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INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

# PARTIAL WAVE ANALYSIS

(Part I)

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AND
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## (Part I)

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February 1967

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*INTRODUCTION*

The inhomogeneous Lorentz group in Wigner's classification possesses essentially four distinct classes of unitary representations; these are

(A) Timelike representations $p^2 > 0$ ; (little group SO(3)).

(B) Spacelike representations $p^2 < 0$ ; (little group SO(2,1)).

(C) Lightlike representations $p^2 = 0$ (little group $T_2 \times O(2)$).

(D) Null representations $p^2 = 0$ , $p_\mu = 0$ (little group SO(3,1)).

In the conventional harmonic (partial wave) analysis of scattering problems the significance of using time-like representations is well appreciated. For a fixed time-like vector - total c.m. energy squared, $s = p^2 > 0$ - an expansion of the (two-body) amplitude $F(s,\theta)$ is made in the associated angle

$$\cos\theta = \frac{s(t-u)}{[s-(m+\mu)^2]\,[s-(m-\mu)^2]}$$

which is the parameter occurring in the representation theory of the appropriate <u>little group</u> SO(3); specifically the expansion employs the complete set of rotation functions $d^J_{\lambda\lambda'}(\theta)$ which correspond to the unitary representations of SO(3).

Less well appreciated has till recently been the use of the other representations (B), (C) and (D). Through the work of Joos, Toller and Sertorio and Hadjioannou, since 1964, it has come to be realized that if the (spacelike) momentum transfer t 0 is held fixed, a partial wave analysis of the same amplitude F( ,t) can be made in the associated (hyperbolic) angle

$$\operatorname{ch}\beta = -t(s-u) \Big/ \left[t(t-4m^2)\{t-(m+\mu)^2\}\{t-(m-\mu)^2\}\right]^{\frac{1}{2}}$$

the expansion employing unitary representations of the corresponding non-compact little group SO(2,1). Specifically it uses functions $d^J_{\lambda\lambda'}(\beta)$ with J complex of the form $J = \frac{1}{2} + i\rho, -\infty < \rho < \infty$. The great merit of this expansion is the direct passage it provides to complex angular momenta.

Its use supplants completely the cumbersome conventional three-steps procedure for passing to complex J representations which uses SO(3) partial wave analysis in the crossed channel, makes a Sommerfeld-Watson transform and then finally continues analytically to physical s and t values.

When momentum transfer vanishes it is clear from the above that the natural group-theoretic procedure for a partial wave analysis should employ representations (C) and (D). For the unequal mass case, as shown in what follows, the appropriate expansion functions for case (C) turn out to be the Bessel functions $J_{\lambda-\chi'}\left[2\rho\sqrt{-t}\,(m^2-\mu^2)^{-1}\right]$. For forward scattering of equal-mass particles, not only does the momentum transfer vanish ($p^2 = 0$), but also each component of $p_\mu = 0$. The little group - the invariance group of the S-matrix - in this exceptional case is the homogeneous Lorentz group SO(3,1) itself - a much larger structure than SO(3). Corresponding to this larger symmetry, the principal unitary representations of SO(3,1) are labelled not by just one quantum number J, but by two numbers, one discrete label ($j_0$) and one continuous pure imaginary number $\sigma$, $-i\infty < \sigma < i\infty$. The corresponding representation functions are $D^{j_0\sigma}_{s\lambda,s\chi'}(\zeta)$ $\left(\operatorname{ch}\zeta = \frac{s-m^2-\mu^2}{2m\mu}\right)$. Group theory would specify a partial wave expansion for forward scattering in terms of these functions. Using these, we pass once again directly to the complex $\sigma$-plane - the variable $\sigma$ now taking over and generalising the role of complex J. This $\sigma$-plane was introduced into the subject by Toller*) in 1965, who noted that if the Regge hypothesis of poles in the complex J plane is carried over to the complex $\sigma$-plane, to one $\sigma$-pole there corresponds an entire family of integrally spaced J-poles - a result foreshadowed earlier in the works of Gribov, Volkov, Domokos and Suranyi and rediscovered by Freedman and Wang in connection with situations involving light-like representations (C).

The present article (Part I) is an attempt at a systematic and self-contained presentation of the group theoretic basis of harmonic analysis using the four types of representations**) (A), (B), (C) and (D). In Part II we extend these results; in particular we show how an expansion of the ampli-

*) All expansions (B), (C) and (D) apply to square integrable functions. In Part II we show how one circumvents this limitation.

**) To our knowledge representations for class (C) have not been previously studied.

tude may be carried through, using the functions $D^{j_0\sigma}_{s\lambda, s'\lambda'}$ , not only in the forward direction but for all momentum transfers and for all values of helicity flip. This type of expansion, with its new separation of the kinematical factors, will allow a more systematic use of analyticity in the $\sigma$-plane for all processes at all momentum transfers, possibly giving a further insight into what may be learnt from a deeper analysis of the Poincaré group.

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The material in this paper is going to be issued in two parts. The contents of the first part are in the nature of a review and are indicated on the next page. This part essentially covers the basis of the group theoretic approach. The second part will deal with generalizations, a study of the complex $\sigma$-plane and applications. The authors would welcome suggestions for improvement of the material.

CONTENTS

## 1. THE UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP

*Definitions and general discussion*

The orthogonal transformations of space-time together with the translations comprise the Poincaré group $\mathcal{P}$. The elements of this group take the form

$$x_\mu \rightarrow x'_\mu = \Lambda_{\mu\nu} x_\nu + a_\nu \tag{1.1}$$

where $\Lambda_{\mu\nu}$ satisfies the orthogonality conditions

$$\Lambda_{\mu\nu} \Lambda_{\rho\nu} = g_{\mu\rho} \tag{1.2}$$

Throughout this paper we use the summation convention $A_\mu B_\mu = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3$ . The metric tensor, $g_{\mu\nu}$, takes the diagonal form $(+ - - -)$. All quantities appearing in (1.1) and (1.2) are real.

We are concerned with properties of the unitary representations of $\mathcal{P}$ denoted by

$$(a_\mu, \Lambda_{\mu\nu}) \rightarrow U(a, \Lambda) \tag{1.3}$$

where $U$ is an operator valued function of $a_\mu$ and $\Lambda_{\mu\nu}$ satisfying

$$U^+ U = U U^+ = 1 .$$

The successive application of two transformations

$$x_\mu \rightarrow x'_\mu = \Lambda_{\mu\nu} x_\nu + a_\mu \rightarrow x''_\mu = \Lambda'_{\mu\nu} (\Lambda_{\nu\rho} x_\rho + a_\nu) + a'_\mu$$

implies the basic requirement

$$U(a', \Lambda') \, U(a, \Lambda) = U(a' + \Lambda' a, \Lambda' \Lambda) \tag{1.4}$$

The infinitesimal transformations of this 10-parameter group may be represented in the form

$$U(a, 1+\epsilon) = 1 + i a_\mu P_\mu - \frac{i}{2} \epsilon_{\mu\nu} J_{\mu\nu} + \dots \tag{1.5}$$

where $a_\mu$ and $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ denote infinitesimal quantities. The hermitian generators $P_\mu$ and $J_{\mu\nu}$ which determine the entire representation satisfy the algebra,

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[J_{\mu\nu}, P_\lambda] &= i\left(g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu\right) \\
[J_{\mu\nu}, J_{\lambda\rho}] &= i\left(g_{\nu\lambda} J_{\mu\rho} - g_{\mu\lambda} J_{\nu\rho} + g_{\mu\rho} J_{\nu\lambda} - g_{\nu\rho} J_{\mu\lambda}\right)
\end{aligned}
\tag{1.6}
$$

These commutation relations which can be deduced* from (1.4) and (1.5) assure, in particular, that under finite homogeneous transformations of the group $P_\mu$ and $J_{\mu\nu}$ transform respectively as a 4-vector and an antisymmetric tensor.

$$
\begin{aligned}
U^{-1}(\Lambda)\, P_\mu\, U(\Lambda) &= \Lambda_{\mu\nu} P_\nu \\
U^{-1}(\Lambda)\, J_{\mu\rho}\, U(\Lambda) &= \Lambda_{\mu\nu} \Lambda_{\rho\sigma} J_{\nu\sigma}
\end{aligned}
\tag{1.7}
$$

For example, corresponding to the space rotation,

$$
\begin{aligned}
x_1 &\to x_1 \cos\alpha - x_2 \sin\alpha , \\
x_2 &\to x_1 \sin\alpha + x_2 \cos\alpha ,
\end{aligned}
\tag{1.8}
$$

we have, through integrating the corresponding infinitesimal trans-

---

* A simple way to derive the commutation rules is to write (1.4) in the form

$$U(\Lambda^{-1})\, U(a', \Lambda')\, U(\Lambda) = U(\Lambda^{-1} a', \Lambda^{-1} \Lambda' \Lambda)$$

Making $(a', \Lambda')$ correspond to an infinitesimal transformation and comparing first order terms yields immediately the relations (1.7). Taking $\Lambda$ infinitesimal in these equations gives the second and third lines of (1.6). The first line of (1.6) is obtained very easily by the same method.

formation, the operator $\exp\left(-i\alpha J_{12}\right)$ and, therefore

$$\begin{aligned} e^{i\alpha J_{12}} P_1 e^{-i\alpha J_{12}} &= P_1 \cos\alpha - P_2 \sin\alpha, \\ e^{i\alpha J_{12}} P_2 e^{-i\alpha J_{12}} &= P_1 \sin\alpha + P_2 \cos\alpha . \end{aligned} \tag{1.9}$$

Similarly, corresponding to the pure Lorentz transformation,

$$\begin{aligned} x_0 &\to x_0 \operatorname{ch}\alpha + x_3 \operatorname{sh}\alpha \\ x_3 &\to x_0 \operatorname{sh}\alpha + x_0 \operatorname{ch}\alpha \end{aligned} \tag{1.10}$$

we have the operator $\exp\left(-i\alpha J_{03}\right)$ and, therefore ,

$$\begin{aligned} e^{i\alpha J_{03}} P_0 e^{-i\alpha J_{03}} &= P_0 \operatorname{ch}\alpha + P_3 \operatorname{sh}\alpha \\ e^{i\alpha J_{03}} P_3 e^{-i\alpha J_{03}} &= P_0 \operatorname{sh}\alpha + P_3 \operatorname{ch}\alpha \end{aligned} \tag{1.11}$$

Relations of this sort will be used repeatedly in the following.

The principal Casimir operators of $\mathcal{P}$ are the two invariants

$$P_\mu P_\mu = m^2 \quad \text{and} \quad W_\mu W_\mu = -m^2 j(j+1) , \tag{1.12}$$

where

$$W_\mu = -\tfrac{1}{2} \epsilon_{\mu\nu\lambda\rho} J_{\nu\lambda} P_\rho \tag{1.13}$$

The vector $W_\mu$ has the useful property of being translation invariant,

$$\left[P_\mu , W_\nu\right] = 0 \tag{1.14}$$

The operators $P^2$ and $W^2$ do not always provide a complete specification of the irreducible representations. When $m^2 \leqslant 0$ several different types of representation are possible. Before going on to their classification we consider some general properties of the representations of $\mathcal{P}$ .

For the construction of representations we follow the method of Wigner. It is advantageous to label, in part, the basis vectors of a representation by the eigenvalues, $p_\mu$, of the Casimir operators of the translation subgroup

$$P_\mu | p, \lambda \rangle = p_\mu | p, \lambda \rangle \tag{1.15}$$

where $\lambda$ denotes those remaining labels which are necessary for a complete specification. In view of the vector behaviour of $P_\mu$ noted above (1.7), we see that under a homogeneous transformation, $\Lambda$, the basis vectors must transform according to a relation of the form

$$U(\Lambda) | p, \lambda \rangle = \sum_\mu | \Lambda p, \mu \rangle \, c_{\mu\lambda} \tag{1.16}$$

which simply states that the transformed state must be an eigenstate of momentum with eigenvalue $p'_\mu = \Lambda_{\mu\nu} p_\nu$. To evaluate the coefficients $C_{\lambda\mu}$ it is necessary to fix, in some conventional fashion, the definition of the basis vectors. This is done most directly by means of the Wigner "boosts".

Let us denote by $L_p$ a 3-parameter family of Lorentz transformations, the boosts. The boost $L_p$ transforms a given momentum vector $\hat{p}_\mu$ - which we take as a standard - into $p_\mu$,

$$(L_p)_{\mu\nu} \, \hat{p}_\nu = p_\mu \qquad \text{with} \qquad p^2 = \hat{p}^2 = m^2 \tag{1.17}$$

There are various alternative specifications of the function $L_p$ which are useful in different circumstances. They are discussed in Sec. 2.

Since, in an irreducible representation, it is by definition possible to obtain any vector in the representation space by applying appropriate transformations of the group to a fixed one, we can formally define the p-dependence of the basis by

$$| p, \lambda \rangle = U(L_p) | \hat{p} \lambda \rangle \tag{1.18}$$

There is a subgroup, $\hat{G}$ , of the homogeneous group SO(3,1) which leaves invariant the manifold of states with $p = \hat{p}$ . This is called the little group. Thus if we define $\hat{G}$ as the set of transformations $R_{\mu\nu}$ satisfying

$$R_{\mu\nu} \hat{p}_\nu = \hat{p}_\mu \tag{1.19}$$

then it follows from (1.16) that

$$U(R) | \hat{p}, \lambda \rangle = \sum_\mu | \hat{p}, \mu \rangle \, C_{\mu\lambda} \tag{1.20}$$

and it is implied that the coefficients $C_{\mu\lambda}$ must belong to a representation of the little group $\hat{G}$ ,

$$C_{\lambda\mu} = D_{\mu\lambda}(R) \tag{1.21}$$

It is now a simple matter to show that the transformation $R(p, \Lambda)$ defined for each p and $\Lambda$ by

$$\Lambda L_p = L_{\Lambda p} R(p, \Lambda) \tag{1.22}$$

is contained in $\hat{G}$ , i.e.,

$$R_{\mu\nu}(p, \Lambda) \hat{p}_\nu = \hat{p}_\mu .$$

This means that

$$U(\Lambda) | p, \lambda \rangle = U(\Lambda L_p) | \hat{p}, \lambda \rangle =$$

$$= U(L_{\Lambda p} R(p, \Lambda)) | \hat{p} \lambda \rangle$$

$$= U(L_{\Lambda p}) \sum_\mu | \hat{p}, \mu \rangle \, D_{\mu\lambda}(R(p, \Lambda))$$

or

$$U(\Lambda)|p,\lambda\rangle = \sum_{\mu} |\Lambda p, \mu\rangle \, D_{\mu\lambda}(R(p,\Lambda)) \tag{1.23}$$

The coefficients $c_{\lambda\mu}$ introduced in (1.16) are thus identified with matrix elements of a representation of the little group. Moreover, the unitarity and irreducibility of $\mathcal{U}(\Lambda)$ is tied to that of $D(R)$.

Firstly, unitarity is guaranteed by the invariance of the (positive) sum over states

$$\int dp \; \delta(p^2-m^2) \sum_{\lambda} |p\lambda\rangle\langle p\lambda|$$

and this follows if

$$\sum_{\lambda} D_{\mu\lambda} D^{*}_{\nu\lambda} = \delta_{\mu\nu}$$

i.e., from the unitarity of D . Obviously the converse also is true.

Secondly, if $U(\Lambda)$ is reducible then so is $D(R)$ (provided of course that $p^2$ takes only one value in the representation space). This follows since $\hat{G}$ is a subgroup of SO(3,1). On the other hand, if $D(R)$ is reducible, it is possible to divide the states $|p,\lambda\rangle$ into two or more sets which do not mix under Lorentz transformations, i.e., $U(\Lambda)$ also is irreducible. It would be a simple matter to spell out in detail proofs for these claims. However, we do not do this but merely state the basic theorem:

<u>The representation of $\mathcal{P}$ carried by the states $|p,\lambda\rangle$ is unitary and irreducible if and only if</u>

(i) <u>the mass' $p^2$ is unique, and</u>

(ii) <u>the associated little group representation $R \to D(R)$ is unitary and irreducible.</u>

So far our Lorentz group includes only the so-called proper, orthochronous transformations. Space and time reflections will be dealt with separately at a later stage.

## 2. CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS.

A consequence of the theorem stated in Sec. 1 is that the unitary representations of $\mathcal{P}$ can be classified by means of the unitary representations of the little groups with which they correspond. There are four distinct types of little group which apply accordingly as $P^2 > 0$, $P^2 < 0$, $P^2 = 0$ or $P_\mu = 0$ which we shall refer to respectively as the timelike, spacelike, lightlike and null cases. This section is devoted primarily to the construction of complete orthogonal basis systems for the representations of $\mathcal{P}$. In the course of doing this we shall have to discuss the little groups and their representations as they arise.

Following the procedure outlined in Sec. 1 we diagonalize the 4-momentum $P_\mu$ in addition to the basic invariants $P^2$ and $W^2$ which of course must be pure numbers in any irreducible representation. In an irreducible representation any vector can be carried into any other by applying a motion of the group. To begin with, the vectors with arbitrary 4-momentum $p_\mu$ can be obtained from a given one with fixed 4-momentum $\hat{p}_\mu$, the "standard momentum". Moreover, all vectors with momentum $\hat{p}_\mu$ can be obtained from the given one by applying transformations of the little group $\hat{G}$ since these are the only motions which leave $\hat{p}_\mu$ unchanged. Evidently, then, the representation of $\hat{G}$ contained in the irreducible representation of $\mathcal{P}$ must itself be irreducible. We shall denote the basis vectors of this irreducible representation by $|\hat{p}\, j\, \lambda\rangle$ or, in the case $p^2 = 0$, by $|\hat{p}\, \rho\, \lambda\rangle$ where $j$ and $\rho$ label the representation of $\hat{G}$ in question and $\lambda$ serves to differentiate the individual basis vectors. Since, by the basic theorem of Sec. 1. the irreducible representations of $\mathcal{P}$ and $\hat{G}$ are correlated, it must be that $j$ and $\rho$ are Poincaré invariants. In fact, as we shall show for each case considered below, $W^2 = -p^2\, j(j+1)$ for $p^2 \neq 0$ and $W^2 = -\rho^2$ for $p^2 = 0$. The remaining label, $\lambda$, is not generally invariant. We find it convenient to associate it always with $J_{12}$, i.e.,

$$J_{12}|\hat{p}\,j\,\lambda\rangle = \lambda|\hat{p}\,j\,\lambda\rangle, \quad \text{or} \quad J_{12}|\hat{p}\,\rho\,\lambda\rangle = \lambda|\hat{p}\,\rho\,\lambda\rangle. \tag{2.1}$$

It happens, however, that when $p^2 = 0$ and $\rho^2 = 0$ then $\lambda$ has invariant significance, namely

$$W_\mu|\,p\,0\,\lambda\rangle = \lambda p_\mu|\,p\,0\,\lambda\rangle, \quad p^2 = 0. \tag{2.2}$$

Generally, then, we have the structure

$$|\,p\,j\,\lambda\rangle = U(L_p)\,|\,\hat{p}\,j\,\lambda\rangle \tag{2.3}$$

so that, under an arbitrary motion of the group $\mathcal{P}$

$$|\,p\,j\,\lambda\rangle \rightarrow U(a,\Lambda)|\,p\,j\,\lambda\rangle = e^{ipa}\sum_{\mu}|\Lambda p\,j\,\mu\rangle\, D^{j}_{\mu\lambda}(R) \tag{2.4}$$

where $R$ belongs to the appropriate $\hat{G}$. The precise choice of boost $L_p$ depends upon what applications are to be made. We shall discuss for each type of $p_\mu$ ($p^2 > 0$, $p^2 < 0$, $p^2 = 0$) three different choices of $L_p$ which serve to diagonalize one of $W_0$, $W_3$ or $W_0 - W_3$. The null case, $p_\mu = 0$, is logically distinct since no $L_p$ is defined for it and we shall have to consider it separately.

Firstly, however, we deal with the subspaces $p = \hat{p}$ and the little group representations contained therein.

(i) <u>Timelike case, $p^2 > 0$</u>

For the standard momentum it is always possible to take

$$\hat{p}_\mu = (\pm\sqrt{p^2}\,,\,0\,,\,0\,,\,0\,) \tag{2.5}$$

where the sign of $\hat{p}_0$ is invariant. There is no transformation in $\mathcal{P}$ which can reverse the sign of $p_0$ when $p^2 > 0$. Such improper transformations belong to the "extended" group which will be considered separately later.

When acting in the subspace $p = \hat{p}$ the components of $W_\mu$ reduce to the form

$$W_\mu = \pm(0,\ J_{23},\ J_{31},\ J_{12})\sqrt{p^2} \tag{2.6}$$

which means that the little group $\hat{G}$ is in this case generated by $J_{23}$, $J_{31}$, $J_{12}$, which obey the commutation rules

$$\begin{aligned} [J_{23}, J_{31}] &= i J_{12} \\ [J_{31}, J_{12}] &= i J_{23} \\ [J_{12}, J_{23}] &= i J_{31} . \end{aligned} \tag{2.7}$$

so that $\hat{G}$ is simply the well-known rotation group SO(3). The irreducible representations of $\mathcal{P}$ are therefore characterized by

$$\begin{aligned} W^2 |\hat{p}\, j\, \lambda\rangle &= -p^2 (J_{12}^2 + J_{23}^2 + J_{31}^2) |\hat{p}\, j\, \lambda\rangle \\ &= -p^2 j(j+1) |\hat{p}\, j \lambda\rangle \end{aligned} \tag{2.8}$$

corresponding to the representations $D^j$ of SO(3) with

$$j = 0, \tfrac{1}{2}, 1, \ldots . \tag{2.9}$$

The representations corresponding to half-integer values of $j$ are of course 2-valued.

(ii) <u>Spacelike case $p^2 < 0$</u>

For the standard momentum we can take

$$\hat{p}_\mu = (0,\ 0,\ 0,\ \sqrt{-p^2}\ ) \tag{2.10}$$

where the root is positive. The sign of $\hat{p}_3$ has no invariant significance. When acting in the subspace $p = \hat{p}$ the components of $W_\mu$ reduce to the form

$$W_\mu = (J_{12}, J_{20}, J_{01}, 0)\sqrt{-p^2} \quad (2.11)$$

which means that the little group $\hat{G}$ is in this case generated by $J_{12}$, $J_{20}$ and $J_{01}$, which obey the commutation rules

$$\begin{aligned} [J_{12}, J_{20}] &= i J_{01} \\ [J_{20}, J_{01}] &= -i J_{12} \\ [J_{01}, J_{12}] &= i J_{20} \end{aligned} \quad (2.12)$$

so that $\hat{G}$ becomes the non-compact rotation group SO(2,1). The irreducible representations of $\mathcal{P}$ are characterised as before by

$$\begin{aligned} W^2 | \hat{p}\, j\, \lambda \rangle &= -p^2 (J_{12}^2 - J_{20}^2 - J_{01}^2) | \hat{p}\, j\, \lambda \rangle \\ &= -p^2 j(j+1) | \hat{p}\, j\, \lambda \rangle \end{aligned} \quad (2.13)$$

corresponding to the representations $D^j$ of SO(2,1). In this case, however, the possible values of $j$ are quite different from (2.9). It is usual to group the unitary representations of SO(2,1) into four distinct families:

(a) **Principal series**

$$Re(j) = -\tfrac{1}{2}, \quad -\infty < Im(j) < \infty \quad (2.14)$$

These representations are all infinite-dimensional with $\lambda$ taking all integer values or all half-integer values between $-\infty$ and $+\infty$.

$$\lambda = 0, \pm 1, \pm 2, \ldots$$

or $$\lambda = \pm \tfrac{1}{2}, \pm \tfrac{3}{2}, \ldots \tag{2.15}$$

Strictly speaking there are additional many-valued representations with $\lambda$ taking fractional values. We are in effect restricting ourselves to the one-valued representations of SU(1,1). Another point to note is that the representations $D^j$ and $D^{-j-1}$ are weakly equivalent. This will be made clear in Sec. 4, where the structure of these representations is discussed in some detail.

(b) Supplementary series

$$-\tfrac{1}{2} < \mathrm{Re}(j) < 0 \quad , \quad \mathrm{Im}(j) = 0 \tag{2.16}$$

These representations also are infinite-dimensional with $\lambda$ taking all integer values,

$$\lambda = 0, \pm 1, \pm 2, \ldots \tag{2.17}$$

(c) Discrete series

$$j = -\tfrac{1}{2}, -1, -\tfrac{3}{2}, \ldots \tag{2.18}$$

These are semi-infinite of two types depending on the sign of $\lambda$,

$$D^{j+}: \quad \lambda = -j, -j+1, -j+2, \ldots \tag{2.19}$$

$$D^{j-}: \quad \lambda = j, j-1, j-2, \ldots \tag{2.20}$$

(d) Scalar representation

$$j = 0 \tag{2.21}$$

This is the only finite-dimensional unitary representation of SO(2,1), $\lambda = 0$.

(iii) Lightlike case $p^2 = 0$

For the standard momentum we can take

$$\hat{p}_\mu = (\omega , 0 , 0 , \omega ) \tag{2.22}$$

where $\omega$ is arbitrary up to sign. As for the timelike case there is an invariant distinction between $\omega > 0$ and $\omega < 0$. When acting on the subspace $p = \hat{p}$ the components of $W_\mu$ reduce to the form

$$W_\mu = ( J_{12} , -\Pi_2 , \Pi_1 , J_{12} ) \omega \tag{2.23}$$

where

$$\begin{aligned} \Pi_1 &= J_{10} - J_{13} \\ \Pi_2 &= J_{20} - J_{23} \end{aligned} \tag{2.24}$$

The group $\hat{G}$ is generated in this case by $J_{12}$, $\Pi_1$ and $\Pi_2$, which obey the commutation rules:

$$\begin{aligned} [ J_{12} , \Pi_1 ] &= i \Pi_2 \\ [ J_{12} , \Pi_2 ] &= -i \Pi_1 \\ [ \Pi_1 , \Pi_2 ] &= 0 \end{aligned} \tag{2.25}$$

so that $\hat{G}$ becomes the Euclidean group in two dimensions, $SO(2) \wedge T(2)$. The irreducible representations of $\mathcal{P}$ are characterized by

$$\begin{aligned} W^2 | \hat{p} \rho \lambda \rangle &= -\omega^2 ( \Pi_1^2 + \Pi_2^2 ) | \hat{p} \rho \lambda \rangle \\ &= -\rho^2 | \hat{p} \rho \lambda \rangle \end{aligned} \tag{2.26}$$

corresponding to the representation $D^{\rho/\omega}$ of $SO(2) \wedge T(2)$. The parameter $\rho$ is of course a Poincaré invariant while the little group Casimir $(\rho / \omega)^2$ is not. This simply reflects the fact that our standard momentum $\hat{p}$ was not specified in terms of invariants as was the case in (i) and (ii). There is in fact a 1-parameter group of transformations - the Lorentz transformations in the 03-plane - which preserves the form of (2.22). Thus

$$e^{-i\zeta J_{03}} \,|\, \hat{p}, \rho\lambda \rangle = | e^{\zeta}\hat{p}, \rho\lambda \rangle \tag{2.27}$$

or

$$\omega \to e^{\zeta}\omega$$

while

$$e^{i\zeta J_{03}}(\Pi_1^2 + \Pi_2^2)e^{-i\zeta J_{03}} = e^{-2\zeta}(\Pi_1^2 + \Pi_2^2) \tag{2.28}$$

so that the product

$$\omega^2(\Pi_1^2 + \Pi_2^2) = \text{invariant}. \tag{2.29}$$

The unitary representations of $SO(2) \wedge T(2)$ can be grouped into two families:

(a.) Principal series, $D^{\rho}$

$$0 < \rho < \infty \;, \quad \mathrm{Im}\,\rho = 0 \tag{2.30}$$

These representations are infinite-dimensional with $\lambda$ taking all integer values or all half-integer values between $-\infty$ and $+\infty$,

$$\lambda = 0, \pm 1, \pm 2, \ldots$$

or

$$\lambda = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots \tag{2.31}$$

(b) Discrete series, $D^{0\lambda}$

These representations are 1-dimensional and correspond to

$$\rho = 0 \;. \tag{2.32}$$

Evidently in this case we have $\Pi_1 = \Pi_2 = 0$ so that the algebra reduces to $J_{12}$ which becomes the Casimir operator. Setting

$\pi_1 = \pi_2 = 0$ in (2.23) gives the relation

$$W_\mu \,|\, \hat{p}\, 0 \lambda \rangle = \hat{p}_\mu J_{12} \,|\, \hat{p}\, 0 \lambda \rangle = \lambda P_\mu \,|\, \hat{p}\, 0 \lambda \rangle \tag{2.33}$$

and the equality

$$W_\mu = \lambda P_\mu \tag{2.34}$$

is evidently Poincaré covariant thus exhibiting $\lambda$ as a Casimir invariant. To each integer or half-integer value of $\lambda$ there corresponds an irreducible representation $D^{0\lambda}$,

$$\lambda = 0, \pm \tfrac{1}{2}, \pm 1, \ldots \tag{2.35}$$

(iv) Null case $p_\mu = 0$

Here the representations of the Poincaré group coincide with those of the homogeneous Lorentz group. There is no standard momentum in this case and the little group $\hat{G}$ becomes SO(3,1) generated by the six $J_{\mu\nu}$. There are two invariants,

$$\tfrac{1}{2} J_{\mu\nu} J_{\mu\nu} = j_0^2 - \sigma^2 + 1 \tag{2.36}$$

$$\tfrac{1}{4} \epsilon_{\mu\nu\lambda\rho} J_{\mu\nu} J_{\lambda\rho} = 2 i j_0 \sigma \tag{2.37}$$

It is possible to label a complete set of basis vectors, $|\, j_0 \sigma j \lambda \rangle$, with two additional quantum numbers $j$ and $\lambda$ defined by

$$(J_{12}^2 + J_{23}^2 + J_{31}^2) \,|\, j_0 \sigma j \lambda \rangle = j(j+1) \,|\, j_0 \sigma j \lambda \rangle \tag{2.38}$$

$$J_{12} \,|\, j_0 \sigma j \lambda \rangle = \lambda \,|\, j_0 \sigma j \lambda \rangle . \tag{2.39}$$

The unitary irreducible representations $D^{j_0\sigma}$ of SO(3,1) come in two series, both infinite-dimensional:

(a) Principal series

$$\mathrm{Re}(\sigma) = 0 \qquad -\infty < \mathrm{Im}(\sigma) < \infty$$
$$j_0 = 0, \tfrac{1}{2}, 1, \ldots \tag{2.40}$$

with $j$ and $\lambda$ taking the values

$$j = j_0, j_0+1, j_0+2, \ldots$$
$$\lambda = -j, -j+1, \ldots, j \tag{2.41}$$

(b) Supplementary series

$$0 < \mathrm{Re}(\sigma) < 1 \quad , \quad \mathrm{Im}(\sigma) = 0 \tag{2.42}$$

$$j_0 = 0 \tag{2.43}$$

with $j$ and $\lambda$ taking the values

$$j = 0, 1, 2, \ldots$$
$$\lambda = -j, -j+1, \ldots, j \tag{2.44}$$

It will be shown in Sec. 4, where these representations are discussed more fully, that the representations $D^{j_0 \sigma}$ and $D^{-j_0 -\sigma}$ are weakly equivalent.

