CONSISTENCY PROBLEMS
IN AN EXTENDED
SCHWINGER-THIRRING-WEEN MODEL

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CONSISTENCY PROBLEMS
IN AN EXTENDED SCHWINGER-THIRRING-WESS MODEL*†

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ABSTRACT

The solution of a Thirring-Wess model describing the interaction of massless fermions with massive vector particles is extended to the case of fermions with a finite mass.

Special care is given to the definition of the conserved current by which the two fields are coupled to each other and all consistency requirements are checked. Also, the contribution to the Green functions is calculated up to the fourth order in coupling constant.
1. INTRODUCTION

A field-theoretical model with one space-one time dimension has been solved exactly by SCHWINGER, THIRRING and WESS \(^1\), \(^2\). This model describes the interaction of massless fermions with "photons" of mass \(\mu\), being coupled to a conserved current.

In the following we want to extend this model to the case of fermions with bare mass \(m_0\). Unfortunately we were not able to obtain an exact solution of this extended model because the boson and fermion Green functions are too complicated.

In the first part we discuss the difficulties connected with the definition of the conserved current. In choosing the well-known definition of the current advocated in particular by SCHWINGER \(^3\) we deduce consistent results.

Then we solve the Schwinger-Dyson equations for the Green functions using an iterative procedure which can be equally well discussed in terms of Feynman diagrams. The contributions to the boson and fermion propagators are calculated up to the fourth order in coupling constant \(e\). The limiting case \(m_0 \to 0\) is also discussed and compared with the exact solution which was expanded in powers of the coupling constant. One finds perfect agreement.

2. FIELD EQUATIONS AND GREEN FUNCTIONS

This two-dimensional field theory is deduced from the following field equations and commutation relations \(^1\), \(^2\):

\[
(\Box + \mu^2)A_\mu(x) = j_\mu(x), \quad \Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2}
\]

(1)
\[
\left( i \gamma^\mu \partial_\mu - m_0 - e A^\mu(x) \right) \psi(x) = 0 \quad , \quad \partial_\mu \equiv \partial / \partial x^\mu \\
\left( i \gamma^\mu + e A^\mu \right) \psi^{+}_{\psi} + m_0 \psi_{\psi} = 0 \quad \bar{\psi} = \psi^{+}_{\psi}
\]

all other commutators and anti-commutators vanish. \( j^\mu(x) \) is the conserved current to which the vector boson is coupled. The usual expression for \( j^\mu(x) \)

\[
j^\mu(x) = \mathcal{E} : \bar{\psi}(x) \gamma^\mu \psi(x) : 
\]
gives rise to difficulties \( ^3 \) as is well known and we adopt the definition given by JOHNSON \( ^4 \):

\[
j^\mu(x) = \lim_{\epsilon \to 0} \left[ j^\mu(x + \epsilon) + j^\mu(x - \epsilon) \right] 
\]

\[
j^\mu(x + \epsilon) = \mathcal{E} \bar{\psi}(x + \epsilon) \gamma^\mu \psi(x) 
\]

for the free field and an even more sophisticated definition

\[
j^\mu(x + \epsilon) = \frac{\mathcal{E}}{2} \left\{ \bar{\psi}(x + \epsilon) \gamma^\mu \psi(x) e^{i e \int \delta^{\mu \nu} (x + \epsilon - x)} - \bar{\psi}(x) \gamma^\nu \psi(x + \epsilon) e^{i e \int \delta^{\mu \nu} (x - \epsilon - x)} \right\}
\]

if an external field is present. Using this definition of the current, one obtains a covariant and gauge invariant expression for the vacuum polarization to all orders of perturbation theory.

The starting points of our perturbation calculation are the Schwinger-Dyson integral equations for the Green functions. We represent them here in a form \( ^* \) derived by MITTER \( ^5 \).

