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INTERNATIONAL ATOMIC ENERGY AGENCY

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ABSTRACT

The discrete and continuous series of infinite-dimensional irreducible representations of Lie algebras of compact unitary groups are derived and their properties are discussed. The possible physical applications are pointed out.

INFINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF LIE ALGEBRAS OF COMPACT UNITARY GROUPS

1. INTRODUCTION

It is rather well known that every representation of a compact topological group G realized on a Hilbert space $\mathcal{H}$ is equivalent to a unitary representation [1] and moreover every irreducible unitary representation is finite dimensional [2]. However, if we extend the class of the linear spaces in which a representation of the group G can be realized, we may obtain new classes of irreducible representations which are no longer finite dimensional. In fact the method of extending a class of irreducible representations of a given group G by extending a class of carrier spaces is well known in the case of non-compact groups. For example, if we consider the irreducible representations of the SL(2, C) group in the linear topological spaces of homogeneous functions we obtain the extended class of irreducible representations in which the irreducible unitary representations correspond to only a small subclass [3].

In this work we show that the infinite-dimensional vector spaces with an indefinite metric appear naturally as carrier spaces of infinite-dimensional irreducible representations of the Lie algebra of the compact unitary groups $U(p)$. The topological and geometrical properties of these spaces are completely determined by the bilinear form related to the indefinite metric.

The problem of an extension of the representation theory of compact groups was raised after the appearance of Regge's classic paper in which the concept of complex angular momenta is introduced [4]. In a series of papers the group theoretical aspects of this problem have been discussed [5] and the existence of local as well as global infinite-dimensional representations of the rotation SO(3) group has been proved [6].

In this work we consider the properties of the irreducible infinite-dimensional representations of the Lie algebras R_p of the unitary groups $U(p), (p=2,3,\dots)$. In Sec. 2 we give a short review of the properties of degenerate irreducible finite-dimensional representations of $U(p)$ groups and the related harmonic functions. The properties of infinite-dimensional irreducible representations of the Lie algebras of $U(p)$ groups are discussed in Sec. 3, which represents the central part of the work. In Sec. 4 we discuss the topological and geometrical properties of carrier spaces. The principal problem of the construction of the invariant bilinear form on carrier spaces of singular functions is solved by applying the classic Riesz-Gel'fand regularization method [7]. In Sec. 5 we briefly discuss possible physical applications and some general properties of the representations considered. Finally, in the four Appendices we present the structure of the Lie algebra of $U(p)$ groups, the classification of the irreducible representations of the $U(2)$ group and some auxiliary calculations.

2. PROPERTIES OF IRREDUCIBLE UNITARY FINITE-DIMENSIONAL REPRESENTATIONS OF THE U(p) GROUPS

In the present section we recall briefly those results of our previous paper [8], which pertain to degenerate series of representations of the compact $U(p)$ group.

Consider a vector space $\mathcal{L}(X)$ of complex-valued functions having as their domain a homogeneous space X of the type

$$X = U(p)/U(p-1) \quad . \tag{2,1}$$

The homogeneous space X has dimension $2p-1$ and can be represented by a "model" space $X^{(p)}$ having the same dimension and the same stability group as X and determined by the equation

$$z^1\bar{z}^1 + \cdots + z^p\bar{z}^p = 1 \quad , \tag{2,2}$$

where $z^k, k = 1,2,\cdots,p$ are points in the p-dimensional complex space C^p.

It is convenient to parametrize the homogeneous space $X^{(p)}$ by the biharmonic co-ordinates (see [8]), which are introduced in the following recursive manner. For $p=1$ put $z^1 = e^{i\varphi^1}$ where $0 \le \varphi^1 \le 2\pi$. Suppose that we have constructed the co-ordinate system $z'^1, \dots, z'^{p-1}$ on the manifold $X^{(p-1)}$; then the co-ordinates $z^1, \dots, z^{p-1}, z^p$ on $X^{(p)}$ are defined as follows :

$$
\begin{aligned}
z^k &= z'^k \sin\vartheta^p \qquad , \quad k = 1,2,\dots,p-1 , \\
z^p &= e^{i\varphi^p} \cos\vartheta^p \qquad ,
\end{aligned}
\tag{2, 3}
$$

where $0 \le \varphi^l \le 2\pi$, $0 \le \vartheta^l \le \pi/2$ $\quad l = 2,3,\dots,p$.

The left-invariant Riemannian metric tensor $g_{\alpha\beta}(X^{(p)})$ on the manifold $X^{(p)}$ is given by

$$
g_{\alpha\beta}(X^{(p)}) = \sum_{k,l=1}^{2p} g_{kl}(E^{2p})\, \partial_\alpha x^k \partial_\beta x^l \quad , \alpha;\beta = 1,2,\dots,2p-1 ,
\tag{2, 4}
$$

where x^k, $k = 1,2,\dots,2p$ are the co-ordinates in the 2p-dimensional Euclidean space (in which the manifold $X^{(p)}$ can be embedded as a hyper-sphere) and ∂_α , $\alpha = 1,2,\dots,2p-1$ denotes the partial differentiation with respect to the angles $\varphi^1, \dots, \varphi^p, \vartheta^2, \dots, \vartheta^p$.

In the case considered the ring of invariant operators is generated by the first order invariant operator

$$
\hat{M}_p = \sum_{\alpha=1}^{p} Z_\alpha \quad ,
$$

(Z_α being generators of the Cartan subalgebra) and by the Laplace-Beltrami operator

$$
\Delta = \frac{1}{\sqrt{|g|}} \partial_\alpha g^{\alpha\beta} \sqrt{|g|}\, \partial_\beta \quad ,
$$

which is proportional to the second order Casimir operator.

It is obvious that the linear envelope of the set of common eigen-functions ψ_M^λ of the invariant operators $\Delta(X^{(p)})$ and $\hat{M}_p$

$$
\begin{aligned}
\Delta \psi_M^\lambda &= \lambda\, \psi_M^\lambda \quad , \\
\hat{M} \psi_M^\lambda &= M\, \psi_M^\lambda \quad ,
\end{aligned}
\tag{2, 5}
$$

creates for a definite λ and M a representation space of the algebra R_p of $U(p)$. It was shown that the explicit form of the eigenfunctions of

Eq. (2, 5) is

$$Y^{J_2,\ldots,J_p}_{m_1,\ldots,m_p}(\omega) = \frac{1}{\sqrt{N_p'}} \exp\left(i\sum_{l=1}^{p} m_l \varphi^l\right) \prod_{k=2}^{p} \sin^{2-k}\vartheta^k \, d^{\frac{1}{2}(J_k+k-2)}_{\alpha_k,\beta_k}(2\vartheta^k) , \tag{2, 6}$$

$$\alpha_k = \frac{1}{2}(m_k + J_{k-1} + k - 2) \quad , \quad k = 2,\ldots,p \quad ,$$

$$\beta_k = \frac{1}{2}(m_k - J_{k-1} - k + 2) \quad , \quad J_1 = m_1 \quad ,$$

$$\omega = \{\varphi^1,\ldots,\varphi^p,\vartheta^2,\ldots\vartheta^p\} \quad , \quad 0 \le \varphi^i \le 2\pi \quad i = 1,2,\ldots,p \ ,$$
$$0 \le \vartheta^k \le \pi/2 \quad k = 2,3,\ldots,p \ ,$$

$$N_p' = 2\pi^p \prod_{k=2}^{p} \frac{1}{J_k + k - 1} . \tag{2, 7}$$

It is sometimes convenient to introduce new azimuthal angles $\phi^1,\ldots,\phi^p$ given by

$$\phi^k = \varphi^k - \varphi^{k+1} \quad , \quad k = 1,2,\ldots,p-1 \ ,$$

$$\phi^p = \sum_{i=1}^{p} \varphi^i \ , \tag{2, 8}$$

$$\Omega \equiv \{\phi^1,\ldots,\phi^p,\vartheta^2,\ldots,\vartheta^p\} \ ,$$

and new representation labels $M_1,\ldots,M_p$,

$$M_l = \sum_{i=1}^{l} m_i \quad , \quad l = 1,2,\ldots,p \ . \tag{2, 9}$$

Then the new harmonic function has the form

$$Y^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega) = \frac{1}{\sqrt{N_p}} \exp\left(i\sum_{l=1}^{p} M_l \phi^l\right) \prod_{k=2}^{p} \sin^{2-k}\vartheta^k \, d^{\frac{1}{2}(J_k+k-2)}_{\alpha_k,\beta_k}(2\vartheta^k) , \tag{2, 10}$$

where

$$
\begin{aligned}
\mu_k &= M_k - \frac{k}{p} M_p \qquad , \qquad k = 1,2,\dots,p-1 , \\
\mu_p &= \frac{1}{p} M_p \qquad , \\
N_p &= 2^{p-1} p N_p' ,
\end{aligned}
\tag{2,11}
$$

and α_k, β_k are expressed in terms of M_l by using (2,9).
On the manifold $X^{(p)}$ the left-invariant Riemannian measure $d\mu(X^{(p)})$ can be defined in the following way:

$$
\begin{aligned}
d\mu(X^{(p)}) &= \sqrt{|g|} \prod_{l=1}^{p} d\varphi^l \prod_{k=2}^{p} d\vartheta^k \\
&= \prod_{k=2}^{p} \sin^{2k-3}\vartheta^k \cos\vartheta^k \, d\vartheta^k \prod_{l=1}^{p} d\varphi^l .
\end{aligned}
\tag{2,12}
$$

The vector space $\mathcal{L}(X^{(p)})$ equipped by this measure becomes a Hilbert space, in which the scalar product is defined by

$$
G_F(f,h) = \int_{X^{(p)}} f(z)\,\overline{h(z)}\, d\mu(X^{(p)}) . \tag{2,13}
$$

The harmonic functions $Y^{J_2,\dots,J_p}_{M_1,\dots,M_p}(\Omega)$ are a complete orthonormal set of functions with respect to the scalar product $G_F(f,h)$ on the manifold $X^{(p)}$. The requirement of the square integrability imposes the following restrictions on $M_1,\dots,M_p$; $J_2,\dots,J_p$

$$
\begin{aligned}
|M_2-M_1| + |M_1| &= J_2 - 2n_2 \qquad , \\
|M_3-M_2| + J_2 &= J_3 - 2n_3 \qquad , \\
&\dots \\
|M_p-M_{p-1}| + J_{p-1} &= J_p - 2n_p \qquad ,
\end{aligned}
\tag{2,14}
$$

where $n_k = 0,1,\dots,\frac{1}{2}(J_k - |M_k|)$, $k = 2,3,\dots,p$.

In our previous work [8] it was shown that the Lie algebra R_p is represented irreducibly in each finite-dimensional space $D^{J_p M_p}(X^{(p)})$

spanned by the set of harmonic functions $Y^{J_2,\dots,J_p}_{M_1,\dots,M_p}(X^{(p)})$ with fixed J_p and M_p. The relation (2,14) between numbers J_p, M_p and J_{p-1}, M_{p-1} induces a definite decomposition of the representation space $D^{J_p M_p}(X^{(p)})$ of $U(p)$ with respect to the irreducible representations $D^{J_{p-1},M_{p-1}}$ of $U(p-1)$. This decomposition is conveniently illustrated by the graphs in Fig. 1.

