THREE-BODY INVESTIGATIONS
WITH REALISTIC POTENTIALS

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The insolubility of the three-body problem in conventional terms has traditionally been associated with that of the helium atom which was probably the first problem in atomic physics to which variational methods were successfully applied. Perhaps one of the reasons for the successful application of variational methods in atomic physics has been the rather precise mathematical knowledge of the physical forces governing atomic structure, mainly electrostatic forces. Since these cannot be changed according to the physicist's will, atomic physics investigators had, perforce, to develop very precise and narrow variational bounds (upper and lower) for physical quantities evaluated with trial functions, in order to make meaningful comparison with experimental numbers of uncanny accuracy facilitated by the highly sophisticated tools of atomic spectroscopy. It would probably be safe to assume that these great atomic investigators (from Hyllaraas down) would not necessarily have taken all the caution they took if they could help it, but both the Coulomb law as well as the experimentalists proved just too strong for them.

The situation is radically different in nuclear physics. The biggest saviour for the average theoretician is perhaps the fact that, within the effective range theory, any potential is just as good or as bad as any other. He has therefore a good deal of freedom to choose his own potential since little discriminating criterion is available to govern his choice. Even in the simplest nuclear two-body problem there is just no quantitative counterpart to the Coulomb force, while the qualitative Yukawa counterpart has enough of play about it. Of course in nuclear theory, as in atomic theory, approximation (especially variational) methods leave little choice of alternatives, but the lack of precise theoretical knowledge of the forces is bound to produce a certain amount of "slackness" in the treatment, versus the corresponding atomic treatment, since it is now possible to lay a good
part of the blame for bad agreement on the input potential! It is this aspect which, in my estimation, has in the past played a big role in producing almost a spectrum of results from the variational treatment of nuclear three-body problems with phenomenological two-body forces, without much tangible difference in the input assumptions.

I believe it would be perfectly honest to put the entire blame for bad agreement on the input assumptions provided one does not make a series of algebraic approximations between the initial assumptions and the final results. Such an attitude would, however, be mathematically impossible to implement if one confines oneself to conventional static forces in the context of the three-body problem. But one could ask oneself the question "to what extent does the traditional choice of local potentials depend on physical necessity, as distinct from time-honoured conventions?" Since the parameters of the two-body force, at the present stage of theoretical development, are largely determined from experiment, it should certainly be logical to make an appropriate choice so long as the latter is not manifestly violated. For example, at low energies, experiment determines only two parameters (range and strength) which could be fitted with almost any potential. Going to higher energies, potentials like Brueckner-Gammel-Thaler, Signell-Marshak, Hamada-Johnston-Yale and their more modern counterparts like Arndt-McGregor are favoured by the phase shifts. However, these high-energy phase shifts do not give any information about the off-energy-shell effects or short-range correlations that form an integral part of the concept of a "potential". It is this feature that one could perhaps exploit to make a phenomenological choice for the potential, the guiding principle being mathematical simplicity and ease of application.

How much freedom does one have in the choice of a potential? If the deuteron wave function (and likewise that of the virtual deuteron state) and the N-N phase shifts for all $\lambda$-values were experimentally known at all distances and all energies, respectively there would be no freedom at all, since these would imply in principle a complete knowledge of

* In the sense that one has to solve a set of two-dimensional integral equations, a prospect which does not yet seem feasible with realistic potentials.
the off- and on-shell elements. However, in practice, one requires other physical systems to gain an indirect experimental knowledge of these off-shell elements. For example, the binding energy of nuclear matter incorporates this knowledge in principle, but it is not a very sensitive method in practice, because of the elaborate approximation techniques involved in the process. Three-body problems, on the other hand, provide an ideal probe, especially since most three-body parameters are sensitive to the two-body potentials. All one needs is a potential simple enough to warrant an exact three-body treatment, provided it does not violate the experimental information on two-body systems (bound and scattering states) that its predictions can be confronted with. Thus, if we regard the potential, or better the T-matrix, as a matrix in energy space, the phase shifts merely determine its diagonal elements, without saying a word about the off-diagonal elements which correspond to short-range correlations, except perhaps as much as could be implied from the binding energy of the deuteron. To look into this latter possibility somewhat more closely, a natural approximation to the potential (or the T-matrix) would be to truncate it in energy space, i.e., to assume that the on- and off-shell elements corresponding to the high-energy part of this matrix do not seriously affect the same elements corresponding to low and moderate energies. While this assumption must really be judged by experiment, it is very similar in spirit to what is done in high-energy physics as well, and at least provides one with a raison d'être on which to base a suitable ansatz for the input potentials.

