IC/67/21

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April 1967

† To be submitted for publication

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1. INTRODUCTION

In a series of recent papers TOLLER[1] has made a fundamental advance in noticing and exploiting the extra $O(3,1)$ invariance possessed by the elastic forward scattering amplitude. The new invariance leads him to an expansion of the amplitude in terms of unitary representations of the group $SO(3,1)$. This is in contrast to the normal partial wave analysis which is an expansion in terms of unitary representations of $SO(3)$ - a much smaller structure. The new expansion - embodying the higher symmetry - leads to newer insights; for example, if the new partial wave amplitude, labelled with the four-dimensional generalized angular momentum σ is assumed to be meromorphic for complex σ, one finds that to each pole in σ-plane there corresponds a family of integrally spaced daughter poles in the complex J plane for the partial wave amplitude a^J. This parent-daughter phenomenon anticipated in the works of GRIBOV and VOLKOV[2], DOMOKOS and SURANYI[3], and rediscovered recently by FREEDMAN and WANG[4], finds its most complete expression in Toller's development insofar as, in contrast to the other authors, Toller takes full account of the very essential complications introduced by spin.

The aim of the present paper is to generalize Toller's work on elastic forward scattering and to expand the general two-body amplitude for all values of momentum transfer in terms of unitary representations of $SO(3,1)$. That such a programme is feasible and that it may be expected to lead to new results has already been demonstrated by OAKES[5] and DOMOKOS[6] for scattering of equal or unequal mass particles when no spins are involved. In this simple case the amplitude is a function of scalar products of incoming and outgoing momenta. Such a function (or rather its analytic continuation to a Euclidean metric) can always be expanded in terms of a complete set of four-dimensional Gegenbauer polynomials.

Our concern in this paper is covariant inclusion of spin. One simple suggestion for doing this would be to separate out all spin-dependent factors and to write out the general amplitude in terms of scalar amplitudes of the type used in Mandelstam's representation. For these one could certainly use the Oakes-Domokos Gegenbauer expansion.

Unfortunately this straightforward procedure suffers from two defects. Firstly the choice of the amplitudes is arbitrary. Secondly, one would lose touch completely with Toller's development for forward scattering which employs the more physically transparent helicity formalism and which must serve as a necessary boundary condition for any $O(3,1)$ expansion of the general amplitude. In this paper we find such an expansion, using conventional helicity formalism as far as possible. Unfortunately the formalism gets involved in a multitude of indices which somewhat obscure the simplicity of the basic ideas.

Even at the risk of repetition we therefore present a résumé of the argument in this introduction.

Consider scattering of two incoming particles of momenta, spins and helicities p_i, S_i, λ_i, $i = 1,2$ to outgoing particles $i = 3,4$. Instead of the conventional S-matrix, we choose to discuss the equivalent transition matrix $\tilde{S}$ which takes p_1, $p_3 \rightarrow p_2$, p_4. This can be expressed in terms of three independent vectors $P = p_1 - p_3 = p_4 - p_2$ (momentum transfer) and the relative momenta $q = p_1 + p_3$, $q' = p_2 + p_4$, in the form

$$(q'; S_2\lambda_2, S_4\lambda_4 | \tilde{S}(P) | q; S_1, \lambda_1, S_3, \lambda_3).$$

The first problem we face is that of covariant composition of S_1, λ_1 and S_3, λ_3. For Toller's case of zero momentum transfer $P = 0$ and $p_1 = p_3$ so that one can simply "add" spins and helicities. In general, of course, the very definition of helicity ties momenta to spins and these must be decoupled before any "addition" is possible. Most of the prepossessing nature of our formulae comes about this conceptually very simple step, with its standard procedure of passing from the helicity basis $|p, \lambda|$ to the well-known[8] (dotted and undotted) spinor basis $|p, A\rangle$ and $|p, \dot{A}\rangle$. We thus employ, as it were, an M-function-like approach and pass from $|q; S_1\lambda_1, S_3\lambda_3\rangle$ to $|q; S_1 A_1, S_3 A_3\rangle$ and then, finally, after a Clebsch-Gordan composition to "states" $|q, J, A\rangle$ reducing the problem to that of an expansion of $\langle q' J' \dot{A}' | \tilde{S}(P) | q J A \rangle$.

This is the step where Toller starts. For the forward scattering situation ($P_\mu = 0$) he notices that the state q' is created from state q by a simple Lorentz transformation. In the frame where we take $q + q'$ along t and $q - q'$ along the z-axis, the q to q' transformation is an O3 rotation through the angle ζ between q and q'.

Thus $$\langle q' J' \dot{A}' | \tilde{S}(0) | q J A \rangle \sim \langle J' \dot{A}' | \tilde{S}(0)\, e^{-i \zeta J_{03}} | J A \rangle \qquad (1.1)$$

so that the expansion problem is simply the problem of writing matrix elements of the Lorentz rotation operator on the right of (1). These matrix elements are well known; they have been computed by STRÖM[7], TOLLER[1], DAO WONG DUC[7] and others and are conventionally written as $D^{j_0\sigma}_{S'A',SA} = \delta_{A'A}\, d^{j_0\sigma}_{S'AS}(\zeta)$. The labels j_0 and σ correspond to the two invariants of SO(3,1) – one discrete (j_0) and the other continuous (σ). The $\tilde{S}$ operator is a sum over j_0 and an integral over σ and there is no helicity flip ($\lambda_1 - \lambda_3 = \lambda_2 - \lambda_4$) as one should expect for forward scattering.

Now, although Toller has not considered this case, his considerations apply equally to the case when $\tilde{S}$ is an invariant function of the four-vector P_μ (i.e., $\tilde{S} = \tilde{S}(P^2)$). This is the key observation in what follows; linked with this is the observation that such matrix elements are flipless. Our general procedure then is simply to express spin-flip amplitudes as a series of kinematic terms*) multiplied into (an equal number of) flipless amplitudes of the form $(\sqrt{-P^2})^{\mu_{\varepsilon_P}} (q' J' \dot{B} \,|\, \tilde{S}(P^2) \,|\, q\ J\ \dot{B})$. These amplitudes are as it were half way between helicity and scalar-invariant amplitudes and are constructed in Sec. 3 so as to give Toller's amplitude naturally in the limit $P_\mu = 0$.

In Sec. 4 is exhibited the parent-daughter phenomenon which arises when the O(3,1) expansion is recast into the conventional expansion based on the functions $d^j_{mm'}(\theta)$. We expect that, besides the phenomena, the new separation of kinematics and dynamics achieved through the use of the new expansion in this paper may prove of more fundamental significance for the analysis of the scattering amplitudes than the separation achieved in the conventional partial wave analysis.

*) A similar but not identical isolation of spin-flip factors from helicity amplitudes was carried out by GELL-MANN, GOLDBERGER, LOW, MARX and ZACHARIASEN (Phys. Rev. 133, B, 145, (1964)) in defining their new amplitudes for reggeisation.

