COUPLING PROBLEM FOR
U(p, q) "LADDER" REPRESENTATIONS - I

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COUPLING PROBLEM FOR U(p,q) "LADDER" REPRESENTATIONS—1

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1. INTRODUCTION

We present here the results of our initial investigation of the coupling of three most degenerate representations of U\((p,q)\). Several authors (Dothan, Gell-Mann, and Ne'eman 1965; Salam and Strathdee 1966; Olszewski 1965) have constructed the Lie algebra \(L(p,q)\) of U\((p,q)\) for the "ladder" representations with appropriate bilinear products of Bose annihilation and creation operators. In this paper, we obtain function bases for the "ladder" representations by realizing the Bose operators, hence the algebra \(L(p,q)\), with differential forms. Further it is shown that these representations can be integrated to irreducible unitary global representations of the universal covering group. Our motivation for obtaining function bases is that we can then employ the well-developed techniques of functional analysis to investigate the construction of invariants - the coupling problem. As an illustrative example, we present the Clebsch-Gordan coefficient for the coupling of two U\((1,1)\) "ladder" representations. For this case, we obtain the interesting result that there is only one "ladder" representation in the direct product of two "ladder" representations.

2. ASSIGNMENT OF PARTICLE REPRESENTATIONS

An important facet of the coupling problem which appears in particle physics is the proper assignment of baryon, antibaryon, and meson representations (Todorov 1966). One possibility is to require the following under the full group: 1) the spectra of the diagonalized commuting operators corresponding to additive quantum numbers for particles are equal in magnitude but opposite in sign to those of the associated anti-particles, 2) the hermitian conjugate of a particle state is the associated anti-particle state, and 3) baryons and anti-baryons correspond to separate representations, while mesons and their associated anti-mesons correspond to the same representation.

Therefore if \(D^B\) is a unitary baryon representation of U\((p,q)\), then \((D^B)^\dagger\) is the associated anti-baryon representation, where the dagger \(\dagger\) represents the hermitian conjugate operation, while for mesons \(D^M\) is a self-conjugate representation. In this way, because of the invariance of the scalar product, we also automatically obtain that we can construct scalar currents of baryons and anti-baryons, or mesons.
Another possibility which has been employed is to require that the conjugate properties refer to some subgroup of \( U(p, q) \), i.e., in the case of \( U(6, 6) \), the assignment is made such that baryons and antibaryons belong to mutually conjugate representations with respect to \( U(6) \), i.e.,

\[
\begin{align*}
\text{Baryons:} & \quad (56, 1) \oplus (126, 6^*) \oplus (252, 21^*) \oplus \cdots \\
\text{Antibaryons:} & \quad (1, 56^*) \oplus (6, 126^*) \oplus (21, 252^*) \oplus \cdots \\
\text{Mesons:} & \quad (1, 1) \oplus (6, 6^*) \oplus (21, 21^*) \oplus \cdots
\end{align*}
\]

where the asterisked unitary \( U(6) \) representation \( d^* \) is conjugate to the \( d \) representation (Dothan, Gell-Mann, and Ne'eman 1965; Salam and Strathdee 1966; Olszewski 1965).

The physical consequences of these alternative assignments will be discussed in a subsequent paper.

3. STRUCTURE OF THE "LADDER" REPRESENTATIONS

We select a basis \( \mathcal{M}_A^B \) (generators) for the Lie algebra \( L(p, q) \) of the group \( U(p, q) \) (Gel'fand and Graev 1966) such that

\[
[\mathcal{M}_A^B, \mathcal{M}_C^D] = \xi_{BC}^D \mathcal{M}_A^D - \xi_{AC}^D \mathcal{M}_A^D,
\]

i.e., the \( \mathcal{M}_A^B \) satisfy the \( GL(n, \mathbb{R}) \) commutation relations and the following hermiticity condition

\[
(\mathcal{M}_A^B)^\dagger = (\gamma_0)^{B'}_{A'} (\mathcal{M}_A^B)^{A'}_{B'},
\]

where \( A, B = 1, \ldots, p, p+1, \ldots, p+q, \) and \( \gamma_0 \) is the matrix

\[
\gamma_0 = \begin{pmatrix}
-I_{p \times p} & 0 \\
0 & I_{q \times q}
\end{pmatrix}.
\]

Here we are following the sign conventions used by Salam and Strathdee (1966). Employing the construction of Dothan, Gell-Mann, and Ne'eman (1965), a realization of the \( \mathcal{M}_A^B \) is obtained with the following array of bilinear products of Bose annihilation \( a, b \) and creation \( a^+, b^+ \) operators.
\[
\begin{pmatrix}
M^+_j \\
M^-_j
\end{pmatrix}
= \begin{pmatrix}
-a^+_j a^-_j \\
-b^+_j b^-_j
\end{pmatrix}
\]
where \( i, j = 1, \ldots, p, \) \( i', j' = p+1, \ldots, p+q, \)
and all other commutators are zero.

The basis \( \langle \alpha^\Lambda, \lambda_1, \lambda_2 \rangle \) of the Hilbert space \( H^\Lambda \) which is a carrier space for irreducible representations of the algebra (3.4) is constructed by the successive application of \( a^+_j \) and \( b^+_j \) to a vacuum state \( |0> \) which is defined such that

\[
[a^-_j, b^-_j] = \delta^\Lambda_{\lambda_1, \lambda_2}, \quad [b^+_j, b^-_j] = \delta^\Lambda_{\lambda_1, \lambda_2},
\]

Thus \( a^+_j |0> = 0 \) and \( b^+_j |0> = 0. \)

The irreducible representations of \( \Lambda \) are characterized by one integral invariant number \( \lambda \) and correspond to a series of discrete most degenerate representations of \( L(p, q) \) known as "ladder" representations.