To summarize the discussion so far, we have found the following classes of unitary irreducible representations of the Poincaré group:

(i) Timelike

$$\mathfrak{D}^j_\pm(p^2): \text{ for } p^2 > 0, \quad \mathrm{sgn}(p_0) = \pm 1, \quad j = 0, \tfrac{1}{2}, 1, \ldots$$

(ii) Spacelike

$$\mathcal{D}^j(p^2): \text{ for } p^2<0, \quad \mathrm{Re}\, j = -\tfrac{1}{2}, \; -\infty < \mathrm{Im}\, j < \infty \qquad \text{(Principal)}$$

$$-\tfrac{1}{2} < \mathrm{Re}\, j < 0 \;, \quad \mathrm{Im}\, j = 0 \qquad \text{(Supplementary)}$$

$$j = 0 \qquad \text{(Scalar)}$$

$$\mathcal{D}^{j\pm}(p^2): \text{ for } p^2<0, \quad \mathrm{sgn}(\lambda) = \pm 1, \; j = -\tfrac{1}{2}, -1, -\tfrac{3}{2}, \ldots \qquad \text{(Discrete)}$$

(iii) Lightlike

$$\mathcal{D}^{\rho}_{\pm}(0): \text{ for } p^2=0, \; \mathrm{sgn}(p_0) = \pm 1, \; 0<\rho<\infty \qquad \text{(Principal)}$$

$$\mathcal{D}^{o\lambda}_{\pm}(0): \text{ for } p^2=0, \; \mathrm{sgn}(p_0) = \pm 1, \; \lambda = 0, \pm\tfrac{1}{2}, \pm 1, \; \ldots \qquad \text{(Discrete)}$$

(iv) Null

$$\mathcal{D}^{j_0\sigma}: \text{ for } p_\mu = 0, \; \mathrm{Re}(\sigma) = 0 \; -\infty < \mathrm{Im}(\sigma) < \infty, \; j_0 = 0, \tfrac{1}{2}, 1, \ldots \qquad \text{(Principal)}$$

$$0 < \mathrm{Re}(\sigma) < 1, \; \mathrm{Im}(\sigma) = 0, \; j_0 = 0 \qquad \text{(Supplementary)}$$

Consider now the problem of defining the functional form of the boost matrices $L_p$ . It is desired, firstly, that the variables in $L_p$ provide a suitable parametrization of the "mass-shell" or orbit of $p$ . Since

$$\begin{aligned} P_\mu |p j \lambda\rangle &= P_\mu U(L_p)|\hat{p} j \lambda\rangle \\ &= (L_p)_{\mu\nu}\hat{p}_\nu \, U(L_p)|\hat{p} j\lambda\rangle \\ &= p_\mu |p j \lambda\rangle \;, \end{aligned} \tag{2.45}$$

we require that

$$p_\mu = (L_p)_{\mu\nu}\hat{p}_\nu \;, \tag{2.46}$$

that is,

$$
\begin{aligned}
p_\mu &= \pm \sqrt{p^2}\,(L_p)_{\mu 0} \quad , \quad p^2 > 0 \\
p_\mu &= -\sqrt{-p^2}\,(L_p)_{\mu 3} \quad , \quad p^2 < 0 \\
p_\mu &= \omega (L_p)_{\mu 0} - \omega (L_p)_{\mu 3} \ , \quad p^2 = 0
\end{aligned}
\tag{2.47}
$$

Secondly, it is desired that $L_p$ be such as to diagonalize one of the operators $W_0$, $W_3$ or $W_0 - W_3$. It turns out that this requirement is met by constructing $L_p$ as the product of a Lorentz transformation in the 03-plane with transformations belonging to one of the little groups SO(3), SO(2,1) or SO(2)$\wedge$T(2). This structure will prove advantageous when we come to the problem of decomposing products of irreducible representations. We shall therefore define three distinct boost functions, $L_p^+$, $L_p^-$ and $L_p^0$, employing operations drawn, respectively, from SO(3), SO(2,1) and SO(2)$\wedge$T(2). They are

$$U(L_p^+) = e^{-i\varphi J_{12}}\ e^{-i\theta J_{31}}\ e^{i\varphi J_{12}}\ e^{-i\alpha J_{03}} \tag{2.48}$$

or

$$
(L_p^+)_{\mu\nu} =
\begin{pmatrix}
\operatorname{ch}\alpha & 0 & 0 & -\operatorname{sh}\alpha \\
\cos\varphi \sin\theta \operatorname{sh}\alpha & -\cos^2\varphi\cos\theta - \sin^2\varphi & -\sin\varphi\cos\varphi(\cos\theta - 1) & -\cos\varphi\sin\theta\operatorname{ch}\alpha \\
\sin\varphi\sin\theta\operatorname{sh}\alpha & \sin\varphi\cos\varphi(\cos\theta - 1) & -\sin^2\varphi\cos\theta - \cos^2\varphi & -\sin\varphi\sin\theta\operatorname{ch}\alpha \\
\cos\theta\operatorname{sh}\alpha & \cos\varphi\sin\theta & \sin\theta\sin\varphi & -\cos\theta\operatorname{ch}\alpha
\end{pmatrix}
\tag{2.49}
$$

$$U(L_p^-) = e^{-i\varphi J_{12}}\ e^{-i\beta J_{01}}\ e^{i\varphi J_{12}}\ e^{-i\gamma J_{03}} \tag{2.50}$$

or $(L_p^-)_{\mu\nu}$ =

$$\begin{pmatrix} ch\beta\, ch\gamma & -sh\beta\cos\varphi & -sh\beta\sin\varphi & -ch\beta\, sh\gamma \\ \cos\varphi\, sh\beta\, ch\gamma & -\cos^2\varphi\, ch\beta-\sin^2\varphi & \sin\varphi\cos\varphi(1-ch\beta) & -\cos\varphi\, sh\beta\, sh\gamma \\ \sin\varphi\, sh\beta\, ch\gamma & \sin\varphi\cos\varphi(1-ch\beta) & -\sin^2\varphi\, ch\beta-\cos^2\varphi & -\sin\varphi\, sh\beta\, sh\gamma \\ sh\gamma & 0 & 0 & -ch\gamma \end{pmatrix} \tag{2.51}$$

$$U(L_p^\circ) = e^{-i\varphi J_{12}}\, e^{-i\xi\Pi_1}\, e^{i\varphi J_{12}}\, e^{-i\chi J_{03}} \tag{2.52}$$

or

$$(L_p^\circ)_{\mu\nu} = \begin{pmatrix} (1+\frac{\xi^2}{2})ch\chi-\frac{\xi^2}{2}sh\chi & \xi\cos\varphi & \xi\sin\varphi & \frac{\xi^2}{2}ch\chi-(1+\frac{\xi^2}{2})sh\chi \\ \xi\cos\varphi(sh\chi-ch\chi) & -1 & 0 & \xi\cos\varphi(sh\chi-ch\chi) \\ \xi\sin\varphi(sh\chi-ch\chi) & 0 & -1 & \xi\sin\varphi(sh\chi-ch\chi) \\ (1-\frac{\xi^2}{2})sh\chi+\frac{\xi^2}{2}ch\chi & \xi\cos\varphi & \xi\sin\varphi & (\frac{\xi^2}{2}-1)ch\chi-\frac{\xi^2}{2}sh\chi \end{pmatrix} \tag{2.53}$$

Thus we have the following parametrisations:

(i) Timelike case $p^2 > 0$

$$\begin{aligned} p_\mu &= \pm\sqrt{p^2}\,(ch\alpha,\ sh\alpha\sin\theta\cos\varphi,\ sh\alpha\sin\theta\sin\varphi,\ sh\alpha\cos\theta) \\ &= \pm\sqrt{p^2}\,(ch\gamma\, ch\beta,\ ch\gamma\, sh\beta\cos\varphi,\ ch\gamma\, sh\beta\sin\varphi,\ sh\gamma) \\ &= \pm\sqrt{p^2}\,(ch\chi+\tfrac{\xi^2}{2}e^{-\chi},\ -\xi e^{-\chi}\cos\varphi,\ -\xi e^{-\chi}\sin\varphi,\ sh\chi+\tfrac{\xi^2}{2}e^{-\chi}) \end{aligned} \tag{2.54}$$

where the parameters take the respective ranges

$$0 \leq \varphi < 2\pi \quad , \quad 0 \leq \theta \leq \pi \quad , \quad 0 \leq \alpha < \infty$$

$$0 \leq \varphi < 2\pi \quad , \quad 0 \leq \beta < \infty \quad , \quad -\infty \leq \gamma \leq \infty \qquad (2.55)$$

$$0 \leq \varphi < 2\pi \quad , \quad -\infty < \xi \leq 0 \quad , \quad -\infty < \chi \leq \infty$$

Comparing the three expressions (2.54) we get the relations between the different parametrizations:

$$\left.\begin{array}{ll} \mathrm{sh}\gamma = \mathrm{sh}\alpha \cos\theta & \mathrm{ch}\alpha = \mathrm{ch}\gamma\,\mathrm{ch}\beta \\ \mathrm{th}\beta = \sin\theta\,\mathrm{th}\alpha & \tan\theta = \dfrac{\mathrm{sh}\beta}{\mathrm{th}\gamma} \end{array}\right\} \qquad (2.56)$$

$$\left.\begin{array}{ll} e^{-\chi} = \mathrm{ch}\gamma\,\mathrm{ch}\beta - \mathrm{sh}\gamma & \mathrm{sh}\gamma = \mathrm{sh}\chi + \dfrac{\xi^2}{2} e^{-\chi} \\ \xi = -\dfrac{\mathrm{ch}\gamma\,\mathrm{sh}\beta}{\mathrm{ch}\gamma\,\mathrm{ch}\beta - \mathrm{sh}\gamma} & \mathrm{th}\beta = -\dfrac{\xi e^{-\chi}}{\mathrm{ch}\chi + \frac{\xi^2}{2} e^{-\chi}} \end{array}\right\} \qquad (2.57)$$

$$\left.\begin{array}{ll} \mathrm{ch}\alpha = \mathrm{ch}\chi + \dfrac{\xi^2}{2} e^{-\chi} & e^{-\chi} = \mathrm{ch}\alpha - \mathrm{sh}\alpha\cos\theta \\ \tan\theta = -\dfrac{\xi e^{-\chi}}{\mathrm{sh}\chi + \frac{\xi^2}{2} e^{-\chi}} & \xi = -\dfrac{\mathrm{sh}\alpha \sin\theta}{\mathrm{ch}\alpha - \mathrm{sh}\alpha\cos\theta} \end{array}\right\} \qquad (2.58)$$

(ii) Spacelike case $p^2 < 0$

$$p_\mu = \sqrt{-p^2}\,(\mathrm{sh}\alpha\,,\ \mathrm{ch}\alpha \sin\theta\cos\varphi\,,\ \mathrm{ch}\alpha\sin\theta\sin\varphi,\ \mathrm{ch}\alpha\cos\theta)$$

$$= \sqrt{-p^2}\,(\mathrm{sh}\gamma\,\mathrm{ch}\beta\,,\ \mathrm{sh}\gamma\,\mathrm{sh}\beta\cos\varphi\,,\ \mathrm{sh}\gamma\,\mathrm{sh}\beta\sin\varphi,\ \mathrm{ch}\gamma\ )$$

$$= \sqrt{-p^2}\,(\mathrm{sh}\chi - \frac{\xi^2}{2} e^{-\chi}\,,\ \xi e^{-\chi}\cos\varphi\,,\ \xi e^{-\chi}\sin\varphi,\ \mathrm{ch}\chi - \frac{\xi^2}{2} e^{-\chi})$$

(2.59)

Comparing these expressions we get

$$\left.\begin{array}{llll} \operatorname{ch}\gamma = \operatorname{ch}\alpha\cos\theta & & \operatorname{sh}\alpha = \operatorname{sh}\gamma\operatorname{ch}\beta \\ \operatorname{th}\beta = \dfrac{\sin\theta}{\operatorname{th}\alpha} & & \tan\theta = \operatorname{sh}\beta\operatorname{th}\gamma \end{array}\right\} \quad (2.60)$$

$$\left.\begin{array}{llll} e^{-\chi} = \operatorname{ch}\gamma - \operatorname{sh}\gamma\operatorname{ch}\beta & & \operatorname{ch}\gamma = \operatorname{ch}\chi - \dfrac{\xi^2}{2}e^{-\chi} \\ \xi = \dfrac{\operatorname{sh}\gamma\operatorname{sh}\beta}{\operatorname{ch}\gamma - \operatorname{sh}\gamma\operatorname{ch}\beta} & & \operatorname{th}\beta = \dfrac{\xi e^{-\chi}}{\operatorname{sh}\chi - \frac{\xi^2}{2}e^{-\chi}} \end{array}\right\} \quad (2.61)$$

$$\left.\begin{array}{llll} \operatorname{sh}\alpha = \operatorname{sh}\chi - \dfrac{\xi^2}{2}e^{-\chi} & & e^{-\chi} = \operatorname{ch}\alpha\cos\theta - \operatorname{sh}\alpha \\ \tan\theta = \dfrac{\xi e^{-\chi}}{\operatorname{ch}\chi - \frac{\xi^2}{2}e^{-\chi}} & & \xi = \dfrac{\operatorname{ch}\alpha\sin\theta}{\operatorname{ch}\alpha\cos\theta - \operatorname{sh}\alpha} \end{array}\right\} \quad (2.62)$$

Evidently only the first parametrization in (2.59) serves to cover the entire orbit $p^2 < 0$ with

$$0 \leq \varphi < 2\pi \;, \quad 0 \leq \theta < \pi \;, \quad -\infty < \alpha < \infty \;. \qquad (2.63)$$

The other two cover only parts of the orbit. Thus

$$0 \leq \varphi < 2\pi \;, \quad 0 \leq \beta < \infty \;, \quad -\infty < \gamma < \infty \qquad (2.64)$$

corresponds to the region $\cos\theta \geq 1/\operatorname{ch}\alpha$ and

$$0 \leq \varphi < 2\pi \;, \quad 0 \leq \xi < \infty \;, \quad -\infty < \chi < \infty \qquad (2.65)$$

corresponds to the region $\cos\theta > \operatorname{th}\alpha$ . These regions are, as it happens, sufficiently large for the applications we shall be making in the next section.

(iii) Lightlike case $p^2 = 0$

$$
\begin{aligned}
p_\mu &= \omega e^{\alpha} ( 1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
&= \omega e^{\gamma} (\mathrm{ch}\beta, \mathrm{sh}\beta \cos\varphi, \sin\theta \sin\varphi, \cos\theta) \\
&= \omega e^{\chi} (1, 0, 0, 1)
\end{aligned}
\qquad (2.66)
$$

This parametrization evidently fails rather drastically in the last line. However, if we apply the boosts $L_p^+$, $L_p^-$ and $L_p^0$ to a different lightlike vector, namely

$$\hat{p}_\mu = (\omega, 0, 0, -\omega) \quad , \qquad (2.67)$$

which is the one that will be concerning us in practice, we get

$$
\begin{aligned}
p_\mu &= \omega(L_p)_{\mu 0} + (L_p)_{\mu 3} \\
&= \omega e^{-\alpha} ( 1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
&= \omega e^{-\gamma} (\mathrm{ch}\beta, \mathrm{sh}\beta\cos\varphi, \mathrm{sh}\beta\sin\varphi, -1) \\
&= \omega e^{-\chi} (1+\xi^2, -2\xi\cos\varphi, -2\xi\sin\varphi, -1+\xi^2)
\end{aligned}
\qquad (2.68)
$$

These expressions contain one parameter too many. To eliminate this redundancy we may regard $\omega$ (say) as a constant scale factor. Then the connections between different parametrizations can be derived as before:

$$
\left.
\begin{aligned}
e^{-\gamma} &= e^{-\alpha}\cos\theta , & e^{-\alpha} &= e^{-\gamma}\mathrm{ch}\beta \\
\mathrm{th}\,\beta/2 &= -\tan\theta/2 & \tan\theta/2 &= -\mathrm{th}\,\beta/2
\end{aligned}
\right\}
\qquad (2.69)
$$

$$
\left.\begin{array}{ll}
e^{-\chi} = e^{-\gamma} ch^2 \beta/2 & e^{-\gamma} = (1-\xi^2)\, e^{-\chi} \\
\xi = - th\, \beta/2 & th\, \beta/2 = -\xi
\end{array}\right\} \qquad (2.70)
$$

$$
\left.\begin{array}{ll}
e^{-\alpha} = (\xi^2+1)\, e^{-\chi} & e^{-\chi} = e^{-\alpha} \cos^2 \theta/2 \\
\tan \theta/2 = \xi & \xi = \tan \theta/2
\end{array}\right\} \qquad (2.71)
$$

The first parametrization serves to cover the orbit $p^2 = 0$ with

$$0 \leq \varphi < 2\pi \quad , \quad 0 \leq \theta \leq \pi \;, -\infty < \alpha < \infty \;. \qquad (2.72)$$

The second parametrisation covers the region $\pi/2 \leq \theta \leq \pi$ with

$$0 \leq \varphi < 2\pi \quad , -\infty < \beta \leq 0 \quad , \quad \infty < \gamma < \infty \;, \qquad (2.73)$$

and the third covers the same region with

$$0 \leq \varphi < 2\pi \quad , \quad 0 \leq \xi < \infty \;, -\infty < \chi < \infty \;. \qquad (2.74)$$

With these parametrizations it is easy to verify that

$$
\begin{aligned}
W_0\, U(L_p^+) &= U(L_p^+)\, (W_0\, ch\alpha + W_3\, sh\alpha) \\
W_3\, U(L_p^-) &= U(L_p^-)\, (W_0\, sh\gamma + W_3\, ch\gamma) \qquad (2.75) \\
(W_0 - W_3)\, U(L_p^0) &= U(L_p^0)\, (W_0 - W_3)\, e^{-\chi} \;.
\end{aligned}
$$

Defining the respective basis systems by

$$
\begin{aligned}
|pj\lambda\rangle^{+} &= U(L_p^{+})\,|\hat{p}\,j\,\lambda\rangle \\
|pj\lambda\rangle^{-} &= U(L_p^{-})\,|\hat{p}\,j\,\lambda\rangle \qquad (2.76) \\
|pj\lambda\rangle^{0} &= U(L_p^{0})\,|\hat{p}\,j\,\lambda\rangle
\end{aligned}
$$

and using the formulae (2.6) and (2.11)and (2.23), which give the action of $W_\mu$ on states with standard momentum

$$
\begin{aligned}
(W_0, W_3)\,|\hat{p}\,j\lambda\rangle &= (0, \pm\lambda\sqrt{p^2})\,|\hat{p}\,j\lambda\rangle, \quad p^2 > 0 \\
(W_0, W_3)\,|\hat{p}\,j\,\lambda\rangle &= (\lambda\sqrt{-p^2}\,, 0)\,|\hat{p}\,j\,\lambda\rangle, \quad p^2 < 0 \qquad (2.77) \\
(W_0, W_3)\,|\hat{p}\,\rho\,\lambda\rangle &= (\omega\lambda, -\omega\lambda)\,|\hat{p}\,\rho\,\lambda\rangle, \quad p^2 = 0
\end{aligned}
$$

we obtain for $p^2 > 0$ and $p^2 < 0$ the eigenvalue equations

$$
\begin{aligned}
W_0\,|pj\lambda\rangle^{+} &= \lambda\epsilon\sqrt{p_1^2 + p_2^2 + p_3^2}\;|pj\lambda\rangle^{+} \\
W_3\,|pj\lambda\rangle^{-} &= \lambda\epsilon\sqrt{p_0^2 - p_1^2 - p_2^2}\;|pj\lambda\rangle^{-} \\
(W_0 - W_3)\,|pj\lambda\rangle^{0} &= -\lambda(p_0 - p_3)\,|pj\lambda\rangle^{0} \qquad (2.78)
\end{aligned}
$$

where $\epsilon$ denotes an invariant sign factor, $\epsilon = \epsilon(p_0)$ for $p^2 > 0$ and $\epsilon = 1$ for $p^2 < 0$. For $p^2 = 0$ the corresponding formulae are

$$W_0 | p \rho \lambda \rangle^{+} = \lambda p_0 | p \rho \lambda \rangle^{+}$$

$$W_3 | p \rho \lambda \rangle^{-} = \lambda p_3 | p \rho \lambda \rangle^{-}$$

$$(W_0 - W_3) | p \rho \lambda \rangle^{\circ} = \lambda (p_0 - p_3) | p \rho \lambda \rangle^{\circ} \qquad (2.78')$$

The $p^2 = 0$ states are defined here by

$$| p \rho \lambda \rangle = U(L_p) | \hat{p} \rho \lambda \rangle$$

$$\quad = U(L_p) e^{-i\pi J_{31}} | \hat{\hat{p}} \rho \lambda \rangle \qquad (2.79)$$

where $L_p$ denotes one of $L_p^+$, $L_p^-$, or $L_p^{\circ}$. The four-momentum $\hat{\hat{p}}$ takes the standard form $(\omega, 0, 0, \omega)$. The helicity $\lambda$ is conventionally defined as the eigenvalue of $J_{12}$ on states with four-momentum $\hat{\hat{p}}$,

$$J_{12} | \hat{\hat{p}} \rho \lambda \rangle = \lambda | \hat{\hat{p}} \rho \lambda \rangle \qquad (2.80)$$

It remains only to construct the unitary matrices which transform one basis system to another, what might be called the "spin rearrangement matrices". Since, for any pair of boosts $L_p^-$ and $L_p^+$, say, we have the equality

$$(L_p^-)_{\mu\nu}\,\hat{p}_\nu = (L_p^+)_{\mu\nu}\,\hat{p}_\nu$$

then it must be that $(L_p^+)^{-1}L_p^-$ belongs to the little group $\hat{G}$ . Thus, for any representation, we have

$$U^{-1}(L_p^+)\,U(L_p^-) = e^{-i\varphi J_{12}}\,e^{i\alpha J_{03}}\,e^{i\theta J_{31}}\,e^{-i\beta J_{01}}\,e^{-i\gamma J_{03}}\,e^{i\varphi J_{12}}$$

$$= \begin{cases} e^{-i\varphi J_{12}}e^{-i\Theta J_{31}}\,e^{i\varphi J_{12}} & , \quad p^2>0 \\ e^{-i\varphi J_{12}}e^{-i\Theta J_{01}}\,e^{i\varphi J_{12}} & , \quad p^2<0 \\ e^{-i\varphi J_{12}}e^{-i\Theta(J_{10}+J_{13})}e^{i\varphi J_{12}} & , \quad p^2=0 \end{cases} \tag{2.81}$$

**where, since the $\varphi$ dependence factors out we have been able to exclude from the form (2.81) those $J_{\mu\nu}$ with $\mu$ or $\nu = 2$ . The angle $\Theta$ depends on $p_\mu$ . Similarly, we have**

$$U^{-1}(L_p^-)\,U(L_p^\circ) = e^{-i\varphi J_{12}}e^{i\gamma J_{03}}e^{i\beta J_{01}}\,e^{-i\xi\Pi_1}\,e^{-i\chi J_{03}}\,e^{i\varphi J_{12}}$$

$$= \begin{cases} e^{-i\varphi J_{12}}e^{-i\Phi J_{31}}\,e^{i\varphi J_{12}} & , \quad p^2>0 \\ e^{-i\varphi J_{12}}e^{-i\Phi J_{01}}\,e^{i\varphi J_{12}} & , \quad p^2<0 \\ e^{-i\varphi J_{12}}e^{-i\Phi(J_{10}+J_{13})}e^{i\varphi J_{12}} & , \quad p^2=0 \end{cases} \tag{2.82}$$

$$U^{-1}(L_p^\circ)\,U(L_p^+) = e^{-i\varphi J_{12}}e^{i\chi J_{03}}e^{i\xi\Pi_1}\,e^{-i\theta J_{31}}\,e^{-i\alpha J_{03}}\,e^{i\varphi J_{12}}$$

$$= \begin{cases} e^{-i\varphi J_{12}}e^{-i\Psi J_{31}}\,e^{i\varphi J_{12}} & , \quad p^2>0 \\ e^{-i\varphi J_{12}}e^{-i\Psi J_{01}}\,e^{i\varphi J_{12}} & , \quad p^2<0 \\ e^{-i\varphi J_{12}}e^{-i\Psi(J_{10}+J_{13})}e^{i\varphi J_{12}} & , \quad p^2=0 \end{cases} \tag{2.83}$$

In terms of the three angles $\Theta$, $\Phi$ and $\Psi$, we can write the sought-after relations in the form

$$
\begin{aligned}
|p\,j\,\lambda\rangle^- &= \sum_\mu |p\,j\,\mu\rangle^+ \, d^j_{\mu\lambda}(\Theta)\, e^{-i(\mu-\lambda)\varphi} \\
|p\,j\,\lambda\rangle^0 &= \sum_\mu |p\,j\,\mu\rangle^- \, d^j_{\mu\lambda}(\Phi)\, e^{-i(\mu-\lambda)\varphi} \qquad (2.84)\\
|p\,j\,\lambda\rangle^+ &= \sum_\mu |p\,j\,\mu\rangle^0 \, d^j_{\mu\lambda}(\Psi)\, e^{-i(\mu-\lambda)\varphi}
\end{aligned}
$$

for the various cases, $p^2 > 0$, $p^2 < 0$ and $p^2 = 0$. The functions $d^j_{\mu\lambda}$ are matrix elements of the little group transformations,

$$
\begin{aligned}
d^j_{\mu\lambda}(\Theta) &= \langle \hat{p}\, j\, \mu | e^{-i\Theta J_{31}} | \hat{p}\, j\, \lambda\rangle \,, \quad p^2 > 0 \\
&= \langle \hat{p}\, j\, \mu | e^{-i\Theta J_{01}} | \hat{p}\, j\, \lambda\rangle \,, \quad p^2 < 0 \qquad (2.85)\\
&= \langle \hat{p}\, j\, \mu | e^{-i\Theta(J_{10}+J_{13})} | \hat{p}\, j\, \lambda\rangle \,, \quad p^2 = 0
\end{aligned}
$$

No confusion can arise from using the same symbol $d^j$ for the different representations since the actual range of $j$ will distinguish them. For the $p^2 = 0$ case, of course, we should read $\rho$ instead of $j$ .

