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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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ABDUS SALAM

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J. STRATHDEE

1967

PIAZZA OBERDAN

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June 1967

* To be submitted to Physical Review Letters.

** On leave of absence from Imperial College, London.

REGGEIZATION OF INTERNAL SYMMETRIES

In this note we suggest a new approach to higher symmetries, and particularly to the particle spectrum associated with them, based on a generalized partial wave analysis of the conventional S-matrix. We exploit the fact that partial wave analyses can be effected using (almost) any complete set of orthogonal functions and in particular (for certain classes of S-matrix elements) using orthogonal sets defined on the higher symmetry groups. The partial wave amplitudes turn out in such a scheme to depend on the Casimir invariants (C) of the symmetry group in question. An assumption of meromorphicity of the amplitudes for complex C then leads to the following results:

1) Reggeization of ordinary spin gives rise also to recurrences in internal symmetries.

2) The generalized Regge recurrences can be arranged in sequences which are in one-one correspondence to rungs of appropriate non-compact group towers.

To illustrate we consider Wigner's higher symmetry, $SU(4) \approx O(6)$. We label the set of 1-particle rest states $|G\rangle = |\lambda_m p_m q_m, \alpha\rangle$ where G denotes the Gel'fand pattern[1)]

$$\begin{pmatrix} \lambda_m & & p_m & & q_m \\ & a & & b & \\ & p & & q & \\ & & I & & \\ & & I_3 & & \end{pmatrix}$$

defined relative to the chain of subgroups

$$O(6) \supset O(5) \supset O(4) \supset O(3) \supset O(2)$$

and α is the reduced sub-pattern $\begin{pmatrix} a & & b \\ p & & q \\ & I & \\ & I_3 & \end{pmatrix}$

In a six-dimensional space $(x_1, \ldots, x_6)$ where $x_1\, x_2\, x_3$ correspond to the physical dimensions and $x_4\, x_5\, x_6$ to internal (I-spin) co-ordinates, we label the representations of

O(6)	rotations	in	$x_1\, x_2\, x_3\, x_4\, x_5\, x_6$	with	Casimirs	$\lambda_m\, p_m\, q_m$,
O(5)	"	"	$x_1\, x_2\quad x_4\, x_5\, x_6$	"	"	a b
O(4)	"	"	$x_2\quad x_4\, x_5\, x_6$	"	"	p q
O(3)	"	"	$x_4\, x_5\, x_6$	"	"	I
O(2)	"	"	$x_5\, x_6$	"	"	I_3

The parameters which enter into the labelling of an irreducible representation of O(6) satisfy the inequalities

$$\lambda \geqslant a \geqslant p_m \geqslant b \geqslant |q_m|$$

$$a \geqslant p \geqslant b \geqslant -q$$

$$p \geqslant I \geqslant |q|$$

$$I \geqslant I_3 \geqslant -I$$

They are all integers or half-integers. As is well known, the dimensionality of the representation is

$$\frac{1}{2!\,3!}\Big[(p_m + q_m + 1)(p_m - q_m + 1)(\lambda_m - p_m + 1)(\lambda_m - q_m + 2)\cdot$$
$$\cdot\,(q_m + \lambda_m + 2)(\lambda_m + p_m + 3)\Big]$$

while $\frac{1}{2}(p_m + q_m)$, $\frac{1}{2}(p_m - q_m)$ take the values 0, $\frac{1}{2}$, 1, 3/2, ... and $\lambda_m - p_m = 0, 1, 2, \ldots$ The spin-isospin (S,I)- content of the multiplet can be worked out: thus, for example, λ_m is the largest value of S_3 (or equally I_3) in the multiplet, p_m is the largest value of I_3 (or equally S_3) for a state with $S_3 = \lambda_m$ (for a state with $I_3 = \lambda_m$).