To prove the irreducibility of the "ladder" representations we show that there is no invariant subspace of \( H^\Lambda \) with respect to the action of the generators \( M^\Lambda_A \) of \( L(p, q) \) defined by (3.4). We find directly that for the orthonormal basis \( \langle \alpha^\Lambda, \lambda_1, \lambda_2 \rangle \),

\[
C^{(\Lambda)} = \sum_{\Lambda = 0}^{p+q} M^\Lambda_A = -\sum_{i=1}^{p} a^+_i a^-_i + \sum_{i, j} b^+_i b^-_j + \sum_{i, j} a^+_i b^+_j + q,
\]

we see that the eigenvalues \( \lambda \) of \( C^{(\Lambda)} \) are integral. Further, in this realization the second order Casimir operator \( C^{(2)} \) is a function of \( C^{(\Lambda)} \), i.e.,

\[
C^{(2)} = C^{(\Lambda)} (C^{(\Lambda)} - p - q + 1).
\]
and the Cartan subalgebra $M_c^i$ and $M_c^j$ is diagonal with eigenvalues $-\lambda_j$ and $\lambda_{j+1}$, respectively.

We see that the space is closed and that starting with any basis function, we can pass to any other basis function by means of a finite number of applications of the generators $M_A^B$. Hence, there is no invariant subspace of $H^A$ with respect to the action of the $M_A^B$ as required for the irreducibility of the representation realized on $H^A$.

Schematically we see in Figure 1 that every point on a line of constant $\Lambda$ corresponds to a representation of $L(p) \overset{+}{L}(q)$, while the totality of points on such a line corresponds to a "ladder" representation of $L(p,q)$. The totality of lines represents the discrete series of "Ladder" representations. The action of the noncompact generators $M_c^i$ and $M_c^j$ is such as to in general move from one point on a line to either of the adjacent points, while the compact generators $M_A^i$ and $M_A^j$ cause no change in the position in Figure 1. The Bose annihilation and creation operators when applied individually cause transitions between points on different lines.
An examination of the commutation relations (3.1) shows that the set of generators $M^i$ commute with the set $M^j$ and that they satisfy the algebra of $U(p)$ and $U(q)$, respectively. These generators then correspond to the maximal compact subalgebra $L(p) + L(q)$, while the set $M^i$ and $M^j$ are the noncompact operators necessary to complete the algebra to that of $L(p,q)$. The decomposition with respect to subgroups of the type

$$U(p-q, q-l), \quad 0 \leq q \leq p, \quad 0 \leq l \leq q,$$

are readily performed (Dothan, Gell-Mann, and Ne'eman 1965; Olszewski 1965) because the set of commuting operators (the Cartan subalgebra $M^i$ and $M^j$) is diagonal in these representations.

4. "LADDER" REPRESENTATIONS OF $L(p,q)$ ON $H(E^{p+q})$

4.1. Realization of $L(p,q)$ on $H(E^{p+q})$ - basis functions

In this section, we present a realization of $L(p,q)$ in terms of $p+q$ real variables (domain $E^{p+q}$); in particular

$$a_{q} = \frac{1}{\sqrt{2}}(\frac{3}{2}z_{q}^{+} + z_{q}), \quad a_{q}^{+} = \frac{1}{\sqrt{2}}(-\frac{3}{2}z_{q}^{+} + z_{q}),$$

$$b_{q} = \frac{1}{\sqrt{2}}(\frac{3}{2}z_{q}^{+} + z_{q}), \quad b_{q}^{+} = \frac{1}{\sqrt{2}}(-\frac{3}{2}z_{q}^{+} + z_{q}),$$

where

$$z_{q}^{+} = z_{q}, \quad \left(\frac{3}{2}z_{q}\right)^{+} = -\frac{3}{2}z_{q}.$$  \hspace{1cm} (4.2)

The hermiticity condition (4.2) is in the sense of functional analysis, i.e., with respect to the appropriate scalar product in $H(E^{p+q})$. The forms (4.1) are the well-known differential forms of the step operators for the harmonic oscillator (Dirac 1947). Bargmann and Moshinsky (1960, 1961) and Moshinsky (1962) note both (4.1) and (3.4) only for the compact $U(n)$ symmetry of the $n$-dimensional isotropic harmonic oscillator which they then treat with the usual Fock-space treatment discussed in Section 4.3. Here we carry through with the forms (4.1) in order to obtain a function basis.

Equation (3.8) implies that the vacuum state $|0\rangle$ is separable in each of the variables $z_{q}$; hence we find that

$$|0\rangle = e^{-\frac{1}{2} \int_{1}^{p} z_{q}^{2} - \frac{1}{2} \int_{q=1}^{p+1} z_{q}^{2}}.$$  \hspace{1cm} (4.3)
This form of the vacuum is given only for the compact case by Bargmann and Moshinsky (1960, 1961).

The method of construction of the basis described in Section 3 directly yields that \( \Phi^{\lambda', \lambda_1, \lambda_2} \) is of the form

\[
\Phi^{\lambda', \lambda_1, \lambda_2} = \sum_{j=1}^{p+q} \frac{\eta_j}{\sqrt{\lambda_j \lambda_1 \lambda_2 \lambda_j}} \left( \frac{\lambda_j}{\lambda_1} \right)^{\frac{\lambda_j^2}{2}} \left( \frac{\lambda_j}{\lambda_1} \right)^{\frac{\lambda_j^2}{2}} \, e^{-\frac{1}{2} x_j^2},
\]

(4.4)

where \( \eta_j \) is a polynomial in \( x \). The solution of the eigenvalue equation for the first order Casimir operator

\[
\mathcal{C}_1 \Phi^{\lambda', \lambda_1, \lambda_2} = \sum_{j=1}^{p+q} \frac{\eta_j}{\sqrt{\lambda_j \lambda_1 \lambda_2 \lambda_j}} \left( \frac{\lambda_j}{\lambda_1} \right)^{\frac{\lambda_j^2}{2}} \left( \frac{\lambda_j}{\lambda_1} \right)^{\frac{\lambda_j^2}{2}} \, e^{-\frac{1}{2} x_j^2},
\]

(4.5)

plus the condition that the \( \Phi^{\lambda', \lambda_1, \lambda_2} \) are square integrable, yields (Szegö 1959)

\[
\mathcal{A} \cdot \Phi^{\lambda', \lambda_1, \lambda_2} = \sum_{j=1}^{p+q} \frac{\eta_j}{\sqrt{\lambda_j \lambda_1 \lambda_2 \lambda_j}} \left( \frac{\lambda_j}{\lambda_1} \right)^{\frac{\lambda_j^2}{2}} \left( \frac{\lambda_j}{\lambda_1} \right)^{\frac{\lambda_j^2}{2}} \, e^{-\frac{1}{2} x_j^2},
\]

(4.6)

where \( \mathcal{A} = -\sum_{j=1}^{p+q} \frac{\lambda_j^2}{2} + \sum_{j=1}^{p+q} \lambda_j \lambda_j \) and the \( \mathcal{H}_n(z) \) are Hermite polynomials.