2. SPIN DECOUPLING

The representation of the Poincaré group associated with a physical particle of mass $m > 0$, spin S, helicity λ, and positive energy is spanned by the set of basis vectors

$$|p S \lambda\rangle = e^{-i\varphi J_{12}} e^{-i\theta J_{31}} e^{i\varphi J_{12}} e^{-i\alpha J_{03}} |\hat{p} S \lambda\rangle = U(L_p)|\hat{p} S \lambda\rangle \tag{2.1}$$

where $$\hat{p}_\mu = (m, 0, 0, 0) \tag{2.2}$$

and $$p_\mu = m(\mathrm{ch}\alpha, \mathrm{sh}\alpha \sin\theta \cos\varphi, \mathrm{sh}\alpha \sin\theta \sin\varphi, \mathrm{sh}\alpha \cos\theta) \tag{2.3}$$

with $$0 \leqslant \varphi < 2\pi, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \alpha < \infty . \tag{2.4}$$

The $J_{\mu\nu}$ operators denote generators of homogeneous Lorentz transformations. The subset of states with $p = \hat{p}$ is defined by the SO(3) conditions,

$$\begin{aligned} \underset{\sim}{J}^2 |\hat{p} S \lambda\rangle &= S(S+1)|\hat{p} S \lambda\rangle, \\ J_{12}|\hat{p} S \lambda\rangle &= \lambda|\hat{p} S \lambda\rangle. \end{aligned} \tag{2.5}$$

Under homogeneous Lorentz transformations we have, from Wigner's well-known theory

$$U(\Lambda)|p S \lambda\rangle = \sum_{\lambda'} |\Lambda p S \lambda'\rangle D^S_{\lambda'\lambda}(L^{-1}_{\Lambda p} \Lambda L_p) \tag{2.6}$$

The crucial point is that D^S is a representation of the SO(3) little group; this form of (2.6) is possible because the transformation $L^{-1}_{\Lambda p} \Lambda L_p$ itself belongs to SO(3).

One of the most important ingredients of the development presented here turned out to be the necessity to decouple the p-dependence of the spin transformations. The formalism for doing this is well known[8]. One modifies the basis system, the connection between the modified basis and the set (2.1) being provided by the (2S + 1)-dimensional representations of the homogeneous group. In order to fix the notation we shall review briefly some of the features of these representations.

It is usual in treating the finite-dimensional representations of SO(3,1) to express the generators, $J_{\mu\nu}$, in the form

$$J_{12} = K_3 + L_3 \ , \ i\, J_{o3} = K_3 - L_3 \ , \ \text{cyclically} \tag{2.7}$$

where the operators $\underline{K}$ and $\underline{L}$ obey the commutation rules of SO(3) x SO(3), i.e.

$$[K_1 \ , K_2] = i\, K_3 \quad ,[L_1 \ , L_2] = i\, L_3 \ \text{cyclically}$$

and

$$[K_i \ , L_j] = 0 \tag{2.8}$$

The irreducible representations, $D^{k\ell}$, are then specified by a pair of integer or half-integer numbers (k,ℓ) where

$$\underline{K}^2 = k(k+1) \qquad , \qquad \underline{L}^2 = \ell(\ell+1) \tag{2.9}$$

There are two alternative sets of basis vectors which are useful for expressing the finite-dimensional representations: one which diagonalizes K_3 and L_3 and one which diagonalizes $\underline{J}^2$ and J_{12}. These are written

$$|A \ , \dot{B}\rangle \ , \quad A = -k, -k+1, \ \ldots \ , k \ \ ; \ \ B = -\ell, -\ell+1, \ \ldots \ , \ell \tag{2.10}$$

and
$$|j\,\lambda\rangle \ \ ; \quad j = |k-\ell|, |k-\ell|+1, \ \ldots \ , k+\ell \ \ ; \ \ \lambda = -j, \ \ldots \ , j \tag{2.11}$$

They satisfy the eigenvalue equations

$$K_3 \,|A\,\dot{B}\rangle = A\,|A\,\dot{B}\rangle \ , \qquad L_3\,|A\,\dot{B}\rangle = B\,|A\,\dot{B}\rangle, \tag{2.12}$$

$$\underline{J}^2|\,j\lambda\rangle = j(j+1)\,|\,j\lambda\rangle, \quad J_{12}\,|j\lambda\rangle = \lambda\,|\,j\lambda\rangle \tag{2.13}$$

Since $\underline{J} = \underline{K} + \underline{L}$, the connection between these basis systems is provided by the ordinary Clebsch-Gordan coefficients.

$$|j\lambda\rangle = \sum_{AB} |A\dot{B}\rangle\langle kA, \ell B|j\lambda\rangle ,$$

$$|A\dot{B}\rangle = \sum_{j\lambda} |j\lambda\rangle\langle j\lambda|kA, \ell B\rangle \tag{2.14}$$

We give examples of finite transformations in the $|A\,\dot{B}\rangle$ system. Firstly, corresponding to a purely spatial rotation,

$$\begin{aligned} U(\varphi, \theta, \psi) &= e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\psi J_{12}} \\ &= e^{-i\varphi K_3}\, e^{-i\theta K_2}\, e^{-i\psi K_3}\, e^{-i\varphi L_3}\, e^{-i\theta L_2}\, e^{-i\psi L_3} , \end{aligned} \tag{2.15}$$

we have

$$U(\varphi, \theta, \psi)|A\,\dot{B}\rangle = \sum_{A'B'} |A'\dot{B}'\rangle\, D^k_{A'A}(\varphi,\theta,\psi)\, D^\ell_{B'B}(\varphi,\theta,\psi) , \tag{2.16}$$

where $D^k_{A'A}$ and $D^\ell_{B'B}$ denote the usual SO(3) rotation matrices:

$$D^k_{A'A}(\varphi, \theta, \psi) = e^{-iA'\varphi}\, d^k_{A'A}(\theta)\, e^{-iA\psi} , \tag{2.17}$$

and similarly for $D^\ell_{B'B}$. It will suffice to exhibit only the pure Lorentz transformation in the 03-plane since any other can be built up from this one together with appropriate spatial rotations. Using $iJ_{03} = K_3 - L_3$ we get

$$e^{-i\alpha J_{03}}\,|A\dot{B}\rangle = |A\dot{B}\rangle\, e^{-\alpha(A-B)} . \tag{2.18}$$

It is a simple matter to express these transformations in the $|j\lambda\rangle$ basis. Thus

$$U(\varphi, \theta, \psi)\,|j\lambda\rangle = \sum_{\lambda'} |j\lambda'\rangle\, D^j_{\lambda'\lambda}(\varphi, \theta, \psi), \tag{2.19}$$

$$e^{-i\alpha J_{03}} |j\lambda\rangle = \sum_{j'} |j'\lambda\rangle \, d^{k\ell}_{j'\lambda j}(\alpha) , \qquad (2.20)$$

where

$$d^{k\ell}_{j'\lambda j}(\alpha) = \sum_{AB} \langle j'\lambda | kA, \ell B\rangle \, e^{-\alpha(A-B)} \langle k\ell, \ell B | j\lambda\rangle . \qquad (2.21)$$