To evaluate the angles $\Theta$, $\Phi$ and $\Psi$, it will be sufficient to work in a two-dimensional representation of the $J_{\mu\nu}$,

$$
J_{ij} = \tfrac{1}{2}\sigma_k \quad , \qquad J_{0k} = \tfrac{i}{2}\sigma_k \qquad (2.86)
$$

where the $\sigma_i$ are Pauli matrices. In this representation we have

$$
e^{-i\alpha J_{03}} = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix}, \qquad e^{i\beta J_{01}} = \begin{pmatrix} \mathrm{ch}\,\beta/2 & \mathrm{sh}\,\beta/2 \\ \mathrm{sh}\,\beta/2 & \mathrm{ch}\,\beta/2 \end{pmatrix},
$$

and

$$e^{-i\theta J_{31}} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} , \quad e^{-i\xi\Pi_1} = \begin{pmatrix} 1 & -\xi \\ 0 & 1 \end{pmatrix} \qquad (2.87)$$

Therefore,

$$e^{i\alpha J_{03}} \; e^{i\theta J_{31}} \; e^{-i\beta J_{01}} \; e^{-i\chi J_{03}} =$$

$$= \begin{pmatrix} e^{-\frac{\alpha-\chi}{2}}(\cos\theta/2 \; ch\beta/2 + \sin\theta/2 \; sh\beta/2) & e^{-\frac{\alpha-\chi}{2}}(\cos\theta/2 \; sh\beta/2 + \sin\theta/2 \; ch\beta/2) \\ e^{\frac{\alpha+\chi}{2}}(-\sin\theta/2 \; ch\beta/2 + \cos\theta/2 \; sh\beta/2) & e^{\frac{\alpha-\chi}{2}}(-\sin\theta/2 \; sh\beta/2 + \cos\theta/2 \; ch\beta/2) \end{pmatrix}$$

$$= \begin{cases} \begin{pmatrix} \cos\Theta/2 & -\sin\Theta/2 \\ \sin\Theta/2 & \cos\Theta/2 \end{pmatrix} , & p^2 > 0 \\ \begin{pmatrix} ch\,\Theta/2 & sh\,\Theta/2 \\ sh\,\Theta/2 & ch\,\Theta/2 \end{pmatrix} , & p^2 < 0 \\ \begin{pmatrix} 1 & 0 \\ -\Theta & 1 \end{pmatrix} , & p^2 = 0 \end{cases} \qquad (2.88)$$

comparing which gives, for $p^2 > 0$ ,

$$\cos\Theta = \cos^2\Theta/2 - \sin^2\Theta/2$$

$$= (\cos\theta/2 \; ch\beta/2 + \sin\theta/2 \; sh\beta/2)(-\sin\theta/2 \; sh\beta/2 + \cos\theta/2 \; ch\beta/2) -$$

$$+ (\cos\theta/2 \; sh\beta/2 + \sin\theta/2 \; ch\beta/2)(-\sin\theta/2 \; ch\beta/2 + \cos\theta/2 \; sh\beta/2) =$$

$$= \left(\cos^2\theta/2\ \mathrm{ch}^2\beta/2 - \sin^2\theta/2\ \mathrm{sh}^2\beta/2\right) + \left(\cos^2\theta/2\ \mathrm{sh}^2\beta/2 - \sin^2\theta/2\ \mathrm{ch}^2\beta/2\right)$$

$$= \left(\cos^2\theta/2 - \sin^2\theta/2\right)\ \left(\mathrm{ch}^2\beta/2 + \mathrm{sh}^2\beta/2\right)$$

$$= \cos\theta\ \mathrm{ch}\,\beta \tag{2.89}$$

and, for $p^2 < 0$

$$\mathrm{ch}\,\Theta = \cos\theta\ \mathrm{ch}\,\beta \tag{2.90}$$

and, for $p^2 = 0$

$$\Theta^2 = e^{\alpha+\gamma}(1-\cos\theta)\,(1+\mathrm{ch}\,\beta)\ . \tag{2.91}$$

With the help of relations (2.56), (2.60) and (2.69) these results may be expressed in the form

$$\tan\Theta = \frac{\tan\theta}{\mathrm{ch}\,\alpha} = \frac{\mathrm{th}\,\beta}{\mathrm{sh}\,\gamma} = \frac{-\xi e^{-\chi}}{(\mathrm{ch}\chi+\frac{\xi^2}{2}e^{-\chi})(\mathrm{sh}\chi+\frac{\xi^2}{2}e^{-\chi})} \quad \text{for } p^2 > 0$$

$$\mathrm{th}\,\Theta = \frac{\tan\Theta}{\mathrm{sh}\,\alpha} = \frac{\mathrm{th}\,\beta}{\mathrm{ch}\,\gamma} = \frac{\xi e^{-\chi}}{(\mathrm{ch}\chi-\frac{\xi^2}{2}e^{-\chi})(\mathrm{sh}\chi-\frac{\xi^2}{2}e^{-\chi})} \quad \text{for } p^2 < 0$$

$$\Theta = e^{\alpha}\tan\theta = -e^{\gamma}\,\mathrm{th}\,\beta = \frac{2\xi e^{\chi}}{1-\xi^4} \quad \text{for } p^2 = 0 \tag{2.92}$$

Similarly,

$$e^{i\gamma J_{03}}\, e^{i\beta J_{01}}\ e^{-i\xi\pi_1}\ e^{-i\chi J_{03}} =$$

$$= \begin{pmatrix} e^{-\frac{\gamma-\chi}{2}}\,\mathrm{ch}\,\beta/2 & -e^{-\frac{\gamma+\chi}{2}}(\xi\,\mathrm{ch}\,\beta/2 + \mathrm{sh}\,\beta/2) \\ -e^{\frac{\gamma+\chi}{2}}\,\mathrm{sh}\,\beta/2 & e^{\frac{\gamma-\chi}{2}}(\xi\,\mathrm{sh}\,\beta/2 - \mathrm{ch}\,\beta/2) \end{pmatrix}$$

$$
= \begin{cases} \begin{bmatrix} \cos \Phi/2 & -\sin \Phi/2 \\ \sin \Phi/2 & \cos \Phi/2 \end{bmatrix} , & p^2 > 0 \\ \begin{bmatrix} ch\ \Phi/2 & sh\ \Phi 2 \\ sh\ \Phi/2 & ch\ \Phi 2 \end{bmatrix} , & p^2 < 0 \\ \begin{bmatrix} 1 & 0 \\ -\Phi & 1 \end{bmatrix} , & p^2 = 0 \end{cases} \tag{2.93}
$$

Thus, for $p^2 > 0$

$$\cos \Phi = \xi\, sh\beta - ch\beta$$

and, for $p^2 < 0$

$$ch\ \Phi = \xi\, sh\beta - ch\beta$$

and, for $p^2 = 0$

$$\Phi = e^{\frac{\gamma+\chi}{2}}\, sh\beta/2$$

$$
\begin{aligned}
\tan \Phi &= \frac{sh\alpha \sin\theta}{sh^2\alpha\cos^2\theta - sh\alpha ch\alpha \cos\theta + 1} = \frac{sh\beta}{ch\gamma - sh\gamma ch\beta} = \frac{-\xi}{ch\chi - \frac{\xi^2}{2} e^{-\chi}} & p^2 > 0 \\
th\ \Phi &= \frac{ch\alpha \sin\theta}{ch^2\alpha\cos^2\theta - sh\alpha ch\alpha\cos\theta - 1} = \frac{sh\beta}{sh\gamma - ch\gamma ch\beta} = \frac{-\xi}{sh\chi + \frac{\xi^2}{2} e^{-\chi}} & p^2 < 0 \\
\Phi &= \frac{e^{\alpha} \tan\theta/2}{\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}} = e^{\gamma}\, th\beta/2 = \frac{\xi\, e^{\chi}}{\xi^2 - 1} & p^2 = 0
\end{aligned} \tag{2.94}
$$

Finally,

$$
e^{i\chi J_{03}}\ e^{i\xi \Pi_1}\ e^{-i\theta J_{01}}\ e^{-i\alpha J_{03}} =
$$

$$
= \begin{pmatrix} e^{-\frac{\chi-\alpha}{2}} (\cos\theta/2 + \xi \sin\theta/2) & e^{-\frac{\chi+\alpha}{2}} (-\sin\theta/2 + \xi\cos\theta/2) \\ e^{\frac{\chi+\alpha}{2}} \sin\theta/2 & e^{\frac{\chi-\alpha}{2}} \cos\theta/2 \end{pmatrix} \tag{2.95}
$$

In the same way as before one gets

$$
\begin{aligned}
\cos\Psi &= \cos\theta + \xi\sin\theta \quad , \quad p^2 > 0 \\
\mathrm{ch}\,\Psi &= \cos\theta + \xi\sin\theta \quad , \quad p^2 < 0 \\
\Psi &= -e^{\frac{\chi+\alpha}{2}}\sin\theta/2 \quad , \quad p^2 = 0
\end{aligned}
\tag{2.96}
$$

which become on using (2.58), (2.62) and (2.71):

$$
\begin{aligned}
\tan\Psi &= \frac{\sin\theta}{\mathrm{sh}\,\alpha - \mathrm{ch}\,\alpha\cos\theta} = \frac{\mathrm{ch}\,\gamma\,\mathrm{sh}\,\beta}{\mathrm{ch}^2\gamma\,\mathrm{ch}^2\beta - \mathrm{sh}\,\gamma\,\mathrm{ch}\,\gamma\,\mathrm{ch}\,\beta - 1} = \frac{\xi}{\mathrm{sh}\,\chi - \frac{\xi^2}{2}e^{-\chi}} \\
\mathrm{th}\,\Psi &= \frac{\sin\theta}{\mathrm{ch}\,\alpha - \mathrm{sh}\,\alpha\cos\theta} = \frac{\mathrm{sh}\,\gamma\,\mathrm{sh}\,\beta}{\mathrm{sh}^2\gamma\,\mathrm{ch}^2\beta - \mathrm{sh}\,\gamma\,\mathrm{ch}\,\gamma\,\mathrm{ch}\,\beta + 1} = \frac{\xi}{\mathrm{ch}\,\chi + \frac{\xi^2}{2}e^{-\chi}} \\
\Psi &= -e^{\alpha}\tan\theta/2 = \frac{e^{\gamma}\,\mathrm{th}\,\beta/2}{\mathrm{ch}^2\beta/2 + \mathrm{sh}^2\beta/2} = -\frac{\xi e^{\chi}}{\xi^2 + 1}
\end{aligned}
\tag{2.97}
$$

This completes the discussion of basis vectors. A check on the computations is provided by the consistency requirement derivable from (2.84), namely,

$$
\Theta + \Phi + \Psi = 0 \tag{2.98}
$$

## 3. REDUCTION OF THE DIRECT PRODUCT.

It is well known that the direct product of two unitary representations of the Poincaré group can be completely reduced into a direct sum of irreducible representations. This reduction has been discussed by many authors using various formalisms. We shall adopt here the physicist's attitude: ignoring any of the more intricate mathematical questions that may arise, proceeding as it were, in a state of innocence.

Before we start, however, it may be worth mentioning that there is at least one important distinction between the finite- and the infinite-dimensional problems. In dealing with a non-compact group one's intuition may fail to warn that the direct product of an infinite-dimensional unitary representation with a finite-dimensional non-unitary one may contain unitary as well as non-unitary irreducible representations. This is indeed the case. That is to say, there do exist invariant couplings between two unitary representations and a non-unitary one. On the other hand, the reduction of the direct product of two unitary representations, as usually formulated, contains only unitary irreducible representations - the non-unitary ones are excluded by convergence requirements. This means that the problem of reducing direct products is not always equivalent to the problem of finding invariant couplings or Clebsch-Gordan coefficients.

Disregarding, for the present, this question of possible finite-dimensional representations, we proceed with the reduction in the light of the formalism developed above. Consider the problem

$$\mathcal{D}_1 \otimes \mathcal{D}_2 = \sum_n \oplus \mathcal{D}_n \tag{3.1}$$

or, in terms of basis vectors,

$$|p_1 j_1 \lambda_1 p_2 j_2 \lambda_2\rangle = \sum_n \int dp \sum_{j\lambda} |n p j \lambda\rangle . \langle n p \lambda | p_1 j_1 \lambda_1 p_2 j_2 \lambda_2 \rangle \tag{3.2}$$

In Eq.(3.2) the index, n , is supposed to comprise all of the necessary labels which are not shown explicitly. An explicit realization of n as indeed of the coupling coefficient, $\langle n\ p\ j\lambda | p_1\ j_1\ \lambda_1\ \ p_2\ j_2\ \lambda_2\rangle$, itself, will be developed in the following.

The basic invariances of the coupling coefficients can be discovered by applying an arbitrary transformation of the Poincaré group to both sides of (3.2) while requiring the index , n , to be invariant. For the states $|n\, p\, j\, \lambda\rangle$ one can assume a transformation law of the form

$$\mathcal{U}(a,\Lambda)\,|n\,p\,j\,\lambda\rangle = e^{ipa}\sum_{\mu}|n\,\Lambda p\,j\,\mu\rangle\; D^{j}_{\mu\lambda}(L^{-1}_{\Lambda p}\,\Lambda\, L_{p}) \qquad (3.3)$$

which assures the invariance of n . Translation invariance alone gives

$$\langle n\,p\,j\,\lambda\,|\,p_1\,j_1\,\lambda_1\,p_2\,j_2\,\lambda_2\rangle \sim \delta(p_1+p_2-p) \qquad (3.4)$$

while invariance under the homogeneous transformations gives

$$\langle n\,p\,j\,\lambda\,|\,p_1\,j_1\,\lambda_1\,p_2\,j_2\,\lambda_2\rangle =$$

$$= D^{j}_{\lambda\mu}(R^{-1})\,\langle n\,\Lambda p\,j\,\mu\,|\,\Lambda p_1\,j_1\,\mu_1\,\Lambda p_2\,j_2\,\mu_2\rangle\; D^{j_1}_{\mu_1\lambda_1}(R_1)\,D^{j_2}_{\mu_2\lambda_2}(R_2) \qquad (3.5)$$

where $R$ , $R_1$ and $R_2$ denote the appropriate little group rotations.

In view of the conditions (3.4) and (3.5) it will be sufficient for us to take $p = p_1 + p_2$ and to fix $p$ in one of the standard directions $\hat{p}$ . Consider now the possible values of $p^2$ corresponding to given $p_1^2$ , $p_2^2$ and, where relevant, sgn $(p_{10})$ and sgn $(p_{20})$. There are ten cases to be distinguished but only four of these need be examined explicitly, the others following rather trivially.

(i) $\underline{\mathcal{D}_+(m_1^2)\otimes\mathcal{D}_+(m_2^2)}$ , where

$$p_1^2 = m_1^2 \geq 0\ ,\quad p_2^2 = m_2^2 \geq 0\ ,\quad \mathrm{sgn}(p_{10}) = \mathrm{sgn}(p_{20}) = +1 \qquad (3.6)$$

There is only one type of representation in this product,

$$\mathcal{D}_+(p^2) \quad \text{with} \quad (m_1+m_2)^2 \leq p^2 < \infty \qquad (3.7)$$

The masses are non-negative in (3.7), $m_1$ , $m_2 \geqslant 0$ .

(ii) $\underline{\mathcal{D}_+(m_1^2) \otimes \mathcal{D}_-(m_2^2)}$ , where

$$p_1^2 = m_1^2 \geqslant 0 \;, \quad p_2^2 = m_2^2 \geqslant 0 \;, \quad \mathrm{sgn}(p_{10}) = +1 \;, \quad \mathrm{sgn}(p_{20}) = -1 \;. \tag{3.8}$$

If $m_1 > m_2$ there are three types of representation in the product

$$\begin{aligned} &\mathcal{D}_+(p^2) \quad \text{with} \quad 0 < p^2 \leqslant (m_1 - m_2)^2 \\ &\mathcal{D}_+(0) \quad \text{with} \quad p^2 = 0 \quad (\text{but } p_\mu \neq 0) \\ &\mathcal{D}(p^2) \quad \text{with} \quad -\infty < p^2 < 0 \;. \end{aligned} \tag{3.9}$$

If $m_1 < m_2$ the $p^2$ content is the same but $\mathcal{D}_+$ becomes $\mathcal{D}_-$ in (3.9). The representations (3.7) and (3.9) must be further classified according to their j-values but this we shall postpone. If $m_1 = m_2$ there are two types,

$$\begin{aligned} &\mathcal{D}^{j_0\sigma} \quad \text{with} \quad p_\mu = 0 \\ &\mathcal{D}(p^2) \quad \text{with} \quad -\infty < p^2 < 0 \end{aligned} \tag{3.10}$$

(iii) $\underline{\mathcal{D}_+(m_1^2) \otimes \mathcal{D}(-m_2^2)}$, where

$$p_1^2 = m_1^2 \geqslant 0, \quad p_2^2 = -m_2^2 < 0 \quad \text{and} \quad \mathrm{sgn}(p_{10}) = +1 \;. \tag{3.11}$$

There are always three types of representation here,

$$\begin{aligned} &\mathcal{D}_+(p^2) \qquad \text{with} \qquad 0 < p^2 < \infty \\ &\mathcal{D}_+(0) \qquad \text{with} \qquad p^2 = 0 \quad (\text{but } p_\mu \neq 0) \\ &\mathcal{D}(p^2) \qquad \text{with} \qquad -\infty < p^2 < 0 \end{aligned} \tag{3.12}$$

(iv) $\underline{\mathcal{D}(-m_1^2) \otimes \mathcal{D}(-m_2^2)}$, where

$$p_1^2 = -m_1^2 < 0 \quad \text{and} \quad p_2^2 = -m_2^2 < 0 \tag{3.13}$$

If $m_1 \neq m_2$ there are five types of representation in this product,

$$\begin{array}{llll} \mathcal{D}_+(p^2) & \text{and} \quad \mathcal{D}_-(p^2) & \text{with} & 0 < p^2 < \infty \\ \mathcal{D}_+(0) & \text{and} \quad \mathcal{D}_-(0) & \text{with} & p^2 = 0 \;\; (\text{but } p_\mu \neq 0) \\ \mathcal{D}(p^2) & & \text{with} & -\infty < p^2 < 0 \end{array} \tag{3.14}$$

If $m_1 = m_2$ there is, in addition, the representation

$$\mathcal{D}^{j_0 \sigma} \quad \text{with} \quad p_\mu = 0 \;. \tag{3.15}$$

The remaining six cases need not be listed explicitly. They are: $\mathcal{D}_- \otimes \mathcal{D}_-$ and $\mathcal{D}_- \otimes \mathcal{D}$ which follow from (i) and (iii) in an obvious way; $\mathcal{D}_\pm \otimes \mathcal{D}^{j_0\sigma}$ and $\mathcal{D} \otimes \mathcal{D}^{j_0\sigma}$ for which $p = p_1$ ; and $\mathcal{D}^{j_0\sigma} \otimes \mathcal{D}^{j_0\sigma}$ for which $p = 0$ .

More difficult to solve is the problem of discovering what values of $j$ can appear in the various cases. This cannot be dealt with merely by considerations involving the basis vectors in isolation. Strictly, one has to use scalar products of them with normalizable states in the Hilbert space - i.e., wave packets - and take careful account of the asymptotic behaviour of these functions. Since, for the applications we have in view, this asymptotic behaviour is not always known in advance we shall have to proceed in a rather formal manner and discard any pretence of rigour. <u>More specifically, we shall assume that any function $f(\hat{G})$ defined over one of the little groups $\hat{G}$ can be expanded as an integral over the unitary representations of $\hat{G}$</u> ,

$$f(\hat{G}) = \int d\mu(j) \sum_{\lambda\mu} f_{\lambda\mu}(j)\, D^j_{\lambda\mu}(\hat{G}) \tag{3.16}$$

with

$$f_{\lambda\mu}(j) = \int d\mu(\hat{G})\, D^j_{\lambda\mu}(\hat{G})^* f(\hat{G}) \tag{3.17}$$

where, in (3.16), the integral extends over the unitary representations $D^j$ with Plancherel measure $d\mu(j)$ and, in (3.17), the integral extends over the group $\hat{G}$ with Haar measure $d\mu(\hat{G})$. The formulae (3.16) and (3.17) are a valid group-theoretical result only if $f$ is square-integrable,

$$\int d\mu(\hat{G})\, |f(\hat{G})|^2 = \int d\mu(j) \sum_{\lambda\mu} |f_{\lambda\mu}(j)|^2 < \infty \quad , \tag{3.18}$$

but we shall apply them formally to <u>basis vectors</u> which are certainly not square-integrable. This can be rigorously justified if the basis vectors are employed only in the specification of those matrix elements which are square-integrable.

In order to be able to define the basis vectors as functions over the little groups it is necessary to take for the states $|p_1\, j_1\, \lambda_1\rangle$ and $|p_2\, j_2\, \lambda_2\rangle$ those defined with the boosts $L^+$, $L^0$ or $L^-$ accordingly as $p_1 + p_2$ is timelike, lightlike or spacelike. This will generally necessitate the introduction of the spin rearrangement matrices (defined in Sec. 2) in order to cover the complete range of representations contained in a given product.

Notice that there are essentially three independent parameters determining the components of $p_1$ and $p_2$ subject to the constraint $p_1 + p_2 = \hat{p}$. These are, for the timelike, lightlike and spacelike cases, the mass $p^2$ and a pair of angles, $(\varphi, \theta)$, $(\varphi, \xi)$ and $(\varphi, \beta)$, respectively. For the null case they are three angles, $(\varphi, \theta, \alpha)$. The unifying characteristic of these parametrizations lie in that while the total momentum $p_1 + p_2$ is fixed in one of the standard directions, the "relative" momentum is specified by angles which relate to the appropriate little group for each case.

(i) <u>Timelike case $(p_1 + p_2)^2 > 0$</u>

The parameters $(\varphi_1\ \theta_1\ \alpha_1)$ and $(\varphi_2\ \theta_2\ \alpha_2)$ associated with the momenta $p_1$ and $p_2$ must be so chosen that the total four-momentum $p_1 + p_2$ is aligned with the zero-axis. This requires

$$m_1 \operatorname{sh}\alpha_1 \cos\theta_1 + m_2 \operatorname{sh}\alpha_2 \cos\theta_2 = 0$$

$$m_1 \operatorname{sh}\alpha_1 \sin\theta_1 \cos\varphi_1 + m_2 \operatorname{sh}\alpha_2 \sin\theta_2 \cos\varphi_2 = 0$$

$$m_1 \operatorname{sh}\alpha_1 \sin\theta_1 \sin\varphi_1 + m_2 \operatorname{sh}\alpha_2 \sin\theta_2 \sin\varphi_2 = 0$$

These conditions are met by taking

$$\varphi = \varphi_1 = \begin{cases} \varphi_2 - \pi\ , & 0 \leqslant \varphi < \pi \\ \varphi_2 + \pi\ , & \pi \leqslant \varphi < 2\pi \end{cases}$$

$$\theta = \theta_1 = \pi - \theta_2$$

$$0 = m_1 \operatorname{sh}\alpha_1 - m_2 \operatorname{sh}\alpha_2 \quad .$$

The angles $\alpha_1$ and $\alpha_2$ are expressible in terms of the total mass $s = (p_1 + p_2)^2$ by

$$\operatorname{ch}\alpha_1 = \frac{s + m_1^2 + m_2^2}{2m_1\sqrt{s}} \quad , \quad \operatorname{ch}\alpha_2 = \frac{s - m_1^2 + m_2^2}{2m_2\sqrt{s}}$$

The product states may be written in the form

$$|p_1\, j_1 \lambda_1\ ,\ p_2\, j_2 \lambda_2 \rangle^{+} = U^{(1)}(L^{+}_{p_1})\, U^{(2)}(L^{+}_{p_2})\, |\, \hat{p}_1\, j_1 \lambda_1\ ,\ \hat{p}_2\, j_2 \lambda_2 \rangle$$

$$= e^{-i\varphi J_{12}^{(1)}} e^{-i\theta J_{31}^{(1)}} e^{i\varphi J_{12}^{(1)}} e^{-i\alpha_1 J_{03}^{(1)}} \cdot$$

$$\cdot\, e^{-i(\varphi\pm\pi) J_{12}^{(2)}} e^{-i(\pi-\theta) J_{31}^{(2)}} e^{i(\varphi\pm\pi) J_{12}^{(2)}} e^{-i\alpha_2 J_{03}^{(2)}} \left|\hat{p}_1 j_1 \lambda_1, \hat{p}_2 j_2 \lambda_2\right\rangle$$

$$= e^{-i\varphi J_{12}} e^{-i\theta J_{31}} e^{-i\alpha_1 J_{03}^{(1)} + i\alpha_2 J_{03}^{(2)}} \cdot$$

$$\cdot\, e^{i\pi J_{31}^{(2)}} e^{i\varphi J_{12}} \left|\hat{p}_1 j_1 \lambda_1, \hat{p}_2 j_2 \lambda_2\right\rangle$$

$$= e^{-i\varphi J_{12}} e^{-i\theta J_{31}} \left|\alpha\right\rangle e^{i(\lambda_1+\lambda_2)\varphi} \tag{3.19}$$

where

$$\left|\alpha\right\rangle = e^{-i\alpha_1 J_{03}^{(1)} + i\alpha_2 J_{03}^{(2)}} e^{i\pi J_{31}^{(2)}} \left|\hat{p}_1 j_1 \lambda_1, \hat{p}_2 j_2 \lambda_2\right\rangle \tag{3.20}$$

The total $J_{\mu\nu}$ is of course defined by

$$J_{\mu\nu} = J_{\mu\nu}^{(1)} + J_{\mu\nu}^{(2)} \tag{3.21}$$

We are now in a position to project out the irreducible representations simply by applying the formula (3.17) to the vectors $U(\hat{G})\left|\alpha\right\rangle$. The result is

$$\left|\lambda_1\lambda_2 ; \hat{p}\, j \lambda\right\rangle \delta_{\mu,\lambda_1-\lambda_2} = \int_{SO(3)} d\mu(\hat{G})\; D^{j}_{\lambda\mu}(\hat{G})^{*}\, U(\hat{G}) \left|\alpha\right\rangle \tag{3.22}$$

where j takes the values $|\lambda_1-\lambda_2|, |\lambda_1-\lambda_2|+1, |\lambda_1-\lambda_2|+2, \dots$ . The integration extends over the compact group SO(3). The Kronecker symbol $\delta_{\mu,\lambda_1-\lambda_2}$ appears as a consequence of the condition

$$J_{12}\left|\alpha\right\rangle = (\lambda_1-\lambda_2)\left|\alpha\right\rangle \tag{3.23}$$

For brevity we have neglected to show in the states $|\lambda_1\lambda_2\,;\hat{p}\,j\,\lambda\rangle$ the invariant labels $p_1^2 = m_1^2$ , $p_2^2 = m_2^2$ and $j_1$ , $j_2$ which are common to all.