This solution is, of course, well known for the one-dimensional harmonic oscillator (Schiff 1949).

The set of all \( \Phi^{\lambda', \lambda_1, \lambda_2} \) for a fixed \( \lambda \) given by (4.6) constitutes an orthonormal basis for a Hilbert space \( \mathcal{H}^{\lambda} (E^{p+q}) \) with respect to scalar product

\[
(\Phi^{\lambda', \lambda_1, \lambda_2}, \Phi^{\lambda', \lambda_1', \lambda_2'}) = \int \Phi^{\lambda', \lambda_1, \lambda_2} \Phi^{\lambda', \lambda_1', \lambda_2'} \, d\mu(E^{p+q})
\]

(4.7)

\[
= \prod_{j=1}^{p+q} \delta_{\lambda_j, \lambda_j'} \delta_{\lambda_1, \lambda_1'} \delta_{\lambda_2, \lambda_2'}. \]
where the measure $d\mu(E^{p+q})$ is given by

$$
\int E^{p+q} = \prod_{j=1}^{q} \int \frac{\rho}{\gamma_j} d\gamma_j \frac{\rho^{p+q}}{\gamma_j^{p+q}} d\gamma_j
$$

(4.3)

The hermiticity condition (4.2) is satisfied by this scalar product.

The representation of $L(p,q)$ realized on $H^{A}$ is irreducible since by direct calculation we verify that the action of the generators $E_A^{B}$ on the basis functions (4.6) is given by (3.9).

4.2. Connection between the hermiticity condition and the measure

The connection between the hermiticity condition contained in (3.5) and the measure on $E^{p+q}$ can be readily seen if we consider another realization of $L(p,q)$:

$$
\alpha_j = \frac{1}{\sqrt{2}} \delta_j^1, \beta_j = \frac{1}{\sqrt{2}} \delta_j^1, \gamma_j = \frac{1}{\sqrt{2}} \delta_j^1, \delta_j = \frac{1}{\sqrt{2}} \delta_j^1.
$$

(4.9)

Proceeding in exactly the same manner as in the preceding sections we find that the vacuum state $|\psi>\rangle$ is given by

$$
|\psi>\rangle = \prod_{j=1}^{q} \left( z^j i \lambda_j (x_j) \gamma_j^{p+q} \right) \mu_j (x_j),
$$

(4.10)

and are orthonormal w.r.t. the scalar product (4.7), but with a measure $d\mu(E^{p+q})$ given by

$$
\int E^{p+q} = \prod_{j=1}^{q} \int \frac{\rho}{\gamma_j} d\gamma_j \frac{\rho^{p+q}}{\gamma_j^{p+q}} d\gamma_j
$$

(4.12)

Therefore, we see that if we express

$$
\int E^{p+q} = \prod_{j=1}^{q} \int \frac{\rho}{\gamma_j} d\gamma_j \frac{\rho^{p+q}}{\gamma_j^{p+q}} d\gamma_j
$$

(4.13)

then the only difference between the basis functions obtained here and those obtained in Section 4.1 lies in the interchange of the functional forms of the vacuum state, and the weight function $W(x)$ for the measure.
This illustrates a general result, namely that the connection between the functional form of the vacuum and that for the weight function in the measure is determined by the hermiticity condition contained in (3.5).

An example of the nonexistence of the required measure is given by

\[ a_j^+ = \frac{\partial}{\partial x_j}, \quad a_j^+ = \frac{\partial}{\partial q_j}, \quad b_j^+ = \frac{\partial}{\partial \bar{q}_j} \quad (4.14) \]

The commutation relations (3.5) are satisfied, but there does not exist a measure such that

\[ \chi^+ = \frac{\partial}{\partial x} \quad (4.15) \]

4.3. Connection with the "covariant oscillator" in a real Minkowski space

As an important aside, we point out the connection between the results of Section 4 for the domain \( E^{p+q} \) and the "covariant harmonic oscillator" (Santilli, 1966) in a real Minkowski space \( M^{p,q} \) with \( p \) space-like coordinates and \( q \) time-like coordinates; i.e., the metric tensor \( g_{\alpha\beta} \) for \( M^{p,q} \) is given by

\[ g_{\alpha\beta} = \gamma_{\alpha\beta} \quad (4.16) \]

where \( \gamma \) is defined by (3.3). This connection is inherent in equation (4.5) and can be readily seen if instead of defining the algebra on \( \mathbb{H}(E^{p+q}) \) we define the algebra on \( \mathbb{H}(\mathbb{R}^{p,q}) \) as follows:

\[ a_j = \frac{1}{\sqrt{2}} \left( -\frac{i}{\sqrt{2}} \cdot \gamma_j \right), \quad a_j^+ = \frac{1}{\sqrt{2}} \left( \frac{i}{\sqrt{2}} \cdot \gamma_j \right), \quad (4.17) \]

\[ b_j = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_j} + \gamma_j \right), \quad b_j^+ = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial q_j} + \gamma_j \right). \]

Then the only nonzero commutators are

\[ [a_j^+, a_j] = \delta_k^j, \quad [b_k, b_k^+] = \delta_k^j. \quad (4.18) \]

