SU$_3$ WIGNER COEFFICIENTS IN ANGULAR MOMENTUM SPACE

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SU$_3$ Wigner coefficients for the decoupling \((\lambda_1 \tau) = (\lambda_1' \mu') \times (\lambda_1'' \mu'')\) and \((\lambda, \sigma) = (\lambda_1' \mu') \times (\lambda_1'' \mu'')\) with \((\lambda'' \mu'') = (2, 0)\); have been obtained using basis functions of given angular momentum.
I. INTRODUCTION

The utilization of SU$_3$ group classifications in nuclear physics and elementary particle physics has made it necessary to obtain Wigner coefficients for direct products of SU$_3$ group representations in order to perform concrete calculations with systems of several particles.

The starting point in the application of the SU$_3$ group to nuclear physics [1] is the invariance of the three-dimensional harmonic oscillator Hamiltonian

$$H = a_x^+ a_x + a_y^+ a_y + a_z^+ a_z$$

under the unitary three-dimensional transformations

$$A_k = \sum_{k=1}^3 U_{kr} a_r; \quad \{ r = x, y, z \}; \quad U^+ U = 1$$

This allows one to classify the many-particle oscillator states by means of the quantum numbers ($\lambda \mu$) of the SU$_3$ irreducible representations.

The states corresponding to an irreducible representation ($\lambda \mu$) are generally labelled either by using the chain of groups SU$_3 \supset$ SU$_2$, (where SU$_2$ are the unitary unimodular transformations in the xy space [2]) or by the chain SU$_3 \supset$ R$_3$, where R$_3$ represents the group of three-dimensional rotations.

While for the first chain it is possible to define a set of commuting operators whose eigenvalues provide a complete labelling of the ($\lambda \mu$) states, the chain SU$_3 \supset$ R$_3$, which leads to states of given angular momentum, introduces an ambiguity, because a representation with given ($\lambda \mu$) generally contains several representations with the same angular momentum $L$. These representations are labelled by an additional quantum number $K$ [1]. In the classification of nuclear states using SU$_3$ group, the eigenvalues corresponding to different $K$ and to the
same \( L \) are interpreted as belonging to different rotational bands; however, the quantum number \( K \) associated with a rotational band has not been found as the eigenvalue of any operator having physical meaning. The functions with different angular momenta belonging to the same rotational band are obtained by a projection from the same function of a given \( K \).

The calculation of matrix elements for functions of given angular momentum involved in [2] is a complicated procedure of stepping up the function obtained by applying the Hamiltonian to the function of maximum weight; in addition to this, the functions with different \( K \) are not orthogonal.

By using Wigner, Racah and fractional parentage coefficients calculated in several papers, e.g., [3] [4] in the \( SU_3 \supset SU_2 \) scheme, the problem of determining rotational energy levels in the Elliott model can be finally solved.

There are, however, cases in which such a complicated procedure can be avoided, namely for the representations \((\lambda, 0)\) and \((\lambda, 1)\) for which only one \( K \) appears. In these cases subsequent diagonalizations are not necessary, nor is it necessary to resort to step-up procedures.

As there exist nuclei whose lowest energy states are described by \((\lambda, 1)\) representations of the \( SU_3 \) group (as, for instance, \( Ne^{21} \), \( Al^{27} \), \( P^{34} \) [5]) we have considered it necessary to determine the Wigner coefficients in angular momentum basis for the decomposition of the \((\lambda, 1)\) state in products of two representations of which one should be either the \((2, 0)\) state describing one-particle sd-shell states or one of the \((4, 0)\) or \((2, 1)\) states which are the only two-particle sd-shell states which can occur in the representation \((\lambda, 1)\). Using the corresponding decomposed wave functions, one can calculate one- and two-body interaction matrix elements.

In this paper we have calculated Wigner coefficients for the decomposition \((\lambda 1) = (\lambda' \mu') x (\lambda'' \mu'')\) for \((\lambda'' \mu'') = (20)\). Calculations for \((\lambda'' \mu') = (40)\) and \((21)\) are under way.
II. THE REPRESENTATION \((\lambda, 0)\) OF SU\(_3\)

Let us consider a three-dimensional space of basis vectors \(x_1, x_o, x_{-1}\) and the unitary unimodular group of transformations \(SU_3\) acting in this space.

The set of homogeneous polynomials of degree \(\lambda\) in \(x_1, x_o, x_{-1}\) constitutes a basis for the irreducible representation \((\lambda, 0)\) of the group \(SU_3\).