The inverse formula to (3.22) is obtained by applying (3.16),

$$U(\hat{G})|\alpha\rangle = \sum_{j\lambda}(2j+1)\,|\lambda_1\lambda_2;\hat{p}\,j\,\lambda\rangle\, D^j_{\lambda,\lambda_1-\lambda_2}(\hat{G}) \tag{3.24}$$

The formulae (3.22) and (3.24) can, with the help of (3.19), be expressed in the more useful form,

$$|\lambda_1\lambda_2;\hat{p}\,j\,\lambda\rangle = \int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\,|p_1 j_1\lambda_1, p_2 j_2\lambda_2\rangle^{+}\, e^{-i(\lambda_1+\lambda_2)\varphi}\, d^j_{\lambda_1-\lambda_2,\lambda}(-\theta)\, e^{i\lambda\varphi} \tag{3.25}$$

$$|p_1 j_1\lambda_1, p_2 j_2\lambda_2\rangle^{+} = \sum_{j\lambda}(2j+1)|\lambda_1\lambda_2;\hat{p}\,j\,\lambda\rangle\, e^{-i\lambda\varphi}\, d^j_{\lambda,\lambda_1-\lambda_2}(\theta)\, e^{i(\lambda_1+\lambda_2)\varphi} \tag{3.26}$$

We have in (3.26) an explicit realisation of the Clebsch-Gordan coefficient introduced in (3.2), viz.,

$$\langle\mu_1\mu_2;\hat{p}\,j\,\lambda|p_1 j_1\lambda_1, p_2 j_2\lambda_2\rangle^{+} =$$

$$= \delta(\hat{p}-p_1-p_2)\,\delta_{\mu_1\lambda_1}\delta_{\mu_2\lambda_2}\,(2j+1)\,e^{-i\lambda\varphi}\,d^j_{\lambda,\lambda_1-\lambda_2}(\theta)\,e^{i(\lambda_1+\lambda_2)\varphi} \tag{3.27}$$

which is valid for $\hat{p}_\mu = (\sqrt{s}, 0 , 0 , 0)$. Its value for any other frame could be obtained by applying the appropriate Lorentz transformation.

The representations $\mathcal{D}^j_+(p^2)$ with $p^2 = (p_1 + p_2)^2 > 0$ can appear not only in the product $\mathcal{D}^{j_1}_+(m_1^2)\otimes\mathcal{D}^{j_2}_+(m_2^2)$ with $m_1^2, m_2^2 > 0$, but also in

$\mathcal{D}_+^{j_1}(m_1^2) \otimes \mathcal{D}_+^{\rho}(0)$ , $\mathcal{D}_+^{j_1}(m_1^2) \otimes \mathcal{D}_-^{j_2}(m_2^2)$ , $\mathcal{D}_+^{j_1}(m_1^2) \otimes \mathcal{D}^{j_2}(-m_2^2)$ ,

$\mathcal{D}^{j_1}(-m_1^2) \otimes \mathcal{D}^{j_2}(-m_2^2)$ and $\mathcal{D}_+^{j_1}(p^2) \otimes \mathcal{D}^{j_0\sigma}$. In each case the Clebsch-Gordan coefficients can be calculated by following the technique outlined above, i.e., by writing the product state with $p_1 + p_2 = \hat{p}$ in the form $U(\hat{G})|\alpha\rangle$ where $\hat{G}$ belongs to the little group and projecting from these the various j values by integration over the group.

(ii) Spacelike case $(p_1 + p_2)^2 < 0$

Let us consider the extraction of representations $\mathcal{D}^j(p^2)$ with $p^2 < 0$ from the product $\mathcal{D}_+^{j_1}(m_1^2) \otimes \mathcal{D}_-^{j_2}(m_2^2)$. Using the -type boosts of Sec. 2 we have

$$p_{1\mu} = m_1 \left( ch\gamma_1\, ch\beta_1 ,\ ch\gamma_1\, sh\beta_1 \cos\varphi_1 ,\ ch\gamma_1\, sh\beta_1 \sin\varphi_1 ,\ sh\gamma_1 \right)$$

$$p_{2\mu} = -m_2 \left( ch\gamma_2\, ch\beta_2 ,\ ch\gamma_2\, sh\beta_2 \cos\varphi_2 ,\ ch\gamma_2\, sh\beta_2 \sin\varphi_2 ,\ sh\gamma_2 \right)$$

and the states with $p_1 + p_2 = \hat{p} = (0, 0, 0, \sqrt{-t})$ are picked out by choosing

$$\varphi_1 = \varphi_2 = \varphi, \quad \beta_1 = \beta_2 = \beta, \quad sh\gamma_1 = \frac{m_2^2 - m_1^2 - t}{2m_1\sqrt{-t}}, \quad sh\gamma_2 = \frac{m_2^2 - m_1^2 + t}{2m_2\sqrt{-t}} \tag{3.28}$$

The product states may be written in the form

$$\begin{aligned} |p_1 j_1 \lambda_1 , p_2 j_2 \lambda_2 \rangle &= U^{(1)}(L^-_{p_1})\, U^{(2)}(L^-_{p_2}) \,|\hat{p}_1 j_1 \lambda_1 , \hat{p}_2 j_2 \lambda_2 \rangle \\ &= e^{-i\varphi J_{12}^{(1)}} e^{-i\beta J_{01}^{(1)}} e^{i\varphi J_{12}^{(1)}} e^{-i\gamma_1 J_{03}^{(1)}} \cdot \\ &\quad \cdot e^{-i\varphi J_{12}^{(2)}} e^{-i\beta J_{01}^{(2)}} e^{i\varphi J_{12}^{(2)}} e^{-i\gamma_2 J_{03}^{(2)}} |\hat{p}_1 j_1 \lambda_1 , \hat{p}_2 j_2 \lambda_2 \rangle \\ &= e^{-i\varphi J_{12}} e^{-i\beta J_{01}} |\gamma\rangle\, e^{i(\lambda_1 + \lambda_2)\varphi} \end{aligned} \tag{3.29}$$

where

$$|\gamma\rangle = e^{-i\gamma_1 J_{03}^{(1)} - i\gamma_2 J_{03}^{(2)}} |\mathring{p}_1 j_1 \lambda_1, \mathring{p}_2 j_2 \lambda_2\rangle \tag{3.30}$$

By applying (3.17) to the states $U(\hat{G})|\gamma\rangle$ we project out the irreducible representations,

$$|\lambda_1\lambda_2; \hat{p}\, j\, \lambda\rangle\, \delta_{\mu,\lambda_1+\lambda_2} = \int_{SO(2,1)} d\mu(\hat{G})\, D^{j}_{\lambda\mu}(\hat{G})^{*}\, U(\hat{G})|\gamma\rangle \quad , \tag{3.31}$$

where the integration extends over the non-compact group SO(2,1). The Kronecker symbol $\delta_{\mu,\lambda_1+\lambda_2}$ appears because

$$J_{12}|\gamma\rangle = (\lambda_1+\lambda_2)|\gamma\rangle . \tag{3.32}$$

The invariant j takes values corresponding to all of the unitary representations of SO(2,1) in the principal series,

$$j + \tfrac{1}{2} = \text{imaginary}$$

and a finite number of those in the discrete series,

$$-j = |\lambda_1+\lambda_2|\, ,\, |\lambda_1+\lambda_2|-1,\ \ldots ,\ (3/2 \text{ or } 1) \tag{3.33}$$

This is exhibited in the inverse formula to (3.31) obtained by applying (3.16),

$$U(\hat{G})|\gamma\rangle = \sum_{-|\lambda_1+\lambda_2|\leq j\leq -1} (2j+1)\, |\lambda_1\lambda_2; \hat{p}\, j\lambda\rangle\, D^{j\pm}_{\lambda,\lambda_1+\lambda_2}(\hat{G}) + \sum_{\lambda} \int_0^{i\infty} dj\ \frac{2j+1}{\tan \pi j}\, |\lambda_1\lambda_2; \hat{p}\, j\lambda\rangle\, D^{j}_{\lambda,\lambda_1+\lambda_2}(\hat{G}) \tag{3.34}$$

The representations $D^{j+}$ appear for $\lambda_1+\lambda_2>0$ and $D^{j-}$ for $\lambda_1+\lambda_2<0$. The formulae (3.31) and (3.34) can, with the help of (3.29), be written in the form

$$|\lambda_1\lambda_2;\hat{p}j\lambda\rangle = \int_0^{2\pi} d\psi \int_0^{\infty} d\beta\, \mathrm{sh}\beta\, |p_1 j_1\lambda_1, p_2 j_2\lambda_2\rangle^{-} e^{-i(\lambda_1+\lambda_2)\varphi} d^{j}_{\lambda_1+\lambda_2;\lambda}(-\beta)\, e^{i\lambda\varphi} \tag{3.35}$$

$$|p_1 j_1\lambda_1, p_2 j_2\lambda_2\rangle^{-} = \sum_{-|\lambda_1+\lambda_2|\leq j\leq -1} \sum_{\pi} (2j+1)|\lambda_1\lambda_2;\hat{p}j\lambda\rangle e^{-i\lambda\varphi} d^{j\pm}_{\lambda,\lambda_1+\lambda_2}(\beta)\, e^{i(\lambda_1+\lambda_2)\varphi} +$$

$$+ \sum_{\lambda} \int_0^{i\infty} dj\, \frac{2j+1}{\tan\pi j} |\lambda_1\lambda_2;\hat{p}j\lambda\rangle e^{-i\lambda\varphi} d^{j}_{\lambda,\lambda_1+\lambda_2}(\beta)\, e^{i(\lambda_1+\lambda_2)\varphi} \tag{3.36}$$

We have in (3.36) an explicit realization of the Clebsch-Gordan coefficient for this case,

$$\langle\mu_1\mu_2;\hat{p}j\lambda|p_1 j_1\lambda_1, p_2 j_2\lambda_2\rangle^{-} =$$

$$= \delta(\hat{p}-p_1-p_2)\,\delta_{\mu_1\lambda_1}\delta_{\mu_2\lambda_2} \frac{2j+1}{\tan\pi j}\, e^{-i\lambda\varphi} d^{j}_{\lambda,\lambda_1+\lambda_2}(\beta)\, e^{i(\lambda_1+\lambda_2)\varphi} \tag{3.37}$$

The representations $\mathcal{D}^{j}(p^2)$ with $p^2<0$ appear also in the products $\mathcal{D}_+^{j_1}(m_1^2)\times\mathcal{D}_-^{\rho}(0)$, $\mathcal{D}_+^{j_1}(m_1^2)\times\mathcal{D}^{j_2}(-m_2^2)$, $\mathcal{D}^{j_1}(-m_1^2)\times\mathcal{D}^{j_2}(-m_2^2)$ and $\mathcal{D}^{j_1}(p^2)\times\mathcal{D}^{j_0\sigma}$. Again the technique for obtaining the Clebsch-Gordan coefficients is the same as for the case outlined here.

## (iii) Lightlike case $(p_1+p_2)^2=0$

The extraction of lightlike representations $\mathcal{D}_\pm^{\rho}(0)$ from the product $\mathcal{D}_+^{j_1}(m_1^2)\otimes\mathcal{D}_-^{j_2}(m_2^2)$ proceeds as follows. Use the O-type boosts and write

$$p_{1\mu} = m_1 (ch\chi_1 + \frac{\xi_1^2}{2} e^{-\chi_1}, -\xi_1 e^{-\chi_1} \cos\varphi_1 , -\xi_1 e^{-\chi_1} \sin\varphi_1 , sh\chi_1 + \frac{\xi_1^2}{2} e^{-\chi_1}) ,$$

$$p_{2\mu} = m_2 (ch\chi_2 + \frac{\xi_2^2}{2} e^{-\chi_2}, -\xi_2 e^{-\chi_2} \cos\varphi_2 , -\xi_2 e^{-\chi_2} \sin\varphi_2 , sh\chi_2 + \frac{\xi_2^2}{2} e^{-\chi_2}) .$$

The states with

$$p_{1\mu} + p_{2\mu} = \hat{p}_\mu = (\omega, 0, 0, \omega) \tag{3.38}$$

are picked out by choosing

$$\varphi_1 = \varphi_2 = \varphi, \quad \xi_1 = \xi_2 = \xi , \quad e^{-\chi_1} = \frac{m_1^2 - m_2^2}{2m_1\omega} , \quad e^{-\chi_2} = \frac{m_1^2 - m_2^2}{2m_2\omega} . \tag{3.39}$$

We shall take $m_1 > m_2$ so that $\omega > 0$. The product states may be written in the form

$$\begin{aligned} |p_1 j_1 \lambda_1 , p_2 j_2 \lambda_2\rangle^0 &= U^{(1)}(L^0_{p_1}) U^{(2)}(L^0_{p_2}) | \hat{p}_1 j_1 \lambda_1 , \hat{p}_2 j_2 \lambda_2 \rangle = \\ &= e^{-i\varphi J_{12}^{(1)}} e^{-i\xi\Pi_1^{(1)}} e^{i\varphi J_{12}^{(1)}} e^{-i\chi_1 J_{03}^{(1)}} \cdot \\ &\quad \cdot e^{-i\varphi J_{12}^{(2)}} e^{-i\xi\Pi_1^{(2)}} e^{i\varphi J_{12}^{(2)}} e^{-i\chi_2 J_{03}^{(2)}} | p_1 j_1 \lambda_1 \hat{p}_2 j_2 \lambda_2 \rangle \\ &= e^{-i\varphi J_{12}} e^{-i\xi\Pi_1} |\chi\rangle e^{i(\lambda_1+\lambda_2)\varphi} \end{aligned} \tag{3.40}$$

where

$$|\chi\rangle = e^{-i\chi_1 J_{03}^{(1)} - i\chi_2 J_{03}^{(2)}} | p_1 j_1 \lambda_1 , \hat{p}_2 j_2 \lambda_2 \rangle \tag{3.41}$$

In the same manner as before we can project out the irreducible representations by integrating over the little group $\hat{G} = SO(2) \wedge T(2)$.

$$|\lambda_1\lambda_2 ; \hat{p}\rho\lambda\rangle \delta_{\mu,\lambda_1+\lambda_2} = \int_{SO(2)\wedge T(2)} d\mu(\hat{G}) \, D^{\rho/\omega}_{\lambda\mu} (\hat{G})^* \, U(\hat{G}) | \chi \rangle , \tag{3.42}$$

the converse relation being

$$U(\hat{G}) | \chi\rangle = \int_0^\infty d\rho^2 \sum_\lambda | \lambda_1\lambda_2 ; \hat{p}\rho\lambda\rangle D^{\rho/\omega}_{\lambda,\lambda_1+\lambda_2} (\hat{G}) . \tag{3.43}$$

The representations $D^{\rho/\omega}$ which appear here belong to the principal series of unitary representations of $SO(2) \wedge T(2)$. Substituting for $|\chi\rangle$ from (3.40) these formulae become

$$|\lambda_1\lambda_2;\hat{p}\rho\lambda\rangle = \int_0^{2\pi} d\varphi \int_0^{\infty} d\xi \, |p_1 j_1 \lambda_1, p_2 j_2 \lambda_2\rangle^{\circ} \, e^{-i(\lambda_1+\lambda_2)\varphi} d^{\rho/\omega}_{\lambda_1-\lambda_2,\lambda}(-\xi) e^{i\lambda\varphi} \tag{3.44}$$

$$|p_1 j_1 \lambda_1, p_2 j_2 \lambda_2\rangle^{\circ} = \int_0^{\infty} d\rho^2 \sum_{\lambda} |\lambda_1\lambda_2;\hat{p}\rho\lambda\rangle \, e^{-i\lambda\varphi} d^{\rho/\omega}_{\lambda,\lambda_1+\lambda_2}(\xi) e^{i(\lambda_1+\lambda_2)\varphi} \tag{3.45}$$

These formulae represent a Bessel transform since, as will be shown in Sec. 4,

$$d^{\rho/\omega}_{\lambda\mu}(\xi) = J_{\lambda-\mu}(\xi\rho/\omega) .$$

Thus we have an explicit realization of the Clebsch-Gordan coefficient coupling $\mathcal{D}^{\rho}_{+}(0)$ to $\mathcal{D}^{j_1}_{+} \times \mathcal{D}^{j_2}_{-}$ :

$$\langle\mu_1\mu_2;\hat{p}\rho\lambda|p_1 j_1 \lambda_1, p_2 j_2 \lambda_2\rangle^{\circ} = \tag{3.46}$$

$$= \delta(\hat{p}-p_1-p_2)\,\delta_{\mu_1\lambda_1}\delta_{\mu_2\lambda_2}\; e^{-i\lambda\varphi}\, J_{\lambda-\lambda_1-\lambda_2}(\xi\rho/\omega)\, e^{i(\lambda_1+\lambda_2)\varphi}$$

expressed in the frame for which

$$\hat{p}_{\mu} = (\omega, 0, 0, \omega) .$$

The same technique can be used for calculating the Clebsch-Gordan coefficients coupling the lightlike representations to other products:

(iv) <u>Null case $(p_1 + p_2)_{\mu} = 0$</u>

This case is in some respects simpler than the previous ones since there is only one independent four-vector in the problem. Also, since the little group compromises the entire homogeneous group SO(3,1), the choice of parametrization is governed by the nature of $p_1 = -p_2$ (timelike, spacelike or lightlike). Again we shall consider only one of the possible situations in which the null representations can appear, viz., the product $\mathcal{D}^{j_1}_{+}(m^2) \otimes \mathcal{D}^{j_2}_{-}(m^2)$. The other situations can be dealt with in similar fashion.

Using the +type boosts we can write

$$p_1 = -p_2 = m(\mathrm{ch}\alpha,\ \mathrm{sh}\alpha \sin\theta\cos\varphi,\ \mathrm{sh}\alpha\sin\theta\sin\varphi,\ \mathrm{sh}\alpha\cos\theta) .$$

The product states with $p_1 + p_2 = 0$ can then be written in the form

$$|p_1 j_1 \lambda_1, p_2 j_2 \lambda_2 \rangle^{+} = U^{(1)}(L^{+}_{p_1})\, U^{(2)}(L^{+}_{p_2})\, |\hat{p}_1 j_1 \lambda_1, \hat{p}_2 j_2 \lambda_2 \rangle$$
$$= e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{i\varphi J_{12}}\, e^{-i\alpha J_{03}}\, |\hat{p}_1 j_1 \lambda_1, \hat{p}_2 j_2 \lambda_2 \rangle$$
$$= e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\alpha J_{03}}\, |\hat{p}_1 j_1 \lambda_1, \hat{p}_2 j_2 \lambda_2 \rangle\, e^{i(\lambda_1+\lambda_2)\varphi}$$

(3.47)

By applying (3.17) to the states $U(\Lambda)\,|\hat{p}_1 j_1 \lambda, \hat{p} j_2 \lambda_2\rangle$ we can project out irreducible representations of SO(3,1). Before doing this, however, it is convenient to define the new states

$$|\hat{p}_1 j_1, \hat{p}_2 j_2 ; J\lambda\rangle = \sum_{\lambda_1 \lambda_2} |\hat{p}_1 j_1 \lambda_1, \hat{p}_2 j_2 \lambda_2 \rangle \langle j_1 \lambda_1, j_2 \lambda_2 | J\lambda\rangle \tag{3.48}$$

where $\langle j_1 \lambda_1, j_2 \lambda_2 | J\lambda\rangle$ denotes an SO(3) Clebsch-Gordan coefficient. The states so defined satisfy two conditions:

$$\begin{aligned} (J_{12}^2 + J_{23}^2 + J_{31}^2)\,|\hat{p}_1 j_1, \hat{p}_2 j_2 ; J\lambda\rangle &= J(J+1)\,|\hat{p}_1 j_1, \hat{p}_2 j_2 ; J\lambda\rangle \\ J_{12}\,|\hat{p}_1 j_1, \hat{p}_2 j_2 ; J\lambda\rangle &= \lambda\,|\hat{p}_1 j_1, \hat{p}_2 j_2 ; J\lambda\rangle \end{aligned} \tag{3.49}$$

which are the analogue for the null case of (3.23) for the timelike case. Application of (3.17) gives

$$|J; j_0 \sigma\, j \lambda\rangle\, \delta_{JJ'}\, \delta_{\mu\mu'} = \int_{SO(3,1)} d\mu(\Lambda)\, D^{j_0 \sigma}_{j\lambda, J'\mu'}(\Lambda)^{*}\, U(\Lambda)\,|\hat{p}_1 j_1, \hat{p}_2 j_2 ; J\mu\rangle\,, \tag{3.50}$$

the converse relation being

$$U(\Lambda)\,|\hat{p}_1 j_1, \hat{p}_2 j_2 ; J\mu\rangle = \sum_{j_0=-J}^{J} \int_0^{i\infty} d\sigma\,(j_0^2 - \sigma^2) \sum_{j\lambda} |J; j_0 \sigma\, j\lambda\rangle\, D^{j_0 \sigma}_{j\lambda, J\mu}(\Lambda) \tag{3.51}$$

The representations $D^{j_0 \sigma}$ which appear here belong to the principal series of unitary representations of SO(3,1):

$\sigma$ = pure imaginary, $j_0 = -J\,,\ -J+1,\ \ldots\,,\ J$. The matrices $D^{j_0 \sigma}_{j'\lambda', j\lambda}(\Lambda)$ are defined in the basis (2.38), (2.39) by

$$U(\Lambda)\,|j_0 \sigma\, j\lambda\rangle = \sum_{j'\lambda'} |j_0 \sigma\, j'\lambda'\rangle\, D^{j_0 \sigma}_{j'\lambda', j\lambda}(\Lambda)$$

The matrix for a transformation in the 03-plane is defined by

$$e^{-i\alpha J_{03}} |j_0 \sigma j \lambda\rangle = \sum_{j'} |j_0 \sigma j' \lambda\rangle \, d^{j_0\sigma}_{j'\lambda j}(\alpha) \, .$$

These functions will be discussed in Sec. 4.

Using (3.47) and (3.48) to eliminate $|\hat{p}_1 j_1 , \hat{p}_2 j_2 ; J\mu\rangle$ these formulae become

$$|J; j_0\sigma j\lambda\rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} d\alpha \, sh^2\alpha \sum_{\lambda_1\lambda_2\mu} |p_1 j_1 \lambda_1 , p_2 j_2 \lambda_2\rangle^+ \cdot$$

$$\cdot \langle j_1\lambda_1 j_2\lambda_2 | J\mu\rangle \, e^{-i\mu\varphi} \, d^{j_0\sigma}_{J\mu j}(-\alpha) \, d^{j}_{\mu\lambda}(-\theta) \, e^{i\lambda\varphi} \qquad (3.52)$$

$$|p_1 j_1 \lambda_1 , p_2 j_2 \lambda_2\rangle^+ = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{j_0=-J}^{J} \int_0^{i\infty} d\sigma (j_0^2-\sigma^2) \sum_{j\lambda} |J; j_0\sigma j\lambda\rangle \cdot$$

$$\cdot \sum_{\mu} e^{-i\lambda\varphi} d^{j}_{\lambda\mu} \, \theta \, d^{j_0\sigma}_{j\mu J}(\alpha) \, e^{i\mu\varphi} \langle J\mu | j_1\lambda_1 j_2\lambda_2\rangle \qquad (3.53)$$

Formula (3.53) provides an explicit realization for the Clebsch-Gordan coefficient coupling $\mathcal{D}^{j_0\sigma}$ to $\mathcal{D}^{j_1}_+ \otimes \mathcal{D}^{j_2}$ :

$$\langle J; j_0\sigma j \lambda | p_1 j_1 \lambda_1 , p_2 j_2 \lambda_2\rangle^+ = \qquad (3.54)$$

$$= \delta(p_1 + p_2) \, (j_0^2 - \sigma^2) \, e^{-i\lambda\varphi} \, d^{j}_{\lambda,\lambda_1+\lambda_2}(\theta) \, d^{j_0\sigma}_{j,\lambda_1+\lambda_2,J}(\alpha) \, e^{i(\lambda_1+\lambda_2)\varphi} \langle J\lambda_1+\lambda_2 | j_1\lambda_1 , j_2\lambda_2\rangle \, .$$

This completes the list of Clebsch-Gordan coefficients that we shall construct explicitly. All of the others with one exception can be found by the methods used here. The exceptional case is $\mathcal{D}^{j_0\sigma} \otimes \mathcal{D}^{j_0'\sigma'}$ which requires more sophisticated techniques. It has been dealt with in great detail in the Russian literature (e.g., M.A. NAIMARK, Am. Math. Soc. Trans. 36, 101 (1964)).

In conclusion it may perhaps be instructive to examine the above manipulations from the viewpoint of complete commuting sets of operators. The problem of decomposing a direct product can of course be looked upon as one of finding the transformation that takes one basis - which diagonalizes certain of the operators $P^{(1)}_\mu$, $J^{(1)}_{\mu\nu}$, $P^{(2)}_\mu$, $J^{(2)}_{\mu\nu}$ - into another which diagonalizes certain of the operators $P_\mu = P^{(1)}_\mu + P^{(2)}_\mu$ and $J_{\mu\nu} = J^{(1)}_{\mu\nu} + J^{(2)}_{\mu\nu}$.

Since all of the states met with above diagonalize the four Casimir operators $(P_\mu^{(1)})^2$, $(P_\mu^{(2)})^2$, $(W_\mu^{(1)})^2$, $(W_\mu^{(2)})^2$,

we can omit them from the following discussion. Of the remaining operators, eight are diagonalized by the product states $|p_1 j_1 \lambda_1, p_2 j_2 \lambda_2\rangle$. These include six independent momentum components contained among $P_\mu^{(1)}$ and $(P_\mu^{2})$ and two helicities, e.g., $W_0^{(1)}$ and $W_0^{(2)}$.

On the other hand, the irreducible states $|\lambda_1 \lambda_2 ; p j \lambda\rangle$ diagonalize the four independent momenta $P_\mu = P_\mu^{(1)} + P_\mu^{(2)}$ and in addition the four operators

$$(W_\mu)^2 \;, \quad W_0 \;, \quad W_\mu^{(1)} P_\mu \;, \quad W_\mu^{(2)} P_\mu \;,$$

again eight in all (at least when $P_\mu \neq 0$, the null case must be treated separately.) The total $W_\mu$ is defined by

$$W_\mu = -\tfrac{1}{2} \epsilon_{\mu\nu\lambda\rho} J_{\nu\lambda} P_\rho = -\tfrac{1}{2} \epsilon_{\mu\nu\lambda\rho} (J_{\nu\lambda}^{(1)} + J_{\nu\lambda}^{(2)}) (P_\rho^{(1)} + P_\rho^{(2)}) .$$

It is a simple matter to verify (for the case $p^2 > 0$) the relations

$$\begin{aligned}
P_\mu |\lambda_1 \lambda_2 ; p j \lambda\rangle &= p_\mu |\lambda_1 \lambda_2 ; p j \lambda\rangle \\
W_\mu W_\mu |\lambda_1 \lambda_2 ; p j \lambda\rangle &= -p^2 j(j+1) |\lambda_1 \lambda_2 ; p j \lambda\rangle \\
W_0 |\lambda_1 \lambda_2 ; p j \lambda\rangle &= \lambda \sqrt{p_1^2 + p_2^2 + p_3^2}\, |\lambda_1 \lambda_2 ; p j \lambda\rangle \\
W_\mu^{(1)} P_\mu |\lambda_1 \lambda_2 ; p j \lambda\rangle &= \lambda_1 \tfrac{1}{2} \sqrt{(p^2 + m_1^2 - m_2^2)^2 - 4 m_1^2 p^2}\, |\lambda_1 \lambda_2 ; p j \lambda\rangle \\
W_\mu^{(2)} P_\mu |\lambda_1 \lambda_2 ; p j \lambda\rangle &= \lambda_2 \tfrac{1}{2} \sqrt{(p^2 - m_1^2 + m_2^2)^2 - 4 m_2^2 p^2}\, |\lambda_1 \lambda_2 ; p j \lambda\rangle
\end{aligned} \tag{3.55}$$

The important point to notice here is that $\lambda_1$, $\lambda_2$ and j are all Poincaré invariants.