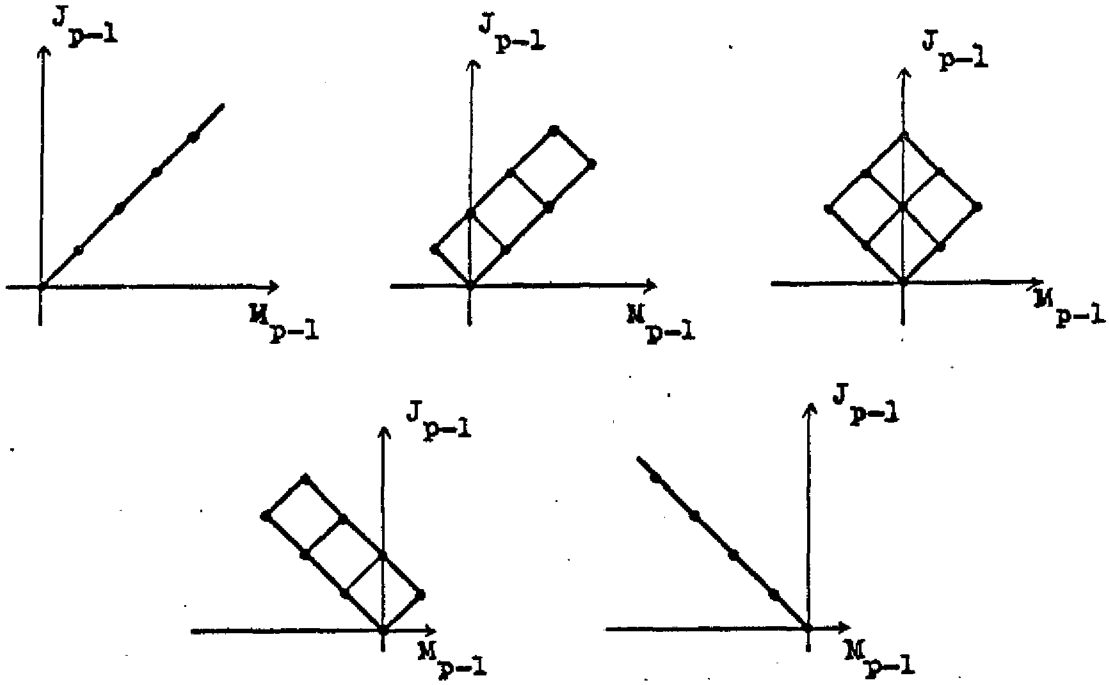

Fig. 1

The action of the $U(p)$ group in the Hilbert space $\mathcal{H}(X^{(p)},\mu)$ is determined by the left translation

$$[T(g)f](X) = f(g^{-1}X) \tag{2,15}$$

Therefore, the unitarity of the representations $D^{J_p M_p}$ follows from the left-invariance of the Riemannian measure $\mu(X)$ on the manifold $X^{(p)}$. The irreducibility of the global representation of $U(p)$ (2,15) follows from the irreducibility of the corresponding Lie algebra.

3. INFINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF THE LIE ALGEBRA OF THE U(p) GROUP

In the previous section we derived the properties of finite-dimensional irreducible representations of the $U(p)$ groups and of the corresponding Lie algebra R_p determined by two integral invariant numbers J_p and M_p. The representation space was spanned by the set of harmonic functions (2, 10) satisfying the equations

$$\begin{aligned} \Delta \mathbf{Y}^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega) &= -J_p(J_p+2p-2)\mathbf{Y}^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega) \ , \\ \hat{M}_p \mathbf{Y}^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega) &= M_p \mathbf{Y}^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega) . \end{aligned} \tag{3, 1}$$

These representations were realized on finite-dimensional vector spaces $\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ spanned by solutions of (3, 1) with J_p and M_p satisfying the conditions

$$\begin{aligned} &J_p = 0, 1, 2, \ldots \ , \\ &J_p \geq |M_p| \ , \\ &J_p + M_p \quad \text{even} . \end{aligned} \tag{3, 2}$$

The set of all values for J_p and M_p satisfying (3, 2) constitutes the full spectrum of the invariant operators $\Delta(X^{(p)})$ and $\hat{M}_p$ which are essentially self-adjoint on the dense linear subspace of $\mathcal{H}(X^{(p)}, \mu)$ determined by the linear envelope of harmonic functions $\mathbf{Y}^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega)$.

A natural question arises as to what class of representations is obtained on the vector space $\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ of eigensolutions of (3, 1) labelled by such values of J_p and M_p which do not satisfy conditions (3, 2), i. e., which are outside the spectrum of the self-adjoint operators Δ and $\hat{M}_p$. We shall consider the case when J_p and M_p are arbitrary complex numbers.

In this case, the harmonic functions satisfying the set of Eqs. (3, 1) differ from (2, 6) only by the labels J_p and M_p , whereas all other labels have the same spectra as before. The conditions (2, 14) are satisfied for $k = 2, 3, \ldots, p-1$ but are not satisfied for $k = p$. Consequently, the dependence of the harmonic functions on angular variables $\varphi^1, \ldots, \varphi^p, \vartheta^2, \ldots, \vartheta^p$ differs from (2,6) only by its φ^p and ϑ^p-dependent part $\Phi_{M_{p-1}M_p}(\varphi^p) \cdot \Theta^{J_{p-1}J_p}_{M_{p-1}M_p}(\vartheta^p)$ which can now be written (including the corresponding part of the normalization constant) in the following form:

$$\Phi_{M_{p-1},M_p}(\varphi^p) = \frac{1}{\sqrt{2\pi}}\, e^{i(M_p - M_{p-1})\varphi^p} , \tag{3,3}$$

$$\Theta^{J_{p-1},J_p}_{M_{p-1},M_p}(\vartheta^p) = (-1)^{\alpha} \frac{\Gamma(n_{\mp}+\alpha+1)\sqrt{2}}{\Gamma(n_{\mp}+1)\,\Gamma(\alpha+1)} \sqrt{G_{\pm}}\, tg^{|J_{p-1}|}\vartheta^p \cos^{\mp\beta}\vartheta^p \,{}_{2}F_{1}(-n_-, -n_+ ; \alpha+1 ; -tg^2\vartheta^p), \tag{3,4}$$

where

$$\alpha = J_{p-1} + p - 2 ,$$

$$\beta = M_p - M_{p-1} ,$$

$$n_{\mp} = \frac{1}{2}\left(J_p \mp M_p - J_{p-1} \pm M_{p-1}\right) ,$$

$$G_{\pm} = (J_p + p - 1) \prod_{k=-\frac{1}{2}(J_{p-1}+M_{p-1})+1}^{\frac{1}{2}(J_{p-1}-M_{p-1})+p-2} \left(\frac{1}{2}(J_p + M_p) + k\right)^{\pm 1} \prod_{l=-\frac{1}{2}(J_{p-1}-M_{p-1})+1}^{\frac{1}{2}(J_{p-1}+M_{p-1})+p-2} \left(\frac{1}{2}(J_p - M_p) + l\right)^{\mp 1}. \tag{3,5}$$

The set of all harmonic functions with both n_- and n_+ being non-negative integers span the Hilbert spaces $\mathcal{H}^{J_p}_{M_p}(X^{(p)})$ in which the finite-dimensional irreducible unitary representations determined by J_p and M_p are realized (see Sec. 2).

If either or both n_- and n_+ in (3, 4) are different from any non-negative integer, we obtain an infinite-dimensional representation of the algebra R_p of $U(p)$ provided that the function $\Theta^{J_{p-1},J_p}_{M_{p-1},M_p}(\vartheta^p)$ is well defined by formula (3, 4). This is the case if the factor $G_\pm$ standing on the right-hand side of (3, 4) is finite and non-vanishing. It is important that, if $G_\pm$ is finite and non-vanishing for some values of $J_p, M_p, J_{p-1}, M_{p-1}$, then it is finite and non-vanishing for all other possible values within the corresponding invariant irreducible subspace. A complete classification of all cases in which $G_\pm$ is finite and non-vanishing is given in Appendix A; here we use only the results of this classification.

Due to the fact that the domain $X^{(p)}$ of the harmonic functions for complex J_p and M_p is the same as in the case of the finite-dimensional representations discussed in Sec. 2, the elements of the Lie algebra R_p of the $U(p)$ group are represented by the same differential operators. The explicit form of the generators in the standard Weyl basis is given in Appendix B. It turns out that the representation of the Lie algebra R_p realized on the vector space $\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ (J_p, M_p complex numbers) is generally reducible on this space, the number and properties of corresponding invariant subspaces being different for different values of the invariant numbers $N_+ = \frac{1}{2}(J_p + M_p)$ and $N_- = \frac{1}{2}(J_p - M_p)$. Nevertheless, all the vector spaces in question can be divided into the following three categories $\mathcal{H}$, I and II:

$\mathcal{H}$: Hilbert space $\mathcal{H}^{J_p}_{M_p}(X^{(p)}, \mu)$ (both n_+ and n_- are non-negative integers). The properties of these spaces were discussed in Sec. 2.

I: infinite-dimensional topological vector space ${}^{I}\mathcal{L}^{J_p}_{M_p}(X^{(p)}, \mu)$ with indefinite metric (either n_+ or n_-, but not both, is a non-negative integer; J_p and M_p are integers). It turns out that on these spaces a global representation of the $U(p)$ group can be realized (see Sec. 4).

II: infinite-dimensional vector space ${}^{II}\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ (neither n_+ nor n_- is a non-negative integer).

The structure of the representation space $\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ in each special case can conveniently be investigated by considering the action of an arbitrary generator of the $U(p)$ group on an arbitrary element of the $\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ space.

As follows from the commutation relations of the Lie algebra R_p (see Appendix B) the action of all generators can be obtained if the action of all basic elements of the subalgebra $R_{p-1} \dot{+} R_1$ and the action of one generator of $U(p)$ not belonging to $R_{p-1} \dot{+} R_1$ is known. Thus, to solve the problem of reducibility of a given representation it is sufficient to select one generator not belonging to the subalgebra $R_{p-1} \dot{+} R_1$ and to see which subspaces of $\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ are invariant under the repeated action of this generator.

In the following, we shall give the solution of the problem in the case when $p \geq 3$. The case $p = 2$ is treated separately in Appendix C.

The generator not belonging to the subalgebra $R_{p-1} \dot{+} R_1$ of R_p is conveniently chosen in the following form:

$$\begin{aligned} E_{\pm(p,p-1)} &\equiv \frac{i}{2\sqrt{p(p-1)}} \left(L^{+}_{p,p-1} \pm i\, L^{-}_{p,p-1} \right) \\ &= \frac{i}{2\sqrt{p(p-1)}} e^{\mp i(\varphi^{p-1} - \varphi^{p})} \left\{ \cos\vartheta^{p-1}\left(-\frac{\partial}{\partial\vartheta^{p}} \pm i \,\mathrm{tg}\,\vartheta^{p} \frac{\partial}{\partial\varphi^{p}}\right) + \mathrm{ctg}\,\vartheta^{p}\left(\sin\vartheta^{p-1}\frac{\partial}{\partial\vartheta^{p-1}} \pm \frac{i}{\cos\vartheta^{p-1}}\frac{\partial}{\partial\varphi^{p-1}}\right)\right\}. \end{aligned} \tag{3,6}$$

The action of $E_{\pm(p,p-1)}$ on an arbitrary element of $\mathcal{L}^{J_p}_{M_p}$ can be expressed as

$$\begin{aligned} E_{\pm(p,p-1)} \mathbf{Y}^{J_2,\dots,J_{p-1},J_p}_{M_1,\dots,M_{p-1},M_p}(\Omega) &= \frac{i}{\sqrt{p(p-1)}} \frac{\sqrt{(A'_{\pm}-1)B'_{\pm}}\sqrt{(A_{\mp}-1)(B_{\pm}+1)}}{\sqrt{(J_{p-1}+p-2)(J_{p-1}+p-3)}} \mathbf{Y}^{J_2,\dots,J_{p-1}-1,J_p}_{M_1,\dots,M_{p-1}\mp 1,M_p}(\Omega) \\ &- \frac{i}{\sqrt{p(p-1)}} \frac{\sqrt{A'_{\mp}(B'_{\mp}+1)}\sqrt{A_{\pm}B_{\mp}}}{\sqrt{(J_{p-1}+p-2)(J_{p-1}+p-1)}} \mathbf{Y}^{J_2,\dots,J_{p-1}+1,J_p}_{M_1,\dots,M_{p-1}\mp 1,M_p}(\Omega), \end{aligned} \tag{3,7}$$

where

$$A_{\pm} = N_{\pm} + N_{\mp}' + p - 1 \quad , \quad A_{\mp}' = N_{\mp}' + N_{\pm}'' + p - 2 \, ,$$
$$B_{\mp} = N_{\mp} - N_{\mp}' \quad , \quad B_{\pm}' = N_{\pm}' - N_{\pm}'' \quad , \tag{3,8}$$

and

$$N_{\pm} = \tfrac{1}{2}(J_p \pm M_p) \quad ,$$
$$N_{\pm}' = \tfrac{1}{2}(J_{p-1} \pm M_{p-1}) \, , \tag{3,9}$$
$$N_{\pm}'' = \tfrac{1}{2}(J_{p-2} \pm M_{p-2}) .$$

As $J_k, M_k, k = 1, 2, \ldots, p-1$, are assumed to satisfy the conditions (2,14), the numbers $N_{\pm}''$ and $N_{\pm}'$ can only be non-negative integers. On the other hand, the invariant numbers N_+ and N_- are considered to assume, in general, arbitrary complex values. The number and the properties of possible invariant subspaces of the representation space $\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ depend on the specific values of N_+ and N_- chosen. We shall treat the different cases separately.

