INFINITE MULTIPLETS AND LOCAL FIELDS

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INTRODUCTION

All familiar examples of relativistic local field theories deal with fields $\psi_\sigma(x)$, $\sigma = 1, 2, \ldots, d$ that transform among themselves according to finite-dimensional representations of the (homogeneous) Lorentz group. This paper explores the novel features -- and there are several -- that arise when infinite-dimensional representations are used instead.

Part I deals with the concept of local interactions between unquantized fields. The main result was already known: a formally local interaction between the fields corresponds to a non-local interaction between particles. The structure is, however, totally unlike those considered in earlier attempts to construct non-local field theories. In fact, all the results seem to be eminently physical, and preliminary attempts at quantization via Feynman rules give additional encouragement.

Part II attempts to set up a theory of local fields. Integral spin fields may be quantized with canonical commutation rules, apparently with no difficulty whatever. The S-matrix may be calculated by means of the usual Feynman rules and local observables are defined. For half-odd-integral spin fields we consider several alternatives. The most conventional approach (Section II, 2) is a direct generalization of the usual quantization procedure for the Dirac field. This leads to the theory first invented by Gel'fand and Yaglom. It has all the requisite locality properties; but it suffers from a rather unphysical mass spectrum. We have tried to quantize the half-odd-integral spin field while keeping all the masses equal, but we have been unable to reach a final conclusion. Briefly, the situation may be summarized as follows: (i) If the relative intrinsic parity of a particle-antiparticle pair is even (odd), then one must introduce Bose (Fermi) statistics. (ii) Fermi quantization of half-odd-integral spin fields (with equal masses) gives a theory that, at first sight, is non-causal. However, the difficult question of the causality of this theory is not settled in this paper.
PART I
LOCAL INTERACTIONS

1. General definition of local interactions.

The only really successful field theory, electrodynamics, is defined by the postulate that the interaction Lagrangian is

$$\bar{\psi}(x) \gamma_{\mu} \psi(x) A^{\mu}(x)$$

where $A^{\mu}(x)$ is the electromagnetic field and $\psi(x)$ is the 4-component Dirac electron-positron field. This postulate is quite mysterious, and its great success does not help us understand:

(i) Why is the Dirac field $\psi(x)$ distinguished from all other fields that are capable of describing free electron states?

(ii) What is the real reason for the absence of derivatives in the interaction Lagrangian?

Let $\psi_\alpha(x)$ denote the components of $\psi(x)$. The generators of Poincaré transformations are

$$P_\mu, \quad and \quad L_{\mu\nu} = L^X_{\mu\nu} + \Lambda_{\mu\nu}$$

where $L^X_{\mu\nu}$ is the orbital part of $L_{\mu\nu}$,

$$L^X_{\alpha\beta} = \frac{i}{2} \left( x_\alpha \frac{\partial}{\partial x_\beta} - x_\beta \frac{\partial}{\partial x_\alpha} \right)$$

and the spin part $s_{\mu\nu}$ commutes with $L^X_{\mu\nu}$. This representation of the Poincaré group is reducible, and may be decomposed into a direct sum of two irreducible representations. The decomposition is, however, non-local, and the irreducible parts do not have the property that $L^X_{\mu\nu}$ commute with $s_{\mu\nu}$. In fact, the Dirac field $\psi(x)$ transforms according to the simplest representation of the Poincaré group that includes half-integral spin and has this property. Similarly, the electromagnetic field $A^{\mu}(x)$ transforms according to the simplest representation of the Poincaré group that includes spin-1 and has this property (that $L^X_{\mu\nu}$ commute with $s_{\mu\nu}$).
By virtue of the fact that $L^\nu_{\mu}$ commute with $s^\nu_{\mu}$, the $\psi_\nu(x)$ and the $A^\mu(x)$ provide bases for representations of a group that is larger than the Poincaré group, namely the group

$$G = P^x \otimes S$$

where $P^x$ is the "orbital Poincaré group", whose generators are $P_{\mu}$ and $L^x_{\mu\nu}$, and $S$ is the group $SL(2, C)$ generated by $s^\nu_{\mu}$. We answer our two questions as follows:

(i) The Dirac field is the basis for an irreducible representation of $G$.

(ii) The interaction Lagrangian is invariant under the transformations of $G$.

The interaction Lagrangian has another important property; it is local. More precisely, it is a "point-like" interaction: the Lagrangian density is a product of fields, all evaluated at the same point. This locality of the interaction is, of course, not preserved by transformation to another basis unless the transformation itself is local. For example, the Dirac interaction is no longer a point interaction after a Foldy-Wouthuysen transformation has been carried out. Since, however, we have just pointed out the distinguished role of the Dirac representation, it seems reasonable to give a general definition of local interactions as follows:

**Definition.** An interaction between fields is called local if the interaction Lagrangian is (i) point-like and (ii) invariant under a group $G = P^x \otimes S$, where $S$ is the spin group $SL(2, C)$.

According to this definition, local interactions can exist only between fields that allow representations of $G$; that is, sets $\psi_\nu(x)$ of fields that transform among themselves according to some representation $D(S)$ of $S$. In conventional local field theories $D(S)$ is always taken to be a finite representation of $S$. In this paper we study field theories based on unitary, infinite-dimensional representations.
2. The group $G$ and some of its representations.

The above considerations lead to a group that is already known to be relevant in several branches of physics. Thus it has long been recognized that the role of spin in the general theory of relativity can be understood only if the spin group $S$ (group of similarity transformations) is divorced from the group $P^x$ of co-ordinate transformations. Spin independence, as introduced in the form of $SU(6)$ invariance of strong interactions, also leads to the consideration of an invariance group $G = P^x \otimes S$, where $S$ is $SL(6, \mathbb{C})$ instead of $SL(2, \mathbb{C})$.

The group $P^x$ is isomorphic to the Poincaré group, and any unitary representation of this group may be used for $P^x$. However, we shall take our clue from the above physical applications, and take $D(P^x)$ to be a representation with real mass and zero spin. The basis is then a set of fields $\psi_\sigma(x)$, such that the operators of $P^x$ act only on the argument, and the group $S$ acts only on the index $\sigma$.

A description of the representations of $SL(2, \mathbb{C})$ that is particularly well suited to the present task has been given elsewhere. Here we only recall that unitary infinite-dimensional representations can be obtained by analytic continuation from non-unitary finite-dimensional representations. In this part of our paper we consider representations $D(N)$ realized on tensors with $N$ four-vector indices. (Other representations are introduced in Section I, 7):

$$\mathcal{L}_{\mu_1 \ldots \mu_N} \psi_{\mu_1 \ldots \mu_N} = i \sum_{\lambda = 1}^{N} \left( q_{\mu_1 \mu_2} \psi_{\mu_3 \ldots \mu_N} - q_{\mu_2 \mu_3} \psi_{\mu_1 \ldots \mu_N} \right) \tag{I-1}$$

The calculations are performed for integer, but otherwise arbitrary values of $N$, and the results are continued analytically to complex values of $N$. The representation is irreducible if the tensor is symmetric and traceless,

$$q^{\mu_1 \mu_2} \psi_{\mu_1 \ldots \mu_N} = 0$$

and it is unitary if

$$(N + 1)^2 < 1 \tag{I-2}$$
In order to discover the physical content of the tensor field (particularly for complex \( N! \)), it is necessary to carry out the reduction according to the Poincaré group. In the rest system we have \( L_{ij} = s_{ij} \), \( i, j = 1, 2, 3 \); therefore, the \( s_{ij} \) generate ordinary rotations of the rest system, and the spin \( j \) of a state at rest is given by \( \frac{1}{2} s_{ij} s_{ij} = j(j+1) \).