In fact, a direct product of \(\lambda\) fundamental representations of the \(SU_3\) group acting in \(\lambda\) different three-dimensional spaces,

\[
U_3(1) \times U_3(2) \times \ldots \times U_3(\lambda)
\]

reduces into a direct sum of irreducible representations of \(U_3\) labelled by Young tableaux with \(\lambda\) boxes and \(r \leq 3\) rows. The base functions of each i.r. has a symmetry type with respect to the permutations of the \(\lambda\) indices characterized by the corresponding Young tableau associated with the irreducible representation of \(U_3\).

It follows then that by considering the direct product of \(\lambda\) fundamental representations of the \(U_3\) group, acting in the same space,

\[
U_3(1) \times U_3(1) \times \ldots \times U_3(1)
\]

the tensor of rank \(\lambda\) that transforms under \([U_3(1)]^\lambda\) will be totally symmetric.

In particular, the set of homogeneous polynomials of order \(\lambda\) in \(x_1, x_o, x_{-1}\) will constitute a basis for the totally symmetric representation of the symmetric group \(S_\lambda\), i.e., a \((\lambda, 0)\) irreducible representation of the \(SU_3\) group.

This representation reduces when we restrict ourselves to the group \(R_3\), i.e., we can construct linearly independent homogeneous polynomials of degree \(\lambda\) in \(x_1, x_o, x_{-1}\), which are eigenfunctions of the total angular momentum \(L^2\) and of its projection \(L_0\).
The homogeneous polynomials of degree $\lambda$ with angular momentum $\ell$ and projection $m$ have (as may easily be checked) the expression

$$\Psi_{\mu, \ell, \ell}^{(\lambda, 0)} = N_{\lambda \ell} B^\mu x_1^\ell$$  \hspace{0.5cm} (II. 3)

where

$$B = x_0^2 - 2x_1 x_{-1}$$  \hspace{0.5cm} (II. 4)

is a scalar with respect to rotations and

$$\lambda = 2n + \ell \quad \text{i.e.,} \quad \ell = \lambda, \lambda - 2, \lambda - 4, \ldots \hspace{0.5cm} (II. 5)$$

The eight generators of the $SU_3$ group that realizes transformations in the three-dimensional space $x_1, x_0, x_{-1}$ can be written as irreducible tensors of degrees one and two with respect to the group $R_3$ [1] [2]:

$$L_q = -\sqrt{2} \sum_{\mu, \nu} C^{1 \mu \nu}_{\mu, \nu} \chi_{\nu}, \frac{\partial}{\partial \chi_{\mu}} ; \quad q = 0, \pm 1 \hspace{0.5cm} (II. 6)$$

$$Q_k = \sqrt{6} \sum_{\mu, \nu} C^{1 \mu \nu}_{\mu, \nu} \chi_{\nu}, \frac{\partial}{\partial \chi_{\mu}} ; \quad k = 0, \pm \frac{1}{2}, \pm \frac{3}{2} \hspace{0.5cm} (II. 7)$$

where the derivatives

$$\frac{\partial}{\partial \chi_{\mu}}$$

transform like the co-ordinates $x_\mu$ under three-dimensional rotations.

The lowering operators $L_-$ enable us to obtain functions of angular momentum $\ell$ and projection $m \neq \ell$. Some of them are, for instance:

$$\Psi_{(\ell, -1)} = N_{\lambda \ell} \sqrt{\ell (\ell + 1)} B^\mu x_1^\ell$$  \hspace{0.5cm} (II. 9)

$$\Psi_{(\ell, -2)} = N_{\lambda \ell} \sqrt{\ell (\ell + 1) (\ell + 2)} B^\mu \left[ x_1^{\ell - 1} x_{-1} + (\ell - 1) x_0^{\ell - 2} \chi_1^{\ell - 2} \right]$$  \hspace{0.5cm} (II. 10)

$$\Psi_{(\ell, -3)} = N_{\lambda \ell} \sqrt{\ell (\ell + 1) (\ell + 2) (\ell + 3)} B^\mu \left[ 3 x_1^{\ell - 2} x_0 x_{-1} + (\ell - 2) x_0^{\ell - 3} x_0^{\ell - 4} \right]$$  \hspace{0.5cm} (II. 11)

$$\Psi_{(\ell, -4)} = N_{\lambda \ell} \sqrt{\ell (\ell + 1) (\ell + 2) (\ell + 3) (\ell + 4)} B^\mu \left[ 3 x_1^{\ell - 3} x_0^{\ell - 2} x_{-1} + (\ell - 3) x_0^{\ell - 4} x_0^{\ell - 5} \right]$$  \hspace{0.5cm} (II. 12)
The set of linearly independent functions $\Psi_{\lambda,0}^{(\alpha, \beta)}$ has $\frac{1}{2}(\lambda+1)(\lambda+2)$ elements, which is exactly the dimension of the irreducible representation $(\lambda,0)$, i.e., they constitute a complete set of basis functions for the representation $(\lambda,0)$.