The irreducible states $|J; j_0 \sigma j \lambda\rangle$ diagonalize the total four-momentum, $P_\mu = 0$; but this accounts for only three independent operators since this case arises only if $(P_\mu^{(1)})^2 = (P_\mu^{(2)})^2$, i.e., $\underline{P} = 0$ implies $P_0 = 0$. The remaining five operators are given by

$$\tfrac{1}{2} J_{\mu\nu} J_{\mu\nu} \;, \quad \tfrac{1}{4} \epsilon_{\mu\nu\lambda\rho} J_{\mu\nu} J_{\lambda\rho} \;, \quad J_{12}^2 + J_{23}^2 + J_{31}^2 = \underline{J}^2 \;, \quad J_{12} \;,$$

and the "total spin"

$$(W_\mu^{(1)} + W_\mu^{(2)})^2$$

The eigenvalues of the total spin operator can be found by applying it to the state (3.60).

$$(W_\mu^{(1)} + W_\mu^{(2)})^2 \,|\, J;\, j_0 \sigma j \lambda \rangle =$$

$$= \int d\mu(\Lambda)\, D^{j_0\sigma}_{j\lambda,\, J\mu}(\Lambda)^* \,(W_\mu^{(1)} + W_\mu^{(2)})^2\, U(\Lambda) \,|\, \hat{p}_1 j_1, \hat{p}_2 j_2 ; J\mu \rangle$$

$$= \int d\mu(\Lambda)\, D^{j_0\sigma}_{j\lambda,\, J\mu}(\Lambda)^* \,U(\Lambda)\,(W_\mu^{(1)} + W_\mu^{(2)})^2 \,|\, \hat{p}_1 j_1, \hat{p}_2 j_2 ; J\mu \rangle$$

but from (2.6)

$$(W_\mu^{(1)} + W_\mu^{(2)})^2 \,|\, \hat{p}_1 j_1, \hat{p}_2 j_2 ; J\mu \rangle = -m^2 (\underline{J}^{(1)} + \underline{J}^{(2)})^2 \,|\, \hat{p}_1 j_1, \hat{p}_2 j_2 ; J\mu \rangle$$

$$= -m^2 J(J+1) \,|\, \hat{p}_1 j_1, \hat{p}_2 j_2 ; J\mu \rangle$$

so that

$$(W_\mu^{(1)} + W_\mu^{(2)})^2 \,|\, J;\, j_0 \sigma j \lambda \rangle = -m^2 J(J+1) \,|\, J;\, j_0 \sigma j \lambda \rangle \quad . \qquad (3.56)$$

This shows that $J$ is a Poincaré invariant.

## 4. ANALYTICAL PROPERTIES OF LITTLE GROUP REPRESENTATIONS

Differential equations

It proves convenient to parametrize all the finite transformations of the little groups in the "Eulerian" manner, namely for

$$\text{SO}(3): \quad U(\varphi,\theta,\psi) = e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\psi J_{12}} ,$$

$$\text{SO}(2,1): \quad U(\varphi,\beta,\psi) = e^{-i\varphi J_{12}}\, e^{-i\beta J_{01}}\, e^{-i\psi J_{12}} ,$$

$$\text{SO}(2)\wedge T(2): \quad U(\varphi,\xi,\psi) = e^{-i\varphi J_{12}}\, e^{-i\xi \pi_1}\, e^{-i\psi J_{12}} ,$$

$$\text{SO}(3,1): \quad U(\varphi,\theta,\psi;\zeta;0,\theta',\psi') = e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{-i\psi J_{12}} .\, e^{-i\zeta J_{03}} .\, e^{-i\theta' J_{31}}\, e^{-i\psi' J_{12}} , \tag{4.1}$$

because all the essential information is contained in the real representations $d$,

$$d^j_{\mu\lambda}(\theta) = \langle U(\theta)\rangle = \langle j\mu | e^{-i\theta J_{31}} | j\lambda\rangle ,$$

$$d^j_{\mu\lambda}(\beta) = \langle U(\beta)\rangle = \langle j\mu | e^{-i\beta J_{01}} | j\lambda\rangle ,$$

$$d^\pi_{\mu\lambda}(\xi) = \langle U(\xi)\rangle = \langle \pi\mu | e^{-i\xi\pi_1} | \pi\lambda\rangle ,$$

$$\delta_{\mu\mu'}\, d^{j_0\sigma}_{j\mu j'}(\zeta) = \langle U(\zeta)\rangle = \langle j_0\sigma j\mu | e^{-i\zeta J_{03}} | j_0\sigma j'\mu'\rangle . \tag{4.2}$$

Thus we require the properties of the d's in order to study the analytical properties of the S-matrix itself.

The normal way of analysing the d-functions is to set up differential equations for them with suitable boundary conditions, like $d(0) = \delta$ . There are several equivalent ways for obtaining these differential equations but the simplest way, from our point of view, is the following:

Express the operators $U(\alpha)\hat{J}$ - where $\alpha = \theta, \beta, \xi$ or $\zeta$ and $\hat{J}$ are all the generators of the little group - in terms of $\frac{dU}{d\alpha}$, $UJ_3$ and $J_3U$ (and $UJ_\pm$, $J_\pm U$ for the case of SO(3,1)). Thereby express the bilinear invariants $U(\alpha)C(\hat{J})$ in those same terms, $C(\hat{J})$ being a Casimir operator for the little group, and take little group representatives. This gives the sought-after differential equations.

Consider each case in turn,

<u>SO(3)</u> : $$U(\theta) = e^{-i\theta J_{31}}$$

$$U J_{31} = i\frac{dU}{d\theta}$$

$$U J_{23} = \frac{1}{\sin\theta}\left[UJ_3 \cos\theta - J_3 U\right]$$

$$U \underline{J}^2 = -\frac{d^2U}{d\theta^2} - \cot\theta\,\frac{dU}{d\theta} + \frac{UJ_3^2 - 2\cos\theta\, J_3 U J_3 + J_3^2 U}{\sin^2\theta}$$

whence the representative equation,

$$\left[\frac{d^2}{d\theta^2} + \cot\theta\,\frac{d}{d\theta} + j(j+1) - \frac{\mu^2 - 2\mu\lambda\cos\theta + \lambda^2}{\sin^2\theta}\right] d^j_{\mu\lambda}(\theta) = 0 \qquad (4.3)$$

<u>SO(2,1)</u> : $$U(\beta) = e^{-i\beta J_{01}}$$

$$U J_{01} = i\frac{dU}{d\beta}$$

$$U J_{02} = -\frac{1}{\mathrm{sh}\beta}\left[U J_3\,\mathrm{ch}\beta - J_3 U\right]$$

$$U\underline{J}^2 = \frac{d^2U}{d\beta^2} + \coth\beta\,\frac{dU}{d\beta} - \frac{UJ_3^2 - 2\mathrm{ch}\beta\, J_3 U J_3 + J_3^2 U}{\mathrm{sh}^2\beta}$$

giving

$$\left[\frac{d^2}{d\beta^2} + \coth\beta\,\frac{d}{d\beta} - j(j+1) - \frac{\mu^2 - 2\mu\lambda\,\mathrm{ch}\beta + \lambda^2}{\mathrm{sh}^2\beta}\right] d^j_{\mu\lambda}(\beta) = 0 \qquad (4.4)$$

Observe that $\beta \to i\theta$ gives the SO(3) equation.

<u>SO(2)∧T(2)</u> : $$U(\xi) = e^{-i\xi\Pi_1}$$

$$U\Pi_1 = i\frac{dU}{d\xi}$$

$$U\Pi_2 = \frac{1}{\xi}(UJ_3 - J_3U)$$

$$U\underline{\Pi}^2 = -\frac{d^2U}{d\xi^2} - \frac{1}{\xi}\frac{dU}{d\xi} + \frac{J_3^2U - 2J_3UJ_3 + UJ_3^2}{\xi^2}$$

The matrix elements of this relation are

$$\left[\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} + \varkappa^2 - \frac{(\mu-\lambda)^2}{\xi^2}\right] d^{\pi}_{\lambda\mu}(\xi) = 0 \;. \tag{4.5}$$

<u>SO(3,1)</u> : $$U(\zeta) = e^{-i\zeta J_{03}}$$

$$UJ_{01} = -\frac{1}{\operatorname{sh}\zeta}\left(\operatorname{ch}\zeta\, UJ_{31} - J_{31}U\right)$$

$$UJ_{02} = \frac{1}{\operatorname{sh}\zeta}\left(\operatorname{ch}\zeta\, UJ_{23} - J_{23}U\right)$$

$$UJ_{03} = i\frac{dU}{d\zeta}$$

$$U\left(\tfrac{1}{2}J_{\mu\nu}J^{\mu\nu}\right) = U\left(\underline{J}^2 - J_{01}^2 - J_{02}^2 - J_{03}^2\right)$$

$$= \frac{d^2U}{d\zeta^2} + 2\coth\zeta\frac{dU}{d\zeta} - \frac{\underline{J}^2U + U\underline{J}^2}{\operatorname{sh}^2\zeta}$$
$$+\left(1 + \frac{2}{\operatorname{sh}^2\zeta}\right)J_3UJ_3 + \frac{\operatorname{ch}\zeta}{\operatorname{sh}^2\zeta}\left(J_+UJ_- + J_-UJ_+\right)$$

and

$$U\left(\tfrac{1}{4}\epsilon_{\mu\nu\kappa\lambda}J^{\mu\nu}J^{\kappa\lambda}\right) = -2U\left(J_{01}J_{23} + J_{02}J_{31} + J_{03}J_{12}\right)$$

$$= -2i\left(\frac{d}{d\zeta} + \coth\zeta\right)UJ_3 + \frac{i}{\operatorname{sh}\zeta}\left(J_+UJ_- - J_-UJ_+\right)$$

Taking matrix elements of these differential relations and using the SO(3) subgroup properties,

$$J_{\pm}|j_0\sigma j\mu\rangle = \sqrt{(j\mp\mu)(j\pm\mu+1)}\;|j_0\sigma j\mu\pm1\rangle$$

we get the coupled equations.

$$\left[\frac{d^2}{d\zeta^2} + 2\coth\zeta\,\frac{d}{d\zeta} - (j_0^2+\sigma^2-1-\mu^2) - \frac{j(j+1)-2\mu^2+j'(j'+1)}{\mathrm{sh}^2\zeta}\right] d^{j_0\sigma}_{j\mu j'}(\zeta)$$

$$= \frac{-\mathrm{ch}\zeta}{\mathrm{sh}^2\zeta}\left[\sqrt{(j+\mu)(j-\mu+1)(j'+\mu)(j'-\mu+1)}\; d^{j_0\sigma}_{j\mu-1 j'}(\zeta) + \right. \tag{4.6}$$

$$\left. + \sqrt{(j-\mu)(j+\mu+1)(j'-\mu)(j'+\mu+1)}\; d^{j_0\sigma}_{j\mu+1 j'}(\zeta)\right]$$

and

$$\left[\mu\left(\frac{d}{d\zeta} + \coth\zeta\right) - j_0\sigma\right] d^{j_0\sigma}_{j\mu j'}(\zeta)$$

$$= \frac{1}{2\mathrm{sh}\zeta}\left[\sqrt{(j+\mu)(j-\mu+1)(j'+\mu)(j'-\mu+1)}\; d^{j_0\sigma}_{j\mu-1 j'}(\zeta) - \right. \tag{4.7}$$

$$\left. - \sqrt{(j-\mu)(j+\mu+1)(j'-\mu)(j'+\mu+1)}\; d^{j_0\sigma}_{j\mu+1 j'}(\zeta)\right]$$

<u>Completeness relations and expansion formulae</u>

The irreducible representative functions $D(\hat{G})$ corresponding to the unitary transformations $U(\hat{G})$ of the little group, obey the orthogonality relations

$$\int \rho(\hat{G})\,d\hat{G}\; D^{j}_{\lambda\mu}{}^{*}(\hat{G})\, D^{j'}_{\lambda'\mu'}(\hat{G}) = \rho^{-1}(j,\lambda,\mu)\,\delta(j,j')\,\delta_{\mu\mu'}\,\delta_{\lambda\lambda'} \tag{4.8}$$

$$\int_{j\lambda\mu} D^{j*}_{\lambda\mu}(\hat{G})\, D^{j}_{\lambda\mu}(\hat{G}')\,\rho(j,\lambda,\mu)\,dj = \rho^{-1}(\hat{G})\,\delta(G,\hat{G}') \tag{4.9}$$

generally. $\rho(\hat{G})\,d\hat{G}$ is the Haar measure over the group $\hat{G}$ while $\rho(j,\lambda,\mu)dj$ is the Plancherel measure over the unitary basis.

If we are given a function $f(\hat{G})$ of the group parameters which is square integrable over the group manifold,

i.e., $\int |f(\hat{G})|^2 \rho(\hat{G})\,d\hat{G} < \infty$ , then it is possible to expand it in terms of the unitary irreducible representations (those which are square integrable) according to

$$f(\hat{G}) = \int_{j\lambda\mu} f^{j}_{\lambda\mu}(\hat{G})\, D^{j}_{\lambda\mu}(\hat{G})\, \rho(j,\mu,\lambda)\, dj \tag{4.10}$$

and the expansion coefficients will be given by

$$f^{j}_{\mu\lambda} = \int f(\hat{G})\, D^{j\,*}_{\mu\lambda}(\hat{G})\, \rho(\hat{G})\, d\hat{G}\ . \tag{4.11}$$

We illustrate these remarks with the specific cases below:

SO(3) :

$$\int D^{j\,*}_{\lambda\mu}(\varphi,\theta,\psi)\, D^{j'}_{\lambda'\mu'}(\varphi,\theta,\psi)\, \frac{d\varphi\, d(\cos\theta)\, d\psi}{8\pi^2} = \frac{\delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\lambda\lambda'}}{2j+1} \tag{4.12}$$

$$\sum_{j\lambda\mu} (2j+1)\, D^{j}_{\lambda\mu}(\varphi,\theta,\psi)\, D^{j\,*}_{\lambda\mu}(\varphi',\theta',\psi') = 8\pi^2\,\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')\,\delta(\psi-\psi') \tag{4.13}$$

$$f(\varphi,\theta,\psi) = \sum_{j\lambda\mu} (2j+1)\, f^{j}_{\mu\lambda}\, D^{j}_{\mu\lambda}\ D^{j}_{\mu\lambda}(\varphi,\theta,\psi) \tag{4.14}$$

$$f^{j}_{\mu\lambda} = \int \frac{d\varphi\, d(\cos\theta)\, d\psi}{8\pi^2}\ .\ f(\varphi,\theta,\psi)\, D^{j\,*}_{\mu\lambda}(\varphi,\theta,\psi) \tag{4.15}$$

providing $f(\theta)$ is regular in the region $-1 \leqslant \cos\theta < 1$ .

SO(2,1) :

Here the situation is somewhat more complicated in that we have three types of representation. However, the supplementary series can be safely neglected because the corresponding representations are not square integrable. Thus, for the principal series:

$$\int D^{j*}_{\lambda\mu}(\varphi,\beta,\psi)\, D^{j'}_{\lambda'\mu'}(\varphi,\beta,\psi)\ \frac{d\varphi\, d(\mathrm{ch}\beta)\, d\psi}{8\pi^2} = \frac{2\tan\pi(j-\mu)}{i(2j+1)}\ \delta(ij-ij')\,\delta_{\lambda\lambda'}\,\delta_{\mu\mu'} \tag{4.16}$$

and for the discrete series:

$$\int D^{j\pm}_{\lambda\mu}(\varphi,\beta,\psi)\, D^{j'\pm *}_{\lambda'\mu'}(\varphi,\beta,\psi)\ \frac{d\varphi\, d(\mathrm{ch}\beta)\, d\psi}{8\pi^2} = \frac{\delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\lambda\lambda'}}{2j+1} \tag{4.17}$$

Integrals over cross representations vanish. Conversely,

$$\sum_{j^\pm\mu\lambda} (2j+1)\, D^{j*}_{\mu\lambda}(\varphi,\beta,\psi)\, D^{j}_{\mu\lambda}(\varphi',\beta',\psi')$$

$$+\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \sum_{\mu\lambda} \frac{i\,dj\,(2j+1)}{2\tan\pi(j-\mu)}\, D^{j*}_{\mu\lambda}(\varphi,\beta,\psi)\, D^{j}_{\mu\lambda}(\varphi',\beta',\psi') = 8\pi^2\,\delta(\varphi-\varphi')\,\delta(\mathrm{ch}\beta-\mathrm{ch}\beta')\,\delta(\psi-\psi') \tag{4.18}$$

Providing that $f(\varphi,\beta,\psi)$ is square integrable in the region $\mathrm{ch}\beta \geq 1$ , i.e., $f = O((\mathrm{ch}\beta)^\alpha)$ ; $\alpha < -\frac{1}{2}$ , then we can proceed with the expansion formula,

$$f(\varphi,\beta,\psi) = \sum_{j^\pm\lambda\mu} (2j+1)\, f^{j\pm}_{\mu\lambda}\, D^{j\pm}_{\mu\lambda}(\varphi,\theta,\psi)$$

$$+\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \sum_{\mu\lambda} \frac{i\,dj\,(2j+1)}{2\tan\pi(j-\mu)}\, f^{j}_{\mu\lambda}\, D^{j}_{\mu\lambda}(\varphi,\beta,\psi) \tag{4.19}$$

Vice versa, we must have $f^{j}_{\mu\lambda}$ square integrable along the imaginary axis $j = -\frac{1}{2} + i\rho$ to write

$$f^{j}_{\mu\lambda} = \int \frac{d\varphi\, d(\mathrm{ch}\beta)\, d\psi}{8\pi^2}\ f(\varphi,\beta,\psi)\, D^{j*}_{\mu\lambda}(\varphi,\beta,\psi)$$

$$f^{j\pm}_{\mu\lambda} = \int \frac{d\varphi\, d(\mathrm{ch}\beta)\, d\psi}{8\pi^2}\ f(\varphi,\beta,\psi)\, D^{j\pm *}_{\mu\lambda}(\varphi,\beta,\psi) \tag{4.20}$$

$\underline{SO(2)\wedge T(2)}$ :

We shall regard the finite-dimensional representations as special cases of the infinite-dimensional ones ( $\pi = \rho/\omega \to 0$ )

$$\int \frac{d\varphi\,\xi d\xi d\psi}{8\pi^2}\, D^{\pi}_{\mu\lambda}(\varphi,\xi,\psi)\, D^{\pi'*}_{\mu'\lambda'}(\varphi,\xi,\varphi) = \delta(\pi^2-\pi'^2)\,\delta_{\mu\mu'}\,\delta_{\lambda\lambda'} \tag{4.21}$$

$$\int \sum_{\mu\lambda} D^{\pi}_{\mu\lambda}(\varphi,\xi,\psi)\, D^{\pi *}_{\mu\lambda}(\varphi',\xi',\psi)\, d\pi^2 = 8\pi^2\,\delta(\varphi-\varphi')\frac{1}{\xi}\,\delta(\xi-\xi')\,\delta(\psi-\psi') \tag{4.22}$$

being essentially the orthogonality properties of Bessel functions. The expansion formulae just correspond to the Hankel transformations

$$f^{\rho}_{\lambda\mu} = \int \frac{d\varphi\,\xi d\xi d\psi}{8\pi^2\omega^2}\, D^{\rho/\omega *}_{\lambda\mu}(\varphi,\xi,\psi)\, f(\varphi,\xi,\psi) \tag{4.23}$$

and $$f(\varphi,\xi,\psi) = \int d\rho^2 \sum_{\lambda\mu} f^{\rho}_{\lambda\mu}\, D^{\rho/\omega}_{\lambda\mu}(\varphi,\xi,\psi) \tag{4.24}$$

and are permissible for all square integrable f.

$\underline{SO(3,1)}$ :

Adhering to the Naimark definitions of the representation functions, the orthogonality properties are

$$\int \frac{d\varphi\, d(\cos\theta)\, d\psi}{8\pi^2}\ \frac{\operatorname{sh}\zeta\, d(\operatorname{ch}\zeta)}{2}\ \frac{d(\cos\theta')\, d\psi'}{4\pi}\, D^{j_0\sigma}_{j\mu,j'\mu'}(\varphi,\dots,\psi')\, D^{j_0'\sigma'*}_{j''\mu'',j'''\mu'''}(\varphi,\dots,\psi')$$

$$= \left[(2j+1)(2j'+1)(j_0^2-\sigma^2)\right]^{-1}\delta_{j_0 j_0'}\ \delta(i\sigma-i\sigma')\,\delta_{jj''}\,\delta_{\mu\mu''}\,\delta_{j'j'''}\,\delta_{\mu'\mu'''} \tag{4.25}$$

and

$$\int_{-i\infty}^{i\infty} -i\,d\sigma \sum_{j\mu j'\mu'} (j_0^2-\sigma^2)(2j+1)(2j'+1)\, D^{j_0\sigma}_{j\mu j'\mu'}(\varphi,\dots,\psi')\, D^{j_0\sigma *}_{j\mu,j'\mu'}(\varphi',\dots,\psi''') \tag{4.26}$$

$$= 64\pi^3\,\frac{1}{\operatorname{sh}\zeta}\,\delta(\operatorname{ch}\zeta-\operatorname{ch}\zeta')\,\delta(\varphi-\varphi')\ \dots\ \delta(\psi'-\psi''')$$

and providing that $f(\zeta)$ is of order $e^{\alpha\zeta}$ for large $\zeta$ with $\alpha < 1$ then it can be expanded in these irreducible representations:

$$f(\varphi,\dots,\psi') = \int_{-i\infty}^{i\infty} -i\,d\sigma \sum_{\substack{j_0,\\ j\mu\, j'\mu'}} (j_0^2-\sigma^2)\, f^{j_0\sigma}_{j\mu,\,j'\mu'}\; D^{j_0\sigma}_{j\mu,\,j'\mu'}(\varphi,\dots,\psi') \qquad \text{with} \tag{4.27}$$

$$f^{j_0\sigma}_{j\mu,j'\mu'} = \int \frac{1}{64\pi^3}\, d\varphi, \dots, \; d\psi' \; f(\varphi,\dots,\psi')\, D^{j_0\sigma\,*}_{j\mu,\,j'\mu'}(\varphi,\dots,\psi') \tag{4.28}$$

Representations of the first kind

The solutions to the differential equations for the d-functions which are regular in the vicinity of the identity transformation will be called the representations of the first kind. It is these group representations which enter basically in the "Fourier expansion" of any function defined over the group manifold and which is non-singular at the "origin". A complete discussion of the differential equation nevertheless requires the "representations of the second kind"; although these functions are singular at the origin, they do have simpler asymptotic characteristics than functions of the first kind, and of course there exist certain integral relations between the two types of representation. (This is analogous to the connection between the $P_\ell$ and $Q_\ell$). For the present let us analyse the properties of the first kind representations.

SO(3) and SO(2,1) functions

These can be treated at once owing to the fact that the substitutions $z = ch\beta = \cos\theta$ lead to identical equations

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + j(j+1) - \frac{\mu^2 - 2\mu\lambda z + \lambda^2}{1-z^2}\right] d^j_{\lambda\mu}(z) = 0 \tag{4.29}$$

with the boundary condition $d^j_{\lambda\mu}(1) = \delta_{\lambda\mu}$.
By displacing singularities to 0,1 and $\infty$ the equation can be cast in hypergeometric form and the solution with the correct boundary value is (for $\mu \geq |\lambda| \geq 0$)

$$d^j_{\mu\lambda}(z) = \left[\frac{\Gamma(j+\mu+1)\,\Gamma(j-\lambda+1)}{\Gamma(j+\lambda+1)\,\Gamma(j-\mu+1)}\right]^{\frac{1}{2}} \left(\frac{1+z}{2}\right)^{\frac{1}{2}(\mu+\lambda)} \left(\frac{1-z}{z}\right)^{\frac{1}{2}(\mu-\lambda)}$$

$$\cdot F\left(-j+\mu, j+\mu+1\,;\, \mu-\lambda+1\,;\, \frac{1-z}{2}\right) \Big/ \Gamma(\mu-\lambda+1) \tag{4.30}$$

for the SO(3) functions and the principal series of SO(2,1). The discrete series of SO(2,1) are obtained by analytic continuation,

$$d^{j+}_{\mu\lambda}(z) = \left[\frac{\Gamma(j+\mu+1)\,\Gamma(\mu-j)}{\Gamma(j+\lambda+1)\,\Gamma(\lambda-j)}\right]^{\frac{1}{2}} \left(\frac{1+z}{2}\right)^{-\frac{1}{2}(\mu+\lambda)} \left(\frac{1-z}{2}\right)^{\frac{1}{2}(\mu-\lambda)} \times$$

$$\times F\left(j-\lambda+1, -j-\lambda\,;\, \mu-\lambda+1\,;\, \frac{1-z}{2}\right) \Big/ \Gamma(\mu-\lambda+1) \quad , \tag{4.31}$$

and $$d^{j-}_{\mu\lambda}(z) \equiv d^{j+}_{-\lambda\,-\mu}(z) \quad . \tag{4.32}$$

It is immediately possible, from the properties of the hypergeometric and gamma functions, to state the equivalence relations and index symmetry properties:

$$d^j_{\mu\lambda}(z) = d^j_{-\lambda,-\mu}(z) = (-1)^{\mu-\lambda}\, d^j_{\lambda\mu}(z) = d^{-j-1}_{-\mu\,-\lambda}(z)\ (-)^{\lambda-\mu} \tag{4.33}$$

$$d^{j+}_{\mu\lambda}(z) = (-1)^{\mu-\lambda}\, d^{j+}_{\lambda\mu}(z) = d^{-j-1\,+}_{-\lambda\,-\mu}(z) \qquad \text{etc.}$$

For all discrete representations ($j-\mu$ = integer) $d$ can be related to the Jacobi polynomials and in that case one may deduce the inversion property

$$d^j_{\mu\lambda}(-z) = \begin{cases} (-1)^{j-\mu}\, d^j_{\mu-\lambda}(z) & \text{if } j-\max(|\mu|,|\lambda|) = 0,1,2,\ldots \\ -(-1)^{j-\mu}\, d^j_{\mu-\lambda}(z) & \text{if } j-\max(|\mu|,|\lambda|) = -1,-2,\ldots \end{cases}$$

More generally however $d^j_{\mu\lambda}(z)$ is a branched function. Thus in the $z$-plane, the branches $(1+z)^{\frac{1}{2}}$ and $(1-z)^{\frac{1}{2}}$ give cuts from $-\infty$ to $-1$ and 1 to $\infty$ while the hypergeometric function gives a cut from $-\infty$ to $-1$ again (defining the principal sheet this way). On the other hand, in the j-plane, the hypergeometric function is an entire function of $j$ and the only branches derive from the $\Gamma$ functions; these j-cuts have a finite extension if we let the principal sheet be defined positive for $j \gg 1$. The asymptotic properties and integral representations must be deferred to later since the second type functions need first to be defined.

SO(2)$\wedge$T(2) functions

The representation function just satisfies Bessel's equation for if we substitute $x = \pi\xi$ the equation reads

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \left\{1 - \frac{(\mu-\lambda)^2}{x^2}\right\}\right] d^{\pi}_{\lambda\mu}(x) = 0 \tag{4.34}$$

i.e., $d^{\pi}_{\lambda\mu}(\xi) = J_{\lambda-\mu}(\pi\xi)$, simply. (4.35)

It is interesting to notice that solutions are just the $p^2 \to 0$ limits of the SO(3) and SO(2,1) functions because

$$\lim_{j\to\infty} d^{j}_{\mu\lambda}(z/j) = J_{\mu-\lambda}(z) \tag{4.36}$$

from Hansen's formula,

$$J_{c-1}(z) = \lim_{a,b\to\infty} (\tfrac{1}{2}z)^c\, F(a,b\,;c\,;-z^2/4ab)\big/\Gamma(c) .$$

In a sense this also dictates the choice of Bessel function.

The symmetry properties are then well-known,

$$\begin{aligned} d^{\pi}_{\mu\lambda}(\xi) &= d^{\pi}_{-\lambda-\mu}(\xi) = (-1)^{\mu-\lambda}\, d^{\pi}_{\lambda\mu}(\xi) \\ &= (-1)^{\mu-\lambda}\, d^{\pi}_{\mu\lambda}(-\xi) = (-1)^{\mu-\lambda}\, d^{-\pi}_{\mu\lambda}(\xi) \end{aligned} \tag{4.37}$$

and the fact that there exists a branch point at $\xi = 0$. Moreover it is known that $d$ is an entire function of $\mu-\lambda$.