This may be written covariantly as follows:

\[
\begin{align*}
  j(j+1) \rho^2 &= \omega^l \\
  \omega^p &= \frac{1}{2} \mathcal{P}^p_{\nu} \mathcal{P}_{\nu} c^{\nu^p \nu^p}
\end{align*}
\]

The \( \omega^p \) generate the little group, which is a maximal compact subgroup of \( S \). The reduction of \( D(N) \) according to this subgroup is carried out exactly as if \( N \) were a positive integer. Thus, for \( N = 1 \),

\[
\psi_{\mu} = \left( \frac{\rho^2}{2} \right)^{\frac{t-N}{2}} \rho^{\mu, j=1} \ldots \rho^{\mu, \nu} \psi_{\nu}
\]

where the first part has spin 1 and the second part has spin zero.

For general \( N \), define

\[
\psi_{\mu, \ldots, \nu} = \left( \frac{\rho^2}{2} \right)^{\frac{t-N}{2}} \rho^\mu_{j=1} \ldots \rho^\nu_{j=1} \psi_{\mu, \ldots, \nu}
\]

and let \( \tilde{\psi}_{\mu, \ldots, \nu} \) denote the transverse projection of \( \psi_{\mu, \ldots, \nu} \). Then it is straightforward to show that

\[
\tilde{\psi}_{\mu_1 \ldots \mu_j} = \Theta_{\mu_1} \ldots \Theta_{\mu_j} \sum_{t=0}^{1+j} \frac{(j+t-1)!!}{(2j-1)!! (j-t)!! t!} \psi_{\nu_1 \ldots \nu_t} \gamma_{\nu_{t+1} \ldots \nu_{2t+1}} \gamma_{\nu_{2t+2} \ldots \nu_{2t+j}}
\]

\[
\psi_{\mu_1 \ldots \mu_N} = \sum_{j=0}^{N} \tilde{\psi}_{\mu_1 \ldots \mu_j} \sum_{t=0}^{j} \frac{a(N, j, t) (2N+j-t)!!}{(N+j+1)!} \gamma_{\nu_1 \ldots \nu_{2t+1}} \gamma_{\nu_{2t+2} \ldots \nu_{2t+j}}
\]

\[
\times \mathcal{P}^{\mu_{i=1} \mu_{i=2}} \mathcal{P}^{\mu_{i=1} \nu_{i=2}} \ldots \mathcal{P}^{\mu_{i=1} \nu_{i=2}} (\rho^2)^{\frac{t-N}{2}}
\]
\[ \psi_{\mu_1\cdots \mu_N} = S \sum \frac{a(N, j, t)}{(\ell + j + 1)!} \times \Theta_{\mu_{i1, i2, \cdots, i\ell}} \cdots \Theta_{\mu_{j1, j2, \cdots, j\ell}} \cdots \Theta_{\mu_N} (\ell^2)^{(t-N)/2} \]  

where \( S \) stands for symmetrization in the indices and

\[ \Theta_{\mu_{\nu}} = q_{\mu_{\nu}} - q_{\nu_{\mu}}, \quad \Theta_{\mu_{\nu}'} = q^{\nu_{\mu}} \Theta_{\mu_{\nu}} \]

\[ a(N, j, t) = \frac{(2j+1)!! \cdot N!}{j!! (t-j)!! (N-t)!} \left((-1)^{(t-j)/2}\right) \]

The transformation properties of \( \tilde{\psi}_{\mu_1, \cdots \mu_j} \) under the Poincaré group are those of an irreducible unitary representation with spin \( j \). The operators \( s_{\mu_{\nu}} \) of \( S \), however, have matrix elements between different \( j \). The effect of \( s_{\mu_{\nu}} \) on \( \tilde{\psi}_{\mu_1, \cdots \mu_j} \) may be calculated from (I-1) and (I-5). In the special case \( \vec{p} = 0 \) the result takes the simple form

\[ \tilde{\psi}_{\mu_1, \cdots \mu_j} = \sum_{k=1}^{j} \left( \delta_{k\mu_r} \tilde{\psi}_{\mu_{r+1} \cdots \mu_j} - \delta_{k\mu_r} \tilde{\psi}_{\mu_1, \cdots, \mu_j} \right) \]

\[ \delta_{\ell \mu_{\nu}} \tilde{\psi}_{\mu_1, \cdots \mu_j} = \left( j - N \right) \tilde{\psi}_{\mu_1, \cdots \mu_j} \]

\[ -\frac{i(N+j+1)}{2j+1} S \left( \delta_{\ell \mu_r} \tilde{\psi}_{\mu_{r+1} \cdots \mu_j} - \delta_{\ell \mu_r} \tilde{\psi}_{\mu_1, \cdots, \mu_j} \right) \]  

Having obtained this result, which is obviously meaningful for any complex value of the parameter \( N \), we could forget about \( \psi_{\mu_1, \cdots \mu_N} \) and avoid manipulations of tensors with complex numbers of indices. This we shall not do, however, as we find the direct use of the tensor method extremely convenient.

In order to construct invariants it is necessary to introduce the contragredient tensor \( \tilde{\psi}_{\mu_1, \cdots \mu_N} \), and the spin projections \( \tilde{\psi}_{\mu_1, \cdots \mu_j} \). This tensor transforms equivalently to \((\psi_{\mu_1, \cdots \mu_N})^*\); comparing the respective transformation laws we find that the equivalence transformation, in the frame \( \vec{p} = 0 \), is
The invariant
\[ \bar{\psi}^{\mu_1 \cdots \mu_i} \psi^{\nu_1 \cdots \nu_i} = (-1)^i \frac{(i-1-N^\ast)!}{(i-1-N)!} \left( \bar{\psi}^{\nu_1 \cdots \nu_i} \right)^\ast \] (I-12)

The representation \( D(N) \) is unitary if and only if all the coefficients are positive; that is, if and only if \( (N+1)^2 < 1 \).


It is easy to write down a divergence-free vector,
\[ \frac{\partial}{\partial x^\mu} \bar{\psi}(x) \delta^\mu_\nu \psi(x) \] (I-16)

that can be used to introduce the electromagnetic interaction, but this does not constitute a theory. First, one needs a prescription for extrapolating \( \psi(p) \) off the mass shell \( p^2 = m^2 \). Secondly, it is known that, when higher spins are involved, a simple electromagnetic interaction sometimes leads to a theory with internal contradictions. Thirdly, we are interested in the possibility of reducing the symmetry by introducing a perturbation on the mass. Invariance under the Poincaré group requires that the eigenstates of the mass operator must coincide with the eigenstates of the spin, and since the projection operators associated with the latter depend in a non-linear fashion on the momentum, it is necessary to take great care not to destroy gauge invariance.
We solve this dilemma by proceeding as is conventional for higher spin theories. That is, we interpret the condition of irreducibility of $D(P^x)$

$$\left( \partial^2 - m^2 \right) \psi_{\mu_1 \ldots \mu_n}(x) = 0 \quad \text{(I-17)}$$

as a differential equation in configuration space, and take as our dynamical postulate the result of adding interaction terms, for example

$$\left[ (\partial_\mu - i e A_\mu)^2 + cg \sum_{\nu=1}^N \delta_{\mu\nu} + m^2 \right] \psi_{\mu_1 \ldots \mu_n}(x) = 0 \quad \text{(I-18)}$$

Suppose, for the moment, that $N$ is a positive integer. Then there exists a well-known and straightforward procedure for constructing a Lagrangian from which (I-18) can be derived, and one may define the usual observables such as the energy density. Asymptotically, these observables may be expressed in terms of the free physical fields $\tilde{\psi}_{\mu_1 \ldots \mu_j}$, and then one finds of course, that the sign of the energy density is not definite, but alternates with $j$. It is therefore necessary, in the case of positive integer $N$, to impose subsidiary conditions: $\tilde{\psi}_{\mu_1 \ldots \mu_j}(p) = 0$ for all odd values of $j$, or for all even values of $j$. Naturally this complicates the theory tremendously, and utterly destroys the invariance under the group $G$, though invariance under the Poincaré group is retained.