Let us now define a scalar product for the set of functions $f(x_1, x_0, x_{-1})$ which admit a Taylor series expansion. We shall start by defining the scalar product for each pair of functions belonging to the complete set

$$\left\{ (x_i)^{\ell} \right\} \text{ with } \ell = 0, 1, 2, \ldots$$

namely

$$\langle x_i^{\ell}, x_j^{\ell'} \rangle = \ell! \delta_{\ell\ell'} \delta_{ij} \quad (\text{II.13})$$

For a function $f(x_1)$, which can be expanded into a Taylor series, we have

$$\langle x_i^{\ell}, f(x_1) \rangle = \sum_{m} \frac{1}{m!} \langle x_i^{\ell}, x_i^{m} \rangle \left( \frac{d^m f}{d x_1^m} \right)_{x_1=0} = \left( \frac{d^m f}{d x_1^m} \right)_{x_1=0} \quad (\text{II.14})$$

If $f(x_1) \equiv x_1^n$ we obtain

$$\langle x_i^{\ell}, x_i^{n} \rangle = \left( \frac{d^n}{d x_i^n} x_i^{\ell} \right)_{x_i=0} = n! \delta_{m,n} \quad (\text{II.15})$$

In this way the scalar product (II.13) receives a meaning, namely

$$\langle x_i^{\ell}, x_j^{n} \rangle = \left( \frac{d^n}{d x_i^n} x_j^{\ell} \right)_{x_i=0} = n! \delta_{ij} \delta_{m,n} \quad (\text{II.16})$$

Using (II.16), the normalization constant $N_{n\ell}$ turns out to be

$$N_{n\ell} = \left[ \frac{(2\ell+1)!!}{(2n)!! (2n+2\ell+1)!!} \right]^{1/2} \quad (\text{II.17})$$
In the three-dimensional space $x_1, x_0, x_3$ we can also define the irreducible tensor of rank zero
\[ H = \sum \left( - \right)^m x_m y_{-m} \] (II.18)

The polynomials $\Psi^{(\lambda, \sigma)}_{n, \ell, m}$ are degenerate eigenfunctions of this operator with eigenvalues that reproduce the spectra of the three-dimensional harmonic oscillator. Reckoning with this behaviour, we can consider that the polynomials $\Psi^{(\lambda, \sigma)}_{n, \ell, m}$ represent $\lambda$ quanta oscillator states.

III. THE DECOUPLINGS $(\lambda, 0) = (\lambda - 1, 0) \times (1, 0)$ AND $(\lambda - 2, 1) = (\lambda - 1, 0) \times (1, 0)$

In our calculation we need also the expression of the functions which form a basis for the irreducible representation $(\lambda, 1)$ of $SU_3$. In order to do that, we first introduce the invariant with respect to $SU_3$,
\[
I(y, x) = \frac{2}{2\gamma_x} + \frac{2}{2\gamma_y} + \frac{2}{2\gamma_z}
\] (III.1)

If we apply this invariant to a function belonging to a given representation $(\lambda, \mu)$ of $SU_3$ we obtain a function belonging to the same irreducible representation $(\lambda, \mu)$ of $SU_3$. The operator $I$ increases the number of variables, as a result of the application of this operator each monomial will contain $\lambda - 1$ variables of type $x$ and one variable of type $y$. The increase of the number of variables leads to an increase of the number of independent polynomials of degree $\lambda$. Due to the analogy between the operator (II.18) and the harmonic oscillator Hamiltonian, written in the second quantization, we shall speak in the following for reasons of convenience, about creation (annihilation) of quanta instead of the appearance (disappearance) of a variable. The operator $I$ thus annihilates a quantum $x$ and creates a quantum $y$, conserving the irreducible representation
\[
I \Psi^{(\lambda, \sigma)}_{n, \ell, m}(x) = N \Psi^{(\lambda, \sigma)}_{n, \ell, m}(x, y)
\] (III.2)
The function on the r.h.s. is normalized and, according to the Wigner-Eckart theorem, the constant $N$ depends only upon the representation $(\lambda, 0)$. Then it is sufficient to calculate it for the wave function having $\ell = m = \lambda$ and the result is:

$$N = \sqrt{\lambda} = \sqrt{2n + \ell}$$  \hspace{1cm} (III. 3)

Applying $I$ to an arbitrary function $\Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda}$ with $\lambda$ and $\ell$ having the same parity, we obtain the linear combination

$$I(y, x) \Psi^{(0,0)}_{m, l, \lambda, \ell} = \sqrt{\lambda} \left\{ \alpha \sum_{n_1, n_2} C_{\lambda, \lambda, \ell, \ell} (x, x) \Psi^{(\lambda, 0)}_{n_1, \ell, \ell, \lambda} + \beta \sum_{n_3, n_4} C_{\lambda, \lambda, \ell, \ell} (x, x) \Psi^{(\lambda, 0)}_{n_3, \ell, \ell, \lambda} \right\}$$  \hspace{1cm} (III. 4)

Here $\alpha$ and $\beta$ are Wigner coefficients for the SU$_3$ group.