**Fermion propagator:**

\[
S^{-A}(p) = \frac{\mathcal{E}}{\mathcal{E}} p - M(p) \quad S_{\tau}^{-A}(p) = \frac{\mathcal{E}}{\mathcal{E}} p - m_0 \\
M(p) = m_0 + \frac{ie^2}{2m} \int d^4q \mathcal{D}(q) \bar{\psi}(p+q) \gamma^\alpha S(p+q) \gamma^\beta \psi(p) 
\]

\( ^* \) A thorough and consistent derivation within the Lagrangian formalism will be given elsewhere.
Photon propagator:

\[ D_{\mu\nu}(k) = D_{F\mu\nu}(k) + D_{F\mu\lambda}(k) S(k) D_{\lambda\nu}(k), \quad D_{F\mu\nu}(k) = \frac{\partial_{\mu\nu}}{\mu_0^2 - k^2} \]

\[ \mathcal{S}_{\mu\nu}(q) = -ie^2 \int d^4x \, \mathcal{J}_\mu \mathcal{F}_{\nu} \left[ S(p+q/2) \mathcal{F}_{\nu}^\dagger(p-q/2) S(p-q/2) + \right] \]

\[ \frac{\partial S(p)}{\partial p^\rho} + \frac{1}{24} \frac{\partial}{\partial p^\rho} \left( q \frac{\partial^2}{\partial p^2} S(p) \right) \]

The equation for the vertex function \( \Gamma_{\alpha} \) can be found in the lectures by MITTER.

Remarkable are the two extra terms in the defining equation of \( \mathcal{S}_{\mu\nu}(q) \):

\[ \mathcal{S}_{\mu\nu}(q) = \frac{\partial S(p)}{\partial p^\rho} + \frac{1}{24} \frac{\partial}{\partial p^\rho} \left( q \frac{\partial^2}{\partial p^2} S(p) \right) \]

These two terms guarantee that we obtain gauge invariant and covariant results. Therefore the polarization tensor \( \mathcal{P}_{\mu\nu}(q) \) must have the following structure:

\[ \mathcal{P}_{\mu\nu}(q) = \left[ q^2 \mathcal{P}_{\mu\nu} - q_{\mu} q_{\nu} \right] \mathcal{P}(q^2) \]

For the zero order vector boson propagator a form was taken which is usually used for the photon propagator in electrodynamics and the mass term \( \mu_0^2 \) only prevents infra-red divergences in the perturbational calculation of the fermion propagator. Because the current is conserved, the term \( \frac{q_{\mu} q_{\nu}}{\mu_0^2} \) in the expected form \( D_{F\mu\nu} = \frac{q_{\mu} q_{\nu}}{\mu_0^2 - k^2} \) does not contribute to \( \mathcal{P}_{\mu\nu}(q) \). This will be shown explicitly.
3. DEFINITION OF THE CURRENT AND GAUGE INVARIANCE

In the following we investigate the problem of gauge invariance and the definition of the current in full detail. We calculate the current induced in the vacuum by an external field (to lowest order in the external field and to lowest order in perturbation theory)\(^4\):

\[
\langle 0 | j^{(2)}_\mu (x) | 0 \rangle = i \int d^4 x \langle 0 | T \left( j^{(n)}_\mu (x) j^{(0)}_\nu (x') \right) | 0 \rangle \delta^{(4)}(x-x')
\]

If we use definition (4) for the current we run into difficulties, that is, we obtain non-covariant terms which destroy the gauge invariance of the theory. We have explicitly to take into account the dependence of the current on the external field.

Invariance of the field equations

\[
\left( \gamma^\nu \partial_\nu - m_0 - e J_\nu A^{\nu} (x) \right) \psi (x) = 0
\]

under the transformation \( \psi (x) \rightarrow e^{i \alpha (x)} \psi (x) \), \( A_\mu (x) \rightarrow A_\mu (x) + \partial_\mu \alpha (x) \) require this. Therefore, we have to write \(^4\)

\[
\langle 0 | j^{(2)}_\mu (x) | 0 \rangle = \int d^4 x' \delta^{(4)}(x-x') \left( \langle 0 | T (j^{(n)}_\mu (x') j^{(0)}_\nu (x')) | 0 \rangle + S^2_{\nu \nu} \delta^{(4)}(x-x') \right)
\]

where \( S^2_{\nu \nu} \delta^{(4)}(x-x') \) accounts for the explicit dependence on the external potential \( A^{\nu} \) ext.