Let us divide the set of all complex numbers into four disjoint parts, C_1, C_2, C_3 and C_4, defined in the following way:

$$\begin{aligned} &C_1 \quad \text{contains the numbers} \quad 0, 1, 2, 3, \ldots \, ; \\ &C_2 \quad \text{contains the numbers} \quad -p+1, -p, -p-1, \ldots ; \\ &C_3 \quad \text{contains the numbers} \quad -1, -2, \ldots, -p+2 \, ; \\ &C_4 \quad \text{contains all other complex numbers.} \end{aligned} \tag{3,10}$$

The various possibilities which are relevant for the structure of the representation space are

$$N_+ \in C_i \quad , \quad N_- \in C_k \, , \quad \begin{matrix} i = 1, 2, 3, 4 \\ k = 1, 2, 3, 4 \end{matrix} \quad . \tag{3,11}$$

1. $\mathcal{N}_+ \in C_3 \cup C_4$, $\mathcal{N}_- \in C_3 \cup C_4$.

The factors $A_\pm$ and $B_\pm$ in formulae (3, 7) and (3, 8) never become zero (for non-negative integral values of $\mathcal{N}_+'$ and $\mathcal{N}_-'$). A successive application of the operators $E_{\pm(p,p-1)}$ makes it possible to pass to every point of the diagram in Fig. 2 starting from any arbitrary point. (In this diagram, every point is determined by the two invariants J_{p-1} , M_{p-1} of the subgroup $U(p-1)$ of $U(p)$ and represents one subspace of the representation space.) The representation is irreducible and the corresponding representation space is the infinite-dimensional vector space of the type II.

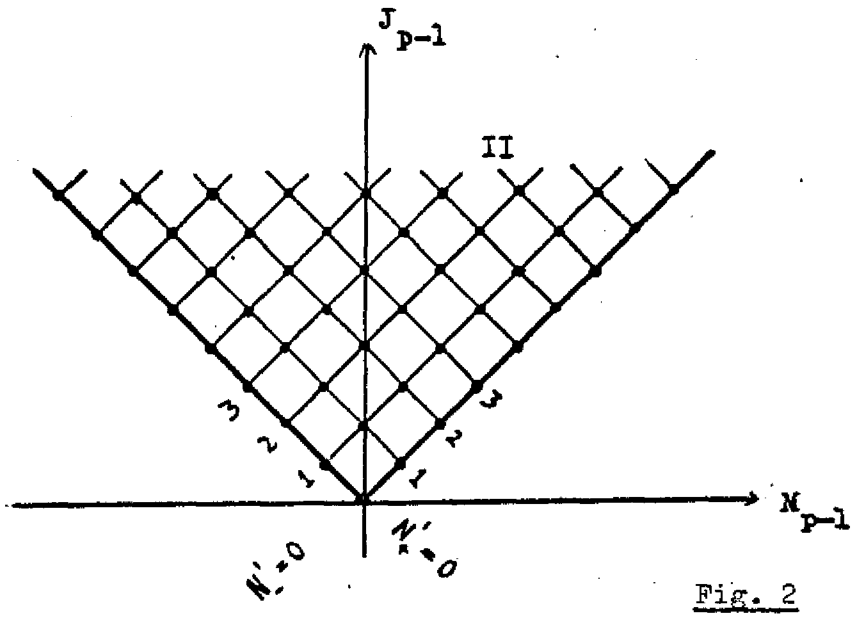

Fig. 2

The boundaries of the representation space in Fig. 2 are given by the lines $J_{p-1} = |M_{p-1}|$ on which the coefficients $B_\pm' = \mathcal{N}_\pm' - \mathcal{N}_\pm''$ become zero. As $\mathcal{N}_\pm'$ and $\mathcal{N}_\pm''$ are non-negative integers the representation space is restricted to the region $|M_{p-1}| \leq J_{p-1}$ in all other cases too.

2. a) $\mathcal{N}_+ \in C_4$, $\mathcal{N}_- \in C_2$.

The factors A_+ and $B_\pm$ never become zero, while factor A_- vanishes at $\mathcal{N}_+' = -\mathcal{N}_- - p + 1$. The two straight lines $A_- - 1 = 0$ and $A_- = 0$ determine a "barrier" in the (J_{p-1}, M_{p-1}) plane which is the boundary of the representation space ${}^{II}\mathcal{L}^{J_p}_{M_p}(X^{(p)})$. The diagram representing the structure of the space ${}^{II}\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ is depicted in Fig. 3.

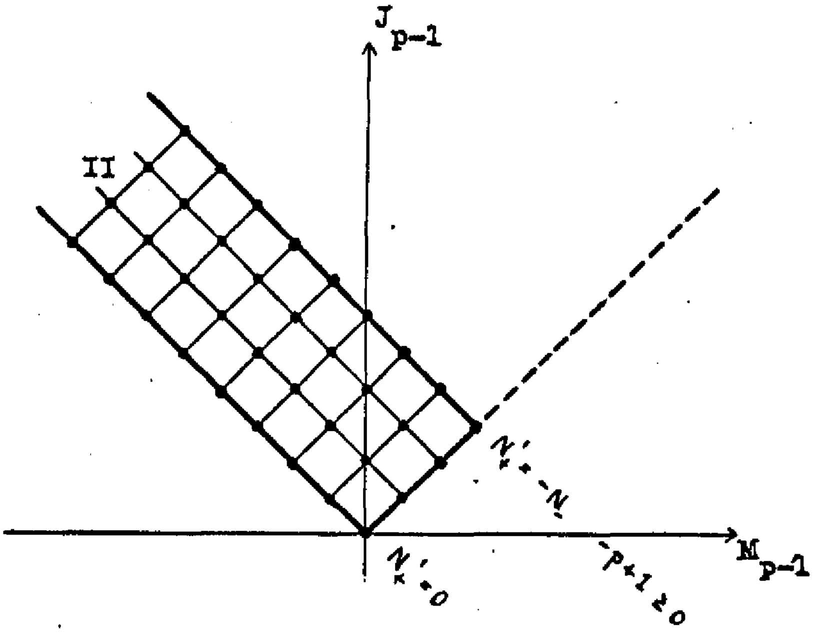

Fig. 3

2.b) $N_+ \in C_3$, $N_- \in C_2$.

The properties of $A_\pm$ and $B_\pm$ are the same as in case 2.a). However, as N_+ is an integer, the space ${}^{\mathrm{I}}\mathcal{L}^{J_p}_{M_p}(X^{(p)})$ has a structure depicted in Fig. 4.

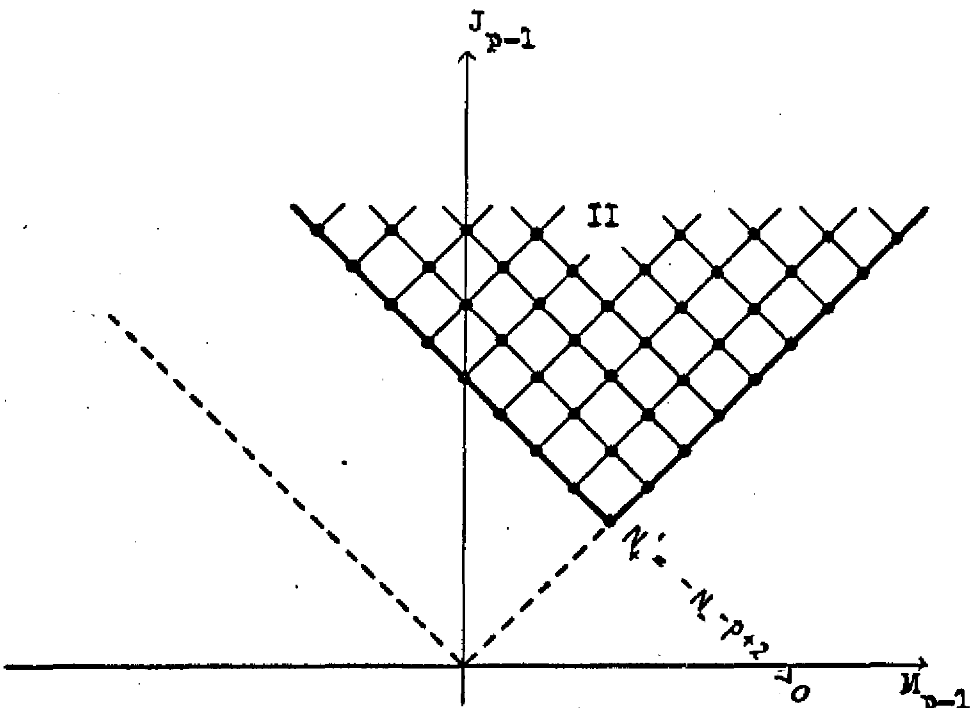

Fig. 4

3.a) $\mathcal{N}_+ \in C_4$, $\mathcal{N}_- \in C_1$.

The factors $A_\pm$ and B_+ can never vanish, while B_- vanishes at $\mathcal{N}_-' = \mathcal{N}_-$. The two straight lines, $B_- + 1 = 0$ and $B_- = 0$, determine a barrier which is the boundary of the representation space ${}^{\mathcal{I}}\mathcal{L}^{J_p}_{M_p}(\chi^{(p)})$ See Fig. 5.

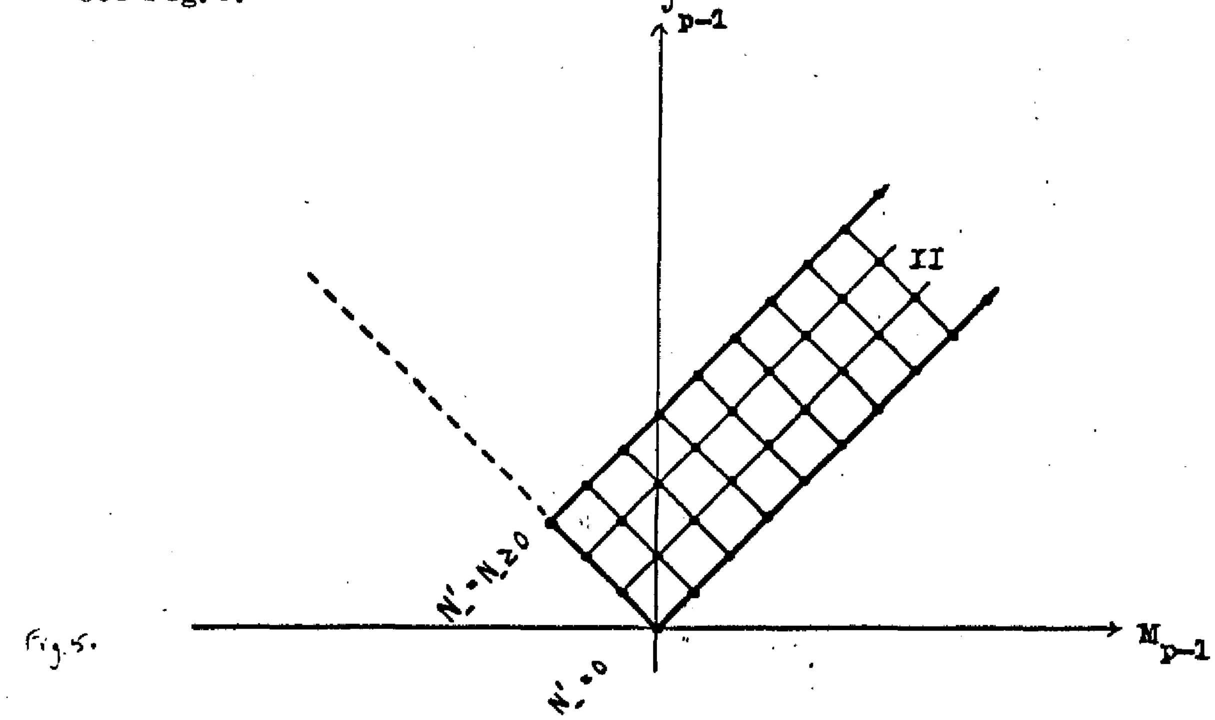

Fig. 5

3.b) $\mathcal{N}_+ \in C_3$, $\mathcal{N}_- \in C_1$.