Calculating the l.h.s. and introducing the expression for $\Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda}$ and $\Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda}$ into the r.h.s., we derive by identification

$$\alpha = \langle (\lambda-1, 0) \ell, \lambda, l | (\lambda, 0) \ell \rangle = \left( \frac{\ell (2\lambda + 2\ell + 1)}{(2n + \ell)(2\ell + 1)} \right)^{1/2}$$  \hspace{1cm} (III. 5)

$$\beta = \langle (\lambda-1, 0) \ell, \lambda, l | (\lambda, 0) \ell \rangle = -\left( \frac{z_{n, \ell+1}}{2n + \ell)(2\ell + 1)} \right)^{1/2}$$  \hspace{1cm} (III. 6)

With these values the wave function* $\Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda}$ becomes

$$\Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda} = \left( \frac{\ell (2\lambda + 2\ell + 1)}{(2n + \ell)(2\ell + 1)} \right)^{1/2} \Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda} \Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda} - \left( \frac{z_{n, \ell+1}}{2n + \ell)(2\ell + 1)} \right)^{1/2} \Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda} \Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda}$$  \hspace{1cm} (III. 7)

The parenthesis $\left( \frac{\ell}{m} \right)$ represent the coupling with respect to $R_3$.

On the other hand, the function $\Psi^{(\lambda-1, 0)}_{\ell, \lambda, \ell, \lambda}$ which appears in the same direct product as $\Psi^{(\lambda, 0)}_{\ell, \lambda, \ell, \lambda}$

$$(\lambda - 1, 0) \times (1, 0) = (\lambda, 0) + (\lambda - 2, 1)$$  \hspace{1cm} (III. 8)

is orthogonal to (III. 7) because they belong to two different irreducible representations of SU$_3$. Using this property we can determine the Wigner coefficients, which appear in the decomposition of

* Whenever a wave function depends upon more than one variable, we shall indicate by an exponent the power of each variable. For example, in (III. 7) the variable $x(x_1, x_0, x_1)$ has the power $\lambda - 1$, while $y(y_1, y_0, y_1)$ has power 1.

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except for their phase which remains to be chosen. The wave function has the expression

\[ \Psi_{\ell, m}^{(\lambda-2, 1)}(x, y) = \left( \frac{2\pi}{(2\pi x + 4)(2\pi y + 4)} \right)^{\frac{1}{2}} \left( \frac{\psi_{\ell, m, -1}^{(\lambda-2, 1)}(x, y)}{m} + \frac{\psi_{\ell, m, 1}^{(\lambda-2, 1)}(x, y)}{m} \right) \]

The functions \( \psi_{\ell, m}^{(\lambda-2, 1)} \), where the angular momentum \( \lambda \) and angular momentum \( \ell \) have different parities, are of the form

\[ \psi_{\ell, m}^{(\lambda-2, 1)} \]

IV. THE SEPARATION OF THE WAVE FUNCTION FOR ONE PARTICLE

The wave functions which describe the states of 2s-1d shell nuclei can be characterized by the group chain \( SU_6 \supset SU_3 \supset R_3 \). In general, additional quantum numbers are necessary to label the \( \ell \) degenerate states which appear in the reduction \( SU_3 \supset R_3 \). In the simple cases \((\lambda, 0)\) and \((\lambda, 1)\) we are considering, degeneracy does not appear. Our aim is to obtain Wigner coefficients that separate from a many-particle wave function that belongs to an irreducible representation \((\lambda, 0)\) or \((\lambda, 1)\) of the \( SU_3 \) group, one 2s-1d shell nucleon which is described by the function belonging to the \((2, 0)\) representation of \( SU_3 \). In this decoupling appear three types of Wigner coefficients

\[ \langle (\lambda-2, 0) \ell' (2, 0) \ell'' | (\lambda, 0) \rangle, \langle (\lambda-2, 0) \ell' (2, 0) \ell'' | (\lambda-2, 1) \ell \rangle \]

and

\[ \langle (\lambda-4, 1) \ell' (2, 0) \ell'' | (\lambda-2, 1) \ell \rangle \].