The generators are then given by the array

\[ \begin{pmatrix} M^i_j \mathbf{M}_j^i \mathbf{M}_j \end{pmatrix} = \begin{pmatrix} a_j^+ a_j^+ & a_j^+ b_j^+ \end{pmatrix} = \begin{pmatrix} -a_j^+ a_j & a_j^+ b_j^+ \end{pmatrix} \]

\[ \begin{pmatrix} a_j^+ a_j^+ & a_j^+ b_j^+ \end{pmatrix} = \begin{pmatrix} -b_j^+ a_j & b_j^+ b_j^+ \end{pmatrix} \quad (4.19) \]

which satisfies conditions (3.1) and (3.2).
Now if we define

$$P^A = -i \partial^A, \quad (4.20)$$

then we obtain

$$[x^B, p^A] = i \delta^A_B, \quad (4.21)$$

which are the usual commutation relations for coordinates and their canonically conjugate momenta. In terms of $p^A$ and $x^A$ the first order Casimir operator becomes

$$c^{(0)} = \sum_{A=1}^{p+q} \left( p^A p_A + x^A x_A \right) + \frac{p+q}{2}, \quad (4.22)$$

which, apart from the additive constant $p+q$, is the Hamiltonian of the "covariant oscillator". Hence, the basis functions given by (4.6) are also eigenfunctions for the covariant oscillator Hamiltonian.

5. "LADDER" REPRESENTATIONS OF $L(p,q)$ ON $H(C^{p+q})$

5.1. Realization of $L(p,q)$ on $H(C^{p+q})$ - basis functions

In this section we present the results of a realization of $L(p,q)$ in terms of $p+q$ complex variables (domain $C^{p+q}$). In particular, this realization is built with

$$a_i = \frac{i}{\hbar} \left( \frac{\partial}{\partial x_i} + z_i \right), \quad a_i^\dagger = \frac{1}{\hbar} \left( -\frac{\partial}{\partial z_i} + \bar{z}_i \right), \quad (5.1)$$

$$b_i = \frac{i}{\hbar} \left( \frac{\partial}{\partial z_i} + \bar{z}_i \right), \quad b_i^\dagger = \frac{1}{\hbar} \left( -\frac{\partial}{\partial z_i} + x_i \right),$$

where

$$\left( \frac{\partial}{\partial x_A} \right)^\dagger = \bar{x}_A, \quad \left( \frac{\partial}{\partial \bar{x}_A} \right)^\dagger = -\frac{\partial}{\partial x_A}. \quad (5.2)$$

Here we use $z$ and $\bar{z}$ in the same sense as Ráczka and Fischer (1966), i.e., the functions we treat are not to be considered as analytical functions of $z$, but rather are to be regarded as differentiable functions of $x$ and $y$, where

$$z = x + iy, \quad \bar{z} = x - iy. \quad (5.3)$$
defines a linear transformation from the variables $x$ and $y$ to the variables $z$ and $w$. According to this definition,
\[
\frac{\partial z}{\partial x} = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x y} \right), \quad \frac{\partial z}{\partial y} = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x y} \right),
\]
and $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ are defined at every point and are identically zero.

The vacuum state $|0\rangle$ is separable in the pairs $(z_A, \omega_A)$ and
we find that
\[
|0\rangle = \prod_j \left( \frac{\partial}{\partial z_j} \right)^{p_j} \left( \frac{\partial}{\partial \omega_j} \right)^{\tilde{p}_j} \prod_j \frac{1}{\lambda_j} \exp \left( \sum_j \frac{z_j}{\lambda_j} \right) \left( \sum_j \frac{\omega_j}{\lambda_j} \right).
\]

The construction of a basis yields that every basis function is a product of monomials in $z_j$ and $\omega_j$ and the vacuum state (5.4). The solution of the eigenvalue equation for the first order Casimir operator
\[
(C^{(0)})_{\Phi}^{\lambda_1, \lambda_2, \lambda_3} = \prod_j \left( \frac{\partial}{\partial z_j} \right)^{p_j} \left( \frac{\partial}{\partial \omega_j} \right)^{\tilde{p}_j} \prod_j \frac{1}{\lambda_j} \exp \left( \sum_j \frac{z_j}{\lambda_j} \right) \left( \sum_j \frac{\omega_j}{\lambda_j} \right),
\]

yields
\[
\Phi_{\lambda_1, \lambda_2, \lambda_3} = \frac{\rho}{\pi^{\lambda_1+\rho} \lambda_1!} \int \prod_{j=1}^{\rho} \left( \frac{\partial}{\partial \lambda_j} \right)^{\lambda_j} \frac{1}{\lambda_j!} \prod_{j=1}^{\rho} \frac{1}{\lambda_j!} \prod_{j=1}^{\rho} \frac{1}{\lambda_j!} \exp \left( \sum_j \frac{z_j}{\lambda_j} \right) \left( \sum_j \frac{\omega_j}{\lambda_j} \right)
\]

where
\[
\lambda = -\sum \frac{\lambda_j}{\lambda_j} + \tilde{\lambda} + \rho + \tilde{\rho}.
\]

The set of all $\left\{ \Phi_{\lambda_1, \lambda_2, \lambda_3} \right\}$ for a fixed $\lambda$ given by (5.6) constitutes an orthonormal basis for a Hilbert space $H^{(\rho+\tilde{\rho})}$ with respect to a scalar product
\[
\langle \Phi_{\lambda_1, \lambda_2, \lambda_3}, \Phi_{\lambda_1', \lambda_2', \lambda_3'} \rangle = \int \prod_{j=1}^{\rho} \frac{1}{\lambda_j!} \prod_{j=1}^{\rho} \frac{1}{\lambda_j!} \delta_{\lambda_j, \lambda_j'} d\lambda_j d\lambda_j' d\lambda_j'' d\lambda_j''\]
\[
= \prod_{j=1}^{\rho} \delta_{\lambda_j, \lambda_j'} \prod_{j=\tilde{\rho}+1}^{\tilde{\rho}} \delta_{\lambda_j, \lambda_j'},
\]

-10-
where the measure \( d\mu(C^{\ast\ast}) \) is given by

\[
d\mu(C^{\ast\ast}) = \prod_{j=1}^{p} d\zeta_j d\bar{\zeta}_j \prod_{j=1}^{q} d\zeta_j d\bar{\zeta}_j , \tag{5.6}
\]

and the connection between \((\zeta, \bar{\zeta})\) and \((x, y)\) is given by (5.3). The hermiticity condition (5.2) is satisfied by the scalar product (5.8).