Using (4) and the formulas for the vacuum expectation values of a time-ordered product (see Appx. I) one finds

\[
i \langle 0 | T (j^{(n)}_\mu (x) j^{(0)}_\nu (x')) | 0 \rangle = -i e^2 \int d^4 x' \langle 0 | T (j^{(n)}_\mu (x') j^{(0)}_\nu (x')) | 0 \rangle S^2_{\nu \nu} \delta^{(4)}(x-x')
\]

\[= i e^2 \{ 2m_0^2 \delta_{\mu \nu} \Delta_F (y) \Delta_F (y) + 4 \Delta_F (y) \Delta_F (y) + 2 q^2 \Delta^P (y) \Delta_F (y) \Delta_F (y) \}
\]
where \( y = x - x' \) and, according to the formulas in Appxs. I and II, one finds

\[
\langle 0 | j^{(o)}_{\mu} (x) j^{(o)}_{\nu} (x') | 0 \rangle = \frac{4 e^2 m_0^2}{\pi} \left[ \int_0^\infty \frac{ds}{2m_0 s^2 \sqrt{s^2 - 4m_0^2}} \left( \partial_\mu \partial_\nu - \Box q_{\mu \nu} \right) \Delta_F (x-x', s) - \right.
\]

\[
- \left. \delta^{(2)} (x-x') \left( \partial_{\mu} \partial_{\nu} - q_{\mu \nu} \right) \right] \]

\[(13)\]

This is manifestly a non-covariant result, because of the term

\[ \delta^{(2)} (x-x') \left( \partial_{\mu} \partial_{\nu} - q_{\mu \nu} \right) \].

In the following it is shown that the counter-term \( S_{\mu \nu} \) cancels this non-covariant term.

Starting from \( \partial^\mu \langle 0 | j^{(o)}_{\mu} (x) | 0 \rangle = 0 \) we obtain

\[
\int d^4 x \left[ \delta(x-x') \langle 0 | \left[ j^{(o)}_{\mu} (x), j^{(o)}_{\nu} (x') \right] | 0 \rangle - i S_{\mu \nu} \partial^\mu \delta^{(2)} (x-x') \right] A^\text{ext} (x') = 0
\]

\[(14)\]

where

\[
\langle 0 | \left[ j^{(o)}_{\mu} (x), j^{(o)}_{\nu} (x') \right] | 0 \rangle = \frac{4 i e^2 m_0^2}{\pi} \left( \partial_\mu \partial_\nu - \Box q_{\mu \nu} \right) \int_0^\infty \frac{ds}{s^2 \sqrt{s^2 - 4m_0^2}} \Delta(x-x', s)
\]

Therefore,

\[
\delta(x-x') \langle 0 | \left[ j^{(o)}_{\mu} (x), j^{(o)}_{\nu} (x') \right] | 0 \rangle = \frac{4 i e^2 m_0^2}{\pi} \delta(x-x') \int_0^\infty \frac{ds}{s^2 \sqrt{s^2 - 4m_0^2}} \partial_\mu \partial_\nu (x-x', s)
\]

\[
= i S_{\mu \nu} \partial_\mu \delta^{(2)} (x-x')
\]

\[(15)\]

Therefore,

\[
S_{\mu \nu} = \left( \partial_{\mu} \partial_{\nu} - q_{\mu \nu} \right) \frac{4 e^2 m_0^2}{\pi} \int_0^\infty \frac{ds}{s^2 \sqrt{s^2 - 4m_0^2}} \Delta_F (x-x', s)
\]

If we combine (13) and (15) we obtain

\[
\langle 0 | j^{(o)}_{\mu} (x) | 0 \rangle = \frac{4 e^2 m_0^2}{\pi} \int d^4 x A^\text{ext} (x') \int_0^\infty \frac{ds}{s^2 \sqrt{s^2 - 4m_0^2}} \left( \partial_\mu \partial_\nu - \Box q_{\mu \nu} \right) \Delta_F (x-x', s)
\]

\[-6-\]

\[(16)\]
which is a gauge-invariant and covariant result for the current induced in the vacuum.