The angular momenta of the separated particle are \( \ell'' = 0, 2 \); \( \ell' \) takes those values allowed by \((\lambda-2, 0)\) or \((\lambda-4, 1)\) and by the coupling \( \ell'' + \ell' = \ell \).

a) The coefficient \( \langle \lambda-2, 0 \ell' (2, 0) \ell'' | (\lambda, 0) \ell \rangle \)

In order to obtain this type of coefficient we apply the invariant...
operator (III.1) either once to the function (III.7) or twice to the function (II.3)

\[ \int \frac{2}{\sqrt{2\lambda(\lambda+1)}} \Psi_{\lambda, \lambda-1, \ell}^{(\lambda, \ell)}(x) = \sqrt{2\lambda(\lambda+1)} \Psi_{\lambda, \lambda-1, \ell}^{(\lambda, \ell)}(x, \lambda-1, \ell) \]  (IV.1)

Here the operator \( I^2 \) destroys two \( x \) quanta and creates two \( y \) quanta. These two quanta are described by the function \( \Psi_{\lambda, \lambda-1, \ell}^{(\lambda, \ell)}(y) \).

The normalized function in the r.h.s. of (IV.1) can then be decoupled in the following way:

\[ \Psi_{\lambda, \lambda-1, \ell}^{(\lambda, \ell)}(x, \lambda-1, \ell) = \sum_{\ell', \ell''} \langle \lambda-2, \ell' | \ell'' \rangle \langle \ell'' | \lambda-2 \rangle \left( \Psi_{\lambda-2, \ell'}^{(\lambda-2, \ell)}(x) \cdot \Psi_{\lambda-2, \ell''}^{(\lambda-2, \ell)}(y) \right) \]  (IV.2)

where \( \ell' \) can take the values \( \ell' = \ell - 2, \ell, \ell + 2 \), allowed by \( (\lambda - 2, 0) \) and \( \ell'' = 0, 2 \). The expressions of \( \Psi_{\lambda-2, \ell'}^{(\lambda-2, \ell)}(x) \) and \( \Psi_{\lambda-2, \ell''}^{(\lambda-2, \ell)}(y) \) are already known and if we introduce them in (IV.1) and calculate the l.h.s. of the same equation we arrive at an identity between two polynomials of order \( \lambda \). By identification of the equal powers of \( x \) and \( y \) we obtain a system of algebraic linear equations for the Wigner coefficient \( \langle \lambda - 2, 0 | \ell' | 2, 0 \rangle \ell'' | \lambda, 0 \rangle \ell \) with a number of equations larger than the number of unknown coefficients. The solution of this system is tabulated in Table I.

b) The coefficient \( \langle \lambda - 2, 0 | \ell' | 2, 0 \rangle \ell'' | \lambda - 2, 1 \rangle \ell \)

We use the same procedure but consider two different situations: wave functions with angular momenta of the same and of opposite parity with respect to \( \lambda \). For the first case we apply the operator \( I \) to the function (III.8):

\[ \int \Psi_{\ell, \ell - 1, \ell}^{(\lambda - 2, \ell)}(x, \lambda - 2, \ell) = \sqrt{\lambda - 2} \Psi_{\ell, \ell - 1, \ell}^{(\lambda - 2, \ell)}(x, \lambda - 2, \ell) \]  (IV.3)
and then we can consider the decoupling

\[ \psi^{(\lambda-2,1)}_{\ell',\ell} (x^{\lambda}, y^{\lambda}) = \sum_{\ell''} \langle (\lambda-2,0) \ell' (2,0) \ell'' \mid (\lambda-2,1) \ell \rangle \left( \psi^{(\lambda-2,0)}_{\ell',\ell} (x) \psi^{(2,0)}_{\ell''} (y) \right) \ell \]  

(IV. 4)

The values for \( \ell' \) are \( \lambda = \lambda-2, \lambda, \lambda+2 \).

For the second case the operator 1 must act upon the function

(III. 10)

\[ \int \psi^{(\lambda-2,1)}_{\ell',\ell} (x^{\lambda}, y^{\lambda}) = \sqrt{\frac{\lambda - \ell}{2\ell + 1}} \psi^{(\lambda-2,1)}_{\ell',\ell} (x^{\lambda}, y^{\lambda}) \quad ; \quad \lambda - 1 = 2\ell + 1 \]  

(IV. 5)

and the corresponding decoupling of the function which contains two \( y \) quanta is similar to (IV. 4),

\[ \psi^{(\lambda-2,1)}_{\ell',\ell} (x^{\lambda}, y^{\lambda}) = \sum_{\ell''} \langle (\lambda-2,0) \ell' (2,0) \ell'' \mid (\lambda-2,1) \ell \rangle \left( \psi^{(\lambda-2,0)}_{\ell',\ell} (x) \psi^{(2,0)}_{\ell''} (y) \right) \ell \]  

(IV. 6)

except for the fact that \( \ell'' \) takes here a single value \( \ell'' = 2 \) and \( \ell' \) only two values \( \ell' = \lambda - 1, \lambda + 1 \). In the same way as in Section IV a), we arrive again at a linear algebraic system, whose solutions are given in Table II for \( \ell \) and \( \lambda \) of the same parity and in Table III for \( \ell \) and \( \lambda \) of different parity.