The formal expressions for the Lagrangian, and for the observables, and especially the asymptotic expressions for the latter in terms of physical particle fields, make sense for complex values of $N$ as well. The indefiniteness of the energy that is encountered for positive integer $N$ is directly associated with the indefiniteness of the invariant (I-14) for non-unitary representations. For unitary representations, that is, for representations of $S$ with $(N + 1)^2 < 1$, the energy density is positive definite. Therefore, theories based on unitary, infinite-dimensional representations are much simpler than the conventional finite-component field theories: there is no need to impose subsidiary conditions, and it is possible to retain the invariance under the whole group $G$.

The theory that we have developed is, in the absence of interactions, equivalent to a direct sum of a countable number of conventional theories.
each conventional theory describing a field with fixed spin $j$, and all having exactly the same mass. However, the equality of the masses is not an a priori requirement for the existence of local interactions; the equations of motion do not have to be invariant with respect to the group $G$.

All the essential features of the above treatment of electromagnetic interactions will be preserved if two essential properties of (1-17) are retained; namely, that of invariance under the Poincaré group, and that of being a second order differential equation in configuration space. We therefore replace (1-17) by

$$
(\Gamma^{\mu\nu} p_{\mu} p_{\nu} - m^2) \psi_{\mu_1...\mu_n}(x) = 0
$$

where $m^2$ is a constant and $\Gamma^{\mu\nu}$ is a constant tensor. It turns out that the most general equation of this form may be rewritten as follows:

$$
(\rho^2 + \alpha \omega^2 - m^2) \psi_{\mu_1...\mu_n} = 0
$$

That is, the most general constant tensor $\Gamma^{\mu\nu}$ is

$$
\Gamma^{\mu\nu} = g^{\mu\nu} + \alpha \left( \frac{1}{2} g^{\mu\nu} \delta_{\lambda\rho} - \delta^{\mu^2} \delta^{\nu^2} q q_{\lambda\rho} \right)
$$

Reduction of $\psi_{\mu_1...\mu_n}$ according to the Poincaré group diagonalizes the wave operator:

$$
[\rho^2 (1 + \alpha j (j + 1)) - m^2] \psi_{\mu_1...\mu_n} = 0
$$

Hence the mass spectrum is essentially the same as that of Gel'fand and Yaglom:

$$
m^2_{\mu} = \frac{m^2}{1 + \alpha j (j + 1)}
$$

This is the only discouraging result; any attempt at splitting the mass degeneracy while retaining a second order field equation leads to a mass spectrum that, at best, requires further study. [It is not impossible to learn to appreciate this type of mass spectrum. If $\alpha$ is small and $< 0$, then it resembles that of rotational bands for small $j$. If $\alpha = \alpha_1 + i \alpha_2$, $\alpha_1 < 0$, $\alpha_2 > 0$, then the inverse of the wave operator has many of the...
physical properties of a two-particle propagator, including a pole on the
real s-axis, resonance poles in the lower half s-plane, and an accumulation
of poles at \( s = 0 \). A dramatic and unusual feature of such a theory is that
"left-hand poles" are smeared out and become left-hand cuts.]

4. Interactions with an external source.

We now consider the Lagrangian

\[
\mathcal{L} = \int d^4 x \left\{ \frac{\bar{\psi}^\mu_1 \ldots \mu_n(x)}{\gamma^2 - m^2 - \not{q} \mathcal{A}(x)} \gamma^\nu \psi_{\mu_1 \ldots \mu_n}(x) + \mathcal{L}_A(x) \right\} \quad (I-23)
\]

where \( \mathcal{A}(x) \) is a scalar field and \( \mathcal{L}_A(x) \) is the Lagrangian density for the
non-interacting \( \mathcal{A}(x) \) field. (There is no difficulty in replacing \( \mathcal{A}(x) \) by
the electromagnetic field, interacting as in Eq. (I-18).) The vertex function
in momentum space, in the first Born approximation, is

\[
\frac{\bar{\psi}^\mu_1 \ldots \mu_n(\mathbf{p}') \gamma^{\nu} \psi_{\mu_1 \ldots \mu_n}(\mathbf{q}) \mathcal{A}(\mathbf{q}' - \mathbf{q})}{(I-24)}
\]

Consider the vertex between \( \bar{\psi}(p), \bar{\psi}(p') \) and \( \mathcal{A}(p'-p) \); that is, the lowest
order interaction between the spin-zero component of \( \psi_{\mu_1 \ldots \mu_n} \) with the
external field. The easiest way to calculate this is to expand \( \bar{\psi}^\mu \ldots \mu_n \)
and \( \psi_{\mu_1 \ldots \mu_n} \) as in Eqs. (I-7) and (I-8), respectively, retaining in each
expansion the \( \mathbf{j} = 0 \) term only. Inserting this into (I-24), and remember-
ing that \( \psi_{\mu_1 \ldots \mu_n} \) is traceless, one finds immediately

\[
\frac{\bar{\psi}^\mu \ldots \mu_n(p') \gamma^\nu \psi_{\mu_1 \ldots \mu_n}(q)}{\mathcal{L}_A(p'-q)} \rightarrow \bar{\psi}^\mu(p') \gamma^\nu(q) \sum_{\ell=0}^{\infty} \frac{x^\ell}{\ell! \mathcal{A}^\ell} \left( \frac{\mathcal{L}_A}{m_{\mu_1} \cdots m_{\mu_n}} \right)^N \frac{1}{N!}
\]

where
\[
m^2 = p^2, \quad m'^2 = q'^2
\]
\[
x = (p \cdot p')^{-1} \left[ (q \cdot p')^2 - p^2 q'^2 \right]^{\frac{1}{2}}
\]
Summing the series we find that our vertex function is

\[ K(t) = \frac{1}{2(N+1)} \left( \frac{p' p}{m' m} \right)^N \frac{\left[ (1+x)^{N+1} - (1-x)^{N+1} \right]}{x} \]

where the factor \( K(t) \) is

\[ K(t) = \frac{m^2}{\sqrt{\Delta}} \frac{2}{N+1} \sinh \left( \frac{N+1}{2} \ln \frac{m^2 + m'^2 - t + \sqrt{\Delta}}{m^2 + m'^2 - t - \sqrt{\Delta}} \right) \]

\[ t = (p' - p)^2 \]

\[ \Delta = m^4 + m'^4 + t^2 - 2m^2 m'^2 - 2t m^2 - 2t m'^2 \]

For positive integer values of \( N \) this function is a polynomial in \( t \).

When \( N \) is in the range \( (N + 1)^2 < 1 \) of unitary representations, then \( K(t) \) has a number of attractive properties:

(i) \( K(t) \) is real, and independent of the sign of \( N + 1 \). [In fact, representations that differ only by the sign of \( N + 1 \) are equivalent.]

(ii) For very large momentum transfers, \( t \to -\infty \), the factor decreases like \( (-t)^{|N+1|} \).

(iii) On the principal sheet of the logarithm, \( K(t) \) is analytic in the entire \( t \) plane. The singular points of the logarithm lie at \( t = \infty \). On all other sheets of the logarithm the function has a singularity at the point \( \Delta = 0 \); this is a well-known non-Landauian (i.e., kinematic) singularity of vertex functions in local field theory, or in S-matrix theory.

We want to emphasize as strongly as possible that \( K(t) \) is not a form factor of the usual dynamical type. In fact, if \( N \) were a positive integer, then \( K(t) \) would easily be recognized as one of the usual kinematical factors that always enter the description of invariant amplitudes in field theory. Since this is often misunderstood we shall calculate one of the dynamical contributions to the form factor. It goes
without saying that $K(t)$ has no relevance when continued analytically to $t > (m' - m)^2$; we shall return to this point below.