SO(3,1) functions

The situation here is more involved than in the previous three cases because we have a set of coupled differential equations to solve in general. However, these equations can be simply combined in the extreme limit where $\mu = \min(j,j')$; supposing for definiteness that $j \leqslant j'$ we get

$$\left[\frac{d^2}{d\zeta^2} + 2\coth\zeta\frac{d}{d\zeta} - \frac{j(j+1)-2j^2+j'(j'+1)}{sh^2\zeta} + (j^2-j_0^2-\sigma^2+1)\right] d^{j_0\sigma}_{jjj'}(\zeta)$$

$$= -2\coth\zeta\left[j\left(\frac{d}{d\zeta}+\coth\zeta\right)-j_0\sigma\right] d^{j_0\sigma}_{jjj'}(\zeta)$$

$$= -\frac{\coth\zeta}{sh\,\zeta}\sqrt{2j(j+j')(j'-j+1)}\; d^{j_0\sigma}_{jj-1j'}(\zeta)\,.$$

or

$$\left[\begin{array}{l}\frac{d^2}{d\zeta^2} + 2(j+1)\coth\zeta\frac{d}{d\zeta} + \left(j(j+1)-j'(j'+1)\right)\coth^2\zeta \\ -2j_0\sigma\coth\zeta - (j_0^2+\sigma^2-1) + j'(j'+1) + j\end{array}\right] d^{j_0\sigma}_{jjj'}(\zeta) = 0 \tag{4.38}$$

which must be solved subject to $d^{j_0\sigma}_{jjj'}(0) = \delta_{jj'}$ .

The substitution $x = e^{-2\zeta}$ allows us to convert the equation into hypergeometric form, and up to normalization factors the solutions which are regular at $x = 1$ are:

$$d^{j_0\sigma}_{jjj'}(x) \propto x^{\frac{1}{2}(j+j_0+\sigma+1)}(1-x)^{j'-j}\; F(j'+j_0+1,\, j'+\sigma+1;\, 2j'+2\,;\, 1-x) \tag{4.39}$$

The general $d^{j_0\sigma}_{j\mu j'}(\zeta)$ can in principle be obtained from $d^{j_0\sigma}_{jjj'}(\zeta)$ by repeated application of the operator $\mu\left(\frac{d}{d\zeta} + \coth\zeta\right) - j_0\sigma$ and leads to a sum of hypergeometric functions; but this is a tedious procedure which is avoided by the integral representation method which gives *

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* We are grateful to Dr. M.A. Rashid for correcting certain errors in the first draft.

$$d^{j_0\sigma}_{j\mu j'}(x) = \frac{\left[(2j+1)(2j'+1)\right]^{\frac{1}{2}}}{\Gamma(j+j'+2)} \cdot \left[\begin{matrix}\Gamma(j+\mu+1)\,\Gamma(j-\mu+1)\,\Gamma(j+j_0+1)\,\Gamma(j-j_0+1)\,\cdot \\ \cdot\,\Gamma(j'+\mu+1)\,\Gamma(j'-\mu+1)\,\Gamma(j'+j_0+1)\,\Gamma(j'-j_0+1)\end{matrix}\right]^{\frac{1}{2}}$$

$$\sum_{k=0}^{j-\mu}\sum_{k'=0}^{j'-\mu} \frac{(-1)^{k+k'}\,\Gamma(k+k'+j_0-\mu+1)\,\Gamma(j+j'-k-k'-j_0+\mu+1)}{\Gamma(j_0-\mu+k+1)\,\Gamma(j_0-\mu+k'+1)\,\Gamma(j-j_0-k+1)\,\Gamma(j'-j_0-k'+1)} \cdot$$

$$\cdot\ \frac{x^{\frac{1}{2}(2k+1+j_0+\sigma-\mu)}\quad F(j+\sigma+1,\ k+k'+j_0-\mu+1;\ j+j'+2;\ 1-x)}{\Gamma(k+1)\,\Gamma(k'+1)\,\Gamma(j'+\mu-k'+1)\,\Gamma(j+\mu-k+1)} \tag{4.40}$$

that correctly satisfies $d^{j_0\sigma}_{j\mu j'}(x=1) = \delta_{jj'}$. Note that the unitarity of the representations gives $d^{j_0\sigma}_{j\mu j'}{}^{*} = d^{j_0 -\sigma}_{j\mu j'}$, so that d is "not quite real", unlike the earlier three cases.

The above form can be used to discuss the analyticity and symmetry properties of the d-functions. Thus it is straightforward, if rather tedious, to obtain the index symmetries

$$d^{j_0\sigma}_{j\mu j'}(\zeta) = d^{-j_0\sigma}_{j-\mu j'}(\zeta) = d^{\mu\sigma}_{jj_0j'}(\zeta) \tag{4.41}$$

and the weak equivalence relation

$$d^{-j_0-\sigma}_{j\mu j'}(\zeta) = \frac{\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)}\ d^{j_0\sigma}_{j\mu j'}(\zeta)\ \frac{\Gamma(j'-\sigma+1)}{\Gamma(j'+\sigma+1)} \tag{4.42}$$

The analytical properties in the $j_0, c$ plane are extremely complicated and we will not describe them here; however in the $\zeta$ plane they are quite simple. In fact, let us define $z = \mathrm{ch}\,\zeta$ as a convenient variable. Then since d is an analytic function of $e^{\zeta}$ cut from $0$ to $-\infty$ say, it will be an analytic function of $z = \frac{1}{2}(e^{\zeta} + e^{-\zeta})$ cut from $-1$ to $-\infty$. (We avoid stating the inversion property $z \rightarrow -z$ as this is only simple for functions of the second kind which will presently be considered).

## Representations of the second kind

In the theory of Legendre's equation we know that the $P_\ell$ must be supplemented by the $Q_\ell$ in any complete discussion. Likewise with the differential equations given earlier we must consider the solutions of the second kind $e$, in addition to the functions, d, of the first kind; and as with $P_\ell$ and $Q_\ell$, $e$ and d are linearly related and connected by integral formulae. Although they are singular in the neighbourhood of the identity transformation the E functions nevertheless have quite simple inversion properties and asymptotic behaviours - in contrast to the D - which makes them very useful in physical applications as we shall subsequently see.

## SO(3) and SO(2,1) functions

A simple rule of thumb method for seeing how the E functions arise is, where possible, to use the continuation property of the hypergeometric function:

$$F(a,b;c,z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,F(a,1+a-c;1+a-b;z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,F(b,1+b-c;1+b-a;z^{-1}) \tag{4.43}$$

which is useful for determining the $z \to \infty$ behaviour, and to identify the e with each of the asymptotic terms. As applied to the $d^j_{\lambda\mu}(z)$ representation this method gives

$$d^j_{\mu\lambda}(z) = \frac{1}{\pi}\tan\pi(j-\mu)\left[e^j_{\mu\lambda}(z) - e^{-j-1}_{-\mu-\lambda}(z)\right] \qquad \text{where} \tag{4.44}$$

$$e^j_{\mu\lambda}(z) = \tfrac{1}{2}\left[\Gamma(j+\mu+1)\,\Gamma(j-\mu+1)\,\Gamma(j+\lambda+1)\,\Gamma(j-\lambda+1)\right]^{\frac{1}{2}}\Big/\Gamma(2j+2)$$
$$\cdot\left(\frac{1+z}{2}\right)^{\frac{1}{2}(\mu+\lambda)}\left(\frac{1-z}{2}\right)^{-\frac{1}{2}(\mu-\lambda)}\cdot\left(\frac{z-1}{2}\right)^{-j-\lambda-1}F\left(j+\mu+1,j+\lambda+1;2j+2;\frac{2}{1-z}\right) \tag{4.45}$$

Again the factors $(1 + z)^{\frac{1}{2}}$ and $(1 - z)^{\frac{1}{2}}$ produce cuts from $-\infty$ to $-1$ and $1$ to $\infty$, and the remaining cut of the $z$ plane, from $1$ to $-\infty$, derives from the hypergeometric function. There is of course the inverse formula which expresses $e$ as a linear function of two $d$ :

$$2 \operatorname{Im} \pi(j-\mu)\, e^{j}_{\mu\lambda}(z) = \pi \left[ e^{\mp i\pi(j-\mu)} d^{j}_{\mu\lambda}(z) - d^{j}_{\mu-\lambda}(-z) \right] \tag{4.46}$$

where $\mp$ is to be taken when $\operatorname{Im} z \gtrless 0$ .

The properties of the e-functions can now be deduced; the inversion property has been simplified at the expense of the equivalence relation

$$\begin{aligned} e^{j}_{\mu\lambda}(z) &= (-1)^{\mu-\lambda} e^{j}_{\lambda\mu}(z) = e^{j}_{-\lambda\,-\mu}(z) , \\ e^{j}_{\mu\lambda}(z) &= -e^{\pm i\pi(j-\mu)} e^{j}_{\mu-\lambda}(-z) \ ; \ \pm \text{ for } \operatorname{Im} z \gtrless 0 . \end{aligned} \tag{4.47}$$

In addition we now have simple asymptotic behaviour,

$$e^{j}_{\mu\lambda}(z) \sim \tfrac{1}{2}\left[ \Gamma(j+\mu+1)\, \Gamma(j-\mu+1)\, \Gamma(j+\lambda+1)\, \Gamma(j-\lambda+1) \right]^{\frac{1}{2}} \Big/ \Gamma(2j+2) \cdot e^{\pm i\pi(\mu-\lambda)/2} \left(\tfrac{1}{2} z\right)^{-j-1} \tag{4.48}$$

as $|z| \to \infty$ and for fixed $j$ , $\mu$ , $\lambda$ . Vice versa, for fixed $z$ , $\mu$ , $\lambda$ and large $|j|$ ,

$$e^{j}_{\mu\lambda}(z) \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{\pm i\pi(\mu-\lambda)/2}}{j^{\frac{1}{2}}} \frac{\left[ z - (z^2-1)^{\frac{1}{2}} \right]^{j+\frac{1}{2}}}{(z^2-1)^{\frac{1}{4}}} \tag{4.49}$$

providing $-\pi + \epsilon < \arg j < \pi - \epsilon$ .

Turning next to the integral connection between $e$ and $d$, we know that in the region $-1 < z < 1$ the square root functions are innocuous, so that the discontinuity across the z-cut comes as

$$e^{j}_{\mu\lambda}(z+i\epsilon) - e^{j}_{\mu\lambda}(z-i\epsilon) = -i\pi\, d^{j}_{\mu\lambda}(z) , \qquad -1 < z < 1 .$$

Also for the region $z > 1$, and taking $\mathrm{Re}\, j \gg \max(|\lambda|, |\mu|)$ in order to circumvent the j-plane singularities, we also find that the e-cut discontinuity is related to e itself upon extracting suitable square root factors. The Cauchy formula then gives the integral connection,

$$
\begin{aligned}
& e^{i\pi(j-\lambda)} \left(\frac{z+1}{2}\right)^{\frac{1}{2}(\lambda+\mu)} \left(\frac{z-1}{2}\right)^{\frac{1}{2}(\lambda-\mu)} e^{j}_{\lambda\mu}(z) \\
&= \frac{\sin\pi(j-\lambda)}{\pi} \int_{1}^{\infty} \frac{\left(\frac{z'+1}{2}\right)^{\frac{1}{2}(\lambda+\mu)} \left(\frac{z'-1}{2}\right)^{\frac{1}{2}(\lambda-\mu)} e^{j}_{\lambda\mu}(z')\, dz'}{z'-z'} \\
&+ \tfrac{1}{2} e^{-i\pi\lambda} \int_{-1}^{1} \frac{\left(\frac{z'+1}{2}\right)^{\frac{1}{2}(\lambda+\mu)} \left(\frac{z'-1}{2}\right)^{\frac{1}{2}(\lambda-\mu)} d^{j}_{\lambda-\mu}(z')\, dz'}{z'-z}
\end{aligned}
\tag{4.50}
$$

which is the simple generalization of the $P_\ell$, $Q_\ell$ formula. In any case, the completeness relation can be restated through the biorthogonality of the d and e functions:

$$
\delta(z-z') = \frac{i}{2\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj\,(2j+1)\, d^{j}_{\lambda\mu}(z)\, e^{j}_{\lambda\mu}(z')\,; \quad 1 < z, z' < \infty \tag{4.51}
$$

as

$$
\begin{aligned}
8\pi^{2}\, \delta(\varphi-\varphi')\, \delta(\mathrm{ch}\beta - \mathrm{ch}\beta')\, \delta(\psi-\psi') &= \sum_{j^{\pm}} D^{j}_{\mu\lambda}(\varphi,\beta,\psi)\, D^{j\,*}_{\mu\lambda}(\varphi',\beta',\psi') \\
&- \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj\,(2j+1)\, D^{j}_{\mu\lambda}(\varphi,\beta,\psi)\, E^{j\,*}_{\mu\lambda}(\varphi',\beta',\psi')
\end{aligned}
\tag{4.52}
$$

where

$$
E^{j}_{\mu\lambda}(\varphi,\beta,\psi) \equiv e^{-i\mu\varphi}\, e^{j}_{\mu\lambda}(\beta)\, e^{-i\lambda\psi} \tag{4.53}
$$

<u>SO(2)$\wedge$T(2)</u>

The ordinary Bessel functions $J_{\lambda-\mu}(z)$ , $z = \pi\xi$ were identified with the representations of the first kind $d^{\pi}_{\lambda\mu}(\xi)$.

It is natural to identify the second kind representations with the complementary Bessel functions $Y_{\lambda-\mu}(z)$ which are logarithmically singular at the origin. Viz.,

$$e^{\pi}_{\mu\lambda}(\xi) = Y_{\lambda-\mu}(z) \tag{4.54}$$

The connections between J and Y can be rewritten as the linear relations between d and e ,

$$e^{\pi}_{\lambda\mu}(\xi) = \frac{\cos\pi(\lambda-\mu)\ \ d^{\pi}_{\lambda\mu}(\xi) - d^{\pi}_{\mu\lambda}(\xi)}{\sin\pi(\lambda-\mu)} \tag{4.55}$$

$$d^{\pi}_{\lambda\mu}(\xi) = \frac{e^{\pi}_{\mu\lambda}(\xi) - \cos\pi(\lambda-\mu)\ \ e^{\pi}_{\lambda\mu}(\xi)}{\sin\pi(\lambda-\mu)} \tag{4.56}$$

to be taken in the limit as $\lambda-\mu \rightarrow$ integer.
The e-functions satisfy the symmetry relations,

$$e^{\pi}_{\lambda\mu}(\xi) = e^{\pi}_{-\mu-\lambda}(\xi) = (-1)^{\mu-\lambda}\ e^{\pi}_{\mu\lambda}(\xi) \tag{4.57}$$

$$e^{\pi}_{\lambda\mu}(-\xi) = e^{-\pi}_{\lambda\mu}(\xi) = (-1)^{\lambda-\mu}\ \left[e^{\pi}_{\lambda\mu}(\xi) + 2i d^{\pi}_{\lambda\mu}(\xi)\right] \tag{4.58}$$

The asymptotic behaviours of d and e as $|z| \rightarrow \infty$ for $|\arg z| < \pi$ are similar since

$$J_{\lambda-\mu}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \tfrac{1}{2}(\lambda-\mu)\pi - \tfrac{1}{4}\pi\right) \tag{4,59}$$

$$Y_{\lambda-\mu}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin\left(z - \tfrac{1}{2}(\lambda-\mu)\pi - \tfrac{1}{4}\pi\right) \tag{4.60}$$

It is only the imaginary Bessel functions which have a more "drastic" asymptotic character.

SO(3,1) functions

The decomposition of a function of the first kind into a linear combination of two representations of the second kind which have simple asymptotic characteristics is a complicated problem which has been solved by Toller directly from the integral representation for $d(\zeta)$. We quote the non-trivial result:

$$d^{j_0\sigma}_{j\mu j'}(\zeta) = e^{j_0\sigma}_{j\mu j'}(\zeta) + (-1)^{j-j'}\, e^{j_0 -\sigma}_{j\mu j'}(\zeta) \tag{4.61}$$

where

$$e^{j_0\sigma}_{j\mu j'}(\zeta) = \left[(2j+1)(2j'+1)\right]^{\frac{1}{2}} \cdot \left[\frac{\Gamma(j+\mu+1)\,\Gamma(j-\mu+1)\,\Gamma(j+j_0+1)\,\Gamma(j-j_0+1)}{\Gamma(j'+\mu+1)\,\Gamma(j'-\mu+1)\,\Gamma(j'+j_0+1)\,\Gamma(j'-j_0+1)}\right]^{\frac{1}{2}}$$

$$\sum_{\kappa\kappa'\lambda\lambda'} \frac{(-1)^{j'-\mu+\lambda'}\,\Gamma(j+j'+\mu-j_0-\kappa-\kappa'+1)\,\Gamma(j_0-\mu+\kappa+\kappa'+1)}{\Gamma(j-j_0-\kappa+1)\Gamma(j-\mu-\kappa+1)\,\Gamma(j_0+\kappa-\mu+1)\,\Gamma(j'-j_0-\kappa'+1)\,\Gamma(j'+\mu-\kappa'+1)\,\Gamma(j_0+\kappa'-\mu+1)}$$

$$\cdot\ \frac{e^{\zeta(j+j'+\mu-j_0-\sigma-2\lambda)} / \Gamma(\kappa+1)\,\Gamma(\kappa'+1)\,(j_0+\lambda'-\sigma)}{\Gamma(\kappa+\lambda+1)\,\Gamma(j'+\kappa-j_0-\lambda-\lambda'+1)\,\Gamma(j+j'+\mu-j_0-\kappa'-\lambda+1)\,\Gamma(2j_0-\mu-j'+\kappa'+\lambda+\lambda'+1)} \tag{4.62}$$

is a polynomial in $e^{\zeta}$. We shall find it convenient to regard it as a function of the variable $z = ch\zeta$ at times.

We note firstly the index symmetries,

$$e^{j_0\sigma}_{j\mu j'}(z) = e^{-j_0\sigma}_{j-\mu j'}(z) = e^{\mu\sigma}_{j j_0 j'}(z) = e^{j_0 -\sigma}_{j\mu j'}(z)^*$$

$$= (-1)^{j-j'}\,\frac{\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)}\; e^{j_0\sigma}_{j'\mu j}(z)\; \frac{\Gamma(j'+\sigma+1)}{\Gamma(j'-\sigma+1)} \tag{4.63}$$

and secondly the inversion property,

$$e^{j_0\sigma}_{j\mu j'}(-z) = e^{-i\pi(j_0+\sigma-\mu+1)}\, e^{j_0\sigma}_{j\mu j'}(z) \tag{4.64}$$

Also we have the asymptotic behaviour

$$e^{j_0\sigma}_{j\mu j'}(z) \sim z^{-(j_0+\sigma-\mu+1)} \quad \text{as} \quad z \to \infty \quad . \tag{4.65}$$

The analyticity of $e$ in the $z$-plane is the same as that of $d$, viz., it has a cut from $-\infty$ to $-1$. The discontinuity across the cut can be obtained from the inversion property:

$$e^{j_0\sigma}_{j\mu j'}(z-i\epsilon) = e^{-2i\sigma\pi}\, e^{j_0\sigma}_{j\mu j'}(z+i\epsilon) \;;\quad z<-1 \;.$$

Upon applying Cauchy's theorem we get the integral relation,

$$\frac{1}{\pi}\int_{-\infty}^{-1} \frac{e^{j_0\sigma}_{j\mu j'}(z')\,dz'}{z'-z-i\epsilon} = \frac{e^{-i\pi\sigma}}{i\sin\pi\sigma}\, e^{j_0\sigma}_{j\mu j'}(z+i\epsilon)$$

or

$$\frac{1}{\pi}\int_{1}^{\infty} \frac{e^{j_0\sigma}_{j\mu j'}(z')\,dz'}{z'-z-i\epsilon} = \frac{e^{j_0\sigma}_{j\mu j'}(-z-i\epsilon)}{\sin\pi(j_0+\sigma-\mu+1)} \tag{4.66}$$

Finally, let us note the inverse linear connection between the first and second kind functions,

$$e^{j_0\sigma}_{j\mu j'}(z) = \mp\frac{i e^{\mp(\mu-1)\pi}}{2\sin\pi(j_0+\sigma)}\left[e^{\pm i(j_0+\sigma+\mu-1)\pi}\, d^{j_0\sigma}_{j\mu j'}(z) - d^{j_0\sigma}_{j\mu j'}(-z)\right] \tag{4.67}$$

where $\pm$ refer to $\mathrm{Im}\, z \gtrless 0$.

## 5. IMPROPER TRANSFORMATIONS

Parity and time reversal

If one adjoins the space and time inversion operations to the homogeneous Lorentz group and the translations, one obtains the so-called extended Poincaré group. This is completely described by the adjoined algebra,

$$
\begin{aligned}
\mathcal{P}\,\underline{P}\,\mathcal{P}^{-1} &= -\underline{P}\,, & \mathcal{P}\,P_0\,\mathcal{P}^{-1} &= P_0 \\
\mathcal{P}\,\underline{J}\,\mathcal{P}^{-1} &= \underline{J}\,, & \mathcal{P}\,\underline{K}\,\mathcal{P}^{-1} &= -\underline{K} \\
\mathcal{T}\,\underline{P}\,\mathcal{T}^{-1} &= -\underline{P}\,, & \mathcal{T}\,P_0\,\mathcal{T}^{-1} &= P_0 \\
\mathcal{T}\,\underline{J}\,\mathcal{T}^{-1} &= -\underline{J}\,, & \mathcal{T}\,\underline{K}\,\mathcal{T}^{-1} &= \underline{K}
\end{aligned}
\tag{5.1}
$$

The effect of improper Lorentz transformations on the representations of the Poincaré group is completely determined from their action on the "standard" states. Since the standard momenta always correspond to linear combinations of $P_0$ and $P_3$ one selects in preference to $\mathcal{P}$ and $\mathcal{T}$, the operators

$$
\mathcal{Y} = e^{-i\pi J_2}\,\mathcal{P}\,, \qquad \mathcal{J} = e^{i\pi J_2}\,\mathcal{T} \tag{5.2}
$$

because they commute with $P_0$ and $P_3$. That these are the "natural" operators from our point of view can be appreciated from the fact that with all of our boosting procedures,

$$
\left.
\begin{aligned}
\mathcal{Y}\,U(L_p)\,\mathcal{Y}^{-1} &= U(L_{\gamma p}) \\
\mathcal{J}\,U(L_p)\,\mathcal{J}^{-1} &= U(L_{\gamma p})
\end{aligned}
\right\}
\qquad \gamma p = (p_0,\; p_1,\, -p_2,\; p_3) \tag{5.3}
$$

Moreover, in contradistinction to $\mathcal{P}$ and $\mathcal{T}$, they always induce outer automorphisms on the little group generators:

$$\mathcal{Y} J_\pm \mathcal{Y}^{-1} = -J_\mp \quad , \quad \mathcal{J} J_\pm \mathcal{J}^{-1} = J_\pm$$

$$\mathcal{Y} K_\pm \mathcal{Y}^{-1} = K_\mp \quad , \quad \mathcal{J} K_\pm \mathcal{J}^{-1} = -K_\pm$$

$$\mathcal{Y} \Pi_\pm \mathcal{Y}^{-1} = \Pi_\mp \quad , \quad \mathcal{J} \Pi_\pm \mathcal{J}^{-1} = -\Pi_\pm$$

$$\mathcal{Y} J_3 \mathcal{Y}^{-1} = -J_3 \quad , \quad \mathcal{J} J_3 \mathcal{J}^{-1} = J_3 \tag{5.4}$$

With suitable phase conventions for the little groups we can choose without loss of generality the phases of our discrete transformations such that

$$\mathcal{J} | s\mu \rangle = | s\mu \rangle \quad , \quad \mathcal{Y} | s\mu \rangle = \eta (-1)^{\nu-\mu} | s\mu \rangle$$

$$\mathcal{J} | j_0 \, \sigma \, j \, \mu \rangle = | j_0 \, -\sigma \, j \, \mu \rangle \; ,$$

$$\mathcal{Y} | j_0 \, \sigma \, j \, \mu \rangle = \eta (-1)^{j-\mu} | -j_0 \, \sigma \, j \, -\mu \rangle \tag{5.5}$$

where $s$ stands for any of the Casimir operators of $SO(3)$ , $SO(2,1)$ or $O(2) \wedge T(2)$. The invariant factor $\nu$ assumes for each of the following cases the values:

| | O(3) | O(2,1) discrete | $O(2) \wedge T(2)$ discrete | O(2,1) continuous or $O(2) \wedge T(2)$ continuous |
|---|---|---|---|---|
| $\nu =$ | s | s | $\lvert\mu\rvert$ | $\begin{cases} 0 & \text{for } \mu \text{ integer} \\ \frac{1}{2} & \text{for } \mu \ \tfrac{1}{2} \text{ odd integer} \end{cases}$ |

$\eta$ is the intrinsic parity of the state and equals $\pm 1$ . The eigenvalue $\mathcal{Y}^2 = \pm 1$ according as we have single-valued or double-valued representations.

The simplicity of the scheme is evident from the general properties:

$$\begin{aligned} \mathcal{Y} | p \, s \, \lambda \rangle &= \mathcal{Y} U(L_p) \mathcal{Y}^{-1} \mathcal{Y} \, | \hat{p} \, s \, \lambda \rangle \\ &= \eta (-1)^{\nu-\lambda} \, | \mathcal{Y} p \, s \, - \lambda \rangle \end{aligned}$$

$$\mathcal{J} | p \, s \, \lambda \rangle = | \mathcal{J} p \, s \, \lambda \rangle \; . \tag{5.6}$$

Note that the complete reversal operator $\mathcal{R} = \mathcal{P}\mathcal{T} = \mathcal{Y}\mathcal{I}$ for which $\mathcal{R} J_{\mu\nu} \mathcal{R}^{-1} = -J_{\mu\nu}$ , $\mathcal{R} P_\mu \mathcal{R}^{-1} = P_\mu$ , gives

$$\mathcal{R} | p\, s\, \lambda \rangle = \eta (-1)^{\nu - \lambda} | p\, s\, -\lambda \rangle$$

$$\mathcal{R} | j_0\, \sigma\, j\, \mu \rangle = \eta (-1)^{j-\mu} | -j_0\, -\sigma\, j\, -\mu \rangle \qquad (5.7)$$

Improper transformations of two-particle states

We have seen that it is most suitable to take the two basic discrete operations as $\mathcal{Y}$ and $\mathcal{I}$ because of their uniformly simple action on all boost operators. Thus we can follow through their action on two-particle states by means of the reduction formulae,

$$|\hat{p}\, s\, \mu\, ,\, s_1\, \lambda_1\, s_2\, \lambda_2 \rangle = \int d\hat{G}\, \rho(\hat{G})\, D^{s*}_{\mu\lambda}(\hat{G})\, U(\hat{G})\, | p_1 s_1 \lambda_1\, ,\, p_2 s_2 \lambda_2 \rangle_z \qquad (5.8)$$

and noticing that only the azimuthal angle in $\hat{G}$ is reversed. We shall treat each case in turn but will simplify the notation by neglecting the obvious labels $\hat{p}\, s_1$ and $s_2$ .