If mathematical simplicity were the only criterion, perhaps the harmonic potential would provide the ideal choice, but for its incompatibility with a simultaneous treatment of structural and scattering problems. Perhaps such potentials are more suited to quarks as a good mathematical excuse for their lack of observation. For the nuclear problem, on the other hand, the way was shown by YAMAGUCHI 1) more than a decade ago, when he solved the deuteron and n-p scattering problems using separable potentials. Experimental
information was not violated at low and moderate energies. We merely proposed the same potential about six years ago \( \text{2)} \) as a practical means of handling the three-body problem which happens to get exactly reduced to the simplicity of a finite number of coupled two-body equations without having to invoke elaborate trial functions, an important requirement for physical quantities requiring accurate wave functions for their evaluation. Later, the utility of these very potentials for three-body problems was re-discovered by LOVELACE \( \text{3)} \) in the context of the FADDEEV theory of three-body systems \( \text{4)} \), and independently by AMADO \( \text{5)} \) in the context of a soluble field-theoretical model for the deuteron.

It may perhaps be in order at this stage to digress a little on the practical philosophy governing the separable approximation to a potential \( \text{2)} \). Especially since such potentials cannot be visualized in terms of traditional concepts, one must clarify at least what one is trying to do, to realize this approximation. Since the arguments are most easily given in momentum space, the Fourier transform of a local potential looks like \( V(p - p') \) which, when expanded in partial waves, has the structure

\[
\sum_{\ell = 0}^{\infty} (2\ell + 1) V_\ell (\hat{p}, \hat{p'}) \hat{P}_\ell (\hat{p}, \hat{p'})
\]

This expression necessarily becomes non-local when the series is truncated after a certain finite value of \( \ell \). Now for low-energy problems, it is only certain low values of \( \ell \) that have by far the most important effect on the physical system. Thus, while working with a local potential, one has, in principle, all the partial waves (that are allowed by the selection rules governing the quantum mechanical system under study), though not all of these \( \ell \)-values are playing the same role in practice. Therefore it makes physical sense to approximate the series by retaining only the low \( \ell \)-terms as dominant, thus defining a non-local approximation to the local potential, of the form
At this stage the original potential $V(p - p')$ has already lost much of its meaning in mathematical terms, though not in physical terms. If now the object is to correlate two-body data with more complicated systems, one could just as well parametrize the radial functions $V^\lambda$ directly, through fits to the appropriate experimental data. It is at this stage that the separable assumption

$$V^\lambda(p, p') = \lambda^\lambda V^\lambda(p) V^\lambda(p')$$

suggests itself in a natural way, merely for the sake of mathematical simplicity, if for no other reason. It does not violate the spirit of the effective range theory to the extent one is prepared to include a range parameter and a strength parameter in the definition of each $V^\lambda$. While from the practical point of view such a requirement is easily fulfilled, it can be shown that the definition can even be mathematically sharpened through a definition of the form

$$V^2_\lambda(p) = \frac{2}{p^2} Q^2_\lambda \left( 1 + \frac{1}{2} \beta^2_\lambda p^{-2} \right),$$

where, in the language of dispersion theory, $\beta_\lambda$ plays the role of the mass of the exchanged particle between the interacting nucleons.

Once the shape has been fixed, with parameters fitted to the data, the off-shell extension has of course been formally achieved, but not tested. It would have been ideal if a mathematically self-consistent criterion were available without having to resort to experiment. Failing such a possibility, one has to fall back on the next best alternative, viz., an appeal to experimental information which must necessarily have to be looked for beyond the two-body system. As said earlier, the three-
body system, being easily handled with such potentials, happens to be the ideal choice, and one must be prepared to calculate as many physical parameters as possible and confront them with experiment.