Also, we have explicitly shown that the commutator

\[
\left[ j^\rho(x), j^\sigma(x') \right]_{x^\rho x'^\sigma}
\]

is not equal to zero:

\[
\langle 0 \left| \left[ j^\rho(x), j^\sigma(x') \right] \right| 0 \rangle = \frac{i e^2}{\hbar} \partial_\rho \delta^{(2)}(x-x')
\] (17)

As is well known\(^{23,6}\), this would contradict the positive definition of energy. But a non-vanishing of the commutator (17) has even further general consequences according to a theorem of DESER and BOULWARE\(^7\), stated recently; the non-vanishing of the equal time commutator is a necessary requirement that the interaction as a whole does not vanish. Therefore, one has to take into account a similar definition of the current (5) when dealing with fields interacting with other fields\(^*\). Eq. (16) can also be written

\[
\langle 0 \left| j^\rho(x) \right| 0 \rangle = \int d^2 \! x' A^\rho(x') \left[ k^{(2)}_{\mu \nu}(x-x') \right]
\]

\[
k^{(2)}_{\mu \nu}(x-x') = \int d^2 \! s \, A(s) \left( \partial_\mu \partial_\nu - \delta_\mu_\nu \gamma^\omega \partial_\omega \right) \Delta_F \left( x-x', s \right)
\]

\[
A(s^2) = \frac{2 e^2 m_o^2 \Theta(s^2 - 4 m_o^2)}{\pi^3} \left[ 1 - \frac{4 m_o^2}{s^2} \right]
\] (18)

The Fourier transform of \( k^{(2)}_{\mu \nu}(x-x') \) is \( P^{(1)}_{\mu \nu}(q) \)

\[
k^{(2)}_{\mu \nu}(x-x') = \frac{1}{(2\pi)^2} \int P^{(2)}_{\mu \nu}(k) e^{-i k \cdot (x-x')} d^2 k
\] (19)

* This will be treated elsewhere.
4. **CALCULATION OF THE VACUUM POLARIZATION UP TO THE FOURTH ORDER IN COUPLING CONSTANT**

Iteration of (6), (7) and the equation for the vertex function gives

\[
S(p) = S_F(p) + \frac{i e^2}{(2\pi)^2} S_F(p) \left[ \frac{d^3k}{D_F} \Gamma_{\alpha,\gamma}^\mu(p, k) \Gamma_{\beta,\gamma}^\nu(p, k) \Gamma_{\alpha,\beta}^\mu(p, k) + O(e^4) \right]
\]

(20)

\[
\Gamma_{\alpha,\gamma}^\mu(p, k) = \frac{i e^2}{(2\pi)^2} \int \frac{d^3k}{D_F} \Gamma_{\alpha,\gamma}^\mu(p, k) \Gamma_{\beta,\gamma}^\nu(p, k) \Gamma_{\alpha,\beta}^\mu(p, k) + O(e^4)
\]

(21)

\[
D_{\mu\nu}(k) = D_{\mu\nu}(k) + \frac{\partial F_{\mu\nu}(k)}{\partial F_{\mu\nu}(k)} + O(e^4)
\]

(22)

\[
S_{\mu\nu}(k) = -\frac{i e^2}{(2\pi)^2} \int d^3p \, \text{Tr} \, \left[ S(p,\frac{k}{2}) \Gamma_{\alpha,\gamma}^\mu(p, k) \Gamma_{\beta,\gamma}^\nu(p, k) \Gamma_{\alpha,\beta}^\mu(p, k) + O(e^4) \right]
\]

(23)

Therefore, we obtain for \( S_{\mu\nu}^{(1)}(k) + S_{\mu\nu}^{(4)}(k) \), if we describe it by Feynman diagrams,

\[
S_{\mu\nu}^{(1)}(k) + S_{\mu\nu}^{(4)}(k) = \frac{p^\mu p^\nu}{p^2} + \frac{p^\mu q^\nu}{q^2} + \frac{q^\mu p^\nu}{p^2} + \frac{q^\mu q^\nu}{q^2} + \frac{q^\mu p^\nu}{p^2}
\]

extra terms which guarantee the gauge invariance of \( S_{\mu\nu}(k) \).