Of particular interest to us in the following are the representations D^{s0} and D^{0s} which are irreducible under SO(3). Since the Clebsch-Gordan coefficients reduce to Kronecker symbols in these cases the distinction between the bases (2.10) and (2.11) disappears. For the finite transformations dealt with above we have

$$D^{s0}: \quad \left.\begin{aligned} U(\varphi,\theta,\psi)\,|s\ A\rangle &= \sum_{A'} |s\ A'\rangle\, D^{s}_{A'A}(\varphi,\theta,\psi) \\ e^{-i\alpha J_{03}}\,|s\ A\rangle &= |s\ A\rangle\, e^{-\alpha A} \end{aligned}\right\} \qquad (2.22)$$

$$D^{0s}: \quad \left.\begin{aligned} U(\varphi,\theta,\psi)\,|s\ \dot{A}\rangle &= \sum_{A'} |s\ \dot{A}'\rangle\, D^{s}_{A'A}(\varphi,\theta,\psi) \\ e^{-i\alpha J_{03}}\,|s\ \dot{A}\rangle &= |s\ \dot{A}\rangle\, e^{\alpha A} \end{aligned}\right\} \qquad (2.23)$$

Finally let us remark that the decomposition of a direct product of two irreducible representations is little more complicated than the corresponding problem in SO(3). Thus,

$$D^{k_1\ell_1} \otimes D^{k_2\ell_2} = \sum_{k=|k_1-k_2|}^{k_1+k_2} \; \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} D^{k\ell} \qquad (2.24)$$

or, in terms of basis vectors,

$$|k_1 A_1\, l_1 \dot{B}_1,\, k_2 A_2\, l_2 \dot{B}_2\rangle = \sum_{kl} |k_1 k_2\, k A,\, l_1 l_2\, l \dot{B}\rangle \langle kA|k_1A_1, k_2A_2\rangle \langle l\dot{B}|l_1\dot{B}_1, l_2\dot{B}_2\rangle$$

or

$$|k_1 l_1\, j_1\lambda_1,\, k_2 l_2\, j_2\lambda_2\rangle = \sum_{klj} |k_1 l_1\, k_2 l_2,\, kl\, j\lambda\rangle \langle j\lambda|j_1\lambda_1, j_2\lambda_2\rangle \cdot \tag{2.25}$$
$$\cdot \langle (k_1 l_1) j_1, (k_2 l_2) j_2; j | (k_1 k_2) k, (l_1 l_2) l; j\rangle$$

where, in the last expression a 9-j symbol has made its appearance.

Returning now to the problem of decoupling the p-dependence in the transformation law (2.6) we see that since D^{SO} and D^{OS} are irreducible under spatial rotations, it is always possible to write

$$\begin{aligned} D^{S}_{\lambda'\lambda}(L^{-1}_{\Lambda p}\,\Lambda\, L_p) &= D^{SO}_{\lambda'\lambda}(L^{-1}_{\Lambda p}\,\Lambda\, L_p) \\ &= \sum_{AA'} D^{SO}_{\lambda' A'}(L^{-1}_{\Lambda p})\, D^{SO}_{A'A}(\Lambda)\, D^{SO}_{A\lambda}(L_p) \end{aligned} \tag{2.26}$$

or, alternatively,

$$\begin{aligned} D^{S}_{\lambda'\lambda}(L^{-1}_{\Lambda p}\,\Lambda\, L_p) &= D^{OS}_{\lambda'\lambda}(L^{-1}_{\Lambda p}\,\Lambda\, L_p) \\ &= \sum_{AA'} D^{OS}_{\lambda' A'}(L^{-1}_{\Lambda p})\, D^{OS}_{A'A}(\Lambda)\, D^{OS}_{A\lambda}(L_p)\,. \end{aligned} \tag{2.27}$$

Let us therefore define the modified basis systems

$$|p\ S\ A\rangle = \sum_{\lambda} |p\,S\lambda\rangle\, D^{SO}_{\lambda A}\ (L^{-1}_{p}) \tag{2.28}$$

and

$$|p\ S\ \dot{A}\rangle = \sum_{\lambda} |p\,S\lambda\rangle\, D^{OS}_{\lambda A}\ (L^{-1}_{p}) \tag{2.29}$$

It follows immediately from (2.6) that under a homogeneous Lorentz transformation these states transform respectively according to the laws

$$U(\Lambda)\,|p\; S\; A\rangle = \sum_{A'} |\Lambda p\; S\; A'\rangle\; D^{so}_{A'A}(\Lambda) \qquad (2.30)$$

$$U(\Lambda)\,|p\; S\; \dot{A}\rangle = \sum_{A'} |\Lambda p\, S\dot{A}'\rangle\; D^{os}_{A'A}(\Lambda) \qquad (2.31)$$

As it happens, the set (2.29) transforms contragrediently to the set (2.28). This follows from the non-unitary nature of the finite-dimensional representations:

$$\left(D^{so}_{AB}(\Lambda)\right)^{*} = D^{os}_{BA}(\Lambda^{-1}) \qquad (2.32)$$

If we adopt the notational convention

$$\langle\, |p\; S\; A\rangle^{*} = \langle p\; S\; \dot{A}|\,\rangle, \quad \langle\, |p\; S\; \dot{A}\rangle^{*} = \langle p\; S\; A|\,\rangle$$

then, corresponding to (2.28) and (2.29), respectively, we must have

$$\langle p\; S\; \dot{A}| = \sum_{\lambda} D^{os}_{A\lambda}(L_p)\,\langle p S \lambda| \qquad (2.33)$$

$$\langle p\; S\; A| = \sum_{\lambda} D^{so}_{A\lambda}(L_p)\,\langle p S \lambda| \qquad (2.34)$$

Thus, in particular, the orthogonality condition reads:

$$\delta(p^2 - m^2)\langle p\; S\; A\,|\, p'\, S\; B\rangle = \delta_{AB}\,\delta(p - p').$$

Let us note finally the explicit form taken by the matrices in (2.28) and (2.29) when L_p refers to the boost defined in (2.1):

$$|p\; S\; A\rangle = \sum_{\lambda} |p\; S\; \lambda\rangle\; e^{-i\lambda(\varphi + i\alpha)}\; d^{s}_{\lambda A}(-\theta)\; e^{iA\varphi}, \qquad (2.35)$$

$$|p\; S\; \dot{A}\rangle = \sum_{\lambda} |p\, S\, \lambda\rangle\; e^{-i\lambda(\varphi - i\alpha)}\; d^{s}_{\lambda A}(-\theta)\; e^{iA\varphi} \qquad (2.36)$$

The application for which the above formalism has been developed concerns the analysis of T-matrix elements. Let us consider the process $\langle p_3\, S_3\, \lambda_3\,,\, p_4\, S_4\, \lambda_4 | T | p_1\, S_1\, \lambda_1\,,\, p_2\, S_2\, \lambda_2 \rangle$ where the λ's denote helicity. From this matrix element we can define a new one by

$$\langle p_3\, S_3\, \lambda_3\,,\, p_4\, S_4\, \lambda_4 | T | p_1\, S_1\, \lambda_1\,,\, p_2\, S_2\, \lambda_2 \rangle =$$

$$= \sum_{A_1 \cdots A_4} D^{S_3 0}_{\lambda_3 A_3}(L^{-1}_{p_3})\, D^{0 S_4}_{\lambda_4 \dot{A}_4}(L^{-1}_{p_4}) \langle p_3 S_3 A_3, p_4 S_4 \dot{A}_4 | T | p_1 S_1 A_1, p_2 S_2 \dot{A}_2 \rangle \cdot$$
$$\cdot\, D^{S_1 0}_{A_1 \lambda_1}(L_{p_1})\, D^{0 S_2}_{\dot{A}_2 \lambda_2}(L_{p_2}) \qquad (2.37)$$