The representation of \( L(p,q) \) realized on \( H(C^{p+q}) \) is irreducible, since by direct calculation we verify that the action of the generators \( X_A^j \) on the basis functions (5.6) is given by (3.9).

### 5.2. Fock-space functions

Exactly as in Section 4.2, we can interchange the functional forms of the vacuum state \( |0\rangle \) and the weight function in the measure with the following realization of the algebra \( L(p,q) \):

\[
a_j^+ = \frac{2}{\pi} \zeta_j^2 , \quad a_j^+ = \bar{\zeta}_j , \quad b_j^+ = \frac{2}{\pi} \zeta_j , \quad b_j^+ = \zeta_j . \tag{5.9}
\]

This then yields a set of basis functions given by equation (5.9) with the exponentials absent (i.e., monomials in \( \zeta_j \) and \( \bar{\zeta}_j \)), a vacuum state

\[
|0\rangle = 1 , \tag{5.10}
\]

and a scalar product given by (5.7), but with a measure \( d\mu(C^{p+q}) \) given by

\[
d\mu(C^{p+q}) = \prod_{j=1}^{p} e^{-\zeta_j^2} d\zeta_j d\bar{\zeta}_j \prod_{j=1}^{q} e^{-\zeta_j^2} d\zeta_j d\bar{\zeta}_j . \tag{5.11}
\]

This approach corresponds to the usual Fock-space treatment (Todorov 1966; Bargmann and Moshinsky 1960, 1961; Moshinsky, 1962).

### 5.3. Connection with a complex Minkowski space

There is a connection between the results of Sections 5.1 and 5.2 for a domain \( C^{p+q} \) and a realization of \( L(p,q) \) on a complex Minkowski space \( \mathcal{M}^{p+q} \) with a metric tensor given by (4.21). In particular, we can obtain a realization of \( L(p,q) \) for the domain \( \mathcal{M}^{p+q} \) with

\[
a_j^+ = \frac{1}{\sqrt{2}} \left( \frac{-2}{\sqrt{2}} \zeta_j + \bar{\zeta}_j \right) , \quad a_j^+ = \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{2}} \bar{\zeta}_j + \zeta_j \right) , \tag{5.12}
\]

\[
b_j^+ = \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{2}} \zeta_j + \bar{\zeta}_j \right) , \quad b_j^+ = \frac{1}{\sqrt{2}} \left( \frac{-2}{\sqrt{2}} \bar{\zeta}_j + \zeta_j \right) .
\]
The first order Casimir operator is then
\[ \mathcal{C}^{(1)} = \sum_{i j} a_i a_j^t + \sum_{i \neq j} b_i b_j^t. \] (5.13)

The important observation here is that the notation employed in Sections 5.1 and 5.2 is also the appropriate notation for \( \mathcal{Y}^{p,q} \).

6. REPRESENTATIONS OF A DIRECT SUM OF TWO \( L(p,q) \) ALGEBRAS ON \( \mathfrak{M}(\mathbb{C}^{p+q}) \)

Another functional class of basis functions can be obtained by considering the direct sum of two \( L(p,q) \) algebras of the type discussed in Section 5. Let us denote the generators of the direct sum as \( Q^B \); then we set
\[ Q^B = M^A + N^A, \] (6.1)

where the \( M^A \) are given by equation (5.1) and the \( N^A \) are given by
\[ \begin{pmatrix} \begin{pmatrix} b_i b_j^t & - b_i a_j^t \\ x_i^t b_j^t & - a_i^t a_j^t \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} b_i b_j^t & - b_i a_j^t \\ x_i^t b_j^t & - a_i^t a_j^t \end{pmatrix} \end{pmatrix}. \] (6.2)

The convention is
\[ a_A = \frac{1}{\sqrt{2}} \left( \frac{\mathbb{A}}{2} + \mathbb{A} \right), \quad a_A^t = \frac{1}{\sqrt{2}} \left( - \frac{\mathbb{A}}{2} + \mathbb{A} \right), \quad b_A = \frac{1}{\sqrt{2}} \left( \frac{\mathbb{A}}{2} + \mathbb{A} \right), \quad b_A^t = \frac{1}{\sqrt{2}} \left( - \frac{\mathbb{A}}{2} + \mathbb{A} \right), \] (6.3)

where \( A, B = i, j, \hat{i}, \hat{j} \) and \( i, \hat{i} = 1, \ldots, p; j, \hat{j} = p+1, \ldots, p+q \).

We see that \( M^A \) are also a realization of \( L(p,q) \) and
\[ [M^A, N^D] = 0 \] (6.4)

The vacuum state \( |0\rangle \) is separable in the pairs \( (Z_A, \bar{Z}_A) \) and we find that
\[ |0\rangle = e^{-\frac{p \epsilon}{2} \bar{Z}_A Z_A \epsilon}, \] (6.5)

where the \( Z_A \) and \( \bar{Z}_A \) are to be treated in the sense discussed in Section 5.1.
According to the construction of the "ladder" representations \( a_\alpha^+ a_\beta, b_\alpha^+ b_\beta \), and hence the first order Casimir operator \( C^{(1)} \), must be diagonal, where

\[
\langle 0| = \sum_{\alpha=1}^{p+q} \left( \frac{\partial}{\partial z_\alpha^2} - \frac{\partial}{\partial \bar{z}_\alpha^2} \right) + 1. \tag{6.6}
\]

Because of these requirements and the separability of the basis functions in the pairs \((z_\alpha^2, \bar{z}_\alpha^2)\), we are then led to the two simultaneous eigenvalue equations

\[
\left( \frac{\partial}{\partial z_\alpha^2} - \frac{\partial}{\partial \bar{z}_\alpha^2} \right) f(z_\alpha^2, \bar{z}_\alpha^2) = (\lambda_\alpha^2 - 1) f(z_\alpha^2, \bar{z}_\alpha^2), \tag{6.7}
\]

and

\[
\left( - \frac{\partial^2}{\partial z_\beta^2 - \partial \bar{z}_\beta^2} \right) f(z_\beta^2, \bar{z}_\beta^2) = \left( \lambda_\beta^2 + 1 \right) f(z_\beta^2, \bar{z}_\beta^2). \tag{6.8}
\]

This identifies \( \lambda_\alpha \) and \( \lambda_\beta \) as the usual occupation numbers.