Interactions between three infinite multiplets are susceptible to a similar treatment.\(^\text{11}\)


A most interesting question is whether the kinematical factors calculated above tend to reduce the number of divergences as compared with conventional local field theory. Since the scalar field $A(x)$ is quite conventional, it may be quantized, and we may calculate the radiative correction of Fig. 1, where

![Fig. 1](image)

the broken lines represent particles of the field $A(x)$. If these particles have mass $\mu$, then the amplitude is proportional to

$$
\overline{\psi}^{\nu_1 \ldots \nu_N} (q') \int d^4k \frac{i}{k^2 - \mu^2} \frac{1}{(q + k)^2 - m^2} \frac{1}{(q' + k)^2 - m^2} \psi^{\nu_1 \ldots \nu_N} (q) \tag{1-27}
$$

Here we have assumed complete mass degeneracy for the particles described by $\psi_{\mu_1 \ldots \mu_N} (x)$, so that the propagator for that field is simply

$$
\frac{1}{q^2 - m^2} \int \frac{\nu_1 \ldots \nu_N}{\mu_1 \ldots \mu_N} \tag{1-28}
$$

where $T$ is a constant invariant tensor.\(^\text{12}\)
The integrand in (1-27) exhibits exactly the same k-dependence as the usual theory of three interacting local scalar fields. Hence the convergence is not improved relative to that (renormalizable) theory. However, as compared with the conventional non-renormalizable theories of particles with higher spin there is a vast improvement. In (1-27) the degree of divergence is independent of the spin; we may project out the spin j states from both external lines and obtain an expression for the radiative correction to the vertex, finding the same degree of divergence as for the case of spin zero.

If one projects out the spin zero state from both external factors in (1-27), then the kinematical factor $K(t)$ appears as an overall factor. If one carries out the renormalization programme to the lowest order, and add the correction to the bare vertex (I-25), then the result has the form

$$K(t) \, \widehat{\psi}^{j}(\varphi') \, \widehat{\psi}(\varphi) \, A(\varphi' - \varphi) \, D(t)$$

where $D(t) = 1 + \text{radiative corrections}$. In fact, this expression remains valid to all orders, and the function $D(t)$ is the vertex function that would be obtained in the conventional theory in which $\psi_{\mu, \ldots, \mu_{n}}(x)$ is replaced by a scalar field. There follows that all the dynamical singularities, whose existence is required by the unitarity of the S matrix, are contained in the dynamical form factor $D(t)$ ; the factor $K(t)$ is purely kinematical.

We have not yet discussed the quantization of $\psi_{\mu, \ldots, \mu_{n}}(x)$. Nevertheless, it may be permissible to use naive Feynman rules to calculate the annihilation process in which two particles of this field are annihilated and one $A(x)$ particle is created. For this we assume that the field $(\psi_{\mu, \ldots, \mu_{n}}(x))^{*}$ describes the anti-particles. This tensor transforms equivalently to $\bar{\psi}^{\mu, \ldots, \mu_{n}}$, and we therefore denote the anti-particle field by $\psi_{\mu, \ldots, \mu_{n}}^{\nu, \ldots, \nu_{n}}$. Carrying out exactly the same calculations as before, we obtain the following expression for the annihilation of two spin zero particles:

$$K'(t) \, \widehat{\psi}_{c}^{\mu}(\varphi') \, \widehat{\psi}^{\nu}(\varphi) \, A(\varphi' + \varphi) \, D(t)$$

Here $\varphi^{\nu}$ is the physical four-momentum of the anti-particle, $t = (\varphi + \varphi')^{2}$ as before, and $K'$ and $K$ are the same function of physical momenta:

-13-
\[ K'(t) \bigg|_{t = (\rho + \rho')^2} = K(t) \bigg|_{t = (\rho - \rho')^2} \]

which is the same as

\[ K'(t) = K(4m^2 - t) \quad (I-29) \]

Thus we see that the singularity of the function \( K(t) \) at the point \( t = 4m^2 \) is not directly relevant for the annihilation process, and that \( K'(t) \), in the relevant region of positive \( t \geq 4m^2 \), has the same desirable properties as has \( K(t) \) in the relevant region of negative \( t \). Thus \( K(t) \) has all the marks of a usual kinematical factor.

Even if we restrict ourselves to the interactions of the spin-zero component of \( \psi_{\mu_i \ldots \mu_n}(x) \), it is clear that the present theory is a radical departure from the usual local field theory. Experimentally, the measured form factor is the product \( K(t)D(t) \), and this quantity is \textbf{not} an analytic function of \( t \).

One may introduce the effect of a breaking of the mass degeneracy into the above calculations by replacing the propagator \( (I-28) \) by the inverse of the wave operator \( (I-19) \). We expect that this will introduce the parameter \( \alpha \) into the results. There is no reason to expect that our conclusion will be radically different for small values of \( \alpha \), since the symmetry limit \( \alpha \to 0 \) is a genuine physical idealization.

6. \textit{Vacuum fluctuations.}

It has been pointed out that, even if a field theory based on unitary representations is less singular than the usual field theories as far as momentum space integrations are concerned, a new type of infinity might arise in Feynman diagrams containing loops.\textsuperscript{13} For if a line running round a closed loop represents the particles of an infinite multiplet, and if each particle (i.e., each value of \( j \)) contributes a finite amount after renormalization, then the fact that an infinite number of particles contribute may
create a difficulty. Actually it is meaningless to separate out the individual contribution of each particle, for the reduction of the field to its physical particle components has only been defined (and need only be defined) for real momentum on the mass shell. However, we may calculate the total contribution without difficulty.

Consider the simplest case: complete mass degeneracy, and the self-energy diagram of the scalar field \( A(x) \), Fig. 2. The corresponding amplitude is

\[
\int d^4k \frac{\gamma_1 \cdots \gamma_n}{k^2 - m^2} \frac{\gamma_1' \cdots \gamma_n'}{(p+k)^2 - m^2} = (N+1)^2 \int d^4k \frac{1}{k^2 - m^2} \frac{1}{(p+k)^2 - m^2} \quad (I-30)
\]

which is not only a finite multiple of the usual result for all \( N \), but it actually vanishes for one value of \( N \). (And a very interesting value of \( N \) it is too!)

This result obviously holds for any closed loop diagram, for it is true for every positive integer value of \( N \), and therefore, for almost all \( N \), (since we are always dealing with algebraic functions or their extensions to \( \Gamma \)-functions.) The result is also unaffected by mass splittings; instead of the propagator (I-28) we could have used the inverse of the wave operator (I-20), although the calculations for this case are much harder.
7. **Half integral spin fields.**

A four-vector index may be replaced by a pair of spinor indices, one dotted and one undotted, so \( \psi_{\mu_1 \ldots \mu_N} \) may be replaced by

\[
\frac{\psi}{A_1 \ldots A_N}.
\]

The symmetry and the tracelessness of \( \psi_{\mu_1 \ldots \mu_N} \) are equivalent to the symmetry of this spinor in upper and lower indices separately. Every irreducible representation \( D(N, k) \) of \( SL(2, \mathbb{C}) \) may be realized on spinors of the following more general kind,

\[
\hat{\psi}_{A_1 \ldots A_{N+k}}, \quad k = \text{positive integer}.
\]

The reduction with respect to the little group is given by

\[
\hat{\psi}_{A_1 \ldots A_{N+k}}(\varphi) = \sum_{\ell=0}^{N+k} \frac{(2\ell+k+1)! \ N! \ (N+k)!}{\ell! \ (N-k)! \ (N+\ell+k)! \ (N+k+\ell+1)!} \ (\rho^2)^{(\ell-k)/2} \times \nabla \hat{\psi}_{A_1 \ldots A_{\ell+k}}(\varphi) \rho_{A_{\ell+k+1}} \ldots \rho_{A_{N+k}}
\]

where

\[
\hat{\psi}_{A_1 \ldots A_{\ell+k}}(\varphi) = \left[ \text{covariant traceless part of} \right] \ (\rho^2)^{(\ell-k)/2} \rho_{A_{\ell+k+1}} \ldots \rho_{A_{N+k}} \hat{\psi}_{A_1 \ldots A_{N+k}}(\varphi)
\]

describes a particle with spin \( t + \frac{k}{2} \), and "covariant traceless" means that

\[
\rho_{A_1 \ldots A_{\ell+k}}(\varphi) \hat{\psi}_{A_1 \ldots A_{\ell+k}}(\varphi) = 0
\]

When \( k \neq 0 \), the unitarity condition is

\[
N = - \frac{k+2}{2} + i \rho
\]

It is only for \( k = 0 \) that there is a "supplementary series" of unitary re-
presentations, in the range $-2 < N < 0$.