The newly defined matrix element, which is a type of M-function, satisfies the invariance condition

$$\langle p_3\, S_3\, A_3\,,\, p_4\, S_4\, \dot{A}_4 | T | p_1\, S_1\, A_1\; p_2\, S_2\, \dot{A}_2 \rangle =$$

$$= \sum_{B_1 \cdots B_4} D^{S_3 0}_{A_3 B_3}(\Lambda^{-1})\, D^{0 S_4}_{\dot{A}_4 \dot{B}_4}(\Lambda^{-1}) \langle \Lambda p_3 S_3 B_3, \Lambda p_4 S_4 \dot{B}_4 | T | \Lambda p_1 S_1 B_1, \Lambda p_2 S_2 \dot{B}_2 \rangle \cdot$$
$$D^{S_1 0}_{B_1 A_1}(\Lambda)\, D^{0 S_2}_{\dot{B}_2 \dot{A}_2}(\Lambda) \qquad (2.38)$$

A more compact and convenient notation for this matrix element can be introduced if we define the three independent vectors

$$P = p_1 - p_3 = p_4 - p_2$$

$$2q = p_1 + p_3$$

$$2q = p_2 + p_4 \qquad (2.39)$$

Let us therefore couple the spins S_1 S_3 and S_2 S_4 writing

$$\langle p_3 S_3\, A_3\,,\, p_4\, S_4\, \dot{A}_4 | T | p_1\, S_1\, A_1\; p_2\, S_2\, \dot{A}_2 \rangle =$$

$$= \sum_{JJ'} \langle S_4 A_4 | J'A', S_2 A_2 \rangle \langle q' J' \dot{A}' | T(P) | q J A \rangle \langle S_3 A_3, JA | S_1 A_1 \rangle \tag{2.40}$$

The matrix element $\langle q' J' \dot{A}' | T(P) | q J A \rangle$ satisfies the invariance condition

$$\langle q' J' \dot{A}' | T(P) | q J A \rangle = \sum_{BB'} D^{oJ'}_{A'B'}(\Lambda^{-1}) \langle \Lambda q J' \dot{B}' | T(\Lambda P) | \Lambda q J B \rangle D^{Jo}_{BA}(\Lambda) \tag{2.41}$$

for any homogeneous Lorentz transformation, Λ. An equivalent formulation of this requirement is contained in the three relations

$$U(\Lambda) | q J A \rangle = \sum_{B} | \Lambda q J B \rangle D^{Jo}_{BA}(\Lambda)$$

$$\langle q' J' A' | U^{-1}(\Lambda) = \sum_{B'} D^{oJ}_{A'B'}(\Lambda^{-1}) \langle \Lambda q' J' \dot{B}' |$$

$$U(\Lambda) T(P) U^{-1}(\Lambda) = T(\Lambda P) \tag{2.42}$$

Let us denote by L_q^{-1} and $L_{q'}^{-1}$ two transformations which carry q and q', respectively, into the time axis

$$L_q^{-1} q = \hat{q} \qquad \text{and} \qquad L_{q'}^{-1} q' = \hat{q}' \tag{2.43}$$

where $\hat{q}$ and $\hat{q}'$ have vanishing space components. With these notations we can write formally

$$\begin{aligned} \langle q' J' \dot{A}' | T(P) | q J A \rangle &= \sum_{BB'} D^{oJ'}_{A'B'}(L_{q'}) \langle \hat{q}' J' \dot{B}' | U(L_{q'}^{-1}) T(P) U(L_q) | \hat{q} J B \rangle D^{Jo}_{BA}(L_q^{-1}) \\ &= \sum_{BB'} D^{oJ'}_{A'B'}(L_{q'}) \langle \hat{q}' J' \dot{B}' | T(L_{q'}^{-1} P) U(L_{q'}^{-1} L_q) | \hat{q} J B \rangle D^{Jo}_{BA}(L_q^{-1}) \end{aligned} \tag{2.44}$$

thereby expressing the matrix element as a function of two Lorentz transformations.

Now our principal aim in this paper is to develop the matrix element in irreducible unitary representations of the homogeneous Lorentz group. This can be done by standard group theoretical techniques provided that that we reduce it to a function of only one Lorentz transformation. In the following section a method is presented for separating out the angular dependence on P_μ and once this has been done we shall have, in effect,

$$T(\Lambda P) = T(P)$$

For the special circumstance of forward scattering ($P_\mu = 0$) this is of course unnecessary - the expression (2.39) already contains only one Lorentz transformation.

Let us assume for the present that $T = T(P^2)$ and so proceed with the development. For the matrices of irreducible unitary representations of the Lorentz group we adopt the notation - in a $|j\lambda\rangle$ basis -

$$D^{j_0\sigma}_{j\lambda,\,j'\lambda'}(\Lambda) \quad ; \quad j\,,\ j' = |j_0|\,,\ |j_0|+1\,,\ |j_0|+2,\ \ldots$$

$$\lambda = -j, \ldots, j$$

$$\lambda' = -j', \ldots, j' \tag{2.45}$$

The labels j_0 and σ are given in terms of the two Casimir invariants by

$$\tfrac{1}{2} J_{\mu\nu} J_{\mu\nu} = j_0^2 + \sigma^2 - 1$$

$$\tfrac{1}{4} \epsilon_{\lambda\mu\nu\rho} J_{\lambda\mu} J_{\nu\rho} = 2\, i\, j_0 \sigma \tag{2.46}$$

The representations of the principal series, which are the only ones we shall need, are given by j_0 and σ in the ranges

$$\pm j_0 = 0\,,\ \tfrac{1}{2}\,,\ 1\,,\ 3/2,\ \ldots$$

$$0 \leqslant -i\sigma < \infty \tag{2.47}$$

The structure of the matrices (2.40) is such that for purely spatial rotations, R, we have

$$D^{j_o\sigma}_{j\lambda,j'\lambda'}(R) = \delta_{jj'}\, D^{j}_{\lambda\lambda'}(R) \tag{2.48}$$

where $D^{j}_{\lambda\lambda'}$ denotes one of the usual SO(3) rotation matrices (2.17). Corresponding to the pure Lorentz transformation in the O3-plane through a hyperbolic angle (cf. (2.20)), we write*)

$$D^{j_o\sigma}_{j\lambda,j'\lambda'}(\alpha) = \delta_{\lambda\lambda'}\, d^{j_o\sigma}_{j\lambda j'}(\alpha) \tag{2.49}$$

The group theoretical result which we shall exploit is the theorem which states that any square integrable function over the Lorentz group may be expanded in the representations of the principal series. Let $f(\Lambda)$ be such a function. If we define the coefficients

$$f^{j_o\sigma}_{j\lambda,j'\lambda'} \doteq \int d\Lambda\ D^{j_o\sigma}_{j'\lambda',j\lambda}(\Lambda)^*\ f(\Lambda) \tag{2.50}$$

where $\int d\Lambda$ denotes the invariant integral over the group, then we can make the expansion

$$f(\Lambda) = \sum_{j_o} \int_0^{i\infty} d\sigma\,(j_o^2-\sigma^2) \sum_{j\lambda,j'\lambda'} f^{j_o\sigma}_{j\lambda,j'\lambda'}\ D^{j_o\sigma}_{j\lambda,j'\lambda'}(\Lambda) \tag{2.51}$$