Some typical representations corresponding to (λ_m , p_m , q_m) with their (2S + 1 , 2I + 1) content are

$$\underset{\sim}{15} = (1\ 1\ 0) = (13)\,,\ (31)\,,\ (33)$$

$$\underset{\sim}{64} = (2\ 1\ 0) = (13)\,,\ (31)\,,\ (33)^2,\ (15)\,,\ (51)\,,\ (35)\,,\ (53)$$

$$\underset{\sim}{20} = (\tfrac{3}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}) = (22)\,,\ (44)$$

$$\underset{\sim}{84} = (\tfrac{5}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}) = (22)\,,\ (24)\,,\ (42)\,,\ (44)\,,\ (46)\,,\ (64)$$

Given two incoming particles in the centre-of-mass frame, $|q\,,\alpha_1,\alpha_2\rangle$, one can use an O(5) Clebsch-Gordan composition to form the state

$$|q\,\alpha\rangle = \sum_{\alpha_1\alpha_2} |q\,\alpha_1\,\alpha_2\rangle\langle\alpha_1\,\alpha_2|\alpha\rangle$$

(The intrinsic labels λ_{1m}, p_{1m}, q_{1m}, λ_{2m}, p_{2m}, q_{2m} are suppressed for brevity). The scattering amplitudes may be expressed in the form:

$$\langle\alpha|\,T(\theta)\,|\alpha'\rangle = \langle\alpha|\,T\,e^{-iJ_{31}\theta}\,|\alpha'\rangle$$

where θ denotes the scattering angle. It is our purpose to expand these amplitudes in a complete set of functions characterized by the labels α,α'. Making the dynamical hypothesis that the scattering operator possesses O(6)-symmetry, the appropriate functions clearly are the O(6) rotation functions

$$\langle G|\,e^{-i\theta J_{31}}\,|G'\rangle = \delta_{I_3 I_3'}\,\delta_{II'}\,\delta_{pp'}\,\delta_{qq'}\quad d^{\lambda_m p_m q_m}_{ab\,(pq)\,a'b'}(\theta)\,.$$

The string of δ-functions express the fact that J_{13} commutes with the I-spin and O(2456) sub-groups. Thus we shall write

$$\langle\alpha|T(\theta)|\alpha'\rangle = \delta_{I_3 I_3'}\,\delta_{II'}\,\delta_{pp'}\,\delta_{qq'} \sum_{\lambda_m p_m q_m} \langle ab|T^{\lambda_m p_m q_m}|a'b'\rangle\, d^{\lambda_m p_m q_m}_{ab(pq)a'b'}(\theta), \tag{1}$$

where

$$\langle ab|T^{\lambda_m p_m q_m}|a'b'\rangle = \sum_{pq} \int_0^{\pi} d\theta \sin^4\theta \; d^{\lambda_m p_m q_m}_{ab(pq)a'b'}(\theta) \;\langle ab\,pq|T(\theta)|a'b'pq\rangle \tag{2}$$

The summations in (1) are restricted by

$$\lambda_m \geqslant a \geqslant p_m \geqslant b \geqslant |q_m|,$$

and

$$\lambda_m \geqslant a' \geqslant p_m \geqslant b' \geqslant |q_m|,$$

so that p_m and q_m cover a finite range while λ_m varies from $\max(a, a')$ to $+\infty$.

We may go further and assume the existence of functions meromorphic in λ_m which interpolate the physical (integer or half-integer) points. With appropriate assumptions about asymptotic behaviour in the λ_m-plane we perform a Sommerfeld-Watson transformation obtaining thereby the expression

$$\langle\alpha|T(\theta)|\alpha'\rangle = \frac{1}{2i}\int \frac{d\lambda_m}{\sin\pi\lambda_m} \sum_{p_m q_m} \langle ab|T^{\lambda_m p_m q_m}|a'b'\rangle\; d^{\lambda_m p_m q_m}_{ab(pq)a'b'}(\theta) + $$

$$+ \text{ pole contributions.} \tag{3}$$

Disregarding the signature complications, the contribution of a pole at $\lambda_m = \alpha(t)$ would be

$$\frac{\langle ab | \beta^{p_m q_m} | a'b' \rangle}{\sin \pi\alpha} \quad d^{\alpha\, p_m q_m}_{ab(pq)a'b'}(\theta) \qquad (4)$$

The trajectory $\lambda_m = \alpha(t)$ would then tie together a sequence of O(6) representations $D^{\alpha\, p_m q_m}$ with

$$\alpha = \alpha_o \,,\ \alpha_o + 1 \,,\ \alpha_o + 2 \,,\ \ldots$$

where α_o is an integer or half-integer.