First \( S_{\mu\nu}^{(1)}(k) \) will be discussed:

\[
S_{\mu\nu}^{(1)}(k) = -\frac{e^2}{(2\pi)^2} \int d^3p \, \text{Tr} \, \left[ \left\{ \frac{p^\mu (p, k) + m}{m} \right\} \left\{ \frac{p^\nu (p, k) - m}{m} \right\} + \frac{2}{q^\nu (q^\nu - m)} \right]
\]

\[
+ \frac{1}{24} \frac{\partial^2}{\partial p^\nu \partial p^\nu} \left\{ \frac{1}{(q^\nu - m)^2} \right\}
\]

We found that the second extra term was identically zero; by standard techniques we obtain
Therefore, we see that the first extra term cancels the non-covariant term, stemming from the straightforward application of the Feynman rules. MARX and SAAVEDRA did not take into account the extra term, which is a consequence of the carefully defined current, and therefore obtained a gauge-variant result.

At last we obtain

\[
\mathcal{F}_{\mu \nu}^{(2)}(k) = - \frac{ie^2}{(2\pi)^3} \int d^4x \left[ \frac{4p_\mu p_\nu - 2p_\mu k_\nu - 4p_\nu k_\mu + 2g_{\mu \nu}(p \cdot k) + 2m_0^2 g_{\mu \nu}}{(p^2 - m_0^2 + i\epsilon)(k^2 - m_0^2 + i\epsilon)} \right] \]

\[
= \frac{e^2}{(2\pi)^3} \left[ \frac{1}{2} (1 - 2) \left( \frac{1}{k^2 - m_0^2 + i\epsilon} \right) + \frac{2g_{\mu \nu} e^2}{(2\pi)^2} - \frac{2g_{\mu \nu} e^2}{(2\pi)^2} \right] \]

Therefore, we see that the first extra term cancels the non-covariant term, stemming from the straightforward application of the Feynman rules. MARX and SAAVEDRA did not take into account the extra term, which is a consequence of the carefully defined current, and therefore obtained a gauge-variant result.

At last we obtain

\[
\mathcal{F}_{\mu \nu}^{(2)}(k) = - \frac{e^2}{\pi} \left( g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) \left( 1 + \frac{2m_0^2}{k^2} \right) \left[ \log \left| \frac{1 + \frac{4m_0^2}{k^2}}{1 - \frac{4m_0^2}{k^2}} \right| - i\pi \right], \quad k^2 > 4m_0^2
\]

\[
= - \frac{e^2}{\pi} \left( g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) \left( 1 + \frac{m_0^2}{k^2} \frac{\log \left| \frac{1}{k^2 - 4m_0^2} \right|}{\log \left| \frac{1 - 4m_0^2}{k^2} \right|} \right), \quad 0 < k^2 < 4m_0^2
\]

\[
= - \frac{e^2}{\pi} \left( g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) \left( 1 + \frac{m_0^2}{k^2} \frac{\arctan \left| \frac{1}{k^2 - 4m_0^2} \right|}{\arctan \left| \frac{1}{k^2 - 4m_0^2} \right|} \right), \quad k^2 < 0
\]

From (24) we deduce

\[
\mathcal{F}_{\mu \nu}^{(2)}(k \cdot l_0) = - \frac{e^2}{\pi} \left( g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right)
\]

The calculation of \( \mathcal{F}_{\mu \nu}^{(4)}(k) \) is much more involved and one encounters all the difficulties of higher order diagrams. Therefore, we restricted the calculation to the case \( \mu = 0 \) and \( k^2 < 4m_0^2 \). In spite of this restriction, the calculation was very time-consuming and therefore
we represent only the result

\[ \Phi^{(4)}_{\mu \nu}(k) \bigg|_{\mu = 0, k^2 < 4m_o^2} = -\frac{e^6}{\pi^2} \left( \frac{1}{\lambda o^2} - \frac{1}{k^2} \right) - \frac{2}{\pi^2} \left( \frac{1}{k^2} \right) - \frac{1}{8\pi^2} \left( \frac{1}{k^2} \right) + \frac{12m_m^4}{(4m_o^2 - k^2)^2} \right] - \frac{1}{(4m_o^2 - k^2)^2} + \frac{1}{4m_o^2 k^2} \left( \frac{1}{k^2} \right)

\]

From (26) one finds \( \Phi^{(4)}_{\mu \nu}(k = 0) = 0 \).