The properties of $A_\pm$ and $B_\pm$ are the same as in case 3.a). The structure of the representation space ${}^{\mathcal{I}}\mathcal{L}^{J_p}_{M_p}(\chi^{(p)})$ is shown in Fig. 6.

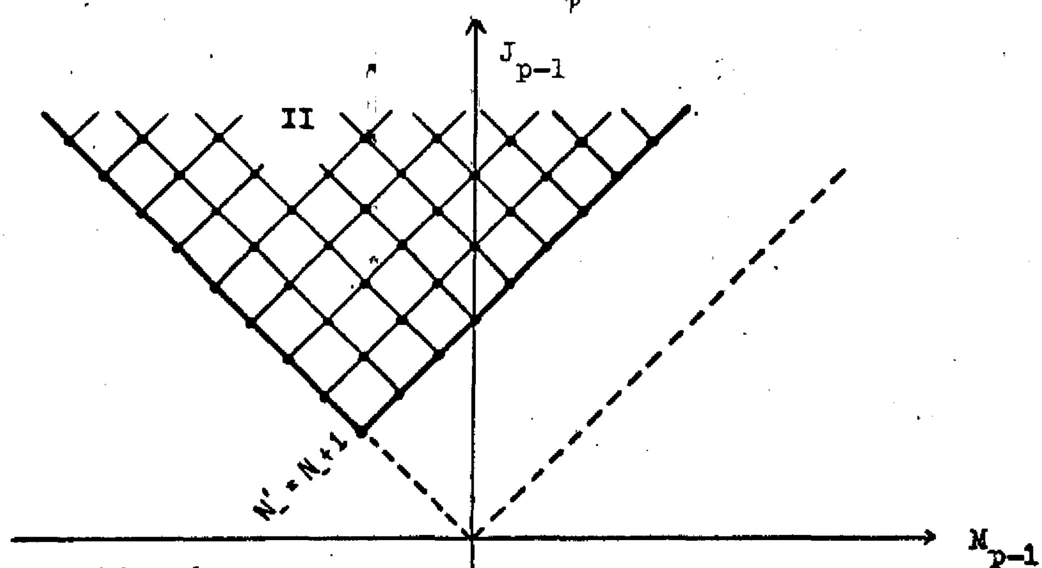

Fig. 6

4\. a) $\mathcal{N}_+ \in C_2$, $\mathcal{N}_- \in C_4$

The factors A_- and $B_\pm$ are non-vanishing while A_+ vanishes for $\mathcal{N}_-' = -\mathcal{N}_+ - p + 1$. The complete description of this case, including the diagram, is obtained from case 2. a), by replacing $\mathcal{N}_\pm$, $\mathcal{N}_\pm'$, $A_\pm$ and $B_\pm$ by $\mathcal{N}_\mp$, $\mathcal{N}_\mp'$, $A_\mp$ and $B_\mp$ respectively.

4\. b) $\mathcal{N}_+ \in C_2$, $\mathcal{N}_- \in C_3$.

This case is obtained from 2. b) by the same change as in 4. a).

5\. $\mathcal{N}_+ \in C_2$, $\mathcal{N}_- \in C_2$.

The factors $B_\pm$ never vanish, while the factors $A_\pm$ give rise to two perpendicular barriers, $A_+ = 0$, $A_+ - 1 = 0$ and $A_- = 0$, $A_- - 1 = 0$, which divide the representation space into two invariant subspaces as shown in Fig. 7. Both subspaces are of type II and are distinguished by the eigenvalues ± 1 of the invariant operator, T_1 , whose action on the harmonic functions is defined as follows:

$$T_1 Y^{J_2, \dots, J_p}_{M_2 \dots M_p}(\Omega) = \operatorname{sign}\,(J_p - M_p + J_{p-1} + M_{p-1} + 2p - 3)\, Y^{J_2, \dots, J_p}_{M_2 \dots M_p}(\Omega) \quad . \quad (3,12)$$

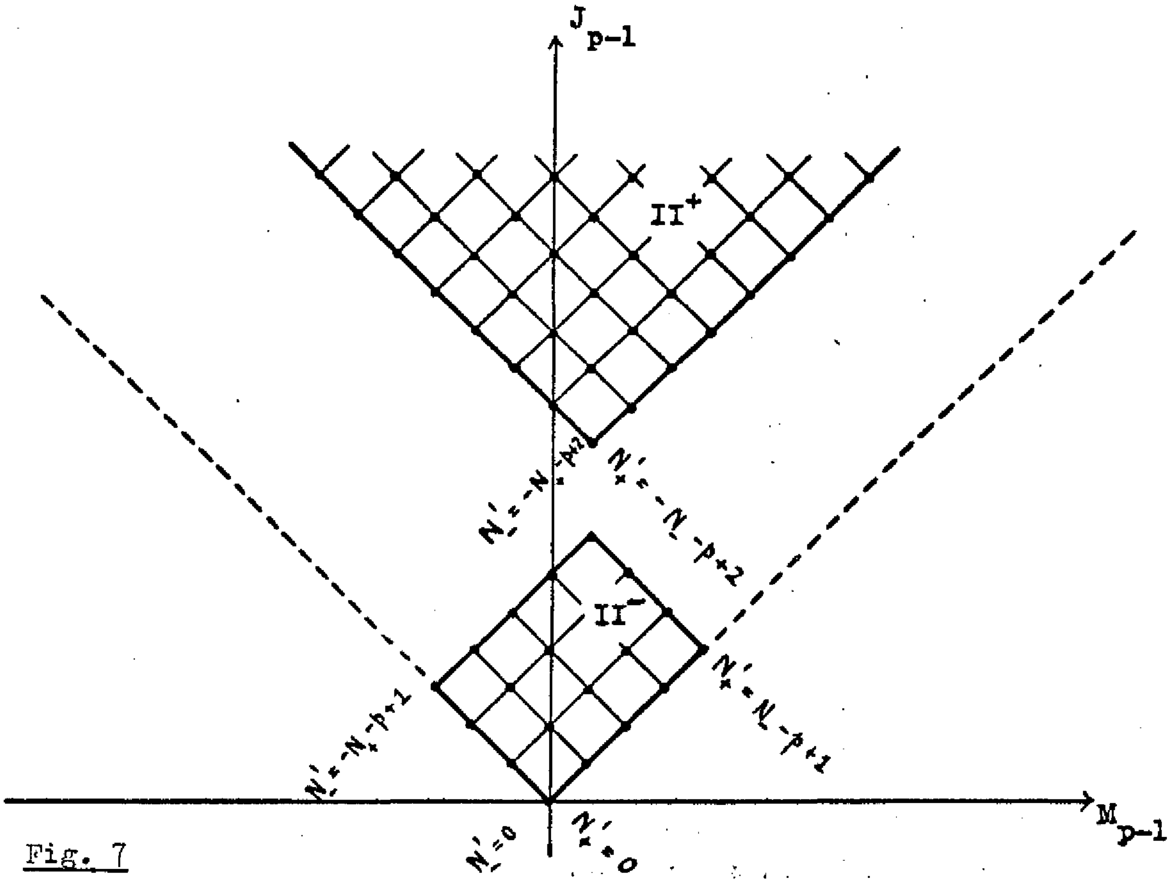

Fig. 7

6. $N_+ \in C_2$, $N_- \in C_1$.

The factors A_- and B_+ cannot vanish, while A_+ and B_- can vanish and give rise to two parallel barriers at a "distance" of $|J_p + p - 1|$ which is not equal to zero. The representation space is divided into two invariant subspaces, ${}^{II}\mathcal{L}^{J_p -}_{M_p}(X^{(p)})$ and ${}^{I}\mathcal{L}^{J_p +}_{M_p}(X^{(p)})$,(see Fig. 8) which can be distinguished by eigenvalues ± 1 of the invariant operator T_2 defined as follows:

$$T_2 Y^{J_2,\dots,J_p}_{M_1,\dots,M_p}(\Omega) = \operatorname{sign}\left(J_p - M_p - J_{p-1} + M_{p-1} + 1\right) Y^{J_2,\dots,J_p}_{M_1,\dots,M_p}(\Omega) \qquad (3,13)$$

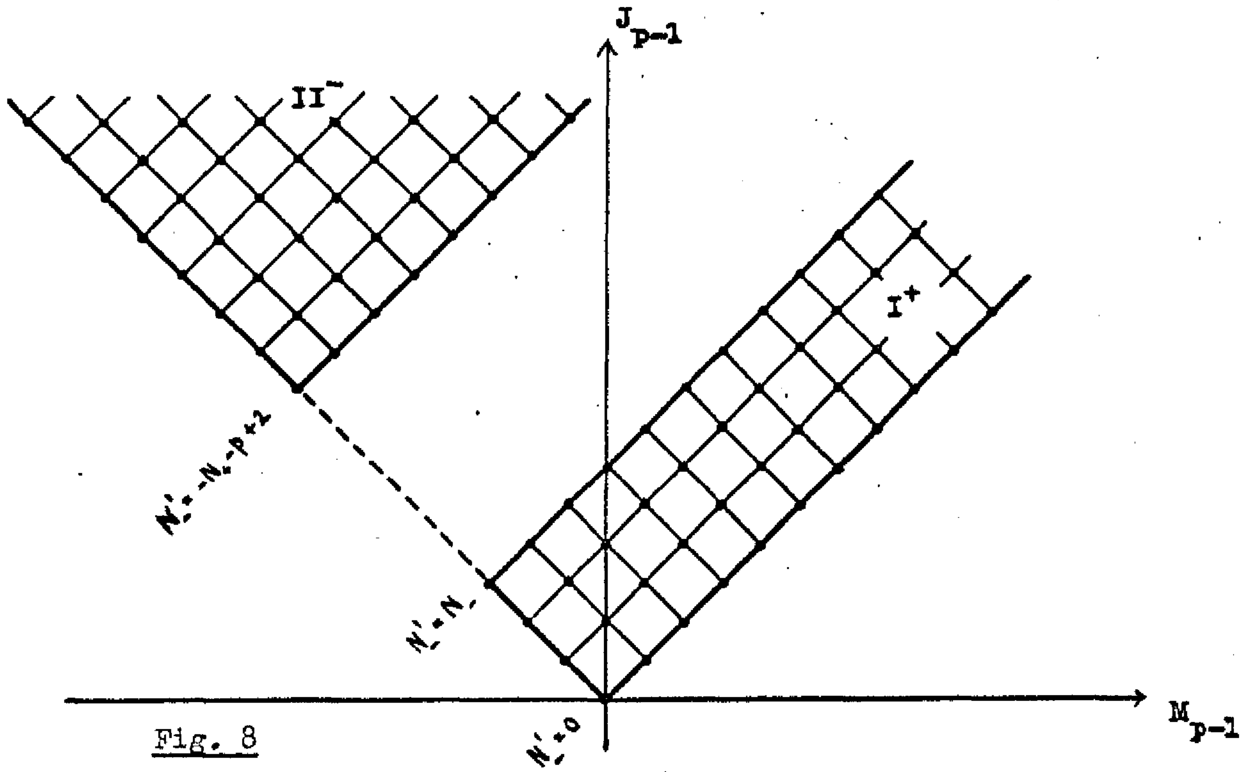

Fig. 8

As is shown in Sec. 4, an indefinite bilinear form is defined on the space I which is invariant under the action of the global representation of the $U(p)$ group.

7. a) $N_+ \in C_1$, $N_- \in C_4$,
   b) $N_+ \in C_1$, $N_- \in C_3$.

These cases and the corresponding diagrams are obtained from the cases 3. a) and 3. b) respectively by performing the changes,

$$N_\pm , N'_\pm , A_\pm , B_\pm \to N_\mp ; N'_\mp , A_\mp , B_\mp .$$

8. $N_+ \in C_1$, $N_- \in C_2$.

This case is obtained from case 6 by performing the same changes as indicated in 7. a) and 7. b). The resulting invariant subspaces can be distinguished by the eigenvalues ± 1 of the operator T_1 and we have I^- and II^+ spaces.

9. $N_+ \in C_1$, $N_- \in C_1$.

The coefficients $A_\pm$ never vanish. Two perpendicular barriers, $B_+ = 0$, $B_+ + 1 = 0$ and $B_- = 0$, $B_- + 1 = 0$, divide the representation space into two invariant subspaces, which can be distinguished by the eigenvalues ± 1 of the operator T_2 (see Fig. 9). The space $\mathcal{H}_{M_p}^{J_p +}$ is the ordinary Hilbert space, in which the finite-dimensional unitary representations of the $U(p)$ group are realized as described in Sec. 2.