The three-body method of testing the off-shell elements of a potential is highly indirect, since it relies entirely on the evaluation of various three-body parameters of experimental interest. What makes the method non-trivial is, however, the fact that most of these parameters are quite sensitive to the input potential. Moreover, such potentials can be made as realistic as one likes in terms of two-body fits, and depend only on the total price (in terms of algebraic manipulations and available machine sizes) that one is prepared to pay. The general structure of the reduced three-body equations is such that one obtains as many coupled one-dimensional integral equations as the "rank" of the input separable potential. The machine sizes available at present limit this rank to not more than four, preferably three. One has thus to make a very economical choice of the potentials so as to get the maximum possible benefit, through an exact inclusion of only those effects which must not be treated perturbatively. In the triplet-even \((T = 0)\) state one must have a minimum of central plus tensor forces both of which are essential for a meaningful three-body calculation. Fortunately, for this purpose one already has Yamaguchi's special factorable potential (including the tensor force) which not only reproduces experiment very satisfactorily, but is ideally suited to a three-body calculation. Next, the reproduction of the \(^1S_0\) phase
shifts to a sufficient degree of accuracy up to, say, 350 MeV or so, requires again a potential of rank two, in order to simulate the short-range repulsion and long-range attraction. Since this already makes a resultant potential of rank four, this seems to be about the limit for machine sizes available at present. Other neglected effects must be taken care of perturbatively. Perhaps this is reasonable for terms like $^1D_2$, $^3P$, $^1P$ or a d-wave L-S force, in the context of the triton problem. However, I wish to emphasize that the repulsive part of the $^1S_0$ force (simulating the hard core) cannot be included perturbatively in order that its full effect (and this is considerable) may be felt on the structure of $H^3$.

Before proceeding to discuss the results of specific applications, it might be of interest to give the essential steps demonstrating the equivalence with the main spirit of Faddeev's theory, viz., to bring about a compact kernel by transforming the potentials to the reaction matrices. The comparative steps of Faddeev's theory and the separable potential method are sketched below, the latter being idealized to the case of three identical, non-relativistic spinless particles interacting through s-waves.

Faddeev theory

$$T = T^{(1)} + T^{(2)} + T^{(3)}; \quad V = V_{23} + V_{31} + V_{12}$$
$$H = H_0 + V; \quad H_0 = K_1 + K_2 + K_3$$
$$T_{23} = V_{23} - V_{23} G_{10} T_{23} T_{23} = (H_0 - E)^{-1} V_{23}$$
$$G_{10} = \left( H_0 - E \right)^{-1}$$

$$T = T^{(1)} + T^{(2)} + T^{(3)} = V_{23} + V_{23} + V_{12} - (V_{23} + V_{31} + V_{12}) G_{10} T$$
$$T^{(1)} = V_{23} - V_{23} G_{10} T^{(1)} - V_{23} G_{10} (T^{(2)} + T^{(3)})$$

** Indeed, a Yamaguchi potential of order one with its range parameter $\delta$ adjusted to fit an effective range value of $R_0 \approx 2.7 F$, is particularly bad for p-p scattering at moderate energies (20-80 MeV).
Separable potentials:
\[ M \langle p | V | p' \rangle = -\lambda g(p) g(p') \]
\[ \langle p_i p_j | V_{ij} | p_i' p_j' \rangle = \delta(p_i - p_i') \langle k_{ij} | V_{ij} | k_{ij}' \rangle \]

\[ p_1 + p_2 + p_3 = 0 ; \quad 2k_{ij} = p_i - p_j \]

\[ D_E \Psi = -M (V_{23} + V_{31} + V_{12}) \Psi \]
\[ D_E = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) - EM \]
\[ D_E \Psi = \lambda \sum_{123} g(k_{23}) \int dk_{23} g(k_{23}) \Psi(p_i - \frac{1}{2} p_i + k_{i23}) \Psi(p_i - \frac{1}{2} p_i - k_{i23}) \]
\[ \equiv \lambda \sum_{123} g(k_{23}) G(p_i) \]
\[ (1 - \lambda h(P)) G(P) = 2 \lambda \int d^3p g(P + \frac{1}{2} q) g(q + \frac{1}{2} p) D(P, q) G(q) \]
\[ h(P) = \int d^3q \left( \frac{3}{4} p^2 + q^2 - EM \right)^{-1} g(q) ; \quad D(P, q) = p^2 + q^2 + p q - EM \]