C) The coefficient \( \langle (\lambda-4,1) \ell' (2,0) \ell'' \mid (\lambda-2,1) \ell \rangle \)

The problem divides also into two parts in terms of the values taken by \( \lambda \), as in Section IV b).

In order to make algebraic calculations easier, let us introduce a second invariant:

\[ \mathcal{I} (x, \chi) = z_1 \frac{2}{\chi_1} + z_0 \frac{2}{\chi_0} + z_{-1} \frac{2}{\chi_{-1}} \]  

(IV. 7)
We first apply the invariant $I^2(z, x)$ to the function (III. 8), which has angular momenta of the same parity as $\lambda$:

$$I^2(z, x) \Psi_{\ell, \ell}^{(\lambda, z)}(x^1, z^1, y) = \sqrt{2(\lambda-\ell)(\lambda+\ell)} \Psi_{\ell, \ell}^{(\lambda-1, z)}(x^1, z^1, y)$$  \hspace{1cm} (IV. 8)

where

$$\Psi_{\ell, \ell}^{(\lambda, z)}(x^1, z^1, y) = \left[ \frac{z^n(x^1, \ell, \ell)}{2^{n+1} \ell(n!)} \right]^{\frac{1}{2}} \left( \Psi_{m, \ell}^{(\lambda-1, z^1, z^1)} \cdot \Psi_{n, \ell}^{(\lambda, z)} \right) + \left[ \frac{z^n(x^1, \ell, \ell)}{2^{n+1} \ell(n!)} \right]^{\frac{1}{2}} \left( \Psi_{m, \ell}^{(\lambda, z^1, z^1)} \cdot \Psi_{n, \ell}^{(\lambda-1, z)} \right)$$  \hspace{1cm} (IV. 9)

is a normalized wave function. The Wigner coefficients which decouple the function $\Psi_{\ell, \ell}^{(\lambda, z)}(x^1, z^1, y)$ in $\Psi_{\ell, \ell}^{(\lambda, z)}(x^1, z^1, y)$ and $\Psi_{\ell, \ell}^{(\lambda-1, z)}(x^1, z^1, y)$ are known from Section IV. a). In order to obtain the Wigner coefficients that we are looking for, it is necessary to make a recoupling in the function $\Psi_{\ell, \ell}^{(\lambda, z)}(x^1, z^1, y)$, i.e., to couple first $\Psi_{\ell, \ell}^{(\lambda, z)}(x^1, z^1, y)$ to $\Psi_{\ell, \ell}^{(\lambda-1, z)}(x^1, z^1, y)$ and then the resulting function to $\Psi_{\ell, \ell}^{(\lambda-2, z)}(x^1, z^1, y)$. We can write $\Psi_{\ell, \ell}^{(\lambda, z)}(x^1, z^1, y)$ as:

$$\left[ \left[ \Psi_{\ell, \ell}^{(\lambda-1, 0)}(x^1, z^1, y) \cdot \Psi_{\ell, \ell}^{(\lambda, 0)}(x^1, z^1, y) \right] \cdot \Psi_{\ell, \ell}^{(\lambda-1, 0)}(x^1, z^1, y) \right]_{\ell, \ell} =$$

$$= \sum_{\lambda', \mu'} \left[ \left[ \Psi_{\ell, \ell}^{(\lambda-1, 0)}(x^1, z^1, y) \cdot \Psi_{\ell, \ell}^{(\lambda-2, 0)}(x^1, z^1, y) \right] \cdot \Psi_{\ell, \ell}^{(\lambda-1, 0)}(x^1, z^1, y) \right]_{\ell, \ell}$$  \hspace{1cm} (IV. 10)

The square brackets symbolize coupling with respect to $SU_3$. The representations $(\lambda', \mu')$ which appear in the recoupling are $(\lambda-2, 0)$ and $(\lambda-4, 1)$. The recoupling is made by the Racah coefficient $U_{[3]}$ which must also be determined.