Later we shall need the matrix $\beta$ that establishes the equivalence between the spinors

$$\psi^* \alpha^{* \gamma} \cdots \alpha^{* \gamma_k} = \left( \begin{array}{c} \hat{\alpha}_1 \cdots \hat{\alpha}_N \\ \beta_1 \cdots \beta_N \end{array} \right)$$

and the contragredient

$$\psi^* \beta^{* \gamma} \cdots \beta^{* \gamma_k} = \psi^* \alpha^* \cdots \alpha^{* \gamma_k} \beta_1 \cdots \beta_N$$

This matrix commutes with the rotation subgroup of SL(2, C); it is therefore sufficient to give the diagonal matrix elements of $\beta$ in a basis where the tensors are reduced out with respect to this subgroup. This reduction is given by (1-31) if we take $p = 0$. The details of the calculation have been given elsewhere, the result is

$$\psi^* \alpha^* \cdots \alpha^{* \gamma_k} = (-1)^t \frac{(t - 1 - N)!}{(t - 1)!} \psi^* \alpha^* \cdots \alpha^{* \gamma_k}$$

The matrix elements of $\beta$ are defined as the numbers $\langle j | \beta | j \rangle$ obtained from the quantity $\psi^* \beta \psi$ after projecting out the spin $j$ parts of both $\psi^*$ and $\psi$. From (1-31) and (1-34) it follows easily that

$$\langle j | \beta | j \rangle = C \frac{(2t+k+1)!}{t!(t+k)!} \frac{(t-1-N)!}{(t+N+1)!} \psi^* \alpha^* \cdots \alpha^{* \gamma_k} \psi_1 \cdots \psi_t$$

where $C$ is a constant that is independent of $t$, and $j = t + \frac{k}{2}$. We observe that these matrix elements are positive definite when $N$ is in the range (1-32). In that case we can normalize the tensors $\psi_{\beta_1 \cdots \beta_t}$ so that $\langle j | \beta | j \rangle = 1$, which makes the representation unitary. The representation (1-34) for the matrix $\beta$ may be generalized to the basis defined by the Poincaré reduction for arbitrary momentum. The result is

-17-
The projection operator $T_{\mu \nu \cdots \nu}$ introduced in Section 1.5 in connection with the propagator is very simple in spinor notation:

$$T_{\mu \nu \cdots \nu}^{\lambda \cdots \lambda} \gamma^{\mu} \cdots \gamma^{\nu} = (-1)^{k} \frac{(t-1-N)!!}{(t-1-N)!} \left( \gamma^{\lambda} \right)_{\delta_{\lambda}^{\mu \cdots \nu}} \cdots \left( \gamma^{\lambda} \right)_{\delta_{\lambda}^{\mu \cdots \nu}}$$

$$\cdots \left( \gamma^{\lambda} \right)_{\delta_{\lambda}^{\mu \cdots \nu}} \cdots \left( \gamma^{\lambda} \right)_{\delta_{\lambda}^{\mu \cdots \nu}}$$

(I-35)
1. Quantization of boson fields.

A local field must contain both positive and negative energy components. The Fourier decomposition is written as follows:

$$\varphi(x) = \int \frac{d^3q}{2\pi^3} \left( P(q) e^{i\phi^q} + A^*(q) e^{-i\phi^q} \right)$$  \hspace{1cm} (II-1)

where $P(p)$ is the annihilation operator for a particle with momentum $p$ and positive energy $E_p$, and $A^*(p)$ is the creation operator for an antiparticle with the same energy and momentum. These operators are assumed to satisfy canonical commutation relations, either with commutators or with anti-commutators:

$$[ P(p), P^*(q) ]_z = [ A(p), A^*(q) ]_z = 2 E_p \delta^{(3)} (p - q)$$

all other commutators or anti-commutators being zero.

We recall WEINBERG's argument for excluding the use of anti-commutators for a scalar field. Define

$$\overline{\psi}(x) = \overline{\psi^*}(x) = \int \frac{d^3q}{2\pi^3} \left( P^*(q) e^{-i\phi^q} + A(q) e^{i\phi^q} \right)$$  \hspace{1cm} (II-2)

and suppose that physical observables (currents) have the form

$$J(x) = \overline{\psi}(x) \overline{\psi}(x) + \bar{h} . c. $$  \hspace{1cm} (II-3)

Physical interpretation requires that the currents be local; that is,

$$[ J(x), J(y) ]_z = 0 \quad \text{when} \quad (x - y)^2 < 0 $$

The easiest way to achieve this is to take the fields to be local; that is

$$[ \varphi(x), \overline{\varphi(y)} ]_z = 0 \quad \text{when} \quad (x - y)^2 < 0 $$
Explicit calculation gives
\[ [-\psi(x), \overline{\psi}(y)]_i = \int \frac{d^3q}{2\pi^3} \left( e^{i\phi(x-y)} + e^{-i\phi(x-y)} \right) \]
which is local with the choice of the lower sign only. Hence, one must use commutators for scalar fields.

It has been pointed out\(^1\) that this line of reasoning is applicable to any field that transforms according to a unitary representation of the spin group \(S\). For the indices on the generalized tensors may safely be ignored, and the above calculation may be repeated step by step; the only point that has to be checked is that the interpretation of \(\overline{\psi}(p)\) as the anti-particle of \(\psi(p)\) is consistent. This interpretation requires that \(\overline{\psi}(p)\) transform contragrediently to \(\psi(p)\), while the addition in (II-1) makes sense if \(\psi^*(p)\) transforms like \(\psi(p)\) only. Hence, it is necessary that \(\psi(p)\) be the basis for a representation \(D\) such that \(D^*\) is equivalent to \(D^{-1}\), which is just the unitarity condition. For non-unitary representations it is necessary to modify the procedure, but for unitary representations the above conclusion would seem to be valid. On the other hand, it is known that quantization with either commutators or anti-commutators is possible in the case of some unitary representations,\(^2\) so clearly, the argument is incomplete. The error lies in the assumption that \(\overline{\psi}(x)\) must be defined as in Eq. (II-2).

In any case it is certainly true that any field that transforms according to a unitary representation may be quantized with commutators. We shall select the representations \(D(N,1)\), which describe particles of half odd integral spins, and quantize, first with commutators and afterwards with anti-commutators. We shall limit the discussion to real values of \(N\).