It may be noticed that the crucial step to make the kernel compact lies in transferring the \( T^{(1)} \)-term from right to left, which has the effect of replacing \( V_{ij} \) by \( T_{ij} \). Exactly the same step is necessary in the other approach, as may be recognized through the appearance of the factor \((1 - \lambda h(P))\) multiplying the spectator function \( G(P) \), which is a result of transferring the "direct term" from right to left in the Schrödinger equation. It may be worth pointing out that, contrary to some beliefs, the Paddeev theory by itself does not simplify the three-body problem in a practical manner. The separable assumption is essential for the simplification, whether it is made directly on the potentials \(^1 \) \(^2 \), \(^7 \) or through a suitable ansatz on the two-body Green function as was done subsequently by Lovelace \(^3 \). As for unitarity requirements, the dynamics provided by the Schrödinger equation has a built-in three-particle elastic unitarity in an exact fashion.
as long as one has hermitian potentials, and the exact Faddeev theory certainly has the same content. However, this old-fashioned potential approach seems to have a practical advantage over the Faddeev-Lovelace formalism insofar as it allows an easier inclusion of important physical effects like the tensor force and the hard core, even though they may have nothing to do with pole dominance in the Green function.

One has a rather simple interpretation of the three-body wave function, strongly reminiscent of Wheeler's famous concept of resonating group structures. For the "ideal case" considered, the complete wave function has the form

\[ \Psi = \sum_{123} \left\{ \frac{g(p_{23})}{\left[ \frac{3}{2} p_1^2 + k_{23}^2 - EM \right]} \right\} G(P) \]

where, according to YAMAGUCHI's definition, the expression in curly brackets is the wave function for the (23) system. Thus \( G(P) \) is the wave function of (1) with respect to the centre of mass of (23). Similar interpretation holds for the other two terms. It looks as though the particle (1) is playing just the role of a "spectator" in relation to (23) in this particular term of \( \Psi \), since the former appears only kinematically as a modification of the energy denominator by just the right amount necessary to compensate for the effect of its kinetic energy. The true dynamical effects are felt by \( G(P) \) through the integral equation it has to satisfy. The boundary conditions can be conveniently incorporated in \( G(P) \) corresponding to the situation one is interested in. Thus for the bound state problem (\( E < 0 \)), the integral equation for \( G(P) \) is strictly homogeneous, since the factor \( (1 - \lambda h(P)) \) does not develop any zero, as it should not. For the scattering problem, on the other hand, one has \( M^2 = -\alpha^2 + 3/4 \frac{k^2}{\pi} \), where \( k \) is the separation momentum between, say, a neutron and a deuteron. This makes \( (1 - \lambda h(P)) \) proportional to \( (\pi^2 - k^2) \) giving rise to a pole in the physical region. The appropriate boundary condition is now\(^7a\)

\[ G_l(p) = (2\pi)^3 \delta(p - k) + \frac{4\pi^2 \alpha(P)}{p^2 - k^2 - \epsilon} \]
where $a(P)$ has the direct interpretation of the scattering amplitude, with the condition $p^2 = k^2$. Since this derivation makes use only of the elementary principles of quantum mechanics based on the interpretation of the wave function, it is probably as simple, or at least not more complicated than the corresponding prescriptions in the Faddeev-Lovelace or Amado formulations. The generalization to include spin and isospin and degrees of freedom is quite straightforward, and is done with an equal amount of ease\(^7a\),\(^8\).

I wish to emphasize this point since it appears that the generally prevalent view is to regard the Faddeev-Lovelace and Amado formulations as more suited to scattering problems than this equally simple formulation based directly on the Schrödinger equation.