In calculating \( \Phi^{(4)}_{\mu \nu}(k) \) we found the following results:

Firstly, the two extra terms, which guarantee the gauge-invariance of \( \Phi^{(4)}_{\mu \nu}(k) \) were identically zero. Therefore, in this two-dimensional theory the contributions to \( \Phi^{(4)}_{\mu \nu}(k) \), stemming from higher order diagrams (e^4 \( k^4 \), etc.) are gauge invariant without using the extra terms (8).

Therefore, one can use the usual Feynman diagrams and their corresponding rules. Only \( \Phi^{(2)}_{\mu \nu}(k) \) needs an extra term, which is \( -\Phi^{(2)}_{\mu \nu}(0) \), to guarantee gauge invariance. Secondly, the terms \( k^2 \mu^2 / \mu_o^2 \) of the zero order propagator \( D_{\mu \nu}(k) = \frac{\gamma_{\mu \nu} - \mu_o^2}{\mu_o^2 - k^2} \) did not contribute to \( \Phi^{(4)}_{\mu \nu}(k) \) because we have a conserved current. Therefore, one is allowed to use \( D_{\mu \nu}(k) = \frac{\gamma_{\mu \nu}}{\mu_o^2 - k^2} \). Thirdly, the terms proportional to \( \log \mu_o \) have dropped out, as it should be.

Using (7) and (25) we can calculate the exact photon propagator in the limit \( m_o = 0 \) and compare it with the exact solution. We obtain

\[ D_{\mu \nu}(k) \bigg|_{m_o = 0} = \frac{\gamma_{\mu \nu} - \mu_o^2}{\mu_o^2 - k^2} + \frac{\gamma_{\mu \nu}}{\mu_o^2 - k^2} \left( \mu_o^2 + \frac{e^2}{\pi} \right) > 0

\]

This is the result obtained by THIRRING and WESS. In the limit \( m_o = 0 \) only the lowest order diagram contributes to the whole \( \Phi^{(4)}_{\mu \nu}(k) \).
5. THE FERMION PROPAGATOR

In this last section we want to give a brief discussion of the fermion propagator.

We start from (6):

\[ S^{-1}(p) = \frac{1}{p^2 - m_o^2} - \frac{i e^\gamma}{\sqrt{2 \pi}} \int dq \mathcal{D}[\phi](q) \gamma^\alpha S(q) S(p+q) \Gamma^{\alpha\beta}(p+q, p) \]

Up to the fourth order in coupling constant \( e \) the iteration procedure gives the following contributions (we only draw the relevant Feynman diagrams):

\[ S(p) = \frac{S_F(p)}{\sum F} + \frac{S_F}{\sum F} \sum \frac{1}{S_F} \]

We only present the calculation of \( \Sigma^{(4)} \) in a short form, because this gives the only contribution in the limit \( m_o = 0 \). The result is then compared with the exact solution:

\[ \Sigma^{(4)}(p) = \frac{e^\gamma}{(2\pi)^3} \int dq \mathcal{D}[\phi](q) \int d^2 k \left[ \text{Tr} \left( \gamma^\sigma S_F(k+q/2) \gamma^\tau S_F(q) \gamma^\mu S_F(p+q) \gamma^\nu \right) \right] \]

This follows straightforwardly from (20), (21), (22) and (23). Because of the gauge-invariant form of \( \gamma^{(2)}(k) \) again the terms \( \frac{k_\mu k_\nu}{\mu_i^2} \) do not contribute. Therefore we can take \[ \frac{1}{\gamma^{(2)}}(\omega) = \frac{\gamma^{(2)}_{\mu\nu}}{\mu_i^2 - k^2} \]

again.
In the limit $m_0 = 0$ we have

$\sum_{\mu}^{(4)} (p) = -\frac{e^4}{(2\pi)^2} \int d^4 p \left\{ -\frac{1}{\mu_0} \frac{1}{(p^2 - \mu_0^2)^2} + \frac{1}{\mu_0} \frac{1}{(p^2 - \mu_0^2)^2} \right\}$

$S_{\mu}^{(4)} (p) |_{m_0 = 0} = -\frac{e^4}{(2\pi)^2} \int d^4 p \left\{ -\frac{1}{\mu_0} \frac{1}{(p^2 - \mu_0^2)^2} + \frac{1}{\mu_0} \frac{1}{(p^2 - \mu_0^2)^2} \right\}$

$\langle 0 | T \psi(x) \overline{\psi}(x') | 0 \rangle = i S_{\mu}^{(4)} (x-x')$  \hfill (29)

$\Delta S_{\mu}^{(4)} (x-x') = \frac{i}{(2\pi)^2} \int d^4 p e^{-i p(x'-x')} S_{\mu}^{(4)} (p)$  \hfill (30)

If one compares (30) with the exact solution $^1$, which was expanded in powers of the coupling constant, one finds, after performing a trivial calculation, perfect agreement.