The particle operators that we need are
\[ p_{\alpha_1 \cdots \alpha_{N+1}} \partial_{\frac{\alpha_i}{\alpha}} \]
the complex conjugate
\[ p^*_{\beta_1 \cdots \beta_{N+1}} = \left( p_{\alpha_1 \cdots \alpha_{N+1}} \right)^* \]
and the contragredients $\bar{P}$ and $\bar{P}^*$ defined by means of the constant matrix $\beta$ (Section I, 7), e.g.,

$$
\bar{P}_{\delta_1 \ldots \delta_n} = \bar{P}^*_{\beta_1 \ldots \beta_{n+1}} C_{\epsilon_1 \ldots \epsilon_n} A_{\epsilon_1 \ldots \epsilon_n}
$$

(II-4)

The anti-particle operator

$$
\frac{A_{\epsilon_1 \ldots \epsilon_{n+1}}}{\delta_1 \ldots \delta_n} (\bar{P})
$$

the complex conjugate $A^*$, and the contragredients $\bar{A}$ and $\bar{A}^*$ are related to each other in a similar way. The main virtue of the matrix $\beta$ is that it is positive definite for a unitary representation. The canonical commutation relations are, for example,

$$
\left[p_{\delta_1 \ldots \delta_n}, p_{\beta_1 \ldots \beta_{n+1}}^{\ast} \right] = 2E_0 \delta_{p^2 - \overline{q}^2} \left( \beta_{\delta_1 \ldots \delta_n} \epsilon_1 \ldots \epsilon_{n+1} \right)
$$

(II-5)

For integral $N$ this is quantization with an indefinite metric (Gupta-Bleuler formalism); in that case subsidiary conditions are needed to exclude negative probabilities. For unitary representations the metric — that is, the matrix $\beta$ — is definite. The fields are

$$
\psi_{\epsilon_1 \ldots \epsilon_{n+1}} (\chi) = \int \frac{d^{1+2} \xi}{2E_0} \left( p_{\delta_1 \ldots \delta_n} (\bar{p}) e^{i\xi^\gamma} + A_{\delta_1 \ldots \delta_n} (\bar{p}) \right) e^{-i\xi^\gamma}
$$

(II-6)

and

$$
\bar{\psi}_{\beta_1 \ldots \beta_n} (\chi) = \bar{\psi}_{\delta_1 \ldots \delta_{n+1}} (\chi) \left( \beta_{\epsilon_1 \ldots \epsilon_{n+1}} C_{\epsilon_1 \ldots \epsilon_n} A_{\epsilon_1 \ldots \epsilon_n} \right)
$$

(II-7)

and the commutators

-21-
or
\[
\left[ \psi_{A_1 \cdots A_{N+1}} (x), \overline{\psi}_{\bar{c}_1 \cdots \bar{c}_N} (y) \right] = \beta \delta^{B_1 \cdots B_N} \Delta_{B_1 \cdots B_N} \int \frac{d^4p}{2\pi^4} (e^{ip(x-y)} - e^{-ip(x-y)})
\]

are local. The local currents are constructed from $\psi$ and $\overline{\psi}$, as in (II-3).

2. Quantization of fermion fields.

We recall Weinberg's argument\textsuperscript{13} for excluding the use of commutators for a spin-$\frac{1}{2}$ field. The particle operators that we need are
\[
P^A (\phi^*)
\]
the complex conjugate
\[
P^A (\phi) = (P^A (\phi^*))^*
\]
and the contragredients
\[
\overline{P}^* = \frac{1}{m} (\cdot \cdot \cdot)_{\cdot \cdot \cdot} B \ P^B
\]
\[
P^A = \frac{1}{m} (\cdot \cdot \cdot)_{\cdot \cdot \cdot} A \ P^A
\]

In this case the bases $\overline{P}^*$ and $P^A$ are not equivalent; that is, no constant matrix transforms one basis into the other. Instead we had to use the momentum-dependent operators $\overline{p}^A$ and $p^A$. These are the same matrices that were used in Section I, 7 with the simpler notation $p^A$ and $\overline{p}^A$; the present notation is more explicit and serves better for the subsequent generalizations.
The fields are

$$\psi_A(x) = \int \frac{d^3p}{2E_p} \left( p_A(p) e^{ipx} + A_A(p) e^{-ipx} \right)$$

$$\bar{\psi}_A(x) = (\psi_A(x))^*$$

We also define

$$\bar{\psi}^*_A(x) = \int \frac{d^3p}{2E_p} \left( \bar{p}_A^*(p) e^{ipx} - A^*_A(p) e^{-ipx} \right)$$

$$\bar{\psi}^*_A(x) = (\bar{\psi}^*_A(x))^*$$

The minus sign in (II-12), though mainly a question of convenience, is motivated by two important considerations: (i) It is useful to define

$$\bar{A}_A = \frac{i}{\hbar} (\varphi \sigma)_A A^*_A$$

with a plus sign (and positive energy, $p_0 = +E_p$), and, at the same time, have a local relation between $\psi_A(x)$ and $\bar{\psi}_A^*(x)$:

$$\bar{\psi}^*_A(x) = \frac{i}{\hbar} (\varphi \sigma)_A A^*_A \bar{\psi}^*_B(x)$$

(ii) Under reflection, $P_A \to P_A^*$ and $A_A \to -A^*_A$. The minus sign here is due to the (experimental) fact that fermions and anti-fermions have opposite parities. The minus sign in (II-12) gives the simple transformation $\psi_A(x, t) \to \bar{\psi}_A(-x, t)$ under space reflection.

The currents are

$$\bar{J}(x) = \bar{\psi}^*_A(x) \partial^A \psi_A(x) + \psi_A^*(x) \partial^A \bar{\psi}^*_A(x)$$

In order that $J(x)$ be local we require that $\bar{\psi}^*_A(x)$ be local relative to $\psi_B(x)$; that is

$$[\bar{\psi}^*_A(x), \bar{\psi}^*_B(y)]_z = 0 \quad \text{when} \quad (x-y)^z < 0$$

From the canonical (anti-) commutation relations
or

\[
\left[ \hat{P}_A (\hat{\rho}^\ast), \hat{P}_B (\hat{\sigma}^\ast) \right]_+ = 2 E_\rho \left( \rho \sigma^\ast \right)_A \mathcal{S} (\hat{\rho}^\ast \hat{\sigma}^\ast)
\]  

(II-16)

and the definitions (II-11) and (II-12), we find

\[
\left[ \psi_A (x), \psi_B (y) \right]_+ = m \mathcal{S}_A \int \frac{d^3 \xi}{2E_\xi} \left( e^{-i\xi(x-y)} - e^{-i\xi(x+y)} \right)
\]  

(II-17)

This is local if we use anti-commutators only. Hence, particles with spin 1/2 are fermions.

We now show that this quantization procedure — with anti-commutators — is possible for the unitary representations \( D(N,1) \). Again we limit the discussion to real \( N \).

Although the constant matrix \( \beta \) used in Eq. (II-4) exists, it does not provide the only possible definition of the contragredient operator. We seek four constant matrices \( \Sigma^\mu \), such that \( p \Sigma = p^\mu \Sigma^\mu \) will do the job; that is,

\[
\hat{P}_A^{A_1 \ldots A_{N+1}} = \hat{P}_B^{\hat{A}_1 \ldots \hat{A}_{N+1}} (\rho \Sigma)^{C_1 \ldots C_N A_1 \ldots A_{N+1}} \frac{\delta_x}{\delta_{\hat{A}_1} \ldots \delta_{\hat{A}_{N+1}}} \frac{\delta_x}{\delta_{A_1} \ldots \delta_{A_{N+1}}}
\]  

(II-18)

Such a matrix four vector exists, for we may take

\[
(\rho \Sigma)^{C_1 \ldots C_N A_1 \ldots A_{N+1}} \frac{\delta_x}{\delta_{\hat{A}_1} \ldots \delta_{\hat{A}_{N+1}}} \delta_{\hat{A}_1} \ldots \delta_{\hat{A}_{N+1}} = \int \beta \frac{C_1 \ldots C_N A_1 \ldots A_{N+1}}{\mathcal{S}_A} \frac{\delta_x}{\delta_{\hat{A}_1} \ldots \delta_{\hat{A}_{N+1}}} (\rho \Sigma)^{A_{N+1}}
\]  

(II-19)

where the symmetrizer \( \mathcal{S} \) acts on the indices \( A_1 \ldots A_{N+1} \) and, independently, on \( \hat{A}_1 \ldots \hat{A}_{N+1} \). The matrix \( \beta \) is a constant matrix, its existence, and in fact its uniqueness, is known from our study of the representations \( D(N,0) \). Provided only that \( p \Sigma \) is positive definite, the quantization procedure carried out above — which corresponds, as it were, to the special case \( N = 1 \) — may be repeated step by step for arbitrary real \( N \).
The conclusion is, of course, the same: only Fermi statistics gives us local fields.