Applying this to the function $\langle \hat{q}\ S'\dot{B}' | T(P^2)\ U(\Lambda) | \hat{q}\ S\ B\rangle$ and using in addition the properties

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*) In the papers of Toller $(j_o,\sigma) \rightarrow (M,\lambda)$. Our notation in (2.49) differs slightly in that Toller writes

$$D^{M\lambda}_{j\mu,j'\mu'}(\alpha) = \delta_{\mu\mu'}\ d^{M\lambda}_{\mu jj'}(\alpha)$$

$$U(R)\,|\hat{q}\ J\ B\rangle = \sum_A |\hat{q}\ J\ A\rangle\ D^J_{AB}\ (R)$$

$$\langle \hat{q}\ J'\ \dot{B}'|\,U^{-1}(R) = \sum_{A'} D^{J}_{B'A'}(R^{-1})\ \langle \hat{q}\ J'\ \dot{A}'| \qquad (2.52)$$

for purely spatial rotations, we get

$$\int d\Lambda \quad D^{j_0\sigma}_{j'\lambda',j\lambda}(\Lambda)^*\ \langle \hat{q}\ J'\ \dot{B}'|\,T(P^2)U(\Lambda)\,|\,\hat{q}\ J\ B\rangle =$$

$$= \delta_{j'J'}\,\delta_{\lambda'B'}\,\delta_{jJ}\,\delta_{\lambda B}\,\langle J'|\,T^{j_0\sigma}(P^2)\,|\,J\rangle \qquad (2.53)$$

and, conversely,

$$\langle \hat{q}\ J'\dot{B}'|\,T(P^2)U(\Lambda)\,|\,\hat{q}\ J\ B\rangle =$$

$$= \sum_{j_0=-M}^{M} \int_0^{i\infty} d\sigma\,(j_0^2-\sigma^2)\ \langle J'|\,T^{j_0\sigma}(P^2)\,|J\rangle\ D^{j_0\sigma}_{J'B',JB}(\Lambda) \qquad (2.54)$$

where M denotes the lesser of J and J'. Inserting the expansion (2.49) into (2.39) we get

$$\langle q'\,J'\,\dot{A}'|\,T(P^2)\,|\ q\ J\ a\rangle =$$

$$= \sum_{j_0=-M}^{M} \int_0^{i\infty} d\sigma\,(j_0^2-\sigma^2)\ \langle J'|T^{j_0\sigma}(P^2)|J\rangle \sum_{BB'} D^{0J'}_{A'B'}(L_{q'})\,D^{j_0\sigma}_{J'B',JB}(L_{q'}^{-1}L_q)\,D^{J0}_{BA}(L_q^{-1}) \qquad (2.55)$$

In a suitably chosen frame it is possible to fix L_q and $L_{q'}$ by

$$U(\ L_q) = e^{-i\frac{\zeta}{2}J_{03}} \quad , \quad U(\ L_{q'}) = e^{i\frac{\zeta}{2}J_{03}}\,e^{-i\pi J_{31}} \qquad (2.56)$$

in which case we have

$$\sum_{BB'} \mathcal{D}^{0J'}_{A'B'}(L_{q'})\, \mathcal{D}^{j_0\sigma}_{J'B',JB}(L^{-1}_{q'} L_q)\, \mathcal{D}^{J0}_{BA}(L^{-1}_{q}) =$$

$$= \delta_{A'A}\; e^{-\zeta/2}\, d^{j_0\sigma}_{J'AJ}(\zeta)\, e^{\zeta/2} \qquad (2.57)$$

In this frame the formula (2.50) becomes

$$\langle q'J'A'|T(P^2)|q\,J\,A\rangle =$$

$$= \delta_{A'A} \sum_{j_0=-M}^{M} \int_0^{i\infty} d\sigma\,(j_0^2-\sigma^2)\, \langle J'|T^{j_0\sigma}(P^2)|J\rangle\, d^{j_0\sigma}_{J'AJ}(\zeta) \qquad (2.58)$$

where ζ denotes the angle between q and q';

$$\mathrm{ch}\,\zeta = \frac{q\,q'}{\sqrt{q^2\,q'^2}} \qquad (2.59)$$

3. REDUCTION OF MATRIX ELEMENTS

In order to carry out the expansion in irreducible representations of the matrix element $\langle q'J'\dot{A}'|T(P)|q\,j\,A\rangle$ it is necessary first to separate out the dependence on the angular co-ordinates of P_μ. This separation corresponds in part to an expansion of the matrix element in invariant amplitudes. However we shall not carry out a complete reduction to invariant amplitudes but rather we shall separate only the powers of P_μ. The coefficients in this expansion we shall denote by $\langle q'j'\dot{B}'|T(P^2)|q\,j\,B\rangle$. It is these simpler objects which depend on P^2 rather than P_μ which can be developed in irreducible representations of the homogeneous Lorentz group. We may expect, of course, that our method of isolating the P_μ dependence has the feature – common to all expansions in invariant amplitudes – of being non-unique.

An expansion which separates the P_μ dependence can always be written in the form

$$\langle q' J' \dot{A}' | T(P) | q J A \rangle = \sum_{jB, j'B'} \langle J' \dot{A}'; j B | \Gamma(P) | J A, j' \dot{B}' \rangle \cdot$$

$$\cdot \langle q' j' \dot{B}' | T(P^2) | q j B \rangle \tag{3.1}$$

where j, B, j', B' cover a finite range and $\Gamma(P)$ is a polynomial in P_μ.

$$\Gamma(P) = \sum_n \Gamma_{\mu_1 \ldots \mu_n} P_{\mu_1} \cdots P_{\mu_n} \tag{3.2}$$

From the invariance conditions,

$$\langle q' J' \dot{A}' | T(P) | q J A \rangle = \sum_{BB'} D^{0J'}_{\dot{A}\dot{B}'} (\Lambda^{-1}) \langle \Lambda q' J' \dot{B}' | T(\Lambda P) | \Lambda q J B \rangle D^{J0}_{BA} (\Lambda)$$

$$\langle q' j' \dot{A}' | T(P^2) | q j A \rangle = \sum_{BB'} D^{0j'}_{\dot{A}\dot{B}'} (\Lambda^{-1}) \langle \Lambda q' j' \dot{B}' | T(P^2) | \Lambda q j B \rangle D^{j0}_{BA} (\Lambda) \tag{3.3}$$

it follows that the coefficients must satisfy

$$\langle J' \dot{A}'; j B | \Gamma(P) | J A, j' \dot{B}' \rangle = \sum D^{0J'}_{\dot{A}'\dot{C}'} (\Lambda^{-1}) D^{j0}_{BD} (\Lambda^{-1})$$

$$\langle J' \dot{C}'; j D | \Gamma(\Lambda P) | J C, j' \dot{D}' \rangle D^{J0}_{CA} (\Lambda) D^{0j'}_{\dot{D}'\dot{B}'} (\Lambda) \tag{3.4}$$

