3.28)

* Capital Latin indices assume the values 1, 2, \ldots, 6.
The Poisson bracket relations of all these generators can be condensed into the following formula:

\[ (g_{IJ}, g_{KL})_B = \delta_{IL} g_{KJ} + \delta_{JK} g_{LI} + \delta_{IK} g_{JL} + \delta_{JL} g_{IK}. \]  

(3.29)

Instead of (3.26) and (2.8), (3.27) one may write

\[ g_{IS} g_{SJ} = 0 \]  

(3.30)

\[ \varepsilon_{IJPQRS} g_{PQ} g_{RS} = 0 \]  

(3.31)

One has thus an example of "purely group-theoretical dynamics".

4. BACKY VARIABLES AND DIRAC BRACKETS

The canonical variables \( b_\alpha, q_\alpha \), related through (3.14-15) to the Backy generators, will be called "Backy variables". They are not all independent; the position vector \( q_\alpha \) and the momentum \( b_\alpha \) obey the identities (3.6-7), which may also be written conveniently in the form

\[ \Theta_I = \pm \frac{m q}{\dot{b} b} - (c_\sigma c_\sigma)^{1/2} = 0, \]  

(4.1)

\[ \Theta_I = \frac{\dot{b} c_\sigma}{(c_\sigma c_\sigma)^{1/2}} = 0. \]  

(4.2)

For such systems involving constraints DIRAC proposed that the usual Poisson bracket be replaced by a new structure, now known as Dirac bracket. The Dirac bracket \( (\cdot, \cdot) \) is defined by

\[ (x, y)^* = (x, y) - (x, \Theta_I) \int_{P\Delta} (\Theta_\Delta, y); \]  

(4.3)
here ( , ) denotes the ordinary Poisson bracket, $\Theta_\Gamma$ are appropriate functions whose vanishing expresses the constraints, $[\mathcal{I}_{\Gamma\Delta}]$ is the matrix inverse to the matrix of the Poisson brackets of the functions $\Theta_\Gamma$:

$$\mathcal{I}_{\Gamma\Sigma}(\Theta_\Sigma, \Theta_\Delta) = \delta_{\Gamma\Delta}. \quad (4.4)$$

In the present case the non-vanishing Poisson brackets (defined in terms of the Bacry variables; cf. (3.13)) of the functions $\Theta_\Gamma$ given by (4.1-2) are the following:

$$\Theta_I, \Theta_J \rangle_S = -\Theta_I, \Theta_J \rangle_S = 1; \quad (4.5)$$

thus

$$[\mathcal{I}_{\Gamma\Delta}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4.6)$$

Consider now the Dirac bracket analogue of (3.29). Since

$$\langle \Theta_I, \Theta_J \rangle_B = 0 \quad (4.7)$$

for any pair of indices $IJ$, eq. (3.29) holds for the Dirac bracket as well:

$$\langle G_{IJ}^*, \Theta_K \rangle_B = \delta_{IL} G_{KJ}^* + \delta_{JK}^* G_{LI} + \delta_{IK} G_{JL} + \delta_{JL} G_{IK} \quad (4.8)$$

* Capital Greek indices will assume the values 1, 2.
In addition to (4.7), the generators $F_{\alpha\beta}$ and $M$ have vanishing Poisson brackets with $\Theta^I$, too:

$$(F_{\alpha\beta}, \Theta^I)_B = (M, \Theta^I)_B = 0 \tag{4.9}$$

It follows that $F_{\alpha\beta}$ and $M$ generate the same transformation via the Dirac bracket as they do via the Poisson bracket. This is not true of $B_{\alpha}$ and $C_{\alpha}$; here only transformations generated via the Dirac bracket preserve both (4.1-2).