To obtain the spectrum of \( \frac{1}{m} p^2 \) it is sufficient to calculate that of \( \Sigma_0 \). From (II-19) and the definition of the matrix \( \beta \) we have

\[
\sum_0 A_{\lambda N}, c, \ldots c_N, \psi^{\dagger}_{\lambda N}, \psi^{\dagger}_{\lambda N}, \psi^{\dagger}_c, \ldots \psi^{\dagger}_c
\]

\[
= \sum_{t=0}^{\infty} \frac{(-1)^{t+N} N! N! (2t+1)!}{t! t! (N-t)! (N+t+1)!} \sum \tilde{\psi}^A_0 \cdots \tilde{\psi}^A_t \cdot \tilde{\psi}^A_{t+1} \cdots \tilde{\psi}^A_N
\]

where the comma preceding \( A_{N+1} \) means that this index is ignored in projecting out the traceless part of \( \tilde{\psi} \). Now we use the formula

\[
\sum \tilde{\psi}^A_0 \cdots \tilde{\psi}^A_t \cdot \tilde{\psi}^A_{t+1} \cdots \tilde{\psi}^A_N
\]

to obtain for the right-hand side of (II-20)

\[
\sum_{t=0}^{\infty} \frac{(-1)^{t+N} N! N! (2t+1)!}{t! t! (N-t)! (N+t+1)!} \sum \tilde{\psi}^A_0 \cdots \tilde{\psi}^A_t \cdot \tilde{\psi}^A_{t+1} \cdots \tilde{\psi}^A_N
\]

Except for the factor \( (t+1)/(N+1) \) this is exactly the same as

\[
\beta_{A_{\lambda N}, c, \ldots c_N} \tilde{\psi}^{\dagger}_{A_{\lambda N}, c, \ldots c_N}
\]

where this \( \beta \) is the matrix that we used in the previous section. Thus

\[
\sum_0 = \frac{j + \frac{1}{2}}{N+1} \beta
\]

which is definite.

It has thus been demonstrated that the field transforming according to the unitary representation \( D(N,1) \) can be quantized either by commutators

\[
[ P(q), \ P^*(q') ] = 2 E \beta \int (q - q')^2
\]
or by anti-commutators

\[
\left[ \mathcal{P}(\vec{q}), \mathcal{P}^* (\vec{q}) \right] = 2E_q \left( \mathcal{P} \Sigma \right) \mathcal{J} (\vec{q} - \vec{q})
\]

(II-23)

That is, both \( \beta \) and \( (pE) \) are positive definite, and both procedures give rise to local fields. There is, nevertheless, something left to be desired in the case of Fermi statistics. The interpretation requires that, in terms of the correctly normalized states, the normalization of the commutator (II-22) or of the anti-commutator (II-23) be the same for all \( j \). In the case of (II-22) this is easy, for the basis that makes \( \beta = 1 \) is precisely the basis in which the operators of \( \text{SL}(2, \mathbb{C}) \) are unitary. In the case of (II-23), normalizing \( \frac{1}{m} pE \) to 1 amounts to taking \( \mathcal{P}^* (\frac{1}{m} pE) \mathcal{P} \) as the inner product instead of \( \mathcal{P} \beta \mathcal{P} \), and this is not invariant under \( S \). Hence, the anti-commutation relations violate the symmetry of the theory under the group \( \mathcal{P} \otimes S \), though the symmetry under the Poincaré group is preserved.

A particularly interesting interpretation has been given by GEL'FAND and YAGLOM. Let us insist that the inner product is the invariant \( \mathcal{P}^* \beta \mathcal{P} \). Then the condition that the matrix elements of

\[
\frac{1}{m^2} p_{ij} \frac{1}{N_{ij}^2} \beta
\]

be equal to unity is a statement about the spectrum of \( p^2 \), namely

\[
\beta^2 \sim \frac{m^2}{(j + \frac{1}{2})^2}
\]

(II-24)

In this case the fields are related by

\[
m \psi^{\mathcal{B} \cdots \mathcal{B}_{N}}_{\mathcal{A} \cdots \mathcal{A}_{N_{ij}}} (x) = (\mathcal{P} \Sigma)_{\mathcal{A} \cdots \mathcal{A}_{N_{ij}}}^{\mathcal{B} \cdots \mathcal{B}_{N}} \psi \mathcal{J} \mathcal{C} \cdots \mathcal{C}_{N} (x)
\]

(II-25)

\[
m \mathcal{P}^{\mathcal{B} \cdots \mathcal{B}_{N}}_{\mathcal{A} \cdots \mathcal{A}_{N_{ij}}} (x) = (\mathcal{P} \Sigma)_{\mathcal{A} \cdots \mathcal{A}_{N_{ij}}}^{\mathcal{B} \cdots \mathcal{B}_{N}} \mathcal{P} \psi \mathcal{J} \mathcal{C} \cdots \mathcal{C}_{N} (x)
\]

which is the Gelfand-Yaglom generalization of the Dirac equation. The spectrum (II-24) is a special case of the spectra studied in Section I, 3.
3. **Fermi statistics without symmetry breaking.**

We have seen that the spin-statistics theorem is invalid when the representation \( D(S) \) of \( SL(2, \mathbb{C}) \) is unitary and infinite dimensional. In particular, we have quantized the theory \( D(N,1) \) with both types of statistics, obtaining close analogues of conventional local field theory in either case. However, in the case of Fermi statistics the anti-commutation relations (II-23) violate the symmetry of the theory with respect to the transformations of the group \( G = P^\infty \otimes SL(2, \mathbb{C}) \). In this section we shall investigate the possibility of Fermi statistics without symmetry breaking.

At first we shall consider a scalar field \( \psi(x) \). The subsequent developments can be repeated for any system of fields that transforms according to a unitary representation of \( SL(2, \mathbb{C}) \), and the infinite dimensionality of the representation plays no essential role here. We are thus, to begin with, trying to quantize a scalar field with anti-commutators, so it is at once clear that the result is not a local field theory in the usual sense. However, the exercise will serve to pinpoint the difficulty.

Let \( P(p) \) and \( A(p) \) be the particle and anti-particle annihilation operators as before, and suppose that

\[
\left[ P(\vec{q}), P(\vec{q}') \right]_+ = \left[ A(\vec{q}), A(\vec{q}') \right]_+ = 2E_\vec{q} \delta(\vec{q} - \vec{q}')
\]

(II-26)

where

\[
\vec{p} = P^\dagger, \quad \vec{A} = A^\dagger
\]

(II-27)

Let

\[
\psi(x) = \int \frac{d^3 \vec{p}}{2E_\vec{p}} \left( P(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + A(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)
\]

(II-28)

The application that we have in mind is, of course, to particles with half-odd-integral spin. We therefore invoke the experimental fact that, in that case, the relative intrinsic parity of a particle-antiparticle pair is negative, and postulate that under space reflection (remember
Eq. (II-27)):

\[ P(\hat{q}^2) \Rightarrow \bar{P}^*(-\hat{q}) \quad \text{but} \quad A(\hat{q}^2) \Rightarrow -\bar{A}^*(-\hat{q}) \]  

(II-29)

This means that the result of a space reflection on \( \psi(x) \) is a new field that we shall call \( \bar{\psi}^*(x) \):

\[-\psi(\hat{x}, t) \rightarrow -\bar{\psi}^*(\hat{x}, t) \]  

(II-30)

Thus, in addition to the fields \( \psi \) and \( \psi^* \), it is necessary to introduce \( \bar{\psi} \) and the conjugate \( \bar{\psi} \).