If we look at (29) we see that the limit $\mu_0 \rightarrow 0$ is not defined.

The exact solution $^1$, on the contrary, shows no infra-red divergence.

This leads us to an old problem of usual perturbation theory. A straightforward expansion in the coupling constant produces infra-red divergences. In fact we must be careful in this case $^9$.

Therefore, we shall give another more sophisticated expansion later, taking into account the infra-particle property of the electrons if $\mu_0 = 0$.

6. CONCLUSION

In this note we started to discuss an extended Thirring-Wess model, which describes the interaction of massive fermions with massive vector bosons.

In the first part we presented the problems connected with the gauge-invariant definition of the current. All consistency requirements
were checked and it was demonstrated again that the current has to depend explicitly on the external field $A_\mu$ to ensure gauge-invariant and covariant results.

The discussion of the polarization tensor $\tilde{\phi}^{(2)}_{\mu,\nu}$ showed that only the lowest order contribution $\tilde{\phi}^{(2)}_{\mu,\nu}$ needed an extra term which guarantees gauge invariance. In the limit $m_0 = 0$ we obtain the result of the exact solution. Also, we compared the contribution to the fermion propagator with the exact solution in the limit $m_0 = 0$ and found perfect agreement.

Still quite a lot of problems remain. First one has to find the Lagrangian from where the field Eqs. (1) can be deduced, taking into account the modified definition of the current. SOMMERFIELD has carried this programme through for the case $m_0 = 0$ and also THIRRING et al. were able to do it for this case.

The discussion of $\tilde{\phi}^{(4)}_{\mu,\nu}(k)$ without imposing the restriction (26) should exhibit some new consequences for the propagator $D_{\mu,\nu}(k)$.

Also, the infra-red problem is not yet completely solved, especially for a perturbation expansion.

From this we can conclude that this extended model shows features which can also be found in the realistic field theories. Therefore, it is worthwhile to study some other problems in the framework of this model.

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Green functions of the free-particle equations: $[\left( p^2 - m^2 \right) \Delta(x)] = 0$. (For a detailed discussion of the algebra see Ref. 8)

\[
\Delta(x) = \Delta^+(x) + \Delta^-(x)
\]

\[
\Delta(0, x) = 0, \quad \Delta(x) = \frac{1}{(2\pi)^3} \int d^3p \ e^{-i p x} \delta(p^2 - m^2)  
\]

\[
\Delta^+(x) = \frac{1}{2} \int d^3p \ e^{-i p x} \Theta(-p_0) \delta(p^2 - m^2), \quad \Delta^-(x) = -\Delta^-(x)
\]

\[
\Delta(x) \bigg|_{m_0^2 = 0} = \frac{1}{4} \int d^3p \ e^{-i p x}  
\]

For the Dirac equation:

\[
\begin{align*}
(S \ S^\dagger, \ S_F) &= -\left( i \gamma^0 - m_0 \right) \left( \Delta, \Delta^\dagger \right) \\
\left(i \gamma^0 - m_0 \right) S_F &= \delta^{(3)}(x)  \\
S_F(x) &= \frac{1}{(2\pi)^3} \int d^3p \ e^{-i p x} \left( \gamma^0 p + m_0 \right)  \\
D_{F, \mu \nu}(x) &= \frac{1}{(2\pi)^2} \int \frac{d^2k}{\mu_0^2 - k^2 - i\varepsilon} 
\end{align*}
\]