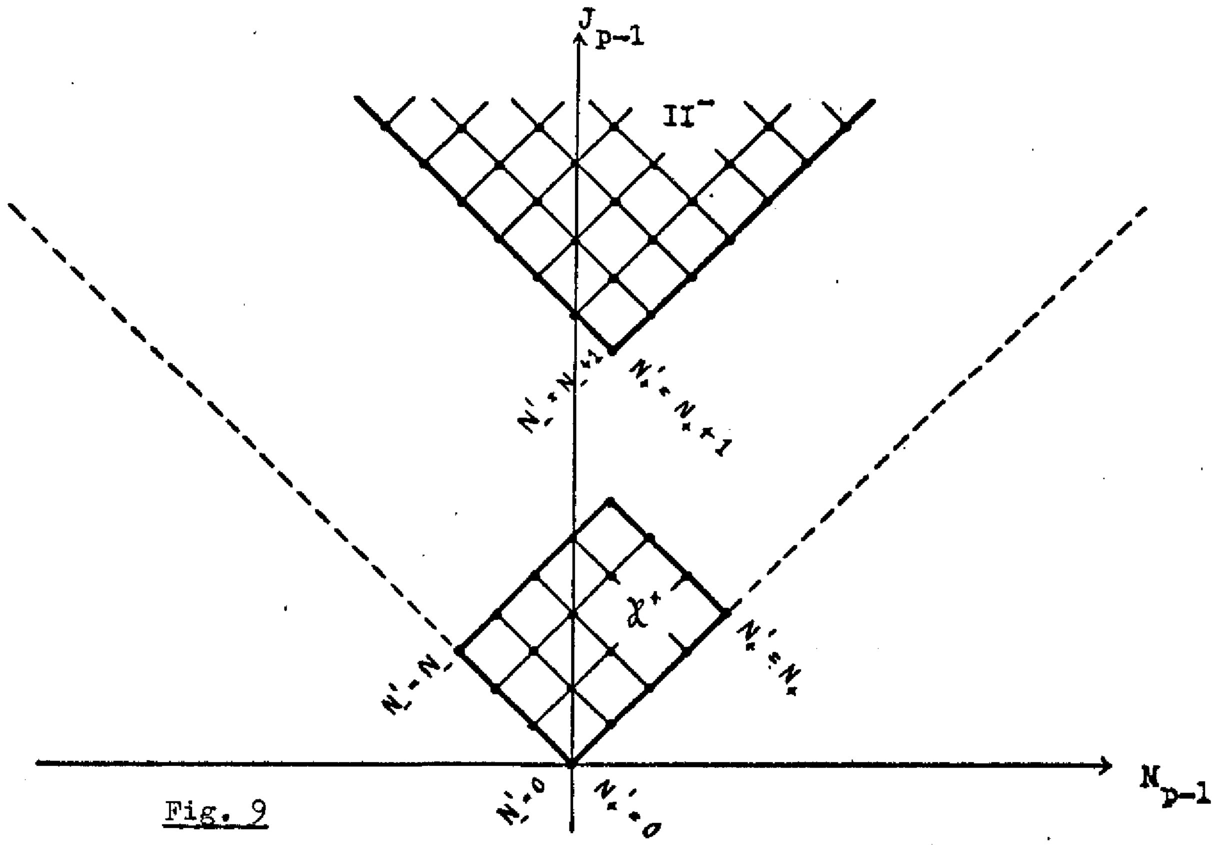

Fig. 9

An examination of Figs. 2-9 shows that there are points appearing in Fig. 2 which do not appear in any of the Figs. 3-9. This is due to the fact that the factor $G_\pm$ given in (3, 5) is infinite or zero in this region. For further details about this question see Appendix A.

4. TOPOLOGICAL AND GEOMETRICAL PEOPERTIES OF CARRIER SPACES

HUREVITSCH and KOOSIS [2] have proved that any irreducible representation of a compact topological group realized on a Hilbert space is equivalent to a finite-dimensional unitary representation. Therefore, we may expect that our irreducible infinite-dimensional representations considered in Sec. 3 will be realized in more general spaces than Hilbert spaces.

The topological and geometrical properties of a vector space $\mathcal{L}^{J_p, M_p}(X^{(p)})$ are completely determined if it is possible to introduce on it a bilinear form $G(x|y)$ fulfilling the following conditions [9] :

i) $$G(\alpha_1 x_1 + \alpha_2 x_2 | \beta_1 y_1 + \beta_2 y_2)$$
$$= \alpha_1 \bar{\beta}_1 G(x_1|y_1) + \alpha_1 \bar{\beta}_2 G(x_1|y_2) + \alpha_2 \bar{\beta}_1 G(x_2|y_1) + \alpha_2 \bar{\beta}_2 G(x_2|y_2) , \tag{4,1}$$

ii) $$G(x|y) = \overline{G(y|x)} ,$$

for every $x, y \in \mathcal{L}^{J_p M_p}(X^{(p)})$ and $\alpha, \beta \in C^1$

In the case of finite-dimensional representations such a bilinear form was given by

$$G_F(f|h) = \int_{X^{(p)}} f(x) \overline{h(x)} \, d\mu(X) , \tag{4,2}$$

where $\mu(X)$ is the left-invariant Riemannian measure on the manifold $X^{(p)}$ determined by the expressions (2,12). The form (4,2), besides conditions i) and ii), fulfills the additional condition

$$G_F(f|f) > 0 ,$$

for every $0 \neq f \in \mathcal{L}^{J_p, M_p}(X^{(p)})$ i.e., it represents the scalar product on the vector space $\mathcal{L}^{J_p, M_p}(X^{(p)})$. However, as harmonic functions (2,10) with the indices

J_p and M_p not obeying condition (3, 2) are, in general, strongly singular at the point $\vartheta^p = \pi/2$, we cannot use directly the expression (4, 2) in the representation space $\mathcal{L}^{J_p,M_p}(X^{(p)})$. The generalization of the metric form (4, 2) on $\mathcal{L}^{J_p,M_p}(X^{(p)})$ should satisfy two additional conditions besides (4, 1):

iii) It should be invariant under the action of the group $U(p)$ on the representation space $\mathcal{L}^{J_p,M_p}(X^{(p)})$, i.e., if

$$(T_g f)(x) = f(g^{-1}x) ,$$

then
$$G(T_g f | T_g h) = G(f|h) . \tag{4, 3}$$

iv) In the special case that J_p and M_p are integers fulfilling conditions (3, 2), then the corresponding generalized metric form $G(f|h)$ should reduce to the usual scalar product (4, 2).

In what follows we shall denote a harmonic function $Y^{J_2,\ldots,J_{p-1},J_p}_{M_1,\ldots,M_{p-1},M_p}(X)$ by the symbol $Y^{J_p,M_p}_{\alpha}(X)$ with α representing the set of indices $J_{p-1}\ldots J_2, M_{p-1}\ldots M_1$. As harmonic functions $Y^{J_p,M_p}_{\alpha}(X)$ span the representation space $\mathcal{L}^{J_p,M_p}(X^{(p)})$ it is sufficient to specify the generalized metric form $G(f|h)$ only for these functions. Let us consider first the case of ${}^{I}\mathcal{L}^{J_p,M_p}(X^{(p)})$ spaces (see Fig. 8, I$^+$). The metric form $G(f|h)$ on these spaces fulfilling the conditions i)...iv) can be taken as the regularized integral of harmonic functions over the manifold $X^{(p)}$

$$G\left(Y^{J_p,M_p}_{\alpha} \middle| Y^{J_p,M_p}_{\alpha'}\right) = Reg \int_{X^{(p)}} Y^{J_2,\ldots,J_{p-1},J_p}_{M_1,\ldots,M_{p-1},M_p} \overline{Y^{J_2',\ldots,J_{p-1}',J_p}_{M_1',\ldots,M_{p-1}',M_p}} , \tag{4, 4}$$

where the regularization is meant in the sense of analytic continuation (see, for example, [7] §3 and Appendix D.) Using the formulae (2, 10) and (3, 4) for the harmonic function $Y^{J_p,M_p}_{\alpha}(X^{(p)})$ and performing the integration over the variables $\phi^1,\ldots,\phi^p$ and $\vartheta^2,\ldots,\vartheta^p$, we finally get (see Appendix D)

$$G\left(Y^{J_2,\ldots,J_{p-1},J_p}_{M_1,\ldots,M_{p-1},M_p} \middle| Y^{J_2',\ldots,J_{p-1}',J_p}_{M_1',\ldots,M_{p-1}',M_p}\right) = \prod_{k=1}^{p-1} \delta_{M_k M_k'} \prod_{l=2}^{p-1} \delta_{J_l J_l'} \; g(J_{p-1}, J_p) , \tag{4, 5}$$

where

$$g(J_{p-1}, J_p) = \begin{cases} (-1)^{J_{p-1}+p-2} & \text{for } J_p + p - 1 > 0 \\ (-1)^{J_{p-1}+p-1} & \text{for } J_p + p - 1 < 0 \end{cases} .$$

We see that the generalized metric form (4, 4) is stongly indefinite and consequently the linear representation space ${}^{I}\mathcal{L}^{J_p M_p}(X^{(p)})$ represents the infinite-dimensional vector space with the indefinite metric.

Let us denote by $\mathcal{L}_+$ and $\mathcal{L}_-$ the linear subspaces of the vector space $\mathcal{L}$ with a positive and negative norm respectively and by $\mathcal{L}_0$ the subspace of so-called isotropic vectors, i.e., the vectors f_0 for which

$$G(f_0|h) = 0 \text{ for every } h \in \mathcal{L}$$

The decomposition of L on to the direct sum

$$\mathcal{L} = \mathcal{L}_0 \dotplus \mathcal{L}_- \dotplus \mathcal{L}_+$$

is called the canonical decomposition. The space $\mathcal{L}$ which contains at least one non-zero isotropic vector is called a degenerate space and $\min[\dim\mathcal{L}_+ , \dim\mathcal{L}_-]$ the rank of the indefiniteness [9] . We see that our infinite-dimensional carrier spaces ${}^{I}\mathcal{L}^{J_p,M_p}(X^{(p)})$ are non-degenerate vector spaces with an infinite rank of indefiniteness.

The existence of the hermitian indefinite metric form (4, 5) on the vector spaces ${}^{I}\mathcal{L}^{J_p,M_p}(X^{(p)})$ does not determine uniquely the topological and geometrical properties of the spaces [9] . It turns out that in these spaces we can introduce at least two different topologies and related geometries which are important from the point of view of group representations.

1° Hilbert's topology.

In order to define this topology we introduce first a positive definite scalar product on the space ${}^{I}\mathcal{L}^{J_p,M_p}(X^{(p)})$ using the indefinite bilinear form (4, 5)

$$(f|h) = G(f_+|h_+) - G(f_-|h_-) ,$$

where

$$f_\pm, h_\pm \in \mathcal{L}_\pm^{J_p, M_p}(X^{(p)}) \quad , \text{ and}$$

$$f = f_+ + f_- \quad ,$$

is the unique decomposition of an arbitrary vector $f \in {}^{I}\mathcal{L}^{J_p, M_p}(X^{(p)})$ into components $f_\pm$ belonging to the subspaces $\mathcal{L}_\pm^{J_p, M_p}$ respectively on which the G -metric (4, 5) has a definite sign.

Then, the topology on the vector space ${}^{I}\mathcal{L}^{J_p, M_p}(X^{(p)})$ will be determined by the norm

$$\| f \| = (f | f)^{1/2} \quad , \tag{4, 6}$$

and the related geometry by the distance

$$\rho(f, h) = \| f - h \| \quad . \tag{4, 7}$$

A sequence χ_n of vectors is convergent to a vector χ if

$$\lim_{n=\infty} \| \chi_n - \chi \| = \lim \rho(\chi, \chi_n) = 0 \quad . \tag{4, 8}$$

2° Freschet's topology

The Freschet topology on a vector space $\mathcal{L}$ is determined by a countable set of semi-norms $p_\alpha(f)$. For this set we can take in the considered spaces ${}^{I}\mathcal{L}^{J_p, M_p}(X^{(p)})$ the collection of semi-norms determined by

$$p_\alpha(f) = | f_\alpha | = | G(\mathbf{Y}_\alpha^{J_p M_p} | f) | \quad . \tag{4, 9}$$

In fact f_α represents the projection of a vector $f \in {}^{I}\mathcal{L}^{J_p, M_p}(X^{(p)})$ on a basis vector $\mathbf{Y}_\alpha^{J_p, M_p}$. The collection of semi-norms defined by formulae (4, 9) is separating because for each $0 \neq f \in {}^{I}\mathcal{L}^{J_p, M_p}(X^{(p)})$ there exists at least one semi-norm $p_\alpha(f) \neq 0$. Using the system of semi-norms $\{ p_\alpha \}$ we can determine the notion of the distance between two vectors

$f, h \in {}^{\mathcal{I}}\mathcal{L}^{J_p, M_p}(X^{(p)})$ in the following way [9]:

$$\rho(f,h) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \frac{P_{\alpha_n}(f-h)}{1+P_{\alpha_n}(f-h)} \quad . \qquad (4,10)$$

This distance is invariant with respect to translations, i.e.,

$$\rho(f+x, h+x) = \rho(f,h)$$

and moreover is determined even in the case when a finite or an infinite number of components f_α are equal to infinity. The topology determined by the metric (4,10) is equivalent to the original Freschet topology, determined by the set of semi-norms (4,9).