Introducing polar coordinates \((r, \theta)\), we obtain from equation (6.7) that

\[
- \frac{1}{r} \frac{\partial}{\partial \theta} \Theta(\theta) = (\lambda_\alpha^2 - \lambda_\alpha) \Theta(\theta), \tag{6.9}
\]

and from (6.8)

\[
\left[ \frac{\lambda_\beta^2}{r^2} + \frac{1}{r} \frac{\partial}{\partial r} \frac{\lambda_\beta^2 - \lambda_\alpha^2}{r^2} - \frac{4}{r^2} \frac{(\lambda_\beta^2 + \lambda_\alpha^2)}{r^2} \right] R(r) = 0, \tag{6.10}
\]

where

\[
\Theta(z_\alpha^2, \bar{z}_\alpha^2) = R(r) \Theta(\theta), \tag{6.11}
\]

The solution of (6.10) is (Kamke 1961; Szegö 1959)

\[
R = \frac{L^{\lambda_\beta - \lambda_\alpha} / \lambda_\alpha!}{L^{\lambda_\beta} / \lambda_\alpha! \lambda_\alpha^2 (2r^2)} \left[ (-1)^{\lambda_\alpha - \lambda_\alpha} \right] \left[ r^{\lambda_\beta - \lambda_\alpha} / \lambda_\alpha! \lambda_\alpha^2 \right] \left[ e^{-r^2 / 2} / \lambda_\alpha! \right], \tag{6.12}
\]

for all integral values of \( \lambda_\alpha - \lambda_\alpha \), where \( L^{\lambda_\alpha} \) is a generalized Laguerre polynomial. From the basic construction \( \lambda_\alpha \) and \( \lambda_\beta \) are non-negative integers.
Hence, the basis functions are given by

\[ \Phi^{\lambda, \lambda_{A}, \lambda_{B}} = \frac{\rho_{+}+\rho_{-}}{\pi} \sqrt{\frac{\lambda_{A!}}{\pi \lambda_{B!}}} \ e^{i \alpha_{A} r_{A}} e^{-r_{A}} \ e^{\lambda_{A} \left( z f_{A} \right)} \]  (6.13)

where

\[ \lambda = \sum_{A=1}^{\rho_{+}+\rho_{-}} \alpha_{A} + \rho + \rho \]

and

\[ \alpha_{A} = \lambda_{B} - \lambda_{A} \]

We note that (6.13) can also be expressed in terms of \( \bar{z}_{A} \) and \( \bar{z}_{A} \).

The set of all \( \{ \Phi^{\lambda, \lambda_{A}, \lambda_{B}} \} \) for a fixed \( \lambda \) given by equation (6.13) constitutes an orthonormal basis for a Hilbert space \( H^{\lambda}(C^{p+q}) \) with respect to the scalar product

\[ (\Phi^{\lambda, \lambda_{A}', \lambda_{A}}, \Phi^{\lambda, \lambda_{B}', \lambda_{B}}) = \int_{C^{p+q}} \Phi^{\lambda, \lambda_{A}', \lambda_{B}} \Phi^{\lambda, \lambda_{A}', \lambda_{B}} d\mu(C^{p+q}) \]

where the measure \( d\mu(C^{p+q}) \) is given by

\[ d\mu(C^{p+q}) = \frac{\rho_{+}+\rho_{-}}{\pi} \ d\alpha_{A} \ dr_{A} \]

(6.15)

The representation of the direct sum realized on \( H^{\lambda}(C^{p+q}) \) is irreducible since by direct calculation we can verify that there is no invariant subspace of \( H^{\lambda}(C^{p+q}) \) under the action of the generators \( A_{A}^{B} \).

We have omitted the explicit expressions for the action of the \( A_{A}^{B} \) on the basis functions given by (6.13) and only point out that here every basis function is transformed into a sum of two basis functions which follows from the separate action of the \( M_{A}^{B} \) and the \( N_{A}^{B} \).
As a variation of the technique of this section, we can realize a "ladder" representation of $L(p,p)$ on $H(C^p)$ as follows:

$$
a_j^+ = \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{2}} z_j + \bar{z}_j \right), \quad a_j^- = \frac{1}{\sqrt{2}} \left( -\frac{2}{\sqrt{2}} z_j + \bar{z}_j \right),$$

$$b_j^+ = b_j^- = \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{2}} z_j + \bar{z}_j \right).$$

The basis functions are then given by (6.13), but where the sum runs only from 1 to $p$. Further, as in Section 4.2, we can interchange the functional forms of the vacuum state and the weight function in the measure with the alternative realization of $L(p,q)$ on $H(C^p)$

$$\hat{a}_j^+ \hat{a}_j^- = \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{2}} z_j + \bar{z}_j \right), \quad b_j^+ b_j^- = \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{2}} z_j + \bar{z}_j \right).$$

7. EXISTENCE OF GLOBAL UNITARY REPRESENTATIONS

Following Gel'fand and Graev (1966), we construct a skew-hermitian basis $X_A^B$ for $L(p,q)$:

$$X_A^k = i M_A^{\ast k}, \quad X_A^\hat{k} = i \hat{M}_A^{\ast k}, \quad (no \ sum)$$

$$X_A^k = i (M_A^k - M_A^{\ast k}), \quad X_A^\hat{k} = M_A^k - M_A^{\ast k}, \quad \hat{k} < k$$

$$X_A^{\hat{k}} = i (\hat{M}_A^k + \hat{M}_A^{\ast k}), \quad X_A^{\hat{\hat{k}}} = \hat{M}_A^k + \hat{M}_A^{\ast k}, \quad \hat{k} < \hat{k}$$

where $k, l = 1, \ldots, p$ and $\hat{k}, \hat{\hat{k}} = p+1, \ldots, p+q$. Since the set $D(X_A^B) = \{ \overrightarrow{X_A^k}, \nu^k, \nu^{\hat{k}} \}$ is dense on the Hilbert space $H^A$, by construction of $H^A$ and because of the hermiticity condition (3.2), and the action of the generators $M_A^B$ given by (3.9), the $X_A^B$ are skew-hermitian on the common dense domain $D(X_A^B)$ which is also invariant.