We rewrite (IV. 10):

$$\left[ \left[ \Psi_{\ell, \ell}^{(\lambda-1, 0)}(x^1, z^1, y) \cdot \Psi_{\ell, \ell}^{(\lambda, 0)}(x^1, z^1, y) \right] \cdot \Psi_{\ell, \ell}^{(\lambda-1, 0)}(x^1, z^1, y) \right]_{\ell, \ell} =$$

$$= U_{[3]} \sum_{\ell' \ell''} \langle (\lambda-2, 0) \ell' | (\lambda-4, 0) \ell'' \rangle \langle (\lambda-2, 0) \ell'' | (\lambda-4, 0) \ell' \rangle \langle (\lambda-2, 0) \ell' | (\lambda-1, 0) \ell'' \rangle \langle (\lambda-2, 0) \ell'' | (\lambda-1, 0) \ell' \rangle$$

$$+ U_{[3]} \sum_{\ell' \ell''} \langle (\lambda-4, 1) \ell' | (\lambda-4, 1) \ell'' \rangle \langle (\lambda-4, 1) \ell'' | (\lambda-4, 1) \ell' \rangle \langle (\lambda-4, 1) \ell' | (\lambda-2, 0) \ell'' \rangle \langle (\lambda-4, 1) \ell'' | (\lambda-2, 0) \ell' \rangle$$  \hspace{1cm} (IV. 11)
The l.h.s. is known. The Wigner coefficients appearing in (IV.11) have already been determined in Section IV.a). In the r.h.s., the first term contains summation over \( J' = J - 2, J, J + 2, \) while in the second term summation extends over \( J' = J - 2, J - 1, J, J + 1, J + 2 \) and \( J'' = 0, 2 \). The unknown Wigner coefficients occur only in the latter term, those appearing in the first term being determined in Section IV.b). By identification of the coefficients of the variables with equal powers we obtain the product \( \mathcal{U}[(\lambda - 3, 0)(2, 0)(\lambda - 2, 1)(10); (\lambda - 1, 0)(\lambda - 2, 0)] \times \langle (\lambda - 2, 0) \lambda' \rangle \lambda'' \rangle (\lambda - 2, 1) \rangle \) and the product \( \mathcal{U}[(\lambda - 3, 0)(2, 0)(\lambda - 2, 1)(10); (\lambda - 1, 0)(\lambda - 4, 1)] \times \langle (\lambda - 4, 1) \lambda' \rangle \lambda'' \rangle (\lambda - 2, 1) \rangle \). From the first product we obtain 
\[ \mathcal{U}[(\lambda - 3, 0)(2, 0)(\lambda - 2, 1)(10); (\lambda - 1, 0)(\lambda - 2, 0)] \] 
the Wigner coefficients being known. From the normalization condition of the total wave function we obtain the second Racah coefficient and then the Wigner coefficient up to the phase which has been chosen as in [3].

The same procedure is applied to the function (III.10) with angular momentum of parity other than \( \lambda \). The results are presented in Tables II and III.

The Racah coefficients are:

\[ \mathcal{U}[(\lambda - 3, 0)(2, 0)(\lambda - 2, 1)(10); (\lambda - 1, 0)(\lambda - 2, 0)] = -\left(\frac{\lambda}{(\lambda - 1)(\lambda - 2)}\right)^{\frac{1}{2}} \] (IV.12)

\[ \mathcal{U}[(\lambda - 3, 0)(2, 0)(\lambda - 2, 1)(10); (\lambda - 1, 0)(\lambda - 2, 0)] = \left(\frac{\lambda}{(\lambda - 1)(\lambda - 2)}\right)^{\frac{1}{2}} \] (IV.13)

where \( \lambda = 2n + \lambda \)

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\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$(\lambda', \mu')$ & $l''$ & $l'$ & $\langle (\lambda', \mu') \ l' \ (2, 0) \ l'' \ | \ (\lambda, 0) \ l \rangle$ \\
\hline
$(\lambda-2, 0)$ & $2$ & $l+2$ & $\left[ \frac{(\lambda-l)(\lambda-\ell-2)(l+1)(l+2)}{\lambda(\lambda-1)(2\ell+1)(2\ell+3)} \right]^{\frac{1}{2}}$ \\
 & & $l$ & $-\left[ \frac{2 \ l(l+1)(\lambda-l)(\lambda+l+1)}{3 \ \lambda(\lambda-1)(2\ell+1)(2\ell+3)} \right]^\frac{1}{2}$ \\
 & & $l-2$ & $\left[ \frac{\ell(l-1)(\lambda+l+1)(\lambda+l-1)}{\lambda(\lambda-1)(2\ell-1)(2\ell+1)} \right]^\frac{1}{2}$ \\
 & $0$ & $l$ & $\left[ \frac{(\lambda-l)(\lambda+l+1)}{3 \ \lambda(\lambda-1)} \right]^\frac{1}{2}$ \\
\hline
\end{tabular}
\end{center}
\end{table}