This means that the matrix $\Gamma_{\mu_1 \cdots \mu_n}$ must transform as a tensor of rank n. No generality is lost by requiring that $\Gamma_{\mu_1 \ldots \mu_n}$ belong to the irreducible representation $D^{n/2,n/2}$. Instead of labelling the components $\Gamma_{\mu_1 \cdots \mu_n}$ we can conveniently use the canonical notation $\Gamma^{n/2,n/2}_{A\dot{B}}$ where A and B range between $-n/2$ and $n/2$. The invariance condition (3.4) allows us to express the matrix elements of $\Gamma^{n/2,n/2}$ in terms of a set of reduced matrix elements as follows:

$$\langle J'\dot{A}',\ j\ B|\Gamma^{n/2,n/2}_{C\dot{D}}|J\ A,\ j'\dot{B}'\rangle =$$

$$= \langle J' j \,\|\, \Gamma^{n} \,\|\, J\ j'\rangle \ \langle J' A'|\,n/2\ D,\ j' B'\rangle \langle j\ B|\,n/2\, C,\ J\ A\rangle \qquad (3.5)$$

It is in the choice of reduced matrix elements that any possible non-uniqueness of the decomposition may lie. We shall adopt what appears to be the simplest possible choice*), viz.,

$$\langle J' j \,\|\, \Gamma^{n} \,\|\, J\ j'\rangle = \delta_{j+j',\,J+J'}\ \delta_{|J-j|,\frac{n}{2}} \qquad (3.6)$$

The ranges of j and j′ must be as follows:

(1) $J > J'$: $J - J' \leqslant 2j \leqslant J + 3J' + 1$, $J - J' - 1 \leqslant 2j' \leqslant J + 3J'$

(2) $J = J'$: $0 \leqslant 2j \leqslant 4J$, $0 \leqslant 2j' \leqslant 4J$

(3) $J < J'$: $J' - J - 1 \leqslant 2j \leqslant J' + 3J$, $J' - J \leqslant 2j' \leqslant J' + 3J + 1$

(3.7)

with $2j$ and $2j'$ taking all possible integer values between these limits. (Notice that because of fermion conservation, $J + J'$ is always an integer). These ranges are fixed essentially by the requirement that the total number of independent amplitudes (unrestricted by T C P considerations) should be $(2J + 1)(2J' + 1)$. This, the reader may easily verify.

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*) We are not considering here the most general possible spin combinations. In addition to the condition $J > J'$ we assume $2J' + 1 \geqslant J - J'$ and similarly when $J < J'$ we assume $J' - J \leqslant 2J + 1$. These restrictions admit the most interesting practical cases. It is of course possible to develop another formula which covers the situations excluded here.

If we introduce the notation $P^n_{C\dot{D}}$ for the traceless part of $P_{\mu_1} \cdots P_{\mu_n}$ then we can write

$$\Gamma(P) = \sum_{n} \sum_{CD} P^n_{C\dot{D}} \Gamma^{n/2 \; n/2}_{C\dot{D}} \tag{3.8}$$

and so, with the help of (3.5) and (3.6), the reduction (3.1) takes the form

$$\begin{aligned} \langle q'J'\dot{A}'|T(P)|qJA\rangle = \sum_{\substack{jB, j'B' \\ (j+j'=J+J')}} P^{2|J-j|}_{C\dot{D}} \langle J'A'||J'-j'|D, j'B'\rangle \cdot \\ \cdot \langle jB||J-j|C, JA\rangle \langle q'j'\dot{B}'|T(P^2)|qjB\rangle \end{aligned} \tag{3.9}$$

This formula, which is explicitly covariant, becomes simpler when referred to a suitable frame. Assuming that P_μ is orthogonal to q_μ and q'_μ we can arrange for P_μ to have the form (0, P, 0, 0) while q and q', being time-like, lie in the 03-plane. In this frame

$$\Gamma(P) = \sum_n P^n \Gamma_{1\ldots1} = \sum_n P^n \sum_{C=-n/2}^{n/2} \Gamma^{n/2 \; n/2}_{C\dot{C}} \tag{3.10}$$

and moreover, $\langle q'j'\dot{B}'|T(P^2)|q\,j\,B\rangle \sim \delta_{BB'}$, so that in (3.9) we have

$$A' - B' = D = C = B - A$$

or $$C = D = \frac{A' - A}{2}$$

and $$B = B' = \frac{A' + A}{2}$$

Thus, finally,

$$\begin{aligned} \langle q'J'\dot{A}'|T(P)|qJA\rangle = \sum_{\substack{jj' \\ (j+j'=J+J')}} P^{2|J-j|} \langle J'A'||J'-j'|\tfrac{A'-A}{2}, j'\tfrac{A'+A}{2}\rangle \cdot \\ \cdot \langle j\tfrac{A'+A}{2}||J-j|\tfrac{A'-A}{2}, JA\rangle \langle q'j'\tfrac{\dot{A}'+\dot{A}}{2}|T(P^2)|qj\tfrac{A'+A}{2}\rangle \end{aligned} \tag{3.11}$$

in the special frame.

To the amplitudes in (3.11) we can apply the previously derived expansion formula (2.53). The result is

$$\langle q'J'A'|T(P)|qJA\rangle = \sum_{jj'} \langle J'A'|\,|J'-j'|\,\tfrac{A'-A}{2},\, j'\,\tfrac{A'+A}{2}\rangle\langle j\,\tfrac{A'+A}{2}\,|\,|J-j|\,\tfrac{A'-A}{2},\, JA\rangle \cdot$$

$$\sum_{j_0=-M}^{M} \int_0^{i\infty} d\sigma\,(j_0^2-\sigma^2)\,(P^2)^{|J-j|}\,\langle J'j'|T^{j_0\sigma}(P^2)|Jj\rangle\, d^{j_0\sigma}_{j'\frac{A'+A}{2}j}(\zeta) \tag{3.12}$$

where M denotes the lesser of j and j'. For the case of elastic forward scattering, P = 0, this reduces to

$$\langle q'J'A'|T(0)|qJA\rangle = \delta_{A'A} \sum_{j_0} \int_0^{i\infty} d\sigma\,(j_0^2-\sigma^2)\,\langle J'J'|T^{j_0\sigma}(0)|JJ\rangle\, d^{j_0\sigma}_{J'AJ}(\zeta) \tag{3.13}$$

which is equivalent to the form deduced by Toller.

The original matrix element of T between helicity states can now be expressed in terms of the amplitudes $\langle J'j'|T^{j_0\sigma}|Jj\rangle$. Substituting (3.12) and (2.40) into (2.37) we get

$$\langle p_3 S_3 \lambda_3\,,\, p_4 S_4 \lambda_4 |\, T\, |\, p_1 S_1 \lambda_1\,,\, p_2 S_2 \lambda_2\rangle =$$

$$= \sum_{JJ'} \sum_{jj'} \sum_{B} \langle \lambda_3\lambda_4\,;\, Jj\,|Z_B|\,J'j'\,;\lambda_1\lambda_2\rangle \cdot$$

$$\sum_{j_0} \int_0^{i\infty} d\sigma\,(j_0^2-\sigma^2)\,(P^2)^{|J-j|}\langle J'j'|T^{j_0\sigma}(P^2)|Jj\rangle\, d^{j_0\sigma}_{jBj'}(\zeta) \tag{3.14}$$