As an example let us consider the elastic scattering of an O(6) singlet by a 15-fold. For this simple case it is of course not necessary to know any O(5) Clebsch-Gordan coefficients. The O(5) contents of the initial and final states are fixed by

$$(a\,,\,b) = (1\,,\,1)\,,\ (1\,,\,0)$$

We have, therefore, the following amplitudes:

$$\langle 10 | T^{\lambda_m p_m q_m} | 10 \rangle \quad \text{with} \quad p_m = 0,1 \,,\quad q_m = 0$$

$$\langle 10 | T^{\lambda_m p_m q_m} | 11 \rangle \quad \text{"} \quad p_m = 1 \,,\quad q_m = 0$$

$$\langle 11 | T^{\lambda_m p_m q_m} | 11 \rangle \quad \text{"} \quad p_m = 1 \,,\quad q_m = 0,1\,,-1$$

leading to the trajectories

(I) : $(0\,,0\,,0) = \underset{\sim}{1}$, $(1\,,0\,,0) = \underset{\sim}{6}$, $(2\,,0\,,0) = \underset{\sim}{20}$, ...
(II) : $(1\,,1\,,0) = \underset{\sim}{15}$, $(2\,,1\,,0) = \underset{\sim}{64}$, $(3\,,1\,,0) = \underset{\sim}{175}$, ...
(III): $(1\,,1\,,1) = \underset{\sim}{10}$, $(2\,,1\,,1) = \underset{\sim}{45}$, $(3\,,1\,,1) = \underset{\sim}{126}$, ...
(IV) : $(1\,,1\,,-1) = \underset{\sim}{10}'$, $(2\,,1\,,-1) = \underset{\sim}{45}'$, $(3\,,1\,,-1) = \underset{\sim}{126}'$, ...

The high-energy behaviour of the scattering amplitudes should be dominated by the λ_m-pole with the largest value of Re(λ_m). This requires computation of the asymptotic behaviour of $d^{\lambda_m p_m q_m}_{ab(pq)a'b'}(\theta)$, a tedious calculation. However, we know that

$$d^{\lambda_m 00}_{a0(00)00}(\theta) \propto (\sin\theta)^a \, C^{a+2}_{\lambda_m - a}(\cos\theta) \sim (\cos\theta)^{\lambda_m} ,$$

and for the general case our conjecture* is that

$$d^{\lambda_m p_m q_m}_{ab(pq)a'b'}(\theta) \sim (\cos\theta)^{\lambda_m - (p_m - p) - |q_m - q|}$$

Thus we expect

$$\langle\alpha|T(\theta)|\alpha'\rangle \sim \frac{\langle ab|\beta^{p_m q_m}|a'b'\rangle}{\sin \pi\alpha} (\cos\theta)^{\lambda_m - (p_m - p) - |q_m - q|} \qquad (5)$$

From this viewpoint one sees that the dominating pole is no longer classified by the I-value of its trajectory but rather by the new quantum numbers p_m and q_m . Such a trajectory gives rise to supermultiplets generally combining several I-values (as well as spin values) when it passes through integer values of λ_m . The asymptotic

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* This conjecture is based on the analogy provided by the mini-universe (see below) in which the rest symmetry is O(4). For this case the relevant functions are $d^{pq}_{j\lambda j'}(\theta)$ and their asymptotic behaviour has been derived by TOLLER[3]:

$$d^{pq}_{j\lambda j'}(\theta) \sim (\cos\theta)^{p - |q - \lambda|}$$

expression above automatically takes account of contributions from all these relevant multiplets. The formalism thus far reggeizes both spin and I-spin, treating both on equal footing.

There is, however, a real sense in which spin is differentiated sharply from I-spin. The physical S-matrix is a sub-matrix operating within the 0123 sub-space and diagonal within 456. This, as is well known, means that physical unitarity must act as a breaker of the exact O(6) symmetry postulated above. Our viewpoint in this note is that the symmetry breaking will occur and will in fact split the super-multiplets according to spin; however the overall pole structure in λ_m-plane may be expected to survive. To study the spin-dependence explicitly and to exhibit the form of the expressions which arise, we consider a model theory in a mini-universe[2)] of two space-dimensions with an I-space of two dimensions also. Thus instead of O(6), take O(4) as a rest-symmetry where in a four-dimensional space $(x_1\, x_2\, x_3\, x_4)$ where $x_1\, x_2$ correspond to the physical dimensions and $x_3\, x_4$ to internal co-ordinates. We adopt the chain of rotation groups

O(4)	in $x_1\, x_2\, x_3\, x_4$	with representations labelled by	p , q
O(3)	" $x_2\, x_3\, x_4$	" " " "	f
O(2)	" $x_3\, x_4$	" " " "	m (= I-spin)