I would now like to describe the results of calculations of the following quantities:

(a) Binding energy of $^3H$ and n-d scattering lengths;
(b) Relativistic corrections;
(c) Coulomb energy and $^3H$-$^3He$ mass difference;
(d) Electromagnetic form factors of $^3H$ and $^3He$;
(e) The trinucleon system.

**Binding energy and scattering lengths:**

As this problem has become sufficiently "old" by now, even by these new techniques, I shall be brief on this point, but indicate the new results. A rather complete formulation of the 3N problem with tensor and hard core effects has been given recently\(^8\) and contains references to the earlier literature. The earliest results were obtained with "effective s-wave forces" of the Yamaguchi type, and naturally gave overbinding. Tensor forces improved the situation considerably but the overbinding with Yamaguchi forces still remained a feature.\(^9\) A slightly different set of parameters found by NAQVI and myself\(^10\) for the singlet s-potential and by NAQVI\(^11\) for the triplet potential, seemed to give a further improvement, but I have now come to believe that this last was probably spurious, and the fault probably lay with the choice of a singlet effective range of about

\*I am not quite sure how precisely this requirement works in the Lovelace formalism, which retains just the pole terms in the Green function.
2.4 F, against the more acceptable value of 2.7 F. As a matter of fact, AMADO and collaborators\textsuperscript{12)} and later PHILLIPS\textsuperscript{13)} found better results even with effective s-wave forces, just by taking $r_{0S} \approx 2.7 F$. We had refrained from doing this because a single Yamaguchi term fitted to a value of $r_{0S}$ higher than 2.4 F gave rather poor fits to p-p scattering even at moderate energies.\textsuperscript{10)} This is really a manifestation of the hard core effect, but in the spirit of this potential approach to the three-body problem, it should be incorporated through a second term in the singlet potential, so that the fits to scattering data, at least at moderate energies, may not be significantly distorted. We were unable to do this in the beginning because of the computational problems associated with four coupled integral equations, but now the results have just started coming. We already got very encouraging results for the doublet scattering length with Yamaguchi’s triplet potential and a second-order separable potential for the $1S_0$ state found recently by GUPTA\textsuperscript{14)} to fit the p-p $1S_0$ phase shifts up to 350 MeV (see Tables I and II).

Using Gupta's parameters (all sets of Table I, except (a)) the best $1S_0$ fit is obtained with the set (a') characterized by $\beta_s = 5.0$, $\beta_o = 10.0$, while for the other sets these fits are not as good. Roughly the trend is as follows. As $\beta_s$ is increased, $\beta_o$ has to be correspondingly decreased for a reasonable fit which is still a compromise over the "best"case, the degree of compromise being proportional to the amount of reduction in $\beta_o$ (and increase in $\beta_s$). In other words, as the core is made "softer" beyond a certain optimum value, the quality of the fit gets poorer. (The Naqvi set (a) does not, of course, conform to this trend, since it uses equal values of $\beta_s$ and $\beta_o$.) Comparison of the doublet scattering length with the Gupta set (a') and the Naqvi set (a) which were tuned to $r_{0S} = 2.7F$ and 2.355F respectively shows the appreciably less sensitivity of this number with $r_{0S}$ when taken in the background of the hard core, than without this background\textsuperscript{13a,8}).

We have no doubt that this potential would automatically give very good results for the binding energy of $^3H$, since the two problems are closely related, but the results for this parameter are still
It seems that both the hard core and the tensor force are playing a decisive role on the three-body system.

As for the quartet n-d scattering length, the case is essentially closed, since the repulsive kernel associated with the quartet n-d system makes it rather insensitive to approximations, and even the earlier results on this parameter agreed excellently with experiment.