SO(3) states

$$|j \mu \lambda_1 \lambda_2 \rangle = \int d\varphi\, d(\cos\theta)\, D^{j*}_{\mu\lambda}(\varphi,\theta,0)\, |\varphi\, \theta\, \lambda_1\, \lambda_2 \rangle$$

$$\begin{aligned} \mathcal{Y} |j \mu \lambda_1 \lambda_2 \rangle &= \int d\varphi\, d(\cos\theta)\, D^{j*}_{\mu\lambda}(\varphi,\theta,0)\, |-\varphi\, \theta\, -\lambda_1 -\lambda_2 \rangle\, \eta_1 \eta_2 (-1)^{s_1+s_2-\lambda} \\ &= \int d\varphi\, d(\cos\theta)\, D^{j*}_{-\mu\, -\lambda}(\varphi,\theta,0)\, |\varphi, \theta, -\lambda_1 -\lambda_2 \rangle\, \eta_1 \eta_2 (-1)^{s_1+s_2-\mu} \\ &= \eta_1 \eta_2 (-1)^{s_1+s_2-\mu}\, |j -\mu\, , -\lambda_1 - \lambda_2 \rangle \end{aligned} \qquad (5.9)$$

$$\begin{aligned} \mathcal{I} |j \mu \lambda_1 \lambda_2 \rangle &= \int d\varphi\, d(\cos\theta)\, D^{j}_{\mu\lambda}(\varphi,\theta,0)\, |-\varphi\, \theta\, \lambda_1\, \lambda_2 \rangle \\ &= \int d\varphi\, d(\cos\theta)\, D^{j*}_{\mu\lambda}(\varphi,\theta,0)\, |\varphi\, \theta\, \lambda_1\, \lambda_2 \rangle \\ &= | j\, \mu\, \lambda_1 \lambda_2 \rangle \end{aligned} \qquad (5.10)$$

$\underline{O(1) \wedge T(2) \text{ states}}$

$$|\rho\mu\lambda_1\lambda_2\rangle = \int d\varphi \frac{\xi}{\omega} d\left(\frac{\xi}{\omega}\right) D^{\rho\,*}_{\mu\lambda}(\varphi,\xi,0)\,|\varphi\xi\lambda_1\lambda_2\rangle \,.$$

$$\begin{aligned} \mathcal{Y}\,|\rho\mu\lambda_1\lambda_2\rangle &= \int d\varphi \frac{\xi}{\omega} d\left(\frac{\xi}{\omega}\right) D^{\rho\,*}_{\mu\lambda}(\varphi,\xi,0)\,|-\varphi\xi\lambda_1\lambda_2\rangle\,\eta_1\eta_2(-1)^{s_1+s_2-\lambda} \\ &= \int d\varphi \frac{\xi}{\omega} d\left(\frac{\xi}{\omega}\right) D^{\rho\,*}_{\mu\lambda}(-\varphi,\xi,0)\,|\varphi\xi\lambda_1\lambda_2\rangle\,\eta_1\eta_2(-1)^{s_1+s_2-\lambda} \\ &= \eta_1\eta_2(-1)^{s_1+s_2-\mu}\,|\rho-\mu_1-\lambda_1-\lambda_2\rangle \end{aligned} \tag{5.11}$$

$$\mathcal{J}\,|\rho\mu\lambda_1\lambda_2\rangle = |\rho\mu\lambda_1\lambda_2\rangle \quad , \quad \text{similarly.} \tag{5.12}$$

$\underline{SO(2,1) \text{ states}}$

The results for the principal series follow exactly the pattern of $O(3)$ since one only makes use of the reality of the $d$ matrices and their symmetry property,

$$d_{\lambda\mu}(z) = (-1)^{\lambda-\mu}\, d_{-\lambda-\mu}(z) \,.$$

Thus we obtain for the principal series,

$$\mathcal{Y}|j\mu\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-\mu}|j-\mu-\lambda_1-\lambda_2\rangle \tag{5.13}$$

$$\mathcal{J}|j\mu\lambda_1\lambda_2\rangle = |j\mu\lambda_1\lambda_2\rangle \tag{5.14}$$

However for the discrete series we must instead use

$d^{j-}_{\mu\lambda}(z) = (-1)^{\mu-\lambda}\; d^{j+}_{-\mu-\lambda}(z)$ because $\mathcal{Y}$ reverses $D^+$ and $D^-$ representations:

$$\mathcal{Y}|j^{\pm}\mu\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-\mu}|j^{\mp}-\mu-\lambda_1-\lambda_2\rangle \tag{5.15}$$

$$\mathcal{J}|j^{\pm}\mu\lambda_1\lambda_2\rangle = |j^{\pm}\mu\lambda_1\lambda_2\rangle, \quad \text{however.} \tag{5.16}$$

SO(3,1) states

$$|j_0\sigma j\mu, J\rangle = \sum_\lambda \int d\varphi\, d(\cos\theta)\, sh\zeta\, d(ch\zeta)\, D^{j*}_{\mu\lambda}(\varphi,\theta,0)\, d^{j_0\sigma}_{j\lambda J}(\zeta) \cdot |\varphi\theta\zeta J\lambda\rangle \quad .$$

The effect of $\mathcal{Y}$ and $\mathcal{I}$ comes directly from the properties

$$d^{j_0\sigma}_{j\lambda j'}(\zeta) = d^{-j_0\sigma}_{j-\lambda j'}(\zeta) \ , \quad d^{j_0\sigma}_{j\lambda j'}(\zeta)^* = d^{j_0\sigma}_{j\lambda j'}(\zeta)$$

$$\mathcal{Y}|j_0\sigma j\mu, J\rangle = \sum_\lambda \int d\varphi \ldots\ldots d^{-j_0\sigma}_{j-\lambda J}(\zeta)\,|-\varphi\,\theta\,\zeta\, J-\lambda\rangle\, \eta_1\eta_2(-1)^{J-\lambda} \qquad (5.17)$$

$$= \eta_1\eta_2(-1)^{J-\mu}\,|-j_0\sigma\, j-\mu, J\rangle$$

and

$$\mathcal{I}|j_0\sigma j\mu, J\rangle = |j_0 -\sigma\, j\mu, J\rangle \quad . \qquad (5.18)$$

Note that from the weak equivalence relation, we have the connection

$$|-j_0 - \sigma\, j\,\mu\, J\rangle = \frac{\Gamma(j-\sigma+1)}{\Gamma(j+\sigma+1)}\ \frac{\Gamma(J+\sigma+1)}{\Gamma(J-\sigma+1)}\ |j_0\, \sigma\, j\mu, J\rangle \qquad (5.19)$$

Parity and time reversal invariance restrictions

Actually it is much simpler not to apply $\mathcal{P}$ and $\mathcal{T}$ as such directly, but instead the equivalent operations $\mathcal{Y}$ and $\mathcal{I}$, to the S-matrix

$$\mathcal{Y}\, S\, \mathcal{Y}^{-1} = S \quad , \quad \mathcal{I}\, S\, \mathcal{I}^{-1} = S^\dagger \qquad (5.20)$$

Without reducing the matrix elements under the Poincaré group, the conditions on the scattering amplitudes (with the scattering confined to the 013 subspace) can be stated immediately:

$$\underline{\mathcal{J}} : \langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | S | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle = \langle p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 | S | p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 \rangle \tag{5.21}$$

$$\mathcal{Y} : \langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | S | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle = \eta_1 \eta_2 \eta_3 \eta_4 \, (-1)^{s_1+s_2+s_3+s_4-\lambda_1+\lambda_2-\lambda_3+\lambda_4} \times \langle p_3 s_3 -\lambda_3 , p_4 s_4 -\lambda_4 | S | p_1 s_1 -\lambda_1 , p_2 s_2 -\lambda_2 \rangle \tag{5.22}$$

The conditions are perhaps easier to analyse when the two-particle states are broken up into irreducible components:

$$\underline{\mathcal{J}} : \langle s_3 \lambda_3 \; s_4 \lambda_4 | T^j | s_1 \lambda_1 \; s_2 \lambda_2 \rangle = \langle s_1 \lambda_1 \; s_2 \lambda_2 | T^j | s_3 \lambda_3 \; s_4 \lambda_4 \rangle \tag{5.23}$$

for SO(2,1) and SO(3) components

$$\langle s_3 \lambda_3 \; s_4 \lambda_4 | T^\rho | s_1 \lambda_1 \; s_2 \lambda_2 \rangle = \langle s_1 \lambda_1 \; s_2 \lambda_2 | T^\rho | s_1 \lambda_1 \; s_2 \lambda_2 \rangle \tag{5.24}$$

$$\langle J' | T^{j_0 \sigma} | J \rangle = \langle J | T^{j_0 -\sigma} | J' \rangle \quad \text{for SO(3,1) components.} \tag{5.25}$$

$$\mathcal{Y} : \langle s_3 \lambda_3 \; s_4 \lambda_4 | T^j | s_1 \lambda_1 \; s_2 \lambda_2 \rangle = \eta_1 \eta_2 \eta_3 \eta_4 (-1)^{s_1+s_2-s_3-s_4} \langle -\lambda_3 - \lambda_4 | T^j | -\lambda_1 - \lambda_2 \rangle \tag{5.26}$$

Note that $j_\pm \to j_\mp$ for the discrete SO(2,1) series and that $j$ has to be interpreted as $\rho$ for the light-like case. Also

$$\langle J' | T^{j_0 \sigma} | J \rangle = \eta_1 \eta_2 \eta_3 \eta_4 (-1)^{J'-J} \langle J' | T^{-j_0 \sigma} | J \rangle \tag{5.27}$$

$$= \eta_1 \eta_2 \eta_3 \eta_4 (-1)^{J'-J} \frac{\Gamma(J' - \sigma + 1)}{\Gamma(J' + \sigma + 1)} \times \langle J' | T^{j_0 -\sigma} | J \rangle \frac{\Gamma(J + \sigma + 1)}{\Gamma(J - \sigma + 1)}$$

by weak equivalence.

It is useful to construct overall eigenstates of fixed Y-parity $\eta$ by taking appropriate linear combinations of the reduced two-particle states. (The connection of this Y-factor $\eta$ with the signature and the conventional definition of the intrinsic parity will be given later.) We can treat all $p \neq 0$ possibilities by using the phase factor $(-1)^{\nu}$ defined earlier for those Poincaré representations.

Define

$$\sqrt{2}\ |\eta j\mu\ \lambda_1\lambda_2\rangle = |j\mu\lambda_1\lambda_2\rangle + \eta\eta_1\eta_2(-1)^{\nu-s_1-s_2}\ |j\mu-\lambda_1-\lambda_2\rangle \quad (5.28$$

whereupon,

$$\mathcal{Y}\sqrt{2}\ |\eta j\mu,\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{\mu-s_1-s_2}\ |j-\mu-\lambda_1-\lambda_2\rangle + $$

$$+\ \eta(-1)^{\nu-\mu}\ |j-\mu\ \lambda_1\lambda_2\rangle$$

$$\mathcal{Y}\ |\eta j\mu,\lambda_1\lambda_2\rangle = \eta(-1)^{\nu-\mu}\ |\eta j-\mu,\lambda_1\lambda_2\rangle \qquad (5.29)$$

showing that the state $|\eta\cdots\rangle$ has Y-parity $\eta(=\pm 1)$. There is of course no change for the operation of "time reversal"

$$\mathcal{J}\ |\eta j\mu,\lambda_1\lambda_2\rangle = |\eta j\mu\lambda_1\lambda_2\rangle \qquad (5.30)$$

No such elaborate construction is needed for the $p = 0$ states which are automatically of the correct type:

$$\mathcal{Y}\ |j_0\ \sigma\ j\ \mu\ J\rangle = \eta(-1)^{j-\mu}\ |-j_0\sigma\ j-\mu,\ J\rangle;\ \eta = \eta_1\eta_2(-1)^{J-j} \quad (5.31)$$

$$\mathcal{J}\ |j_0\ \sigma\ j\ \mu\ J\rangle = |j_0-\sigma\ j\mu\ J\rangle$$

$$= \frac{\Gamma(j+\sigma-1)}{\Gamma(j-\sigma+1)}\ \frac{\Gamma(J-\sigma+1)}{\Gamma(J+\sigma+1)}\ |-j_0\ \sigma\ j\ \mu\ J\rangle\ . \qquad (5.32)$$

Inversely we can of course express the basic two-particle states as linear combinations of parity eigenstates,

$$\sqrt{2}\,|j\mu\lambda_1\lambda_2\rangle = \sum_\eta |\eta j\mu\lambda_1\lambda_2\rangle$$

$$\langle\lambda_3\lambda_4|S^j|\lambda_1\lambda_2\rangle = \frac{1}{2}\sum_\eta \langle\lambda_3\lambda_4|S^{j\eta}|\lambda_1\lambda_2\rangle \qquad (5.33)$$

with

$$\langle\lambda_3\lambda_4|S^{j\eta}|\lambda_1\lambda_2\rangle = \langle\lambda_1\lambda_2|S^{j\eta}|\lambda_3\lambda_4\rangle \qquad (5.34)$$

by $\mathcal{J}$-invariance and $\eta$ diagonal by $\mathcal{Y}$ invariance.

## 6. PARTIAL WAVE ANALYSIS OF S-MATRIX ELEMENTS

The main purpose in developing the theory of Poincaré group representations is the application to partial wave analysis. Everyone is familiar with the ordinary partial wave decomposition of scattering amplitudes into angular momentum components $S(j,E)$ by using an expansion in terms of Legendre functions $P_j$ or generalizations thereof. However, it is not always well appreciated that this analysis has a straightforward group theoretic meaning, being precisely the decomposition of the S-matrix into irreducible components of the Poincaré group. Basically, the quantities of physical interest contained in an S-matrix element such as, for example, $\langle p_3\, s_3\, \lambda_3\,,\, p_4\, s_4\, \lambda_4 | S |\, p_1\, s_1\, \lambda_1\,,\, p_2\, s_2\, \lambda_2 \rangle$, are the relativistic invariants. The isolation of these invariant quantities corresponds, group theoretically, to the extraction of scalars from a direct product of irreducible representations,

$$\mathcal{D}_+^{s_1}(m_1^2)^* \otimes \mathcal{D}_+^{s_2}(m_2^2)^* \otimes \mathcal{D}_+^{s_3}(m_3^2) \times \mathcal{D}_+^{s_4}(m_4^2) \ ,$$

the operator $S$ itself being a scalar operator. Under the usual procedure the two-particle in-states, $|p_1\, s_1\, \lambda_1\,,\, p_2\, s_2\, \lambda_2\rangle$, which transform according to a direct product representation, are decomposed into a sum of components which belong to irreducible representations, $\mathcal{D}_+^{i}((p_1+p_2)^2)$. A similar decomposition is made for the out-states, $\langle p_3\, s_3\, \lambda_3\,,\, p_4\, s_4\, \lambda_4|$. Now, the matrix elements of the scalar operator, $S$, between states which belong to irreducible representations of the Poincaré group are simply expressible in terms of a "reduced matrix element". Thus

$$\langle n';\, p_3+p_4\, j'\mu' | S | n;\, p_1+p_2\, j\,\mu\rangle =$$

$$= \delta(p_3+p_4-p_1-p_2)\,\delta_{j'j}\,\delta_{\mu'\mu}\,\langle n' \| S(j, (p_1+p_2)^2) \| n\rangle \ , \qquad (6.1)$$

where $n$ and $n'$ denote the additional quantum numbers which are necessary to distinguish among the representations $\mathcal{D}^j$ which generally occur in the decomposition with some multiplicity. Using the Clebsch-Gordan coefficients which were evaluated in Sec. 3 we can express the original matrix element,

$\langle p_3\, s_3\, \lambda_3\,,\ p_4\, s_4\, \lambda_4 \,|\, S \,|\, p_1\, s_1\, \lambda_1\,,\ p_2\, s_2\, \lambda_2 \rangle$, in terms of the invariant quantities $\langle n' \| S(j, p^2) \| n \rangle$. It is these quantities which are physically significant.

Having recognized the group-theoretical nature of the problem one can immediately contemplate some generalizations. Firstly, it is evident that the techniques for extracting scalars from products of representations may be applied to matrix elements of $S$ involving any number of particles. Secondly, they can be applied to form factor decompositions where the scalar operator $S$ is replaced by, for example, the vector $j_\mu(q)$. For the present, however, we shall be concerned with yet another generalization: one which exploits the variety of coupling schemes between products of four representations.

As is well known from the theory of angular momentum, the procedure for extracting invariants is not unique. Thus it is possible to couple $\mathcal{D}_1$ firstly with any one of $\mathcal{D}_2$, $\mathcal{D}_3^*$ or $\mathcal{D}_4^*$ and secondly to couple together the remaining pair and then extract the scalars. There are three such schemes which we indicate by

$$(\mathcal{D}_1 \times \mathcal{D}_2)\,\mathcal{D}_{12} \times (\mathcal{D}_3^* \times \mathcal{D}_4^*)\,\mathcal{D}_{34}$$

$$(\mathcal{D}_1 \times \mathcal{D}_3^*)\,\mathcal{D}_{13} \times (\mathcal{D}_2 \times \mathcal{D}_4^*)\,\mathcal{D}_{24}$$

$$(\mathcal{D}_1 \times \mathcal{D}_4^*)\,\mathcal{D}_{14} \times (\mathcal{D}_2 \times \mathcal{D}_3^*)\,\mathcal{D}_{23}$$

Beyond this, we could couple $\mathcal{D}_3^*$ to $\mathcal{D}_{12}$ to give $\mathcal{D}_{123}$ and then extract the scalars from the product of $\mathcal{D}_{123}$ with $\mathcal{D}_4$. This scheme we indicate by

$$((\mathcal{D}_1 \mathcal{D}_2)\,\mathcal{D}_{12}\ \mathcal{D}_3^*)\,\mathcal{D}_{123}\ \mathcal{D}_4^*$$

There are many such schemes. It is of course necessary that the set of invariants obtained under one coupling scheme should be expressible in terms of those obtained under any other. The connection is made by means of recoupling coefficients. The recoupling coefficients of the Poincaré group are better known as <u>crossing matrices</u>. Some of these will be computed in a later section.

A problem which should be faced here and which does not arise in the familiar theory of angular momentum is that of convergence. The procedures developed in Sec. 3 for expressing the product states $|p_1\ s_1\ \lambda_1\ ,\ p_2\ s_2\ \lambda_2\rangle$ in irreducible components was valid only insofar as these states were employed specifying square-integrable functions. It is not clear that, in the applications we shall be making, this condition can always be met. We shall simply assume that there exists some region in the complex planes of s and t where the amplitudes can be expanded in the relevant irreducible representations and that this expansion can be continued to the physical region. This attitude can always be justified by relating the expansion in question to the well-known partial wave expansion by means of recoupling coefficients.

In the following we shall deal with only two coupling schemes for the process $1 + 2 \to 3 + 4$. These are

$$(\mathcal{D}_1\ \mathcal{D}_2)\ \mathcal{D}_{12}\ (\mathcal{D}_3^*\mathcal{D}_4^*)\ \mathcal{D}_{34} \tag{6.2}$$

and

$$(\mathcal{D}_1\ \mathcal{D}_3^*)\ \mathcal{D}_{13}\ (\mathcal{D}_2\ \mathcal{D}_4^*)\ \mathcal{D}_{24}\ , \tag{6.3}$$

of which the first corresponds to the well-known partial wave expansion.

In order to apply the formalism of Sec. 3 it is necessary for us first to demonstrate the equivalence

$$\mathcal{D}_+^j(m^2)^* \approx \mathcal{D}_-^j(m^2)\ . \tag{6.4}$$

This follows with the help of the equivalence between the SO(3) representations $D^j$ and $D^{j*}$, viz.,

$$\mathcal{D}^j_{\lambda\lambda'}(R)^* = (-)^{j-\lambda}\,D_{-\lambda,-\lambda'}(R)\,(-)^{j+\lambda'}\ , \tag{6.5}$$

We can set up a correspondence between the basis vectors of $\mathfrak{D}^j_+(m^2)^*$ and those of $\mathfrak{D}^j_-(m^2)$. Since for $\mathfrak{D}^j_+(m^2)$ we have

$$|p\,j\,\lambda\rangle \to e^{ipa}\sum_{\lambda'} |\Lambda p\,j\,\lambda'\rangle\, D^j_{\lambda'\lambda}(R)\,, \quad p_0 > 0 \tag{6.6}$$

then for $\mathfrak{D}^j_+(m^2)^*$ we must have

$$|p\,j\,\lambda\rangle^* \to e^{-ipa}\sum_{\lambda'} |\Lambda p\,j\,\lambda'\rangle^*\, D^j_{\lambda'\lambda}(R)^*\,, \quad p_0 > 0\,. \tag{6.7}$$

Defining a new set of basis vectors by

$$|-p\,j\,-\lambda\rangle' = |p\,j\,\lambda\rangle^*\,(-)^{j-\lambda} \tag{6.8}$$

it follows that the transformation becomes

$$|p\,j\,\lambda\rangle' \to e^{ipa}\sum_{\lambda'} |\Lambda p\,j\,\lambda'\rangle'\, D^j_{\lambda'\lambda}(R)\,, \quad p_0 < 0 \tag{6.9}$$

which is characteristic of $\mathfrak{D}^j_-(m^2)$. Thus the equivalence is proved.

Consider now the coupling scheme (7.2). We choose the reference frame such that $p_1 + p_2 = p_3 + p_4$ is aligned with the time axis and all momenta are contained in the 0 1 3 subspace. Specifically we choose

$$\begin{aligned}
p_1 &= m_1(\mathrm{ch}\,\alpha_1\,,\ \mathrm{sh}\,\alpha_1 \sin\theta\,,\ 0\,,\ \mathrm{sh}\,\alpha_1 \cos\theta\,)\\
p_2 &= m_2(\mathrm{ch}\,\alpha_2\,,\ -\mathrm{sh}\,\alpha_2 \sin\theta\,,\ 0\,,\ -\mathrm{sh}\,\alpha_2 \cos\theta\,)\\
p_3 &= m_3(\mathrm{ch}\,\alpha_3\,,\ 0\,,\ 0\,,\ \mathrm{sh}\,\alpha_3\,)\\
p_4 &= m_4(\mathrm{ch}\,\alpha_4\,,\ 0\,,\ 0\,,\ -\mathrm{sh}\,\alpha_4\,)
\end{aligned} \tag{6.10}$$

where the angles satisfy the conditions

$$m_1 \operatorname{ch}\alpha_1 + m_2 \operatorname{ch}\alpha_2 = m_3 \operatorname{ch}\alpha_3 + m_4 \operatorname{ch}\alpha_4 = \sqrt{s}$$

$$m_1 \operatorname{sh}\alpha_1 - m_2 \operatorname{sh}\alpha_2 = m_3 \operatorname{sh}\alpha_3 - m_4 \operatorname{sh}\alpha_4 = 0 \qquad (6.11)$$

which are solved by

$$\operatorname{ch}\alpha_1 = \frac{s + m_1^2 - m_2^2}{2 m_1 \sqrt{s}}, \quad \operatorname{ch}\alpha_2 = \frac{s - m_1^2 + m_2^2}{2 m_2 \sqrt{s}}, \quad \operatorname{ch}\alpha_3 = \frac{s + m_3^2 - m_4^2}{2 m_3 \sqrt{s}}, \quad \operatorname{ch}\alpha_4 = \frac{s - m_3^2 + m_4^2}{2 m_4 \sqrt{s}} \qquad (6.12)$$

The angle $\theta$ is fixed in terms of the momentum transfer, $t$, by

$$t = (p_1 - p_3)^2 = m_1^2 - 2 m_1 m_3 (\operatorname{ch}\alpha_1 \operatorname{ch}\alpha_3 - \operatorname{sh}\alpha_1 \operatorname{sh}\alpha_3 \cos\theta) + m_3^2 \qquad (6.13)$$

so that

$$\cos\theta = \frac{(s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2) - 2s(t - m_1^2 - m_3^2)}{\left[\left((s + m_1^2 - m_2^2)^2 - 4 m_1^2 s\right)\left((s + m_3^2 - m_4^2)^2 - 4 m_3^2 s\right)\right]} \qquad (6.14)$$

With the momenta (7.10) we can exhibit the decompositions of the in- and out-states respectively as

$$|p_1 s_1 \lambda_1, p_2 s_2 \lambda_2\rangle^+ = \sum_{j \geq |\lambda_1 - \lambda_2|} \sum_{\lambda} (2j+1) |\lambda_1 \lambda_2 ; \hat{p}\, j \lambda\rangle\, d^j_{\lambda, \lambda_1 - \lambda_2}(\theta) \qquad (6.15)$$

$$^+\langle p_3 s_3 \lambda_3, p_4 s_4 \lambda_4 | = \sum_{j \geq |\lambda_3 - \lambda_4|} (2j+1) \langle \lambda_3 \lambda_4 ; \hat{p}\, j\, \lambda_3 - \lambda_4 | \qquad (6.16)$$

where we have left implicit some of the invariants, $m_1^2$ $S_1$, $m_2^2$ $S_2$, ... etc., showing only the important numbers $\lambda_1$, $\lambda_2$ and $\lambda_3$, $\lambda_4$ which - themselves invariant - serve to enumerate the multiplicity with which the representation $\mathcal{D}^j$ appears in the decomposition of $\mathcal{D}^{s_1} \times \mathcal{D}^{s_2}$ and $\mathcal{D}^{s_3} \times \mathcal{D}^{s_4}$ respectively. The total four-momentum of the resulting states is denoted by

$$\hat{p} = ( \sqrt{s}, 0, 0, 0 ) . \tag{6.17}$$

Defining the matrix elements of the T operator by extracting the four-momentum conserving $\delta$-function from S-1, we have, since T is a scalar operator,

$$\begin{aligned} &\langle \lambda_3 \lambda_4 ; \hat{p}\, j' \lambda' | T | \lambda_1 \lambda_2 ; \hat{p}\, j \lambda \rangle = \\ &= \delta_{j'j}\, \delta_{\lambda'\lambda} \frac{1}{2j+1} \langle \lambda_3 \lambda_4 \| T^j(s) \| \lambda_1 \lambda_2 \rangle \end{aligned} \tag{6.18}$$

where the reduced matrix element, $\langle \lambda_3 \lambda_4 \| T^j(s) \| \lambda_1 \lambda_2 \rangle$ , is a Poincaré invariant. From (7.15), (7.16) and (7.17) we obtain the required decomposition

$$\begin{aligned} &{}^{+}\langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle^{+} = \\ &= \sum_{j \geq M} (2j+1) \langle \lambda_3 \lambda_4 \| T^j(s) \| \lambda_1 \lambda_2 \rangle\, d^{\,j}_{\lambda_3 - \lambda_4 , \lambda_1 - \lambda_2}(\theta) \end{aligned} \tag{6.19}$$

where M denotes the maximum of $|\lambda_1 - \lambda_2|$ and $|\lambda_3 - \lambda_4|$.

Consider the coupling scheme (7.3). This can be dealt with in a similar fashion to (7.2) if we exploit the equivalence (7.4) to define a "transposed" T operator by

$$\begin{aligned} &\langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle = \\ &= (-)^{s_2 \lambda_2} \langle -p_2 s_2 -\lambda_2 , p_4 s_4 \lambda_4 | \tilde{T} | p_1 s_1 \lambda_1 , -p_3 s_3 \lambda_3 \rangle (-)^{s_3 - \lambda_3} \end{aligned} \tag{6.20}$$

The states between which $\tilde{T}$ is sandwiched transform according to the representations $\mathcal{D}_-^{s_2}(m_2^2)^* \otimes \mathcal{D}_+^{s_4}(m_4^2)^*$ and $\mathcal{D}_+^{s_1}(m_1^2) \otimes \mathcal{D}_-^{s_3}(m_3^2)$ and our problem is to express them as sums of irreducible components. This problem differs from the previous one in that the decomposition of $\mathcal{D}_+ \otimes \mathcal{D}_-$ contains more than one type of irreducible representation. The invariant "total mass$^2$" of these states, $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ can range from $-\infty$ up to the lesser of $(m_1 - m_3)^2$ and $(m_2 - m_4)^2$. Since the representations appropriate to $t > 0$, $t = 0$ and $t < 0$ are entirely different, there will result distinct types of expansion for those ranges of $t$. It will be necessary to impose some continuity requirements at $t = 0$ where these ranges merge.