A sequence x_n of vectors is convergent to the vector x in the Freschet topology if, for an arbitrary fixed N,

$$\lim_{n=\infty} P_N(x_n - x) = 0 \ . \qquad (4,11)$$

The Freschet topology on the vector space ${}^{\mathcal{I}}\mathcal{L}^{J_p, M_p}(X^{(p)})$ is weaker than the Hilbert topology, which implies that not every sequence x_n convergent in the Freschet topology is also convergent in the Hilbert one.

The definition (4,4) of the hermitian indefinite bilinear form satisfying conditions i)...iv) can be used also for complex values of J_p and M_p (i.e., also for some spaces of type ${}^{\mathcal{I}}\mathcal{L}^{J_p, M_p}(X^{(p)})$ provided that in the ϑ^p dependent part of the function (3,4) the hypergeometric function ${}_2F_1(-n_-, -n_+; \alpha+1; -t_g{}^2\vartheta^p)$ represents a polynomial. This holds, for example, if J_p is an arbitrary complex number and $M_p = J_p - 2n, n=0,1,2,\ldots$ The sign of the corresponding norm depends on the choice of cuts in the N_+ or N_- complex plane (see Appendix D). However, as we cannot determine a single-valued global representation of the $U(p)$ in these spaces, we shall not consider them in detail.

In the topological vector spaces ${}^{\mathcal{I}}\mathcal{L}^{J_p, M_p}(X^{(p)})$, the global representation $g \to T(g)$ of $U(p)$ may be determined by

$$\left(T(g)f\right)(X) = f(g^{-1}X), \quad f \in {}^{I_p}\mathcal{L}^{J_p,M_p}(X^{(p)}) \ .$$

The operator $T(g)$ conserves the indefinite metric form (4, 4) because of the left-invariance of the measure $\mu(X)$ on the homogeneous manifold $X^{(p)}$ Moreover, the conditions

$$T(g_1 g_2) = T(g_1)\,T(g_2) \quad \text{and} \quad T(e) = I \quad ,$$

are also formally satisfied. However, in order that a mapping $g \to T(g)$ represents a global representation of a topological group G in a topological vector space $\mathcal{L}$, two additional conditions must be satisfied [3] : namely, for an arbitrary $g \in G$

1° $T(g)$ is a continuous operator in the space $\mathcal{L}$,

2° $T(g)$ is a continuous function of g , i. e., if $\lim_{n\to\infty} g_n = g$, then for an arbitrary $f \in \mathcal{L}$

$$\lim_{n\to\infty} T(g_n)f = T(g)f$$

The fulfilment of these continuity conditions, as well as a concrete and not merely formal fulfilment of Eq. (4, 3), essentially depends on the topologies introduced in the spaces ${}^{I_p}\mathcal{L}^{J_p,M_p}(X^{(p)})$, and in the space of linear maps of the space $\mathcal{L}^{J_p M_p}(X^{(p)})$. The detailed analysis of these problems will be given elsewhere.

5. CONCLUDING REMARKS

The linear space $\mathcal{L}^{J_p,M_p}(X^{(p)})$ generated by a set of simultaneous eigenfunctions (3, 1) of the invariant operators $\Delta(p)$ and $\hat{M}_p$ provides a carrier space for the Lie algebra R_p of $V(p)$. We have selected three classes of such spaces. The first class contains the finite-dimensional Hilbert spaces $\mathcal{H}^{J_p M_p}(X^{(p)}, \mu)$ spanned by the set of eigensolutions (3, 1) with eigenvalues J_p and M_p satisfying condition (3, 2), i.e., belonging to the spectrum of the self-adjoint invariant operators $\Delta(p)$ and $\hat{M}_p$. The second class contains infinite-dimensional vector spaces ${}^{I}\mathcal{L}^{J_p,M_p}(X^{(p)})$ with the indefinite metric (4, 5). In these spaces, convergence can be determined by means of the Hilbert or Freschet topologies. The eigenvalues J_p and M_p which determine these spaces satisfy the condition 6 or 8 of Sec. 3. The third class contains infinite-dimensional vector spaces ${}^{II}\mathcal{L}^{J_p,M_p}(X^{(p)})$ corresponding to arbitrary eigenvalues of J_p and M_p from the complex plane C^2 which do not satisfy conditions (3, 2). or 6 or 8 of Sec. 3. The algebraic structures of all spaces are drawn in Figs. 1 - 9.

In the carrier spaces $\mathcal{L}^{J_p,M_p}(X^{(p)})$, the maximal set of commuting operators in the enveloping algebra of the Lie algebra R_p of $U(p)$ contains the following operators:

$$\Delta(p)\,,\hat{M}_p\,,\ \Delta(p-1)\,,\ M_{p-1}\,,\ \ldots\,,\ \Delta(2)\,,\ \hat{M}_2 \text{ and } \hat{M}_1\ . \tag{5,1}$$

In the vector spaces ${}^{I}\mathcal{L}^{J_p,M_p}(X^{(p)})$ and ${}^{II}\mathcal{L}^{J_p,M_p}(X^{(p)})$ all operators from the set (5, 1) related to the subgroup $U(p-1)$, $U(p-2)$, ..., $U(1)$, are unbounded operators. We may use the formalism developed here for finding continuous series of irreducible infinite-dimensional representations of other compact Lie algebras, e. g., of $SO(n)$ and $Sp(n)$ groups, using as the main tool the sets of harmonic functions for corresponding groups [10] .

The continuous series of representations of Lie algebras of compact groups may be applicable in some physical problems; e. g., for investigation of the symmetry properties of the hydrogen atom. In this case we usually use the compact Lie algebra of the $SO(4)$ group for the description of the discrete spectrum of energy and the non-compact Lie algebra of the $SO(3,1)$ group for the continuous spectrum [11] . If, for different states of the same physical system described by the Hamiltonian, we really have to use two different groups as symmetry groups, then a concept of a higher symmetry group becomes extremely unclear. Previously, it was not possible to use the Lie algebra of the compact group for the description of the symmetry of both bound and scattering states because the irreducible representations of these algebras determined by continuous invariant numbers were not available. Therefore, the introduction of continuous series of irreducible representation of compact Lie algebras may clarify the hydrogen atom problem as well as other problems of higher symmetries in elementary particle physics.

Moreover, the possibility of the existence of infinite irreducible multiplets which can be related to continuous representations of a compact Lie algebra, seems very attractive, as compact groups appear most naturally as internal higher symmetry groups in elementary particle physics.

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APPENDIX A

List of cases in which relation (3, 4) determines a harmonic function

As stated in Sec. 3, relation (3, 4) can be used to determine a harmonic function $\mathbf{Y}^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega)$ if and only if the factor $G_\pm$ is finite and non-vanishing. The factor $G_\pm$ is given by formula (3, 5) and can be written in the following form:

$$G_\pm = (J_p+p-1)\left(\frac{(N_+-N_+'+1)(N_+-N_+'+2)\ldots N_+(N_++1)\cdots(N_++N_-'+p-3)(N_++N_-'+p-2)}{(N_--N_-'+1)(N_--N_-'+2)\ldots N_-(N_-+1)\cdots(N_-+N_+'+p-3)(N_-+N_+'+p-2)}\right)^{\pm 1}. \tag{A, 1}$$

The factors in the numerator as well as in the denominator form increasing sequences. It is easy to see that $G_\pm$ is finite and non-vanishing if any one of the following conditions is fulfilled:

1. Both N_+ and N_- are not integers, and

$$N_+ + N_- \neq -p+1 . \tag{A, 2}$$

2. If N_+ is an integer and N_- is not an integer, but either

$$N_+ - N_+' = 0, 1, 2, \ldots , \tag{A, 3a}$$

or

$$N_+ + N_-' + p - 1 = 0, -1, -2, \ldots . \tag{A, 3b}$$

3. If N_- is an integer and N_+ is not an integer, but either

$$N_- - N_-' = 0, 1, 2, \ldots , \tag{A, 4a}$$

or

$$N_- + N_+' + p - 1 = 0, -1, -2, \ldots . \tag{A, 4b}$$

4. If both N_+ and N_- are integers, $G_\pm$ is non-vanishing and finite in the following cases:

a)

$$N_+ - N_+' = 0, 1, 2, \ldots ,$$
$$N_- - N_-' = 0, 1, 2, \ldots . \tag{A, 5a}$$

b)

$$N_+ - N_+' = 0, 1, 2, \ldots ,$$
$$N_- + N_+' + p - 1 = 0, -1, -2, \ldots , \tag{A, 5b}$$
$$N_+ + N_- \neq -p + 1 .$$

c)

$$N_- - N_-' = 0, 1, 2, \ldots ,$$
$$N_+ + N_-' + p - 1 = 0, -1, -2, \ldots , \tag{A, 5c}$$
$$N_+ + N_- \neq -p + 1 .$$

d)

$$N_+ + N_-' + p - 1 = 0, -1, -2, \ldots ,$$
$$N_- + N_+' + p - 1 = 0, -1, -2, \ldots . \tag{A, 5d}$$

e)

$$\left.\begin{array}{l} N_+ - N_+' < 0 , \\ N_+ + N_-' + p - 1 > 0 \end{array}\right\} \text{ if and only if } \left\{\begin{array}{l} N_- - N_-' < 0 , \\ N_- + N_+' + p - 1 > 0 \end{array}\right. \tag{A, 5e}$$

We can easily check that condition (A, 5a) leads to the Hilbert space $\mathcal{H}^{J_p}_{M_p}(\chi^{(p)}, \mu)$ (see Fig. 9) whereas conditions (A, 5c) and (A, 5b) lead to the vector spaces ${}^{I}\mathcal{D}^{J_p}_{M_p}(\chi^{(p)}, \mu)$ (see Fig. 8 and a diagram obtained from Fig. 8 by reflection in the J_{p-1} -axis). Condition (A, 5d) leads to the space Π^- in Fig. 7.

APPENDIX B

Properties of the Lie algebra of the compact unitary group

The Lie algebra R_p of the $U(p)$ group consists of p^2 basic elements which can be subdivided into the following categories:

a) $\binom{p}{2}$ operators which represent by themselves generators of the subgroup $SO(p)$ of the $U(p)$ group. We shall denote them by

$$L^{+}_{ij} \ , \ i,j = 1,2,\dots,p \ , \ i<j \ ;$$

b) $\binom{p+1}{2}$ generators L^{-}_{ij} $(i,j = 1,2,\dots,p, i \leq j)$ which do not correspond to the $SO(p)$ group. Among them there are "diagonal" operators L^{-}_{ii} $(i=1,2,\dots,p)$ which belong to a Cartan subalgebra of R_p .

The commutation relation for these operators have the following form:

$$\left[L^{\pm}_{ij} , L^{\pm}_{kl} \right] = \pm \left(\delta_{kj} L^{\pm}_{il} - \delta_{il} L^{\pm}_{kj} \right) - \delta_{ki} L^{\pm}_{jl} + \delta_{jl} L^{\pm}_{ki} \ , \qquad (B,1)$$

where it is convenient to introduce the following symmetry properties:

$$L^{\pm}_{ji} = \mp L^{\pm}_{ij} \ , \qquad (B,2)$$

for all $i,j = 1,2,\dots,p$.