for $\lambda$ and $l$ having the same parity
TABLE II

**SU₃ Wigner Coefficients**

\( \langle \lambda', \mu' | \ell' (20) \ell'' | (\lambda - 2, 1) \ell \rangle \)

for \( \lambda \) and \( \ell \) having the same parity

| \((\lambda', \mu')\) | \(\ell''\) | \(\ell'\) | \(\langle \lambda', \mu' | \ell' (20) \ell'' | (\lambda - 2, 1) \ell \rangle\) |
|-------------------|---------|---------|------------------------------------------------|
| \((\lambda - 2, 0)\) | \(\ell\) | \(l + 2\) | \(- \frac{2 \ell (l + 2)(\lambda - l - 2)(\lambda + l + 1)}{\lambda (\lambda - 2)(2l + 1)(2l + 3)}\) |
| \- | \(\ell\) | \(-l - 2\) | \(\frac{1}{3 \lambda (\lambda - 2)(2l + 1)(2l + 3)}\) |
| \- | \(0\) | \(l\) | \(- \frac{2 \ell (l + 1)}{3 \lambda (\lambda - 2)}\) |
| \((\lambda - 4, 1)\) | \(\ell\) | \(l + 2\) | \(\frac{\ell (l + 3)(\lambda - l - 2)(\lambda - l - 4)}{(\lambda - 2)(\lambda - 1)(2l + 1)(2l + 3)}\) |
| \- | \(\ell\) | \(-l - 1\) | \(\frac{2 (\lambda - l - 2)}{(\lambda - 3)(2l + 1)}\) |
| \- | \(\ell\) | \(-l - 2\) | \(- \frac{2 (\lambda - l - 2)(\lambda + l - 1)}{3 (\lambda - 2)(\lambda - 3)(2l + 1)(2l + 3)}\) |
| \- | \(0\) | \(l\) | \(\frac{2 (\lambda + l - 1)}{3 (\lambda - 3)(2l + 1)}\) |
### TABLE III

**SU$_3$ Wigner Coefficients**

\[
\langle (\lambda', \mu') \ell' | (2, 0) \ell'' | (\lambda - 2, 1) \ell \rangle
\]

for $\lambda$ and $\ell$ having different parities

| $(\lambda', \mu')$ | $\ell''$ | $\ell'$ | $\langle (\lambda', \mu') \ell' | (2, 0) \ell'' | (\lambda - 2, 1) \ell \rangle$ |
|-------------------|---------|---------|------------------------------------------------------------------|
| $(\lambda - 2, 0)$ | $2$     | $\ell + 1$ | $- \left[ \frac{(\lambda - \ell - 1)(\ell + 2)}{(\lambda - 2)(2\ell + 1)} \right]^{\frac{\ell}{2}}$ |
|                   | $2$     | $\ell - 1$ | $\left[ \frac{(\ell - 1)(\lambda + \ell)}{(\lambda - 2)(2\ell + 1)} \right]^{\frac{\ell}{2}}$ |
| $(\lambda - 4, 1)$ | $2$     | $\ell + 2$ | $\left[ \frac{\ell(\ell + 3)(\lambda - \ell - 3)(\lambda - \ell - 1)}{\lambda (\lambda - 3)(2\ell + 1)(2\ell + 3)} \right]^{\frac{\ell}{2}}$ |
|                   | $2$     | $\ell + 1$ | $\left[ \frac{2(\lambda - \ell - 1)(\lambda - \ell - 3)(\lambda + \ell)}{\lambda(\lambda - 2)(\lambda - 3)(\ell + 1)(2\ell + 1)} \right]^{\frac{\ell}{2}}$ |
|                   | $2$     | $\ell$    | $-(\ell' + \ell - 3) \left[ \frac{2(\lambda - \ell - 1)(\lambda + \ell)}{3(\lambda - 3) \lambda(\ell + 1)(2\ell - 1)(2\ell + 3)} \right]^{\frac{\ell}{2}}$ |
|                   | $2$     | $\ell - 1$ | $\left[ \frac{2(\lambda - \ell - 1)(\lambda + \ell - 2)(\lambda + \ell)}{\lambda(\lambda - 2)(\lambda - 3)\ell(2\ell + 1)} \right]^{\frac{\ell}{2}}$ |
|                   | $2$     | $\ell - 2$ | $\left[ \frac{(\ell - 2)(\ell + 4)(\lambda + \ell - 2)(\lambda + \ell)}{\lambda(\lambda - 3)(2\ell - 1)(2\ell + 1)} \right]^{\frac{\ell}{2}}$ |
|                   | $2$     | $0$       | $\left[ \frac{(\lambda - \ell - 1)(\lambda + \ell)}{3 \lambda(\lambda - 3)} \right]^{\frac{\ell}{2}}$ |
REFERENCES