Important vacuum expectation values:

\[
\begin{align*}
\langle 0 | T \bar{\psi}_\alpha \left( x \right) \psi_\beta \left( y \right) | 0 \rangle &= -i \langle 0 | T \bar{\psi}_\alpha \left( x \right) \psi_\beta \left( y \right) | 0 \rangle = -i S_{\alpha \beta}(x-y) \\
\langle 0 | \psi_\alpha \left( x \right) \bar{\psi}_\beta \left( y \right) | 0 \rangle &= i S_{\alpha \beta}(x-y)  \\
\langle 0 | \psi_\beta \left( y \right) \bar{\psi}_\alpha \left( x \right) | 0 \rangle &= i S_{\beta \alpha}(x-y) \\
\langle 0 | T \delta \left( x - y \right) | 0 \rangle &= -i \left( \Theta(-x_0 - y_0) \Delta^+(x-y) - \Theta(-x_0 - y_0) \Delta^+(y-x) \right)  \\
&= -i \Delta_F(x-y)  \\
\langle 0 | T \mathcal{A}_\mu \left( x \right) \mathcal{A}_\nu \left( y \right) | 0 \rangle &= i \mathcal{D}_{\mu \nu}(x-y) 
\end{align*}
\]
APPENDIX II

Products of invariant functions \(^{15}\).

\[ \Delta^+(x,m_0)\Delta^-(x,m_0) = -\frac{1}{(2\pi)^{2}} \int e^{-i\mathbf{p}\cdot\mathbf{x}} d\mathbf{p} \int e^{-i\mathbf{q}\cdot\mathbf{x}} d\mathbf{q} \delta(q^2-m_0^2)\delta(p^2-m_0^2) \Theta(q) \Theta(p) \]

Introducing the variable \( k = p + q \), we are led to the integral:

\[ \int d^4q \Theta(q) \Theta(k-q) \delta(q^2-m_0^2) \delta((q-k)^2-m_0^2) = \frac{\Theta(k) \Theta(k^2-4m_0^2)}{k \sqrt{k^2-4m_0^2}} \]

which we evaluate in the rest frame of \( k \). This gives us:

\[ \Delta^+(x,m_0)\Delta^-(x,m_0) = -\frac{1}{2\pi^2} \int \frac{d^4k}{k} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\Theta(k) \Theta(k^2-4m_0^2)}{k \sqrt{k^2-4m_0^2}} = \frac{i}{\pi} \int_0^\infty \frac{ds}{2m_0} \Delta^+(x,s) \]

Similarly we have

\[ \Delta^-(x,m_0)\Delta^-(x,m_0) = -\frac{i}{\pi} \int_0^\infty \frac{ds}{2m_0} \Delta^-(x,s) \]

Taking into account the following relations:

\[ \partial^\lambda \partial_\lambda [\Delta^+ \Delta^+] = 2(\Box \Delta^+) \Delta^+ + 2 \Delta^+ \Delta^+ \Delta^+ + \frac{1}{2\chi^2} \Delta^+ \]

and \( \Box \Delta^+ = -m_0^2 \Delta^+ \)

we arrive at

\[ \Delta^+(x,m_0)\Delta^+(x,m_0) = \frac{i}{\pi} \left[ \frac{\Box}{2} + m_0^2 \right] \int_0^\infty \frac{ds}{2m_0} \Delta^+(x,s) \]

\[ \Delta^-(x,m_0)\Delta^-(x,m_0) = -\frac{i}{\pi} \left[ \frac{\Box}{2} + m_0^2 \right] \int_0^\infty \frac{ds}{2m_0} \Delta^-(x,s) \]

On similar lines we also calculate \( \Delta^+_\mu^\nu \Delta^-_\mu^\nu \) and find

\[ \Delta^+_\mu^\nu(x,m_0)\Delta^+_\mu^\nu(x,m_0) = -\frac{i}{4\pi} \int \frac{ds}{2m_0} \Theta^{\mu^\nu}(s^2-4m_0^2) - 4 \Theta^{\mu^\nu}(s^2-4m_0^2) \Delta^+(x,s) \]

\[ \Delta^-\mu^\nu(x,m_0)\Delta^-\mu^\nu(x,m_0) = \frac{i}{4\pi} \int \frac{ds}{2m_0} \Theta^{\mu^\nu}(s^2-4m_0^2) - 4 \Theta^{\mu^\nu}(s^2-4m_0^2) \Delta^-(x,s) \]

We also mention the following relations:

\[ \Theta^{\mu^\nu} \delta x = \partial^{\mu} \delta x \]
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