All the generators can be represented in the space $\mathcal{L}(X^{(p)})$ in terms of co-ordinates z^i and their derivatives $\frac{\partial}{\partial z^i}$ in the form

$$\begin{aligned} L^{+}_{kl} &= z^k \frac{\partial}{\partial z^l} - z^l \frac{\partial}{\partial z^k} + \bar{z}^k \frac{\partial}{\partial \bar{z}^l} - \bar{z}^l \frac{\partial}{\partial \bar{z}^k} \ , \\ L^{-}_{kl} &= i \left(z^k \frac{\partial}{\partial z^l} + z^l \frac{\partial}{\partial z^k} - \bar{z}^k \frac{\partial}{\partial \bar{z}^l} - \bar{z}^l \frac{\partial}{\partial \bar{z}^k} \right) . \end{aligned} \qquad (B,3)$$

As we have chosen the group parameters to be real, the generators are skew-adjoint.

The algebra R_p has an abelian centre, which is formed by one operator $\sum_{i=1}^{p} L^{-}_{ii}$. If this operator is dropped from R_p we obtain the algebra of $SU(p)$. The most suitable way to do this is to define the Weyl basis

$$H_l = i\, l \sqrt{\frac{2}{l(l+1)}} \sum_{k=1}^{l} \left(L^-_{l+1,\,l+1} - L^-_{kk} \right) \ , \quad l = 1, 2, \ldots, p-1 \ ,$$

$$E_{(k\,l)} = \frac{1}{2\sqrt{p(p-1)}} \left(i\, L^+_{kl} - L^-_{kl} \right) \ . \tag{B,4}$$

If $E_{(k\,l)}$ is denoted by E_α , and $E_{-\alpha}$ is given by $E_{-\alpha} = E_{(lk)} = E^+_{(kl)}$ then the set of generators H_l , E_α and $E_{-\alpha}$ fulfil the standard commutation relation of the Lie algebra of the $SU(p)$ group.

APPENDIX C

Classification of representation spaces for the algebra of the $U(2)$ group

The classification of representation spaces given in Section 4 is valid for a general $U(p)$ group, $p = 3, 4, \dots$. In the case $p = 2$ the classification is similar but differs in some points due to the fact that the $U(p-1)$ subgroup of the $U(p)$ group is abelian in this case. In this Appendix, we shall point out these differences.

The generators $E_{\pm(p,p-1)}$ given by formula (4,5) have now the following form:

$$E_{\pm(2,1)} \equiv \frac{i}{2\sqrt{2}} \left(L^{+}_{2,1} \pm i\, L^{-}_{2,1} \right)$$
$$= \frac{i}{2\sqrt{2}}\, e^{\mp i(\varphi^1-\varphi^2)} \left(-\frac{\partial}{\partial\vartheta} \pm i \operatorname{tg}\vartheta \frac{\partial}{\partial\varphi^2} \pm i \operatorname{cotg}\vartheta \frac{\partial}{\partial\varphi^1} \right). \qquad (C,1)$$

The action of $E_{\pm(2,1)}$ on a general harmonic function can be expressed as follows :

$$E_{\pm(2,1)} Y^{J_2}_{M_1M_2}(\phi^1,\phi^2,\vartheta) = \frac{i}{\sqrt{2}} \sqrt{D_{\mp}(D_{\pm}+1)}\; Y^{J_2}_{M_1\mp 1, M_2}(\phi^1,\phi^2,\vartheta), \qquad (C,2)$$

where $$D_{\pm} = \frac{1}{2}(J_2 \pm M_2) \mp M_1 . \qquad (C,3)$$

In accordance with (4,8) we introduce

$$N_{\pm} = \frac{1}{2}(J_2 \pm M_2),$$
$$N'_{+} = \frac{1}{2}(J_1 + M_1) = M_1, \qquad (C,4)$$
$$N'_{-} \equiv 0 .$$

In this definition, N'_+ is a general (non-negative or negative) integer, whereas N'_- is identically zero. If we had defined J_1 as $|M_1|$ instead of M_1, we would obtain that both N'_+ and N'_- are non-negative integers. However, the definition $J_1 = M_1$ appears to be more convenient for the classification of representation spaces and for the definition of phase factors of the harmonic functions.

The linear spaces to be dealt with in the subsequent classification can, again, be divided into three categories, $\mathcal{H}, \mathcal{I}$ and $\mathcal{II}$, which are determined by the value of the numbers $n_\pm$

$$n_\pm = \frac{1}{2}\left(J_2 \pm M_2 - |M_1| \mp M_1\right). \qquad (C,5)$$

The criteria of appurtenance to a given category are the same as those discussed in Section 3 for $p = 3, 4, \ldots$.

We divide, again, all complex numbers into the categories C_1, C_2, C_3 and C_4 according to (3,10). Note that the set C_3 in the case $p = 2$ is empty. The classification of all cases indicated in (3,11) can be performed in an analogous way.

1. $N_+ \in C_4$, $N_- \in C_4$.

The analysis of formula (C,1) leads to the same result as for $p \geq 3$. We have only one irreducible representation space, which belongs to the category II. For given values J_2 and M_2, the number M_1 can be an arbitrary integer.

2. $N_+ \in C_4$, $N_- \in C_2$.

The coefficient D_+ can never vanish; D_- vanishes at $M_1 = -N_- > 0$. The barrier, $M_1 = -N_-$, $M_1 = -N_- - 1$ restricts the representation space to the region of M_1 smaller than $-N_-$. The position of the representation space on the M_1-axis is shown in Fig. 10.
For $M_1 \geq -N_- - 1$ there exists no representation space because the factor $G_\pm$ in (3,5) is not well defined.

which can be distinguished by the eigenvalues ± 1 of the invariant operator τ_1 defined as follows:

$$\tau_1 Y^{J_z}_{M_1,M_z}(\Omega) = \mathrm{sign}\,(J_z - M_z + 2M_1 + 1)\, Y^{J_z}_{M_1,M_z}(\Omega).$$

The corresponding scheme is $(\mathrm{II}^-, \emptyset, \mathrm{I}^+)$.

7. $N_+ \in C_1$, $N_- \in C_4$.

The coefficient D_- cannot vanish; the vanishing of D_+ creates a barrier restricting the representation space ${}^{\mathrm{I}}\mathcal{L}^{J_z}_{M_z}(X^{(2)})$ to the region $M_1 \leq -N_+$ according to scheme $(\mathrm{II}, \emptyset)$.

8. $N_+ \in C_1$, $N_- \in C_2$.

D_- vanishes for $M_1 = -N_- > 0$; D_+ vanishes for $M_1 = N_+ \geq 0$. The mutual position of these two barriers depends, as in case 6, on the value of $N_+ + N_- \equiv J_z$. The corresponding scheme is $(\mathrm{I}^-, \emptyset, \mathrm{II}^+)$ and the spaces are distinguished by the eigenvalues ± 1 of the invariant operator τ_1

9. $N_+ \in C_1$, $N_- \in C_1$.

D_- vanishes at $M_1 = -N_- \leq 0$; D_+ vanishes at $M_1 = N_+ \geq 0$. These two barriers select from the M_1-line one representation space, $\mathcal{H}^{J_z}_{M_z}(X^{(2)}, \mu)$, M_1 being restricted by condition $-N_- \leq M_1 \leq N_+$. The scheme is $(\emptyset, \mathcal{H}, \emptyset)$.

APPENDIX D

<u>Normalisation of the harmonic functions</u>

In this Appendix we shall study the normalisation properties of the harmonic functions $Y^{J_2,\dots,J_p}_{M_1,\dots,M_p}(\Omega)$. The bilinear form is determined by the expression

$$\int \overline{Y^{J_2',\dots,J_{p-1}',J_p}_{M_1',\dots,M_{p-1}',M_p}}(\Omega)\, Y^{J_2,\dots,J_{p-1},J_p}_{M_1,\dots,M_{p-1},M_p}(\Omega)\, d\mu(X^{(p)}) \quad , \qquad (D,1)$$

where the bar means complex conjugation. Since we have assumed $J_2,\dots,J_{p-1}$ to be non-negative and integral and $M_1,\dots,M_{p-1}$ to be integral in the present paper, we can put $\overline{J_k'} = J_k'$ and $\overline{M_\ell'} = M_\ell'$ $(k = 2,3,\dots,p-1\ ;\ \ell = 1,2,\dots p-1)$ in (D,1) Furthermore, as global representations of the $U(p)$ group have been obtained by us only for integral values of J_p and M_p, we put also $\overline{J_p} = J_p$ and $\overline{M_p} = M_p$ in (D,1).

The integration in (D,1)over the variables $\phi^1,\dots,\phi^p$ and $\vartheta^2,\dots,\vartheta^{p-1}$ can be performed in the same way as in the case of the finite-dimensional unitary representations of $U(p)$ discussed in Section 2, because the integrability conditions

$$|M_k - M_{k-1}| + |J_{k-1}| = J_k - 2n_k \quad , \quad n_k = 0,1,2,\dots \ , \qquad (D,2)$$

are fulfilled for $k = 2,3,\dots,p-1$ On the other hand, as the validity of (D.2) for $k = p$ has not been assumed by us in this work, the integration (D,1) in the variable ϑ^p is, generally speaking, not defined.

The ϑ^p-dependent part $\Theta^{J_{p-1},J_p}_{M_{p-1},M_p}(\vartheta^p)$ of the general harmonic function $Y^{J_2,\dots,J_p}_{M_1,\dots,M_p}(\Omega)$ can be written in the form given by formula (3,4) The integrability properties of

$\Theta^{J_{p-1}\, J_p}_{M_{p-1}\, M_p}(\vartheta^p)$ are determined by the values of the parameters $n_\mp = N_\mp - N'_\mp$. If both n_- and n_+ are some non-negative integers the corresponding harmonic function belongs to the Hilbert space $\mathcal{H}^{J_p}_{M_p}(X^{(p)}, \mu)$. If only one of these numbers is a non-negative integer the function $\Theta^{J_{p-1},J_p}_{M_{p-1},M_p}(\vartheta^p)$ can be expressed in terms of Jacobi polynominals $P^{\alpha,\beta}_n(\cos 2\vartheta^p)$ in the following way:

$$\Theta^{J_{p-1},J_p}_{M_{p-1},M_p}(\vartheta^p) = (-1)^\alpha 2^{-\frac{1}{2}(J_{p-1}\pm\beta-1)}\sqrt{G_\pm}\,(1+x)^{\pm\frac{1}{2}\beta}(1-x)^{\frac{1}{2}J_{p-1}} P^{\alpha,\pm\beta}_{n_\mp}(x), \qquad x = \cos 2\vartheta^p, \tag{D.3}$$

where all symbols have the same meaning as in (3,4)and the upper or the lower sign pertains to the case when n_- or n_+ is a non-negative integer, respectively.

The normalization integral

$$I = \int_0^{\pi/2} \overline{\Theta^{J_{p-1},J_p}_{M_{p-1},M_p}(\vartheta^p)}\;\Theta^{J_{p-1},J_p}_{M_{p-1},M_p}(\vartheta^p)\,\sin^{2p-3}\vartheta^p \cos\vartheta^p\, d\vartheta^p , \tag{D,4}$$

can be written in the form

$$I_\pm = 2^{-\alpha\mp\beta-1}\,\overline{\sqrt{G_\pm}}\,\sqrt{G_\pm}\int_{-1}^{1}(1+x)^{\pm\beta}(1-x)^\alpha\,\overline{P^{\alpha,\pm\beta}_{n_\mp}(x)}\,P^{\alpha,\pm\beta}_{n_\mp}(x)\,dx . \tag{D,5}$$

Since the properties of I_+ and of I_- are quite analogous, it is sufficient to discuss one case only, I_+, say. The quantity I_+ depends on three independent variables, α, β and n_- ; α and n_- being some non-negative integers. β is in general complex but we have only obtained global representations of the $U(p)$ group for integral values of β.