The relativistic correction to the triton binding energy was evaluated by using Bhakar’s exact wave function to calculate the expectation value of the second order \((v/c)\) correction to the potential and kinetic energies on the basis of a theory given by SHIROKOV et al. 15

The relevant formulae are (for equal masses)

\[
\langle \frac{p p}{\omega} | \Delta V | \frac{p' p'}{\xi} \rangle = -\frac{1}{4} \left[ p^2 + \frac{1}{2} (p \cdot p) \left( p \cdot \frac{2}{2} \right) + \frac{1}{2} (p' \cdot p) \left( p' \cdot \frac{2}{2} \right) \right] \langle \frac{p' V}{\xi} \rangle
\]

where \(p\) and \(p'\) are the relative momenta, and \(\omega\) is the total c.m. momentum of a 2N system;

\[
\Delta T_i = -\frac{1}{8} P_i^4 M^{-3} \quad (i = 1, 2, 3).
\]

Thus the total correction \(\Delta E\) is defined by

\[
\Delta E = \langle \Delta T \rangle + \langle \Delta V \rangle,
\]

where

\[
\langle \Delta T \rangle = 3 \left( \psi, \Delta T_i \psi \right) / \left( \psi, \psi \right),
\]

\[
\langle \Delta V \rangle = 3 \left( \psi, \Delta V_{13} \psi \right) / \left( \psi, \psi \right),
\]

\(\psi\) being the triton wave function.

* We have just obtained a binding energy of 8.9 keV to go with a doublet scattering length of -135F using the Yamaguchi-triplet and the Naqvi-singlet potential(A.N. Mitra and G.L. Sohrenk to be published).
The kinetic energy correction was about 1.0 MeV (attractive) and the potential energy correction about 0.5 MeV (repulsive). This gave a nett effect of \( \sim 0.5 \) MeV towards higher binding. This leaves scope for reduction by hard core effects to the extent of about 10-15\%, in agreement with TABAKIN's model estimate.

The Coulomb correction for \( \text{He}^3 \) was also evaluated by using Bhakar's exact wave function. For a "point" proton, the Coulomb energy varied between 0.95 MeV and 1.06 MeV, according to the choice of the potentials, while for a "Hofstadter" proton, the variation was only from 0.85 MeV to 0.83 MeV, giving an average of 0.84 MeV. This is only about 10\% larger than the experimental mass difference of 0.764 MeV and seems to be just of the expected order of magnitude available for reduction by hard core effects. This is rather satisfactory, since in the context of the more modern value of \( a_{nn} \approx -18 \) F, there seems to be little scope for other effects to bridge the gap between theory and experiment.

It took us a considerable amount of time to work out the electromagnetic form factors of \( \text{H}^3 \) and \( \text{He}^3 \) within this formalism, and that too without hard core effects. The analysis was made closely on the lines of SCHIFF's pioneering work on the subject relating to the experiments of HOFSTADTER and collaborators. Our wave function is characterized by a relatively small \( S^f_1 \) state (\( \sim 1\% \)), a 4-5\% D-state and about 95\% totally symmetric S-state. This magnitude of \( S^f_1 \) state is compatible with the Gamow-Teller matrix elements for \( \text{H}^3 \) decay, the thermal neutron capture rate in deuterium and inelastic electron scattering from the triton. The features of direct experimental interest in this investigation are perhaps the near equality of the three radii \( a_{ch}(\text{H}^3) \), \( a_{mag}(\text{H}^3) \) and \( a_{mag}(\text{He}^3) \) at about 1.71 F, and a somewhat bigger value of \( a_{ch}(\text{He}^3) \) at 1.87 F. Our analysis indicates that while the agreement with experiment is poor with pure s-wave forces, inclusion of the tensor force improves the picture considerably and gives values for these various radii which fall short of experiment by about 10\%, once again within the scope of the hard core effect (in magnitude and sign). The difference between
the charge radii of H\(^3\) and He\(^3\) requires a positive slope for the neutron charge form factor, a result in agreement with the analysis of electron-deuteron scattering\(^{26}\) and bridges about 0.1 \(\text{F}\) of the difference 0.17 \(\text{F}\), leaving the balance of 0.07 \(\text{F}\) once again to hard core effects!

Finally, I would like to report on some results on the trineutron system\(^{27}\) which has been of considerable experimental interest since its possible existence was reported by SLAUS and collaborators.\(^{28}\)

From the theoretical point of view, this is a very simple system, in which neutrons are restricted by the Pauli principle to interact mainly in \(^1\text{S}_0\) and \(^3\text{P}\) states, and is ideally suited to this kind of exact three-body treatment. Actually the \(^1\text{S}_0\) force plays only a negative role, providing a mild repulsion in the state \((0 \frac{1}{2} \frac{1}{2})\) in an (LSJ) classification. The main attraction is provided by the \(^3\text{P}\) force which, however, is considerably weaker than the forces responsible for triton binding.