The 1-particle rest states are written $|\alpha\rangle$ where α now denotes the Gel'fand pattern

$$\begin{pmatrix} p & & q \\ & f & \\ & m & \end{pmatrix}$$

$$p \geqslant f \geqslant |q| \quad , \quad f \geqslant m \geqslant -f \; .$$

The dimensionality of the O(4) representations is given by

$$(p + q + 1)(p - q + 1) \quad .$$

The scattering amplitude takes the form

$$\langle fm | T e^{-i\theta J_n} | f'm' \rangle = \delta_{mm'} \sum_{pq} \langle f | T^{pq} | f' \rangle \, d^{pq}_{fmf'}(\theta) . \qquad (6)$$

The analogue in this theory of a familiar partial wave expansion takes the form

$$\langle fm | T e^{-i\theta J_n} | f'm' \rangle = \delta_{mm'} \sum_{\lambda} \langle fm | T^{\lambda} | f'm \rangle \; d^{\lambda}(\theta) \qquad (7)$$

where $d^{\lambda}(\theta) = e^{-i\lambda\theta}$.

The connection between the amplitudes T^{λ} and the T^{pq} is obtained by Fourier expanding $d^{pq}_{fmf'}(\theta)$ in terms of $d^{\lambda}(\theta)$,

$$d^{pq}_{fmf'}(\theta) = \sum_{\lambda} (pq\,fm|\lambda) \, d^{\lambda}(\theta) \, (\lambda | pq\,f'm)$$

There are a number of ways of expressing the coefficients, $(pq\ fm\,|\,\lambda)$, for example, as a Clebsch-Gordan symbol of O(3).

$$(pq\,fm|\lambda) = \left\langle fm \,\middle|\, \tfrac{p+q}{2}\,\tfrac{m+\lambda}{2},\ \tfrac{p-q}{2}\,\tfrac{m-\lambda}{2} \right\rangle$$

With the help of this expansion we can write

$$\langle f\, m | T^{\lambda}(t) | f' m \rangle = \sum_{pq} (pq\ fm | \lambda) \langle f | T^{pq}(t) | f' \rangle (\lambda | pq\ f' m) \qquad (8)$$

Presumably an analogous (though much more complicated) expression could be written down which connects the O(6) amplitudes $\langle ab | T^{\lambda_m p_m q_m} | a'b' \rangle$ to the ordinary partial wave amplitudes $\langle ab\ pq | T^{J} | a'b'\ pq \rangle$. On the assumption that such relations can be continued to complex values of λ_m and J (or p and λ for the mini-theory), it would follow that a simple pole in the λ_m plane would correspond to a sequence of satellite poles in the J-plane (likewise for the p and λ poles). <u>We must emphasize that the residues of these satellite poles are fixed relative to the parent by the coefficients $(pq\ fm | \lambda)$ $(\lambda | pq\ f'm)$</u>.

The unitary requirement on the physical S-matrix states that $\mathrm{Im}\, T^{\lambda} = k |T^{\lambda}|^2$. Using (8) we thus derive a non-linear relation between the amplitudes T^{pq}. This relation is unlikely to be satisfied without symmetry breaking. It is a problem for future investigation how such a symmetry breaking will affect the location (and the existence) of the poles postulated in the p-plane.

It is amusing that in the development up to now we have not needed any embedding of the (static) SU(4) symmetry in a relativistic group structure. However, the implications of our higher symmetry as regards particle spectra are not exhausted by the above discussion. In line with the recent developments in Regge theory which have revealed the necessity of grouping Regge trajectories into infinite families of daughters, we may expect to find analogous effects in the present scheme. Thus[3] at

$t = 0$ (in elastic scattering) the disappearance of the momentum transfer vector gives rise to an additional symmetry which in our model is the non-compact symmetry O(6,1). This means that instead of expanding the t-channel amplitudes in representations of the rest symmetry O(6) we may, at the unphysical point $t = 0$, expand in representations of O(7).[4] The O(6) amplitudes $T^{\lambda_m p_m q_m}$ at $t = 0$ (and at all t for the special[5] "flipless case" $\langle ab | T^{\lambda_m p_m q_m}_{(pq)} | a' b' \rangle$ where the cross-channel incoming and outgoing values of p,q equal p', q') could be expressed in terms of the O(7) amplitudes $T^{\Lambda PQ}$ with

$$\Lambda = \lambda_m + n$$

$$P = p_m + n'$$

$$Q = |q| + n''$$

where (n , n' , n") takes integer values; (the range of n' , n" are restricted by the Gel'fand inequalities , see below) . Assuming that such an expression can be continued in the complex λ_m plane, then a pole in $T^{\Lambda PQ}$ at ΛPQ would correspond to the set of generalized daughter poles in $T^{\lambda_m p_m q_m}$.