For real values of β the integral on the right-hand side of (D,5) coincides with

$$A^{\alpha}_{n_-}(\beta) = \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \left(P^{\alpha,\beta}_{n_-}(x)\right)^2 dx . \qquad \text{(D,6)}$$

The integral (D,6) defines a function $A^{\alpha}_{n_-}(\beta)$ which is analytic in the complex β-plane. For $Re\,\beta > -1$ the function $A^{\alpha}_{n_-}(\beta)$ can be evaluated directly by performing the integration in (D,6) leading to the result (see [12])

$$A^{\alpha}_{n_-}(\beta) = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+1+2n)\,\Gamma(\alpha+\beta+n+1)} . \qquad \text{(D,7)}$$

This leads to the following value of $I_{\pm}$

$$I_{\pm} = \frac{\sqrt{G_{\pm}}\,\overline{\sqrt{G_{\pm}}}}{G_{\pm}} = G_{\pm}^{-\frac{1}{2}}\,\overline{G_{\pm}^{\frac{1}{2}}} \qquad \text{(D,8)}$$

For $Re\,\beta \leq -1$ the integral (D,6) does not exist in the usual sense. However, it can be defined in the regularised sense of Riesz and Gel'fand. In fact, representing $(P^{\alpha,\beta}_{n_-}(x))^2$ as a polynomial in $(1+x)$ with coefficients $a^{\alpha,\beta}_{\ell}$, we express $A^{\alpha}_{n_-}(\beta)$ as a finite sum of beta functions:

$$\begin{aligned} A^{\alpha}_{n_-}(\beta) &= \sum_{\ell=0}^{2n_-} a^{\alpha,\beta}_{\ell} \int_{-1}^{1} (1+x)^{\beta+\gamma}(1-x)^{\alpha} dx \\ &= \sum_{\ell=0}^{2n_-} a^{\alpha,\beta}_{\ell}\, 2^{\alpha+\beta+\ell+1} \int_0^1 y^{\beta+\gamma}(1-y)^{\alpha} dy \\ &= \sum_{\ell=0}^{2n_-} a^{\alpha,\beta}_{\ell}\, 2^{\alpha+\beta+\ell+1}\, B(\beta+\ell+1,\ \alpha+1) . \end{aligned}$$

The regularization procedure for such functions is given explicitly in the book by Gel'fand and Shilov, Chapter I (see [6]).
We obtain

$$A^{\alpha}_{n_-}(\beta) = \sum_{l=0}^{2n_-} a_l^{\alpha,\beta}\, 2^{\alpha+\beta+l+1} \Big\{ \int_0^{1/2} y^{\beta+l} \Big[(1-y)^{\alpha} - \sum_{r=0}^{k} (-1)^r \frac{\alpha!\, y^r}{r!\,\Gamma(\alpha-r+1)}\Big] dy$$

$$+ \int_{1/2}^{1} (1-y)^{\alpha} y^{\beta+l}\, dy + \sum_{r=0}^{k} (-1)^r \frac{\alpha!}{2^{r+\beta+l+1}\, r!\,\Gamma(\alpha-r+1)} \quad \frac{1}{r+\beta+l+1} \Big\}$$

This formula defines $A^{\alpha}_{n_-}(\beta)$ for all complex values of β for which $Re\,(\beta+1) \geq -k$ (except $\beta = 0, -1, -2, \ldots$) where k is an arbitrary non-negative integer. In the half-plane $Re\,\beta > -1$, its value coincides with the integral in the usual sense.

Taking k sufficiently large, $k \geq \alpha$, we obtain an expression which does not depend on k at all:

$$A^{\alpha}_{n_-}(\beta) = \sum_{l=0}^{2n_-} a_l^{\alpha,\beta}\, 2^{\alpha+\beta+l+1} \sum_{r=0}^{\alpha} (-1)^r \binom{\alpha}{r} \frac{1}{r+\beta+l+1} \; . \qquad \text{(D.9)}$$

We see that the regularised value of the integral (D,6) is an analytic function in β in the whole complex β -plane except a finite number of points where $A^{\alpha}_{n_-}(\beta)$ has poles.

By (D,9) the function $A^{\alpha}_{n_-}(\beta)$ is represented in the form of a finite series of rational functions. The sum of this series can easily be obtained by remembering that its value for $Re\,\beta > -1$ is given by the right-hand side of (D,7). Since both expressions are analytic in β , we obtain

$$A^{\alpha}_{n_-}(\beta) = \frac{2^{\alpha+\beta+1}\,(\alpha+n)!\;\Gamma(\beta+n+1)}{n!\;(\alpha+\beta+2n+1)\;\Gamma(\alpha+\beta+n+1)} \; , \qquad \text{(D,10)}$$

in the whole complex β- plane. Thus, $A^{\alpha}_{n_-}(\beta)$ has poles only at the following points:

$$\beta = -n_- - 1, -n_- - 2, \; \ldots \; , -\alpha - n_- \;\; \text{and} \;\; \beta = -\alpha - 2n_- - 1 . \qquad \text{(D,11)}$$

For real values of β the integral on the right-hand side of (D,5) coincides with

$$A_{n_-}^{\alpha}(\beta) = \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \left(P_{n_-}^{\alpha,\beta}(x)\right)^2 dx . \qquad \text{(D,6)}$$

The integral (D,6) defines a function $A_{n_-}^{\alpha}(\beta)$ which is analytic in the complex β-plane. For $Re\,\beta > -1$ the function $A_{n_-}^{\alpha}(\beta)$ can be evaluated directly by performing the integration in (D,6) leading to the result (see [12])

$$A_{n_-}^{\alpha}(\beta) = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+1+2n)\,\Gamma(\alpha+\beta+n+1)} . \qquad \text{(D,7)}$$

This leads to the following value of $I_{\pm}$

$$I_{\pm} = \frac{\sqrt{G_{\pm}}\,\overline{\sqrt{G_{\pm}}}}{G_{\pm}} = G_{\pm}^{-\frac{1}{2}}\,\overline{G_{\pm}^{\frac{1}{2}}} \qquad \text{(D,8)}$$

For $Re\,\beta \leq -1$ the integral (D,6) does not exist in the usual sense. However, it can be defined in the regularised sense of Riesz and Gel'fand. In fact, representing $(P_{n_-}^{\alpha,\beta}(x))^2$ as a polynomial in $(1+x)$ with coefficients $a_{\ell}^{\alpha,\beta}$, we express $A_{n_-}^{\alpha}(\beta)$ as a finite sum of beta functions:

$$\begin{aligned} A_{n_-}^{\alpha}(\beta) &= \sum_{\ell=0}^{2n_-} a_{\ell}^{\alpha,\beta} \int_{-1}^{1} (1+x)^{\beta+\gamma}(1-x)^{\alpha}\,dx \\ &= \sum_{\ell=0}^{2n_-} a_{\ell}^{\alpha,\beta}\, 2^{\alpha+\beta+\ell+1} \int_0^1 y^{\beta+\gamma}(1-y)^{\alpha}\,dy \\ &= \sum_{\ell=0}^{2n_-} a_{\ell}^{\alpha,\beta}\, 2^{\alpha+\beta+\ell+1}\, B(\beta+\ell+1,\ \alpha+1) . \end{aligned}$$

The regularization procedure for such functions is given explicitly in the book by Gel'fand and Shilov, Chapter I (see [6]).
We obtain

$$A_{n_-}^{\alpha}(\beta) = \sum_{l=0}^{2n_-} a_l^{\alpha,\beta}\, 2^{\alpha+\beta+l+1} \left\{ \int_0^{1/2} y^{\beta+l} \left[(1-y)^{\alpha} - \sum_{r=0}^{k} (-1)^r \frac{\alpha!\, y^r}{r!\, \Gamma(\alpha-r+1)}\right] dy \right.$$

$$\left. + \int_{1/2}^{1} (1-y)^{\alpha} y^{\beta+l}\, dy + \sum_{r=0}^{k} (-1)^r \frac{\alpha!}{2^{r+\beta+l+1}\, r!\, \Gamma(\alpha-r+1)} \; \frac{1}{r+\beta+l+1} \right\}$$

This formula defines $A_{n_-}^{\alpha}(\beta)$ for all complex values of β for which $Re\,(\beta+1) \geq -k$ (except $\beta = 0, -1, -2, \ldots$) where k is an arbitrary non-negative integer. In the half-plane $Re\,\beta > -1$, its value coincides with the integral in the usual sense.

Taking k sufficiently large, $k \geq \alpha$, we obtain an expression which does not depend on k at all:

$$A_{n_-}^{\alpha}(\beta) = \sum_{l=0}^{2n_-} a_l^{\alpha,\beta}\, 2^{\alpha+\beta+l+1} \sum_{r=0}^{\alpha} (-1)^r \binom{\alpha}{r} \frac{1}{r+\beta+l+1} . \qquad \text{(D.9)}$$

We see that the regularised value of the integral (D,6) is an analytic function in β in the whole complex β-plane except a finite number of points where $A_{n_-}^{\alpha}(\beta)$ has poles.

By (D,9) the function $A_{n_-}^{\alpha}(\beta)$ is represented in the form of a finite series of rational functions. The sum of this series can easily be obtained by remembering that its value for $Re\,\beta > -1$ is given by the right-hand side of (D,7). Since both expressions are analytic in β, we obtain

$$A_{n_-}^{\alpha}(\beta) = \frac{2^{\alpha+\beta+1}(\alpha+n)!\, \Gamma(\beta+n+1)}{n!\, (\alpha+\beta+2n+1)\, \Gamma(\alpha+\beta+n+1)} , \qquad \text{(D,10)}$$

in the whole complex β- plane. Thus, $A_{n_-}^{\alpha}(\beta)$ has poles only at the following points:

$$\beta = -n_- -1, -n_- -2, \ldots, -\alpha - n_- \text{ and } \beta = -\alpha - 2n_- - 1. \qquad \text{(D,11)}$$

The conditions (C,5c)

$$N_- - N_-' = 0, 1, 2, \dots \quad ,$$
$$N_+ + N_-' + p - 1 = 0, -1, -2, \dots ,$$
$$N_+ + N_- \neq -p + 1 \quad ,$$

under which the representation space ${}^{\mathcal{I}_p}\mathcal{L}_{M_p}^{J_p}(X^{(p)})$ can be realised, are equivalent, respectively, to the following conditions:

$$\begin{aligned} &n_- = 0, 1, 2, \dots \quad , \\ &\beta = -\alpha - n_- - 1, -\alpha - n_- - 2, \dots , \\ &\beta \neq -\alpha - 2n - 1 \; . \end{aligned} \qquad \text{(D,12)}$$

Comparing (D,12) with (D,11) we see that in the representation space ${}^{\mathcal{I}_p}\mathcal{L}_{M_p}^{J_p}(X^{(p)})$ characterised by conditions (C,5c) , a metric can be defined by using $\mathcal{I}_+$ given by formula (D,8).

In a completely analogous way, the case of the lower sign in (D,5) is treated. We come to the result that in the representation space ${}^{\mathcal{I}_p}\mathcal{L}_{M_p}^{J_p}(X^{(p)})$ characterised by conditions (C,5b), a metric can be defined by using $\mathcal{I}_-$ given by formula (D,8).

Let us now determine the sign of $\mathcal{I}_\pm$. From the form (C,1) of $G_\pm$ we see that if both $N_+ - N_+'$ and $N_- - N_-'$ are non-negative integers, all factors of the numerator as well as of the denominator are positive, i.e., $G_\pm^{1/2}$ is real and the metric is positive definite. On the other hand, if conditions (C,5c) or (C,5b) are satisfied we obtain

$$\operatorname{sign} G_\pm = (-1)^{N_+' + N_- + p - 2} = (-1)^{J_{p-1} + p - 2} \; ,$$

for $J_p + p - 1 > 0$ and

$$\operatorname{sign} G_\pm = (-1)^{J_{p-1} + p - 1} \; ,$$

for $J_p + p - 1 < 0$. Inserting these expressions into (D,8), we get finally

$$\mathcal{I}_{\pm} = (-1)^{J_{p-1}+p-2} ,$$

for $J_p + p - 1 > 0$ and

$$\mathcal{I}_{\pm} = (-1)^{J_{p-1}+p-1} ,$$

for $J_p + p - 1$. In both cases, the sign of the norm depends on the invariant J_{p-1} of the subgroup $\mathcal{U}(p-1)$ of $\mathcal{U}(p)$ and is different for different harmonic functions belonging to the same representation space ${}^{\mathcal{I}}\mathcal{L}^{J_p}_{M_p}(X^{(p)}, \mathcal{M})$. Thus the metric form on the space is indefinite.

Let us mention that the definition (D,1) of the bilinear form can also be used for complex values of J_p and M_p provided that the function $P^{\alpha, \pm\beta}_{n_{\mp}}$ *in* (D,3) is of the polynomial type. The sign of the corresponding norm depends on the choice of cuts in the $\mathcal{N}_+$ or $\mathcal{N}_-$ complex plane. However, as we cannot determine a single-valued global representation of the $\mathcal{U}(p)$ group for complex J_p or M_p , we shall not consider these cases in detail.

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