where $|S_1 - S_3| \leqslant J \leqslant S_1 + S_3$, $|S_2 - S_4| \leqslant J' \leqslant S_2 + S_4$, $j + j' = J + J'$ (with ranges fixed by (3.7)) and $-\min(j,j') \leqslant A$, $j_0 \leqslant \min(j,j')$. The coefficient Z_B is defined by

$$\langle \lambda_3\lambda_4 ; Jj | Z_B | J'j' ; \lambda_1\lambda_2 \rangle =$$

$$= \sum_{\substack{A_1 \cdots A_4 \\ (A+A'=2B)}} \Big[D^{0s_4}_{\lambda_4 A_4}(L^{-1}_{p_4}) \langle s_4 A_4 | J'A', s_2 A_2 \rangle D^{0s_2}_{A_2\lambda_2}(L_{p_2}) \cdot$$

$$\langle J'A' | | J'_z j' | \tfrac{A'-A}{2}, j' \tfrac{A'+A}{2} \rangle \langle j \tfrac{A'+A}{2} | | J_z j | \tfrac{A'-A}{2}, JA \rangle D^{s_3 0}_{\lambda_3 A_3}(L^{-1}_{p_3}) \langle s_3 A_3, JA | s_1 A_1 \rangle D^{s_1 0}_{A_1\lambda_1}(L_{p_1}) \Big]$$

(3.15)

The boost matrices here are defined by

$$D^{s0}_{AB}(L_p) = e^{-iA\varphi} d^s_{AB}(\theta)\, e^{iB(\varphi+i\alpha)}$$

$$D^{0s}_{AB}(L_p) = e^{-iA\varphi} d^s_{AB}(\theta)\, e^{iB(\varphi-i\alpha)} \qquad (3.16)$$

with the angles (φ, θ, α) as in (2.3). It is of course necessary to express these angles as functions of the Mandelstam variables s and t consistently with our choice of frame. The results of this calculation will be contained in a later publication.

Finally, one must realize that for inelastic scattering we have $P\cdot q$ and $P\cdot q'$ non-vanishing according to the definitions (2.39). This means that these definitions must be modified. There appear to be two courses one can follow:

(1) Choose the frame of reference such that the momentum transfer $p_4 - p_3 = p_4 - p_2 = P$ is aligned with the 1-axis while defining q and q' by

$$q = p_1 + p_3 - \lambda P \quad , \quad q' = p_2 + p_4 - \lambda' P \qquad (3.17)$$

with λ and λ' given by

$$\lambda = \frac{m_1^2 - m_3^2}{P^2} \qquad \lambda' = \frac{m_4^2 - m_2^2}{P^2} \qquad (3.18)$$

so that $Pq = P'q' = 0$.

(2) Choose the frame of reference such that $q = p_1 + p_3$ and $q' = p_2 + p_4$ lie in the O3-plane while defining P by

$$P = p_1 - p_3 - \alpha q - \beta q' \tag{3.19}$$

where
$$\alpha = \frac{(m_1^2 - m_3^2)q'^2 - (m_4^2 - m_2^2)qq'}{q^2 q'^2 - (q q')^2}$$

$$\beta = \frac{(m_4^2 - m_2^2)q^2 - (m_1^2 - m_3^2)qq'}{q^2 q'^2 - (q q')^2} \tag{3.20}$$

4. POLES IN THE σ-PLANE

A sufficient condition for the existence of the SO(3,1) expansion employing only the principal series of unitary representations $D^{j_0 \sigma}$ is that the amplitude as a function of energy s should fall faster than s^{-1} (the square integrability condition). For forward scattering this condition is not satisfied. For the general case however it would appear, purely empirically, that for a region of large momentum transfer (−t) values, the amplitude is likely to decrease fast enough for the square integrability criteria to be satisfied and the SO(3,1) expansion in terms of only the principal series to be valid.

Let us fix t large and negative so that this is the case. Assume that the amplitude can be continued for complex values of σ and is meromorphic in a strip $|\mathrm{Re}\,\sigma| < L$. With Toller, if we now shift the integration path to the left, we may write*) for large negative t

$$T = \int_{-i\infty}^{i\infty} d\sigma = \int_{-M-i\infty}^{-M+i\infty} + \sum \quad \sigma\text{-plane pole terms}$$

and then the pole contributions dominate asymptotically the contribution along the new shifted path. The poles lie on trajectories $\sigma = a_\tau(j_0, t)$ in the σ-plane and clearly provide a generalization of Regge trajectories. The important remark is that moving analytically along a trajectory from large $(-t)$ up to $t = 0$, together with the assumption that $T \approx \sum$(pole contributions), means that we shall also reach those values of t for which the amplitude was not square integrable in the first place. Thus the possibility of the formulae holding while we ride along the trajectory implies that no non-unitary representations need to be added to the SO(3,1) expansion and all we require is the analytic continuation of the principal series representations. But before shifting the integration path arbitrarily to the left it is important that we expand the amplitude into representation functions of the second kind $e^{j_0\sigma}$, in place of the $d^{j_0\sigma}$, since these have well defined asymptotic behaviours[9]. More precisely the e are defined by the split

$$d^{j_0\sigma}_{j'\mu j}(z) = e^{j_0\sigma}_{j'\mu j}(z) + \frac{\Gamma(j-\sigma+1)}{\Gamma(j+\sigma+1)}\, e^{-j_0-\sigma}_{j'\mu j}(z)\, \frac{\Gamma(j'-\sigma+1)}{\Gamma(j'+\sigma+1)}$$

with the asymptotic property

$$e^{j_0\sigma}_{j'\mu j}(z) \sim z^{-(\sigma+1+|j_0-\mu|)}$$

Note the independence of this term from j and j'.

The factorization hypothesis leads us to suppose that the pole structure is

$$T^{j_0\sigma}_{J'j'Jj} = \sum_\tau \frac{b_{J'j'\tau}(t)\, b_{Jj\tau}(t)}{\sigma - a_\tau(j_0,t)}$$

where τ labels the set of Toller poles. Introducing a single pole into the amplitude in (3.11) and retaining the leading term,

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*) Because the $d^{j_0\sigma}$ functions satisfy the weak equivalence relation

$$d^{-j_0-\sigma}_{j\mu j'} = \frac{\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)}\, d^{j_0\sigma}_{j\mu j'}\, \frac{\Gamma(j'-\sigma+1)}{\Gamma(j'+\sigma+1)}$$

the integration over σ defined in Sec. 3 over positive imaginary values can be extended over the entire imaginary axis provided also that we restrict the range j_0 to non-negative values up to min (j, j'). Hereafter we adhere to this reinterpretation.

$$\langle q'j'\mu | T(p^2) | q j \mu \rangle \sim b_{j'}(t)\, b_j(t)\, [j_0^2 - a^2(j_0,t)]\, z^{-a(t)-1-|j_0-\mu|}$$

where $z = \hat{q}\cdot\hat{q}' = \dfrac{2(s-u)}{[(4M^2-t)(4M'^2-t)]^{1/2}}$

It is the structure of the daughter-parent relationship which is important, and this has been worked out by SCIARRINO and TOLLER[1] by expanding SO(3,1) representations in terms of SO(2,1) representations. The latter are based on the subgroup of SO(3,1) which operates in the 0 2 3 subspace having states labelled by the two eigenvalues $J_{23} = m$ $\tilde{j}(\tilde{j}+1) = J_{23}^2 - J_{02}^2 - J_{03}^2$. On the other hand the O(3) subgroup employed in the SO(3,1) representations above use the eigenvalues $J_{12} = \mu$ and $j(j+1) = J_{12}^2 + J_{23}^2 + J_{31}^2$. We thus wish to pass from

$$\langle j_0\sigma j'\mu | e^{-i\zeta J_{03}} | j_0\sigma j\mu \rangle \quad \text{to} \quad \langle \tilde{j} m' | e^{-i\zeta J_{01}} | \tilde{j} m \rangle .$$