Returning to the example, $1 + 15 \to 1 + 15$, if we assume that the 15 and 1 have the same mass then at $t = 0$ we have pure 15-folds for initial and final states. These correspond to the values

$$(\lambda_m, p_m, q_m) = (1, 1, 0)$$

and so if we expand the amplitude in O(7) representations, $D^{\Lambda PQ}$ then we need to include in the sum only those representations with

$$\Lambda \geqslant 1 \geqslant P \geqslant 1 \geqslant Q \geqslant 0$$

i.e., $Q = 1, 0$, $P = 1$ and $\Lambda = 1 + n$.

We can make a table to indicate the content of these representations. Corresponding to P = 1, Q = 0, we have

$$\Lambda = 1 \quad \text{containing} \quad \lambda_m = 1$$
$$\Lambda = 2 \quad \text{"} \quad \lambda_m = 2 , 1$$
$$\Lambda = 3 \quad \text{"} \quad \lambda_m = 3 , 2 , 1$$
$$\ldots$$
$$\Lambda = n \quad \text{"} \quad \lambda_m = n , n-1 , \ldots , 1$$

The λ_m values listed here may be arranged into columns - each column having the same content as an irreducible (non-degenerate) representation of the non-compact group O(6,1). Thus, the set $(\lambda_m , p_m , q_m) = (1 , 1 , 0) + (2 , 1 , 0) + (3 , 1 , 0) + \ldots$ corresponding to the O(6) supermultiplets $\underset{\sim}{15}$, $\underset{\sim}{64}$, $\underset{\sim}{175}$,..., which is repeated infinitely many times, has the content of an irreducible representation of O(6,1). Likewise P = 1 , Q = 1 reproduce the content of a tower with rungs of $\underset{\sim}{10}$, $\underset{\sim}{45}$, $\underset{\sim}{126}$,..., particles. The "towers" we are thus generating through a process of reggeizing of course do not correspond to one <u>single irreducible</u> representation of the non-compact O(6,1). Rather, each rung belongs to a different irreducible representation which is labelled through a t-dependent Casimir operator $\Lambda(t)$. This is a situation fully familiar for the archetypal example of all non-compact ideas, the hydrogen atom whose O(4)-symmetric levels can be arranged in a tower which has the same content as a representation of the non-compact O(4,1). However there is no <u>single irreducible</u> representation of O(4,1) to which this tower can be supposed to correspond except in the highly degenerate idealization where all levels collapse on to one mass value.

The major value of the theory presented here, in our view, lies in the definite prediction of a possible particle spectrum and in the implicit internal symmetry-dependent high-energy behaviour (relation (5)). We have not worked out the case of SU(6) in detail but our conjecture is that the appropriate Casimir which gets reggeized is the quark number.

We intend to return to this problem. The present theory is to be contrasted with the predictions of groups like U(6) x O(3), the distinction of the predicted particles lying, for example, in the existence or non-existence of SU(3) 10's and 27's among the higher meson resonances. Presumably one can also make statements in the present theory regarding high-energy behaviour of I-spin flipping amplitudes in photon-induced interactions.

ACKNOWLEDGMENT

Our appreciation is due to Drs. R.L. Anderson, J. Fischer and M.A. Rashid who introduced us to Gel'fand patterns and to Dr. R. Rączka who also clarified some points in their use.

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2) R. DELBOURGO, ABDUS SALAM and J. STRATHDEE, Physics Letters (to be published).

3) M. TOLLER (CERN preprints, 1967).

4) We follow FREEDMAN and WANG (Phys. Rev. Letters 18, 863 (1967)) in making the expansion in that part of the Mandelstam plane where the non-compact group O(6,1) becomes effectively the compact O(7). This enables us to employ finite-dimensional representations.

5) R. DELBOURGO, ABDUS SALAM and J. STRATHDEE, (Imperial College, London, preprint, 1967).

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