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According to Nelson's Theorem 5 (1959), if \( \Delta = \sum_{A_B=1}^{p+q} \lambda_A^B \lambda_A^B \) is essentially self-adjoint, then a representation of the \( \lambda_A^B \) on \( H^A \) can be integrated to a unique unitary strongly continuous representation of the universal covering group having the \( \lambda_A^B \) as its Lie algebra. By direct calculation we find that every basis function \( \Phi^{\lambda_1,\lambda_2} \) is an eigenfunction of \( \Delta \), i.e.,

\[
\Delta \Phi^{\lambda_1,\lambda_2} = -\frac{1}{2} \left( (\lambda_1,\lambda_2,\lambda_3) \right) \Phi^{\lambda_1,\lambda_2,\lambda_3},
\]

where

\[
f(\lambda_1,\lambda_2,\lambda_3) = \sum_{j=1}^{p+q} \lambda_j^2 + \frac{\sum_{j=1}^{p+q} \lambda_j}{\lambda_j} (\lambda_1+1) + \frac{\sum_{j=1}^{p+q} \lambda_j}{\lambda_j} (\lambda_2+1) + \frac{\sum_{j=1}^{p+q} \lambda_j}{\lambda_j} (\lambda_3+1) + \frac{\sum_{j=1}^{p+q} \lambda_j}{\lambda_j} (\lambda_4+1) \geq q(2p+1).
\]

Hence, we see that the range of \( \Delta \) is again the dense set \( \mathcal{D}(\lambda_A^B) \), \( \Delta \) is symmetric on this dense set, and \( f \) is semi-bounded from below.

Here we consider \( p,q > 1 \). There is a lemma (Doebner and Heilsheimer 1966) which states that the operator \( \Delta \) which is symmetric on an invariant dense domain \( \mathcal{D}(\lambda_A^B) \) on \( H^A \) is essentially self-adjoint because the range of \( \Delta \) is dense on \( H^A \) and \( (\Delta \Phi, \Phi) > a(\Phi, \Phi) \) for all \( \Phi \in \mathcal{D}(\lambda_A^B) \) where \( a > 0 \) and \( \Phi \) is independent of \( \Phi \). Thus, by virtue of Nelson's Theorem 5, our "ladder" representations can be integrated to global unitary strongly continuous representations of the universal covering group with \( \lambda_A^B \) as its Lie algebra. The irreducibility is a consequence of the irreducibility of the representations of the \( \mathbb{N}_A^B \) and hence the \( \lambda_A^B \) in the sense that \( \mathcal{D}(\lambda_A^B) \) contains no invariant subspace.

5. COUPLING OF TWO \( U(1,1) \) "LADDER" REPRESENTATIONS

For the discussion of the coupling of \( U(1,1) \) "ladder" representations, it is convenient to adopt the Fock-space approach given in Section 5.2, but with a renormalized measure

\[
d\mu(c^2) = e^{-\frac{1}{2}c^2} \frac{1}{c^2} dc^2 dc_x^2 dc_y^2 dc_\eta^2 d\eta^2.
\]
and hence renormalized eigenfunctions

\[
\Phi_{\lambda_a, \lambda_b} = \frac{\lambda_a^\ast \lambda_b}{\sqrt{\pi \lambda_a} \sqrt{\pi \lambda_b}},
\]

where

\[
\lambda = -\lambda_a + \lambda_b + 1.
\]

Further, we have for \( L(1,1) \)

\[
\begin{pmatrix}
M_1^1 & M_2^1 \\
M_1^2 & M_2^2
\end{pmatrix} = \begin{pmatrix}
-a^\ast_a & a^\dagger b^\dagger \\
-b a & bb^\dagger
\end{pmatrix} = \begin{pmatrix}
-\frac{2}{\sqrt{2}} & \sqrt{2} \gamma \\
-\frac{2}{\sqrt{2}} & \sqrt{2} \gamma
\end{pmatrix}.
\]

We now consider the direct product \( T^{\lambda_1} \otimes T^{\lambda_2} \) of two irreducible representations of \( U(1,1) \). In order not to obscure the simplicity of the result, let us consider \( \lambda_1, \lambda_2 \geq 1 \); then an orthonormal basis for the product space \( \mathcal{H}^{\lambda_1} \otimes \mathcal{H}^{\lambda_2} \) is

\[
\left\{ \Phi_{\lambda_1, \lambda_2, \lambda_a, \lambda_b}, \Phi_{\lambda_1, \lambda_2, \lambda_a, \lambda_b} \right\} \quad (8.5)
\]

where equation (8.3) is satisfied. This apparent limitation, i.e., \( \lambda_1, \lambda_2 \geq 1 \) can be removed, and will be discussed in our subsequent work. Since the direct product \( T^{\lambda_1} \otimes T^{\lambda_2} \) corresponds to the direct sum of the algebras, we have

\[
M^B_\lambda = M^A_\lambda \otimes I_2 + I_2 \otimes M^B_\lambda \quad (8.6)
\]

where \( \lambda, \beta = 1, 2 \), hence, if there exists a "ladder" representation in the product space

\[
\lambda_a = \lambda_{a_1} + \lambda_{a_2} = 0, 1, 2, \ldots
\]

and

\[
\lambda_b = \lambda_{b_1} + \lambda_{b_2} + 1 = \lambda_{b_{a_1}} + \lambda_{b_{a_2}} + 1 = \lambda_{b_{a_1}} + \lambda_{b_{a_2}} + 2, \ldots
\]

and

\[
\lambda = \lambda_1 + \lambda_2
\]

(8.8)
Equation (8.8) states that in the direct product there exists at most one "ladder" representation (apart from considerations of multiplicity). Note we have not excluded the possibility of the existence of less degenerate representations in the direct product.