This connection is obtained from the identity

$$e^{-i\zeta J_{03}} = e^{+i\pi/2 J_{31}}\, e^{-i\zeta J_{01}}\, e^{-i\pi/2 J_{31}}$$

$$d^{j_0\sigma}_{j'\mu j}(\zeta) = \sum_{\nu\nu'} d^{j'}_{\mu\nu'}(-\pi/2)\, \langle j'\nu' | e^{-i\zeta J_{01}} | j\nu \rangle\, d^{j}_{\nu\mu}(\pi/2)$$

Once it is recognized that $\langle \tilde{j}\, m' = J_{23} | e^{-i\zeta J_{02}} | \tilde{j}\, m = J_{23} \rangle$

$$= \langle \tilde{j}\, m' = J_{12} | e^{-i\zeta J_{01}} | \tilde{j}\, m = J_{12} \rangle$$

the introduction of a complete set of O(2,1) states gives

$$d^{j_0\sigma}_{j'\mu j}(\zeta) = \sum_{\nu\nu'} \int_{\tilde{j}} d^{j'}_{\mu\nu'}(-\pi/2)\, K^{j_0\sigma}_{\nu'}(\tilde{j},j')^{*}\, d^{\tilde{j}}_{\nu'\nu}(\zeta)\, K^{j_0\sigma}_{\nu}(\tilde{j},j)\, d^{j}_{\nu\mu}(\pi/2)$$

where K denotes the overlap function from O(3) to O(2,1) base vectors and $\int_{\tilde{j}}$ stands for the sum over the discrete and principal series of O(2,1).

K can be regarded as the expectation value of $e^{-\frac{\pi}{2}J_{03}}$ and equals $d^{j_0\sigma}_{\tilde{j}\mu j}\left(i\frac{\pi}{2}\right)$.

Consider the contribution of a single Toller pole to the amplitude

$$\langle j'\mu'|T|j\mu\rangle \sim b_{j'}b_j\left(j_0^2-a^2(j_0)\right)d^{j_0 a}_{j'\mu j}(z)$$

$$= b_{j'}b_j\left(j_0^2-a^2\right)\sum_{\nu\nu'}\int_{\tilde{j}} d^{j'}_{\mu\nu'}(-\pi/2)\,K^{j_0 a}_{\nu'}(\tilde{j}j')^{*}\,d^{\tilde{j}}_{\nu'\nu}(z)\,K^{j_0 a}_{\nu}(\tilde{j},j)\,d^{j}_{\nu\mu}(\pi/2)$$

The detailed properties of the SO(3,1) $\rightarrow$ SO(2,1) reduction formulae have been derived by Sciarrino and Toller. The crux of the argument is the fact that the integrand K^* K as a function of $\tilde{j}$ exhibits poles at

$$\tilde{j} = \sigma - n - 1 \qquad ; \qquad n = 0,\ 1,\ 2,$$

This means that the Regge poles (identified with $\tilde{j}$) occur at the points $\alpha_n(t) = a(j_0,\ t) - 1 - n$ where n is the daughter number. The amplitude thereby decomposes as a sum of Regge terms.

$$\langle j'\mu|T|j\mu\rangle = b_{j'}b_j\left(j_0^2-a^2\right)\sum_{n\nu\nu'} d^{j}_{\mu\nu'}(-\pi/2)\,d^{a-n-1}_{\nu'\nu}(z)\,d^{j}_{\nu\mu}(\pi/2)\cdot$$

$$\cdot\left[W^{j_0 a n\,*}_{j'\nu'}\,K^{j_0 a n}_{\nu'}(a-n-1,j) + W \leftrightarrow K\right]$$

where $W^{j_0 a n}_{j\nu}$ is the residue of K at the daughter pole (n). It is evident that the residues of the daughters are completely determined in terms of the residue of the parent Toller poles and the universal factors W and K.

An important place where physical consequences of the scheme presented above might be looked for are the zeroes of the amplitude associated*) with e^{-j_0-a}. These occur at $a(t) = -\mu + 1 + n$ and $j' + 1 + n$ (for $j_0 \leqslant \mu$). At those particular values of t the amplitude is expected to vanish.

We have not discussed C P T transformations and the related question

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*) The asymptotic form of $e^{j_0\sigma}_{j\mu j}(z)$ is as follows:

$$e^{j_0\sigma}_{j\mu j'}(z) \sim \left[(2j+1)(2j'+1)\right]^{1/2}\left[\frac{\Gamma(j-\mu+1)\Gamma(j+j_0+1)\Gamma(j'-\mu+1)\Gamma(j'+j_0+1)}{\Gamma(j+\mu+1)\Gamma(j-j_0+1)\Gamma(j'+\mu+1)\Gamma(j'-j_0+1)}\right]^{1/2}\cdot$$

$$\cdot\ \frac{(-)^{j'+\mu}\,\Gamma(\sigma+j'+1)\Gamma(-j_0-\sigma)}{\Gamma(j_0-\mu+1)\Gamma(\sigma-\mu+1)\Gamma(-\sigma+j'+1)}\,z^{-(\sigma+1+j_0-\mu)}\ ,\ j_0 \geqslant \mu\ .$$

of signature as this paper has been devoted principally to exhibiting the methods. In a subsequent publication we intend to remedy these deficiencies and make use of the formulae in specific physical situations.

ACKNOWLEDGMENTS

Our thanks are due to M.A. Rashid and J.F. Boyce for considerable help.

REFERENCES

1. M. TOLLER, Nuovo Cimento 37, 631 (1965);
Internal Reports University of Rome Nos. 76 (1965) and 84 (1965);
M. TOLLER and A. SCIARRINO, Internal Report University of Rome No. 108 (1966)

2. R.J.N. VOLKOV and V.N. GRIBOV, Sov. Phys.-JETP 17, 720 (1963)

3. G. DOMOKOS and P. SURANYI Nucl. Phys. 54, 529 (1964);
Ya. A. SMORODINSKI, M. ULHIR and P. WINTERNITZ (Dubna Preprint E-1591)

4. P.Z. FREEDMAN and J.M. WANG, Phys. Rev. 153, 1596 (1967)
M.L. GOLDBERGER and E. JONES Phys. Rev. Letters 17, 105 (1966)

5. R. OAKES, Phys. Letters 24B, 154 (1967)

6. G. DOMOKOS, Berkeley Preprint 1967

7. S. STRÖM, Arkiv för Fysik, 29, 467 (1965);
DAO WONG DUC, NGUYEN VAN HEIU, Dubna Preprint P-2886

8. H. JOOS, Fortschritte der Physik, 10, 65 (1962)

9. J. STRATHDEE, J.F. BOYCE, R. DELBOURGO and ABDUS SALAM, ICTP, Trieste preprint IC/67/9