Taking the case $t < 0$ we choose the reference frame such that all momenta lie in the 0 1 3 subspace with $p_1 - p_3 = p_1 - p_2$ aligned with the three-axis. Specifically we choose

$$
\begin{aligned}
p_1 &= m_1(\mathrm{ch}\,\gamma_1\,\mathrm{ch}\,\beta\ ,\ \mathrm{ch}\,\gamma_1\,\mathrm{sh}\,\beta\ ,\ 0\ ,\ \mathrm{sh}\,\gamma_1) \\
p_2 &= m_2(\mathrm{ch}\,\gamma_2\ ,\ 0\ ,\ 0\ ,\ \mathrm{sh}\,\gamma_2) \\
p_3 &= m_3(\mathrm{ch}\,\gamma_3\,\mathrm{ch}\,\beta\ ,\ \mathrm{ch}\,\gamma_3\,\mathrm{sh}\,\beta\ ,\ 0\ ,\ \mathrm{sh}\,\gamma_3) \\
p_4 &= m_4(\mathrm{ch}\,\gamma_4\ ,\ 0\ ,\ 0\ ,\ \mathrm{sh}\,\gamma_4)
\end{aligned}
\tag{6.21}
$$

where the angles satisfy the conditions

$$
\begin{aligned}
m_1\,\mathrm{ch}\,\gamma_1 - m_3\,\mathrm{ch}\,\gamma_3 &= m_4\,\mathrm{ch}\,\gamma_4 - m_2\,\mathrm{ch}\,\gamma_2 = 0 \\
m_1\,\mathrm{sh}\,\gamma_1 - m_3\,\mathrm{sh}\,\gamma_3 &= m_4\,\mathrm{sh}\,\gamma_4 - m_2\,\mathrm{sh}\,\gamma_2 = \sqrt{-t} > 0
\end{aligned}
\tag{6.22}
$$

which are solved by

$$
\mathrm{sh}\,\gamma_1 = \frac{-t - m_1^2 + m_3^2}{2m_1\sqrt{-t}}\ ,\quad \mathrm{sh}\,\gamma_2 = \frac{t - m_4^2 + m_2^2}{2m_2\sqrt{-t}}\ ,\quad \mathrm{sh}\,\gamma_3 = \frac{t - m_1^2 + m_3^2}{2m_3\sqrt{-t}}\ ,\quad \mathrm{sh}\,\gamma_4 = \frac{-t - m_4^2 + m_2^2}{2m_4\sqrt{-t}}
\tag{6.23}
$$

The angle $\beta$ is fixed in terms of the energy, $s$, by

$$s = (p_1 + p_2)^2 = m_1^2 + 2m_1 m_2 \left(\mathrm{ch}\gamma_1\,\mathrm{ch}\gamma_2\,\mathrm{ch}\beta - \mathrm{sh}\gamma_1\,\mathrm{sh}\gamma_2\right) + m_2^2 \tag{6.24}$$

so that

$$\mathrm{ch}\beta = \frac{(m_2^2 - m_4^2 + t)(m_3^2 - m_1^2 - t) - 2t(s - m_1^2 - m_2^2)}{\left[\left((m_2^2 - m_4^2 + t)^2 - 4m_2^2 t\right)\left((m_3^2 - m_1^2 - t)^2 - 4m_1^2 t\right)\right]^{\frac{1}{2}}} \tag{6.25}$$

According to the fomulae of Sec. 3 the product states with momenta defined by (7.21) decompose respectively as

$$|p_1 s_1 \lambda_1, -p_3 s_3 - \lambda_3 \rangle^- = \sum_{-|\lambda_1 - \lambda_3| \leq j \leq -1} (2j+1)\,|\lambda_1, \lambda_3; \hat{p}\, j \lambda\rangle\, d^{j\pm}_{\lambda, \lambda_1 - \lambda_3}(\beta) + \tag{6.26}$$
$$+ \sum_{\lambda} \int_{-\frac{1}{2}}^{-\frac{1}{2}+i\infty} dj\, \frac{2j+1}{\tan \pi j}\, |\lambda_1, \lambda_3; \hat{p}\, j\, \lambda\rangle\, d^{j}_{\lambda, \lambda_1 - \lambda_3}(\beta)$$

and

$$\langle -p_2 s_2 - \lambda_2, p_4 s_4 \lambda_4 | = \sum_{-|\lambda_2 - \lambda_4| \leq j \leq -1} (2j+1)\,\langle \lambda_2 \lambda_4; \hat{p}\, j\, \lambda_4 - \lambda_2 | + \tag{6.27}$$
$$+ \int_{-\frac{1}{2}}^{-\frac{1}{2}+i\infty} dj\, \frac{2j+1}{\tan \pi j}\, \langle \lambda_2 \lambda_4; \hat{p}\, j\, \lambda_4 - \lambda_2 |$$

where the superscript (–) on the states (7.26) and (7.27) indicates that they are defined relative to the – type boosts of Sec. 2 and the appearance in (7.26) of the discrete representations $d^{j+}$ or $d^{j-}$ is determined by the sign of $\lambda_1 - \lambda_3$ . The "total" four-momentum is denoted by

$$\hat{p} = (0, 0, 0, \sqrt{-t}) \tag{6.28}$$

Since $\tilde{T}$ is a scalar operator we have the result

$$\langle \lambda_2 \lambda_4 ; \hat{p}\, j' \lambda' \,|\, \tilde{T} \,|\, \lambda_1 \lambda_3 ; \hat{p}\, j \lambda \rangle =$$

$$= \delta(j'-j)\, \delta_{\lambda' \lambda} \frac{\tan \pi j}{2j+1} \langle \lambda_2 \lambda_4 \| T^j(t) \| \lambda_1 \lambda_3 \rangle \tag{6.29}$$

for the principal series components with analogous expressions for the discrete components. Putting together (7.26), (7.27), (7.29) and (7.20) we obtain the decomposition for $t < 0$,

$$^{-}\langle p_3 s_3 \lambda_3 ,\ p_4 s_4 \lambda_4 \,|\, T \,|\, p_1 s_1 \lambda_1 ,\ p_2 s_2 \lambda_2 \rangle^{-} =$$

$$= (-)^{s_2 - \lambda_2} \Big[ \sum_{-M \leq j \leq -1} (2j+1) \langle \lambda_2 \lambda_4 \| T^{j\pm}(t) \| \lambda_1 \lambda_3 \rangle d^{j\pm}_{\lambda_4 - \lambda_2 , \lambda_1 - \lambda_3}(\beta) +$$

$$+ \int_{-\frac{1}{2}}^{-\frac{1}{2}+i\infty} dj \, \frac{2j+1}{\tan \pi j} \langle \lambda_2 \lambda_4 \| T^j(t) \| \lambda_1 \lambda_3 \rangle d^{j}_{\lambda_4 - \lambda_2 , \lambda_1 - \lambda_3}(\beta) \Big] (-)^{s_3 - \lambda_3} \tag{6.30}$$

In order to compare the two decompositions (7.19) and (7.30) it is necessary to apply a Lorentz transformation to (7.30) so as to carry it from the brick wall frame specified by (7.21) to the centre-of-mass frame specified by (7.10) and then to transform from the –type states to the +type by applying the spin-rearrangement factors of Sec. 2.

Consider next the case, $t = 0$. There are two possibilities, firstly $m_1 = m_3$ and $m_2 = m_4$ in which case $t = 0$ implies $p_1 = p_3$ and $p_2 = p_4$ or, secondly, $m_1 \neq m_3$ and $m_2 \neq m_4$ in which case $t = 0$ implies that $p_1 - p_3 = p_4 - p_2$ is lightlike. Taking first the lightlike case we choose the frame such that

$$p_1 = m_1 (\mathrm{ch}\chi_1 + \frac{\xi^2}{2} e^{-\chi_1},\ -\xi e^{-\chi_1},\ 0\ ,\ \mathrm{sh}\chi_1 + \frac{\xi}{2} e^{-\chi_1})$$

$$p_2 = m_2 (\mathrm{ch}\chi_2 ,\quad 0\quad ,\quad 0\quad ,\ \mathrm{sh}\chi_2)$$

$$p_3 = m_3(\mathrm{ch}\chi_3 + \frac{\xi^2}{2}\, e^{-\chi_3}, \quad 0 \quad , \quad \mathrm{sh}\chi_3 + \frac{\xi^2}{2}\, e^{-\chi_3})$$

$$p_4 = m_4(\mathrm{ch}\chi_4 , \quad 0 \quad , \quad 0 \quad , \quad \mathrm{sh}\chi_4) \tag{6.31}$$

where the angles satisfy the conditions

$$m_1 \,\mathrm{ch}\chi_1 - m_3 \,\mathrm{ch}\chi_3 = m_4 \,\mathrm{ch}\chi_4 - m_2 \,\mathrm{ch}\chi_2 = \omega$$

$$m_1 \,\mathrm{sh}\chi_1 - m_3 \,\mathrm{sh}\chi_3 = m_4 \,\mathrm{sh}\chi_4 - m_2 \,\mathrm{sh}\chi_2 = \omega \tag{6.32}$$

where $\omega$ is an arbitrary parameter (positive when $m_1 > m_3$ , $m_4 > m_2$ and negative when $m_1 < m_3$ , $m_2 < m_4$). These angles are given in terms of $\omega$ by

$$e^{-\chi_1} = \frac{m_1^2 - m_3^2}{2m_1\omega} \;, \quad e^{-\chi_2} = \frac{m_4^2 - m_2^2}{2m_2\,\omega} \;, \quad e^{-\chi_3} = \frac{m_1^2 - m_3^2}{2m_3\omega} \;, \quad e^{-\chi_4} = \frac{m_4^2 - m_2^2}{2m_4\,\omega} \tag{6.33}$$

The angle $\xi$ is fixed in terms of s by

$$s = (p_1+p_2)^2 = m_1^2 + 2\, m_1\, m_2 \,\mathrm{ch}(\chi_1 - \chi_2) + m_2^2 + \xi^2\, m_1\, m_2 e^{-(\chi_1+\chi_2)} \tag{6.34}$$

so that

$$\left(\frac{\xi}{\omega}\right)^2 = \frac{4}{(m_1^2 - m_3^2)(m_4^2 - m_2^2)} \left[ s - m_1^2\left(1 + \frac{m_4^2 - m_2^2}{m_1^2 - m_3^2}\right) - m_2^2\left(1 + \frac{m_1^2 - m_3^2}{m_4^2 - m_2^2}\right)\right] \tag{6.35}$$

According to the formulae of Sec. 3 the product states with momenta defined by (7.31) decompose respectively as,

$$|p_1 s_1 \lambda_1, -p_3 s_3 -\lambda_3\rangle^0 = \sum_{\lambda} \int_0^{\infty} d\rho^2 |\lambda_1 \lambda_3 ; \hat{p}\rho\lambda\rangle \, d^{\rho/\omega}_{\lambda, \lambda_1 - \lambda_3}(\xi) , \qquad (6.36)$$

and

$$^0\langle -p_2 s_2 -\lambda_2 , p_4 s_4 \lambda_4 | = \int_0^{\infty} d\rho^2 \langle \lambda_2 \lambda_4 ; \hat{p}\rho \lambda_1 - \lambda_2 | \qquad (6.37)$$

where the superscript (0) on the states (7.36) and (7.37) indicates that they are defined relative to the 0-type boosts of Sec. 2. The "total" four-momentum is defined by

$$\hat{p} = (\omega , 0 , 0 , \omega) \qquad (6.38)$$

Since $\tilde{T}$ is a scalar operator we have the result

$$\langle \lambda_2 \lambda_4 ; \hat{p}\rho' \lambda' | \tilde{T} | \lambda_1 \lambda_3 ; \hat{p}\rho\lambda\rangle = \delta(\rho^2 - \rho'^2)\, \delta_{\lambda'\lambda} \langle \lambda_2 \lambda_4 \| T^{\rho} \| \lambda_1 \lambda_3 \rangle \qquad (6.39)$$

Putting together (7.36), (7.37), (7.39) and (7.20) we obtain the decomposition for t = 0,

$$^0\langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle^0 =$$
$$= (-)^{s_2 - \lambda_2} \int_0^{\infty} d\rho^2 \langle \lambda_2 \lambda_4 \| T^{\rho} \| \lambda_1 \lambda_3 \rangle \, d^{\rho/\omega}_{\lambda_4 - \lambda_2 , \lambda_1 - \lambda_3}(\xi) \; (-)^{s_3 - \lambda_3} \qquad (6.40)$$

and, using the explicit form ( ) for $d^{\rho/\omega}$ this becomes,

$$^0\langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle^0 =$$
$$= (-)^{s_2 - \lambda_2} \int_0^{\infty} d\rho^2 \langle \lambda_2 \lambda_4 \| T^{\rho} \| \lambda_1 \lambda_3 \rangle \, J_{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4}(\rho\xi/\omega) \, (-)^{s_3 - \lambda_3} \qquad (6.41)$$

where $\xi/\omega$ is to be expressed in terms of s by (7.35). Again it would be necessary to apply the appropriate rearrangement matrices and Lorentz transformation in order to compare this with the other expressions.

The other important case at $t = 0$ arises when $m_1 = m_3$ and $m_2 = m_4$ (i.e. elastic forward scattering). Here the momentum transfer is null and we can choose the frame such that

$$
\begin{aligned}
p_1 = p_3 &= M\,(\,\mathrm{ch}\,\alpha/2,\ 0\ ,\ 0\ ,\ \mathrm{sh}\,\alpha/2\,) \\
p_2 = p_4 &= m\,(\,\mathrm{ch}\,\alpha/2,\ 0\ ,\ 0\ ,\ -\mathrm{sh}\,\alpha/2\,)
\end{aligned}
\tag{6.42}
$$

It is just the brick wall frame. The angle $\alpha$ is given by

$$
s = M^2 + 2\,M\,m\,\mathrm{ch}\,\alpha + m^2 \tag{6.43}
$$

The product states decompose respectively as

$$
\begin{aligned}
\left| p_1 s_1 \lambda_1 , -p_3 s_3 - \lambda_3 \right\rangle^{+} = \sum_{J=|s_1-s_3|}^{s_1+s_3} \sum_{j_0=-J}^{J} \int_0^{i\infty} d\sigma \,(j_0^2-\sigma^2) \sum_{j\lambda} \\
\times \left| J ; j_0 \sigma j \lambda \right\rangle d^{j_0\sigma}_{j\lambda J}\left(\tfrac{\alpha}{2}\right) \left\langle J\lambda \right| s_1 \lambda_1 , s_3 - \lambda_3 \rangle
\end{aligned}
\tag{6.44}
$$

$$
\begin{aligned}
{}^{+}\!\left\langle -p_2 s_2 - \lambda_2 , p_4 s_4 \lambda_4 \right| = \sum_{J=|s_2-s_4|}^{s_2+s_4} \sum_{j_0=-J}^{J} \int_{j_0=-J}^{J} d\sigma \,(j_0^2-\sigma^2) \sum_{j\lambda} \\
\times \left\langle s_2 - \lambda_2 , s_4 \lambda_4 \right| J\lambda \rangle\, d^{j_0\sigma}_{J\lambda j}\left(\tfrac{\alpha}{2}\right) \left\langle J ; j_0 \sigma j \lambda \right|
\end{aligned}
\tag{6.45}
$$

For the scalar operator $\tilde{T}$ we have

$$
\left\langle J' ; j_0' \sigma' j' \lambda' \right| \tilde{T} \left| J ; j_0 \sigma j \lambda \right\rangle = \delta_{j_0' j_0} \frac{\delta(\sigma' - \sigma)}{j_0^2 - \sigma^2}\, \delta_{j' j}\, \delta_{\lambda' \lambda} \left\langle J' \right\| T^{j_0 \sigma} \left\| J \right\rangle \tag{6.46}
$$

Hence the decomposition for $t = 0$ (null) is given by

$$
{}^{+}\langle p_3 s_3 \lambda_3 , p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1 , p_2 s_2 \lambda_2 \rangle^{+} =
$$

$$
= \sum_{JJ'} \sum_{j_0=-J_m}^{J_m} \int_0^{i\infty} d\sigma (j_0^2 - \sigma^2) \langle J' \| T^{j_0\sigma} \| J \rangle \cdot \tag{6.47}
$$

$$
\cdot \langle s_4 \lambda_4 | s_2 \lambda_2 , J'\lambda \rangle d^{j_0\sigma}_{J'\lambda J}(\alpha) \langle J\lambda , s_3 \lambda_3 | s_1 \lambda_1 \rangle
$$

where $J_m$ denotes the lesser of $J$ and $J'$.

Finally we take the $0 < t < (m_1-m_3)^2$ (or $(m_4-m_2)^2$). The reference frame can be chosen so that $p_1-p_3=p_4-p_2$ is aligned with the zero-axis. In this frame the momenta are given by

$$
\begin{aligned}
p_1 &= m_1(\mathrm{ch}\,\alpha_1 , \mathrm{sh}\,\alpha_1 \sin\theta , 0 , \mathrm{sh}\,\alpha_1 \cos\theta ) \\
p_2 &= m_2(\mathrm{ch}\,\alpha_2 , 0 , 0 , \mathrm{sh}\,\alpha_2 ) \\
p_3 &= m_3(\mathrm{ch}\,\alpha_3 , \mathrm{sh}\,\alpha_3 \sin\theta , 0 , \mathrm{sh}\,\alpha_3 \cos\theta ) \\
p_4 &= m_4(\mathrm{ch}\,\alpha_4 , 0 , 0 , \mathrm{sh}\,\alpha_4 )
\end{aligned} \tag{6.48}
$$

where the angles satisfy the conditions

$$
\begin{aligned}
m_1 \,\mathrm{ch}\,\alpha_1 - m_3 \,\mathrm{ch}\,\alpha_3 &= m_4 \,\mathrm{ch}\,\alpha_4 - m_2 \,\mathrm{ch}\,\alpha_2 = \pm\sqrt{t} \\
m_1 \,\mathrm{sh}\,\alpha_1 - m_3 \,\mathrm{sh}\,\alpha_3 &= m_4 \,\mathrm{sh}\,\alpha_4 - m_2 \,\mathrm{sh}\,\alpha_2 = 0 \cdot
\end{aligned} \tag{6.49}
$$

which are solved by

$$
\mathrm{ch}\,\alpha_1 = \frac{t + m_1^2 - m_3^2}{2 m_1 \sqrt{t}} , \quad \mathrm{ch}\,\alpha_2 = \frac{-t + m_4^2 - m_2^2}{2 m_2 \sqrt{t}} , \quad \mathrm{ch}\,\alpha_3 = \frac{-t + m_1^2 - m_3^2}{2 m_3 \sqrt{t}} , \quad \mathrm{ch}\,\alpha_4 = \frac{t + m_4^2 - m_2^2}{2 m_4 \sqrt{t}} \tag{6.50}
$$

The angle $\theta$ is fixed in terms of the energy $s$ by

$$s = (p_1+p_2)^2 = m_1^2 + 2\, m_1\, m_2(\mathrm{ch}\,\alpha_1\, \mathrm{ch}\,\alpha_2 - \mathrm{sh}\,\alpha_1\, \mathrm{sh}\,\alpha_2 \cos\theta) + m_2^2 \qquad (6.51)$$

so that

$$\cos\theta = \frac{(m_1^2 - m_3^2 + t)(m_4^2 - m_2^2 - t) - 2t(s - m_1^2 - m_2^2)}{\left[\left((m_1^2 - m_3^2 + t)^2 - 4m_1^2 t\right)\left(m_4^2 - m_2^2 - t\right)^2 - 4m^2 t\right)\right]^{\frac{1}{2}}} \qquad (6.52)$$

The product states decompose respectively as

$$|p_1 s_1 \lambda_1, -p_3 s_3 \lambda_3 \rangle^{+} = \sum_j (2j+1) \sum_\lambda |\lambda_1 \lambda_3 ; \hat{p}\, j\, \lambda \rangle\, d^j_{\lambda, \lambda_1 - \lambda_3}(\theta) \qquad (6.53)$$

$${}^{+}\langle -p_2 s_2 -\lambda_2, p_4 s_4 \lambda_4 | = \sum_j (2j+1) \; \langle \lambda_2 \lambda_4 ; \hat{p}\, j\, \lambda_4 - \lambda_2 | \qquad (6.54)$$

where the "total" four-momentum is given by

$$\hat{p} = (\sqrt{t}, 0, 0, 0) \qquad (6.55)$$

Thus we get the decomposition for $t > 0$.

$${}^{+}\langle p_3 s_3 \lambda_3, p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1, p_2 s_2 \lambda_2 \rangle^{+} =$$

$$= (-)^{s_2 - \lambda_2} \sum_j (2j+1) \langle \lambda_2 \lambda_4 \| T^j(t) \| \lambda_1 \lambda_3 \rangle\, d^j_{\lambda_4 - \lambda_2, \lambda_1 - \lambda_3}(\theta)\; (-)^{s_3 - \lambda_3} \qquad (6.56)$$

This completes the treatment of the coupling scheme (7.3).

Let us summarize the formulae so far obtained and refer them all to bases defined with +type boosts. (The + will therefore be omitted)

1) <u>Conventional coupling $(\mathcal{D}_+^{s_1}\mathcal{D}_+^{s_2})\mathcal{D}_{12}(\mathcal{D}_+^{s_3 *}\mathcal{D}_+^{s_4 *})\mathcal{D}_{34}^*$</u>:

$$\langle p_3\lambda_3, p_4\lambda_4 | T | p_1\lambda_1, p_2\lambda_2\rangle = \sum_{j\geq M}(2j+1)\langle\lambda_3\lambda_4\| T^j_{(s)} \|\lambda_1\lambda_2\rangle d^j_{\lambda_3-\lambda_4,\lambda_1-\lambda_2}\theta$$

$$M = \max(|\lambda_3-\lambda_4|, |\lambda_1-\lambda_2|)$$

$$\cos\theta = \frac{(s+m_1^2-m_2^2)(s+m_3^2-m_4^2)-2s(t-m_1^2-m_3^2)}{\left[\left((s+m_1^2-m_2^2)^2-4m_1^2 s\right)\left((s+m_3^2-m_4^2)^2-4m_3^2 s\right)\right]^{\frac{1}{2}}}$$

Frame: $p_1 + p_2 = p_3 + p_4 = (\sqrt{s}, 0, 0, 0)$

2) <u>Crossed coupling $(\mathcal{D}_+^{s_1}\mathcal{D}_-^{s_3})\mathcal{D}_{13}(\mathcal{D}_-^{s_2 *}\mathcal{D}_+^{s_4 *})\mathcal{D}_{24}^*$</u>:

Four cases must be distinguished according to the value of $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$. These are

(a) <u>Timelike</u>: $0 < t \leq \min\left((m_1 - m_3)^2, (m_4 - m_2)^2\right)$

$$\langle p_3\lambda_3 p_4 | T | p_1\lambda_1 p_2\lambda_2\rangle = (-)^{s_3-\lambda_3}\sum_{j\geq M}(2j+1)\langle\lambda_2\lambda_4\| T^j(t) \|\lambda_1\lambda_3\rangle d^j_{\lambda_4-\lambda_2,\lambda_1-\lambda_3}(\theta)(-)^{s_2\lambda_2}$$

$$M^- = \max(|\lambda_1-\lambda_3|, |\lambda_4-\lambda_2|)$$

$$\cos\theta_t = \frac{\left((m_1^2-m_3^2+t)^2-4m_1^2 t\right)-2t(s-m_1^2-m_2^2)}{\left[\left((m_1^2-m_3^2+t)-4m_1^2 t\right)\left((m_4^2-m_2^2-t)^2-4m_2^2 t\right)\right]}$$

Frame: $p_1 - p_3 = p_4 - p_2 = (\sqrt{t}, 0, 0, 0)$

(b) <u>Lightlike</u>: $t = 0$ and $m_1 > m_3$, $m_4 > m_2$ (or $m_1 < m_3$, $m_4 < m_2$)

$$\langle p_3\lambda_3 p_4\lambda_4 | T | p_1\lambda_1 p_2\lambda_2 \rangle = \sum_{\mu_1\mu_3} d^{s_3}_{\lambda_3\mu_3}(\Psi_3)\,(-)^{s_3-\mu_3}\, d^{s_1}_{\mu_1\lambda_1}(\Psi_1)\,(-)^{s_1-\lambda_1} \times$$

$$\times \int_0^\infty d\rho^2 \langle \lambda_2\lambda_4 \| T^\rho \| \lambda_1\lambda_3 \rangle\, J_{\mu_1+\lambda_2-\mu_3-\lambda_4}(\rho\xi/\omega)$$

$$\left(\frac{\xi}{\omega}\right)^2 = \frac{4}{(m_1^2-m_3^2)(m_4^2-m_2^2)} \left[ s - m_1^2 \left(1 + \frac{m_4^2-m_2^2}{m_1^2-m_3^2}\right) - m_2^2\left(1 + \frac{m_1^2-m_3^2}{m_4^2-m_2^2}\right)\right]$$

$$\tan\Psi_1 = \frac{1}{m_1}\,\frac{m_1^2-m_3^2}{1-\left(\frac{m_1^2-m_3^2}{2m_1}\right)^2\left(\frac{1}{\omega^2}+\frac{\xi^2}{\omega^2}\right)}\,\frac{\xi}{\omega}\,, \quad \tan\Psi_3 = \frac{1}{m_3}\,\frac{m_1^2-m_3^2}{1-\left(\frac{m_1^2-m_3^2}{2m_3}\right)^2\left(\frac{1}{\omega^2}+\frac{\xi^2}{\omega^2}\right)}\,\frac{\xi}{\omega}$$

**Frame:** $p_1 - p_3 = p_4 - p_2 = (\omega, 0, 0, \omega)$

(c) <u>Null</u>: $t = 0$ and $m_1 = m_3 = M$, $m_4 = m_2 = m$

$$\langle p_3\lambda_3 p_4\lambda_4 | T | p_1\lambda_1 p_2\lambda_2 \rangle = \sum_{JJ'} \sum_{j_0=-J_m}^{J_m} \int_0^{i\infty} d\sigma\,(j_0^2-\sigma^2)\,\langle J' \| T^{j_0\sigma} \| J \rangle \times$$

$$\times \langle s_4\lambda_4 | s_2\lambda_2 J'\lambda \rangle\, d^{j_0\sigma}_{J'\lambda J}(\omega)\, \langle J\lambda\, s_3\lambda_3 | s_1\lambda_1 \rangle$$

$$|s_1-s_3| \leq J \leq s_1+s_3 \qquad \text{and} \qquad |s_4-s_2| \leq J' \leq s_4+s_2$$

$$J_m = \min(J, J')$$

$$\cosh\omega = \frac{s - M^2 - m^2}{2Mm}$$

**Frame:** $p_1 + p_2 = p_3 + p_4 = (\sqrt{s}, 0, 0, 0)$

(d) Spacelike: $t < 0$

$$\langle p_3\lambda_3 p_4\lambda_4 | T | p_1\lambda_1 p_2\lambda_2 \rangle = \sum_{\mu_1\mu_3} d^{s_3}_{\lambda_3\mu_3}(-\Theta_3)(-)^{s_3-\mu_3}\, d^{s_1}_{\mu_1\lambda_1}(-\Theta_1)(-)^{s_1-\lambda_2} \times$$

$$\times \Big[ \sum_{-M\le j\le -1} (2j+1) \langle \lambda_2\lambda_4 \| T^{j\pm}(t) \| \lambda_1\lambda_3 \rangle\, d^{j\pm}_{\lambda_4-\lambda_2,\, \mu_1-\mu_3}(\beta) +$$

$$+ \int_{-\frac{1}{2}}^{-\frac{1}{2}+i\infty} dj\, \frac{2j+1}{\tan\pi j} \langle \lambda_2\lambda_4 \| T^{j}(t) \| \lambda_1\lambda_3 \rangle\, d^{j}_{\lambda_1-\lambda_2,\, \mu_1-\mu_3}(\beta) \Big]$$

$$M = \min\left( |\mu_1-\mu_3| , |\lambda_4-\lambda_2| \right)$$

$$ch\beta = \frac{(m_1^2-m_3^2+t)(m_4^2-m_2^2-t) - 2t(s-m_1^2-m_2^2)}{\left[ \left( (m_1^2-m_3^2+t)^2 - 4m_1^2 t \right) \left( (m_4^2-m_2^2-t)^2 - 4m_2^2 t \right) \right]^{\frac{1}{2}}}$$

$$\tan\Theta_1 = \frac{2m_1\sqrt{-t}}{m_3^2-m_1^2-t}\, th\beta \quad , \quad \tan\Theta_3 = \frac{2m_3\sqrt{-t}}{m_3^2-m_1^2+t}\, th\beta$$

**Frame:** $p_1 - p_3 = p_4 - p_2 = (0, 0, 0, \sqrt{-t})$.

(In this decomposition the discrete representation $d^{j+}$ appears when both $\lambda_1 - \lambda_2$ and $\mu_1 - \mu_3$ are positive while $d^{j-}$ appears when both $\lambda_1 - \lambda_2$ and $\mu_1 - \mu_3$ are negative. These are the only situations in which the discrete representations can contribute.)

It remains only to transform all of these expressions (a)...(d) to a common frame of reference following normal convention, for example, to the centre-of-mass frame.