The Baez variables themselves satisfy the following Dirac bracket relations:

\begin{align*}
(\zeta_{\alpha}, \zeta_{\beta})_B &= \frac{1}{c_{\sigma}c_{\tau}} F_{\alpha\beta}^1, \\
(\xi, \zeta_{\beta})_B &= \delta_{\alpha\beta} - 2 \frac{\zeta_{\alpha} \xi}{\zeta_{\tau}} + \frac{c_{\alpha} c_{\beta}}{c_{\sigma} c_{\tau}}, \\
(\xi, \zeta_{\beta})_B &= -\frac{2}{\zeta_{\sigma} \zeta_{\tau}} F_{\alpha\beta}^1, \tag{4.10}
\end{align*}

Eq. (4.5) provides a sufficient condition that the Dirac bracket determined by the $\Theta^I$, $\Theta^H$ correspond to "freezing" one pair of canonical variables. In fact one finds that the Poisson bracket relations written down in terms of the three-dimensional variables $\vec{x}(x_1, x_2, x_3)$ and $\vec{p}(p_1, p_2, p_3)$ (cf. (2.34)) have precisely the same form as the corresponding $(\cdot, \cdot)_B^*$ Dirac bracket relations; the transformations generated via the Dirac bracket are contact transformations of the three-dimensional canonical variables. One may verify, e.g., the validity of the following relations (analogous to (4.8) and (4.10), respectively):

$$\begin{align*}
(q_{ij}, q_{KL})_N &= \delta_{ij} G_{K3}^I + \delta_{ij} G_{L3}^I + \delta_{IK} G_{j3}^I + \delta_{IL} G_{3j}^I, \\
(\zeta_{\alpha}, \zeta_{\beta})_N &= \frac{1}{c_{\sigma}} \frac{1}{c_{\tau}} F_{\alpha\beta}^1, \\
(\xi, \zeta_{\beta})_N &= \delta_{\alpha\beta} - 2 \frac{\zeta_{\alpha} \xi}{\zeta_{\tau}} + \frac{c_{\alpha} c_{\beta}}{c_{\sigma} c_{\tau}}, \\
(\xi, \zeta_{\beta})_N &= -\frac{2}{\zeta_{\sigma} \zeta_{\tau}} F_{\alpha\beta}^1.
\end{align*}$$
ACKNOWLEDGMENT

The author takes pleasure in expressing his deep gratitude to Professors Abdus Salam, P. Budini and to the IAEA for the possibility to work at the International Centre for Theoretical Physics, Trieste, and to Professor P. Budini for his kind interest in this work. Thanks are due to Professors L.C. Biedenharn, R. Raczka and I.T. Todorov for illuminating discussions, to Professors C. Frocdal and G. Marx for having read the manuscript, and to Dr. B. Koltay for numerous conversations on the subject. Financial support of the IAEA is gratefully acknowledged.
REFERENCES

1) V. FOCK, Z. Physik 38, 145 (1935).
2) L. HULTHEN, Z. Physik 86, 21 (1933).
3) W. PAULI, Z. Physik 36, 336 (1926).
14) L. C. BIEDENHARN, Lectures delivered at the Summer Institute for Theoretical Physics, Univ. of Colorado, Boulder (1962); New York Interscience, p. 384 (1963).


29) R. MUSTO, Generators of O(4,1) for the quantum mechanical hydrogen atom, NYO-3399-50, Syracuse Univ. preprint.


31) A.O. BARUT, Calculation of transition probabilities from non-compact dynamical groups, U COL-66-10 preprint, Univ. of Colorado.

32) A.O. BARUT and H. KLEINERT, Transition probabilities of the hydrogen atom from non-compact dynamical groups, Univ. of Colorado preprint.

33) A.O. BARUT and H. KLEINERT, Current operator and Majorana equation for the hydrogen atom from dynamical groups, U COL-66-13 preprint, Univ. of Colorado.
34) A. O. BARUT and H. KLEINERT, Transition form factors in H atom, University of Colorado preprint (1967).


39) P. A. M. DIRAC, Lectures on Quantum Mechanics, New York, Belfer Graduate School of Science, Yeshiva Univ. (1964).