Because of the minus sign in (II-31) it is clear that the conclusion about the statistics is reversed if we replace \( \psi^* \) by \( \bar{\psi} \). That is,

\[ \left[ \psi(x), \bar{\psi}(y) \right]_+ = \int \frac{d^3q}{2E} \left( e^{i\varphi(x-y)} - e^{-i\varphi(x-y)} \right) \]  

(II-32)

is causal, while the commutator would not be. Thus \( \psi(x) \) is causal relative to \( \bar{\psi}(x) \) and \( \bar{\psi}^*(x) \) is causal relative to \( \psi^*(x) \). However,

\[ \left[ \psi(x), \psi^*(y) \right]_+ = \int \frac{d^3q}{2E} \left( e^{i\varphi(x-y)} + e^{-i\varphi(x-y)} \right) \]  

(II-33)

is not causal. The question is whether this "partial locality" of the fields is sufficient to make the currents local.

In order to construct the correct currents we first remark that the total charge \( Q \) obtained from the charge density \( \rho \sim \psi^* \partial_\nu \psi \) is positive definite, whereas the energy, constructed in the usual way from \( \psi^* \) and \( \psi \), is indefinite. This is, of course, a consequence of using the wrong statistics. But if we define the charge density as \( \bar{\psi} \partial_\nu \bar{\psi} \), and similarly replace \( \psi^*(x) \) by \( \bar{\psi}(x) \) in the expression for the energy density, then the energy becomes definite and the charge indefinite. We take our clue from this, and take the general expression for a physical observable
to be
\[ J(x) = \overline{\Psi}(x) \mathcal{O}_1 \Psi(x) + \psi^*(x) \mathcal{O}_2 \overline{\Psi}^*(x) \] (II-34)

If \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are local operators, then it is obvious that the commutator
\[ \left[ \overline{\Psi}(x) \mathcal{O}_1 \Psi(x), \overline{\Psi}(y) \mathcal{O}_2 \psi^*(y) \right] \]
is local because the anti-commutator (II-32) is local. The trouble arises from the appearance of \( \psi^*(x) \) in (II-34), and this can probably not be avoided.

In all conventional field theories \( \psi(x) \) is local with respect to both \( \overline{\psi}(x) \) and \( \psi^*(x) \). This is possible because there is a local relationship between these fields, so that locality of \( \psi(x) \) with respect to one of them implies locality with respect to the other. In fact, in the case of spin zero fields one takes \( \overline{\psi} = \psi^* \), in the case of higher integral spin there is a constant tensor that relates \( \overline{\psi} \) to \( \psi^* \), in the case of spin \( \frac{1}{2} \) \( \overline{\psi} \) is related to \( \psi^* \) by a first order differential operator. In the model studied here we have, instead,
\[ \overline{\Psi}(x) = \psi^*(x) \in (\mathcal{C}_o) \] (II-35)
\[ \in (\mathcal{C}_o) = \mathcal{P}_o / |\mathcal{P}_o| \]
which is not local.

To summarize: the requirement that the relative intrinsic parity of a particle-antiparticle pair be negative forces us to introduce a pair of fields \( \psi(x) \) and \( \overline{\psi^*}(x) \). Unless we introduce symmetry breaking, in the form of a differential connection between these two fields, it is impossible to construct a local current with requisite physical properties, neither with commutators nor with anti-commutators.

So much for the scalar field. In the case of the representation \( D(N,1) \) we may repeat all of the above by simply attaching the indices as in Section II, 1. Thus Eq. (II-4), \( \overline{\mathcal{F}} = P^* \beta \) where \( \beta \) is a constant matrix, replaces the first of Eqs. (II-27); the commutation relation (II-5) becomes an anti-commutator but is otherwise unchanged. The fields \( \psi \) and \( \overline{\psi} \) are
defined as in (II-6) and (II-7), except for the novelty of the minus sign in the definition of $\tilde{\psi}$, as in Eq. (II-31). All the conclusions are unchanged, in particular we have the non-local equivalent of (II-35):

$$\tilde{\psi}^{A_{\cdots B_{\cdots}}}_{\tilde{\alpha}_{\cdots} \tilde{\beta}_{\cdots}}(x) = \int \frac{d^3y}{(2\pi)^3} \frac{dy}{(2\pi)^3} \cdots \frac{d^{3N}}{(2\pi)^{3N}} \psi^{A_{\cdots B_{\cdots}}}_{\alpha_{\cdots} \beta_{\cdots}}(y) \in (\rho_0)$$

(II-36)

It would thus appear that Fermi statistics without symmetry breaking is inconsistent with locality.

Nevertheless, we are still entitled to ask whether this precludes a causal theory. The interactions that we have called local in the first part of this paper are actually not so local when expressed in terms of the physical fields. Is it not possible that the violation of local commutativity of $J(x)$ that we have found above actually represents a less severe departure from micro-causality in the interaction between the physical fields? This hope is nourished by the following observation: when the non-local relation (II-36) is expressed in terms of the physical field, then it becomes a local equation for each physical field, namely

$$\tilde{\psi}^{A_{\cdots B_{\cdots}}}_{\tilde{\alpha}_{\cdots} \tilde{\beta}_{\cdots}}(x) = (-1)^t \left( \frac{\partial}{\partial \mu} \right)^{\tilde{\alpha}_1} \cdots \left( \frac{\partial}{\partial \mu} \right)^{\tilde{\alpha}_t} \tilde{\psi}^{A_{\cdots B_{\cdots}}}_{\alpha_{\cdots} \beta_{\cdots}}(x)$$

The projected field $\tilde{\psi}^{A_{\cdots B_{\cdots}}}_{\tilde{\alpha}_{\cdots} \tilde{\beta}_{\cdots}}(x)$ is local with respect to $\tilde{\psi}^{A_{\cdots B_{\cdots}}}_{\alpha_{\cdots} \beta_{\cdots}}(x)$ as well as $\tilde{\psi}^{A_{\cdots B_{\cdots}}}_{\tilde{\alpha}_{\cdots} \tilde{\beta}_{\cdots}}(x)$; in fact the anti-commutation relations are the same as those obtained by Weinberg. Each projected physical field is therefore, in the absence of interactions, a conventional field for a spin $t + \frac{1}{2}$ particle. From our analysis it seems that the interesting and attractive features that are found by adding together all the physical Fermi fields into the single generalized tensor field, are obtained at the price of the causality of the $S$-matrix. Further study is certainly needed before this question can be settled. In the least favourable case one has to accept the symmetry-breaking quantization procedure of Section II, 2 for fermions. For bosons we are not aware of any need, from the purely theoretical standpoint, to break the symmetry.
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REFERENCES


5. C. FRONSDAL, "The representations of SL(n, C)", ICTP, Trieste, preprint IC/66/51, (especially Section VII) and earlier work quoted there.


9. The function $K(t)$ is a "spherical function" for the group SL$(2, C)$. See I. M. GEL'FAND and M. A. NAIMARK, "Unitäre Darstellungen der Klassischen Gruppen", Akademie-Verlag, Berlin 1957.
10. Some of these properties have also been noted by B. ZUMINO and J. WESS, and by Y. NAMBU, reference 2, and by W. RÜHL, Proceedings of the Third Coral Gables Conferences p. 59, (published by W. H. Freeman and Co., San Francisco, 1966).


12. The tensor $T^\nu_{\mu...}$ is the projection operator that projects out the traceless part of $\psi_{\mu...}$; thus it is completely determined by three properties: (i) symmetry in upper indices and in lower indices, (ii) $g^{\nu\sigma} T^{\lambda...}_{\mu\nu...} = 0$, and similarly in the upper indices, (iii) $T\psi = \psi$. The exact expression for $T$ is easily written down by inspection of Eq. (I-6). In actual calculations we prefer to use the spinor formalism of Section I, 7; then the expression for $T$ is very simple. See the end of Section I, 7.

13. S. COLEMAN, Private communication.

14. It would appear that it is better, in making quick estimates of numerical magnitudes, to take the number of particles to be $(N+1)^2$ even if $N$ is complex, rather than infinite.