Therefore, we proceed in the usual way (e.g. Gel'fand, Minlos, and Shapiro 1963), i.e., let

\[ \Phi_{\lambda_1 + \lambda_2, \lambda} = \sum_{\lambda_1', \lambda_2'} \beta_{\lambda_1', \lambda_2'; \lambda_1', \lambda_2', \lambda} \Phi_{\lambda_1', \lambda_2', \lambda}, \]

(8.9)

and then we must determine the Clebsch-Gordan coefficients \( \beta_{\lambda_1', \lambda_2'; \lambda_1', \lambda_2', \lambda} \).

Here the problem is simple because if there is a "ladder" representation in the direct product, then there is a vector state such that

\[ M_2 \Phi_{\lambda_1 + \lambda_2, 0} = 0, \]

(8.10)

and moreover

\[ \Phi_{\lambda_1 + \lambda_2, 0}^{\lambda_1', \lambda_2', 0} = \Phi_{\lambda_1', \lambda_2', 0}^{\lambda_1, \lambda_2, 0} = \frac{(x_1^0 y_1^{\lambda_1-1}) (x_2^0 y_2^{\lambda_2-1})}{f \sigma_1! f \sigma_2! f \sigma_3! f \sigma_4!}, \]

(8.11)

where

\[ 10_1 = 1, \quad \text{and} \quad 10_2 = 1, \]

i.e.,

\[ \beta_{\lambda_1 + \lambda_2, 0}^{\lambda_1', \lambda_2', 0} = 1 \quad \text{(within a phase factor)}. \]

(8.12)

For the group \( U(1,1), \) \( \lambda_1^2 \) and \( \lambda_2^1 \) are respectively the raising and lowering operators between adjacent \( U(1) \otimes U(1) \) representations, hence

\[ (M_1^2)^{\lambda} \Phi_{\lambda_1 + \lambda_2, 0} \propto \Phi_{\lambda_1 + \lambda_2, \lambda} \]

(8.13)

or

\[ (x_1^0 y_1^{\lambda_1-1} + 1, \sigma x_2^0 y_2^{\lambda_2-1})^{\lambda} \frac{(x_1^0 y_1^{\lambda_1-1}) (x_2^0 y_2^{\lambda_2-1})}{f \sigma_1! f \sigma_2! f \sigma_3! f \sigma_4!} \propto \Phi_{\lambda_1 + \lambda_2, \lambda} \]

(8.14)
Therefore, the orthonormal set \( \left\{ \Phi_{\lambda_1,\lambda_2,\lambda_3} \right\} \) is given by

\[
\Phi_{\lambda_1,\lambda_2,\lambda_3} = \sum_{n=0}^{\infty} \mathcal{B}_{\lambda_1,\lambda_2-n;\lambda_3,n} \Phi_{\lambda_1,\lambda_2-n;\lambda_3,n},
\]

where the Clebsch-Gordan coefficients are given by

\[
\mathcal{B}_{\lambda_1,\lambda_2-n;\lambda_3,n} = \frac{\left( \frac{\lambda_1-1+\lambda_2-n}{2} \right) \left( \frac{\lambda_2-1}{2} \right)}{\left( \frac{\lambda_1-1}{2} \right) \left( \frac{\lambda_2-1}{2} \right)} \frac{\lambda_3}{\lambda_3-n}. \tag{8.16}
\]

Note for \( \lambda_1 = \lambda_2 = 1 \), equation (8.16) greatly simplifies, i.e.,

\[
\mathcal{B}_{\lambda_1,\lambda_2-n;\lambda_3,n} = \frac{1}{\lambda_3-n}. \tag{8.17}
\]

From (8.15), we conclude that the entire product space is a carrier space for one "ladder" representation and, therefore, that this "ladder" representation occurs only once in the direct product of two \( U(1,1) \) "ladder" representations. There exist other less degenerate representations and a manifestation of this is that each \( F_{\lambda_1,\lambda_2,\lambda_3} \) appears in only one basis vector \( \Phi_{\lambda_1,\lambda_2,\lambda_3} \).

The coupling of three \( U(1,1) \) representations can then be accomplished, if with the direct product of any two of the representations, a representation contragredient to the third can be formed.

9. REMARKS AND CONCLUSIONS

The maximal set of commuting operators for the "ladder" representations of \( U(p,q) \) consists of \( p+q \) operators, only the generators of the Cartan subalgebra, in contrast to the set for the non-degenerate representations which consists of \( (p+q)(p+q+1) \) operators, e.g., for \( U(6,6) \), we have 12 instead of 78 commuting operators. The "ladder" representations are a series of most degenerate representations, hence the number of commuting operators is maximally reduced and this is important for physical applications, since it minimizes the number of operators that must be interpreted as physical observables (Raczka 1965). The spectra of the commuting operators for the "ladder" representations is discrete; further, the decomposition with respect to the maximal compact subgroup \( U(p) \oplus U(q) \) is readily obtained from Figure 1.
In fact, the representations of $U(p) \otimes U(q)$ are totally symmetric representations.

In Sections 4 and 5, we have determined explicit function bases for irreducible "ladder" representations of the algebra $L(p,q)$ of the group $U(p,q)$, and in Section 7 we have shown that these representations can be integrated to irreducible unitary global strongly continuous representations of the universal covering group.

An interesting result of this initial investigation of the $U(p,q)$ coupling problem is the fact that there exists only one "ladder" representation in the direct product of two "ladder" representations.

In the general case of the coupling of $U(p,q)$ representations, the direct product of two "ladder" representations contains less degenerate non-"ladder"-type representations and at most one "ladder" representation. These will be considered in future papers.

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