The state of strongest attraction turns out to have an axial vector \((L^P = 1^+)\) structure and from the magnitudes of the \(^3\text{P}_0\) versus \(^3\text{P}_2\) phase shifts in p-p scattering\(^{29}\), the \(J^P = \frac{1}{2}^+\) seems to have a slight edge over \(J^P = 3/2^+\). Our initial analysis on the basis of a comparison of the minimum strength of the \(^3\text{P}\) force needed to give zero binding, with the strength required to produce the \(^3\text{P}_0\) phase shifts of Bryan and Scott, suggested the possibility of a binding in this system. However, we have now come to believe that perhaps there was too much of a central force in our p-wave potential, since the three-body analysis was made on the basis of a dominant central force with a small spin-orbit admixture.\(^{30}\) We are now looking afresh at the problem, having first made a more accurate determination of the parameters to fit all the 3 p-wave phase shifts of Bryan and Scott. Unfortunately we find a dominant spin-orbit effect which complicates the analysis considerably. The outcome is not yet clear, but we consider the field still open. We are aware of a similar calculation by DAVIES and OKAMOTO\(^{31}\) and possibly by others, using variational methods.
with negative results. While we have nothing as such against variation-
al methods, we believe that such calculations involving $p$-wave orbital
structures like $L^P = 1^+$ require, for comparable accuracy, much more
elaborate trial functions than are needed, for example, for the binding
energy of the triton.

Although this talk was confined mainly to the work done by our
group in the past few years, I should like to mention that similar work
on somewhat complementary physical systems has been carried out by
AMADO and collaborators using pure $s$-wave potentials. Especially
interesting is their work on $n$-$d$ scattering above break-up threshold 32)
and deuteron break-up reactions 33) for which the inelasticity effect is
more important than tensor forces. It is also possible to use three-
body techniques for stripping reactions. 34)

A good deal of work using separable techniques has been
carried out by a number of workers in hypernuclear physics, especially
the hypertriton 35), $K^+ - d$ scattering 36) and double hyper-
fragments. 37)

Finally, one might hopefully add that if the non-relativistic
idea of quarks as the basic building stones for hadrons makes any
sense at all, here is a field representing vast possibilities 38) for
doing "high-energy" physics by such "low-energy" methods.
TABLE I

SINGLET PARAMETERS

\[-M < p \mid V \mid p' > = \lambda_{13} [f(p) f(p') - f_1(p) f_1(p')]\]

\[f(p) = (p^2 + \beta_s^2)^{-1}, \quad f_1(p) = np^2 (p^2 + \beta_0^2)^{-2}\]

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<th>$\lambda_{13}/\alpha^3$</th>
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<td>6.2</td>
<td>6.6</td>
<td>3.30</td>
<td>25.32</td>
<td>18F</td>
<td>2.7F</td>
</tr>
</tbody>
</table>

TABLE II

DOUBLET n-d SCATTERING LENGTHS

<table>
<thead>
<tr>
<th>Set</th>
<th>(Singlet parameters)</th>
<th>(Triplet parameters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>8.0 , 8.0</td>
<td>+.135F</td>
</tr>
<tr>
<td>(a')</td>
<td>5.5 , 10.0</td>
<td>+.180F</td>
</tr>
<tr>
<td>(b)</td>
<td>6.0 , 6.8</td>
<td>+.561F</td>
</tr>
<tr>
<td>(c')</td>
<td>6.0 , 6.8</td>
<td>+.606F</td>
</tr>
<tr>
<td>(f)</td>
<td>6.2 , 6.6</td>
<td>+.716F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Set</th>
<th>Set I (Y)</th>
<th>Set III (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>-.262F</td>
<td></td>
</tr>
<tr>
<td>(a')</td>
<td>-.216F</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>+.235F</td>
<td></td>
</tr>
<tr>
<td>(c')</td>
<td>+.353F</td>
<td></td>
</tr>
</tbody>
</table>

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