INFLUENCE OF STATIC, RADIAL ELECTRIC FIELDS ON TRAPPED PARTICLE INSTABILITIES IN TOROIDAL SYSTEMS

A. A. GALEEV
R. Z. SAGDEEV
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H. V. WONG

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A.A. GALEEV*
R.Z. SAGDEEV*
and
H.V. WONG

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* Permanent address: Institute for Nuclear Physics, Novosibirsk, USSR.
The drift motion of trapped particles in toroidal systems leads to a new branch of the gravitational instability. This is stabilized by radial electric fields. However, a new unstable mode then appears, which is similar to that due to ion impurity density gradients.
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I. INTRODUCTION

Recently B.B. KADOMTSEV and O. POGUTSE \(^1\) derived a new branch of the gravitational instability which is immune to shear stabilization. The instability arises in toroidal confining systems and is due essentially to the drift motion of the "trapped" particles.

In this paper we show that this instability can be stabilized by radial electric fields arising in the plasma \(^2\), in other words by plasma rotation. The effect of the electric field is to push the "trapped" ions to the tail of the ion distribution function so that the contribution of the trapped ions to the dispersion relation becomes small. A new unstable mode then becomes possible. This mode is analogous to that due to ion impurity density gradients, the trapped electrons taking the role of the impurity ions.

II. PARTICLE ORBITS

We consider a symmetric toroidal system with dimensions as shown in Fig. 1.

![Diagram of Toroidal Co-ordinates](Image)
\( R \) is the major radius, \( a \) is the minor radius, and \( a/R \ll 1 \). We choose co-ordinates \((r, \varphi, \zeta)\) to describe the geometry of the system. The \((r, \varphi)\) plane contains the major axis, and the position of this plane is defined by the angle \( \varphi \).

The main magnetic field is due to a current flowing along the major axis and is given by \( B_0 = B_0 (1 - (r/R_0) \cos \varphi) \). A rotational transform is about the minor axis is produced by the field \( B_\varphi = -i (r/R_0) B_0 \ll B_0 \).

The equations of motion of the particle-guiding centre in zero order approximation are:

\[
\frac{d}{dt} = 0, \quad \frac{d \varphi}{dt} = -i + \frac{c}{B_0} \frac{\partial \Phi^{(0)}}{\partial t} = -e v_n + \frac{c}{B_0} \frac{\partial \Phi^{(0)}}{\partial t} \Rightarrow \frac{d \varphi}{dt} = v_n
\]

and we have:

\[
\zeta = -i \left[ \varphi - \frac{c}{B_0} \frac{\partial \Phi^{(0)}}{\partial t} \right] = -q \left[ \varphi - \frac{c}{B_0} \frac{\partial \Phi^{(0)}}{\partial t} \right]
\]

where we denote by \( v_n \) and \( v_\varphi \) the components of particle velocity parallel and perpendicular to the minor axis. We assume the presence of a static, radial electric field \(-\frac{d \Phi^{(0)}}{dr}\).

The particles experience drifts due to curvature and gradients in the magnetic field. It is convenient to introduce a new quantity \( \xi \):

\[
\xi = \zeta + q \left[ \varphi - \frac{c}{B_0} \frac{\partial \Phi^{(0)}}{\partial t} \right]
\]

which describes the variation from the zero-order motion of the \( \varphi \) co-ordinate. For particles with charge \( e \) and mass \( m \), the equations of motion of the guiding centre, including drifts, can be written:

\[
\frac{d r}{dt} = -v_\varphi \sin \varphi
\]

\[
\frac{d \varphi}{dt} = -\varphi v_n + \varphi - v_\varphi \cos \varphi
\]

\[
\frac{d \xi}{dt} = -\frac{v_\varphi}{r} \left[ q \cos \varphi + q' \varphi + \sin \varphi \right]
\]

where \( v_\varphi = (v_\varphi^2 + v_n) / \Omega_j R_0 \), the magnitude of the curvature and gradient drifts, \( v_\varphi = (e / m) \Omega_j (\partial \Phi^{(0)}/\partial r) \) the drift due to a static, radial
electric field \( - (\partial E/\partial t) \), \( \Omega_j = (e_j B_0/m_0) \) the cyclotron frequency, and \( q' = dq/dr = (d/dr) (1/1) \).

The particle trajectories corresponding to Eqs. (4) - (6) have been discussed by BERK et al. Two types of orbits exist: the untrapped particles which circulate around the minor axis, and the trapped particles which do not. The particles are trapped when

\[
\left( \Omega_j - \frac{V_{\perp}}{\theta} \right)^2 < 4 \frac{V_{\parallel} + \frac{1}{2} \left( \frac{dV_{\parallel}}{dt} + \frac{1}{2} \frac{dV_{\perp}^2}{dt} \right)}{\Omega_j} \tag{7}
\]

Thus the trapped particles have velocity \( V_{\parallel} = V_{\parallel}/\theta \) and for large \( V_{\parallel} \) or small \( \theta \) they could lie in the tail of the distribution function. In the absence of electric fields, \( V_{\parallel} < \sqrt{r/r_0} V \) and they are in the main part of the distribution function.

We can estimate the period of oscillation of the trapped particles by using the zero order equation:

\[
\frac{d\varphi}{dt} = - \Theta V_{\parallel} + V_{\parallel} \tag{8}
\]

\( v_e \cos \varphi \) is of order \( \frac{r_0}{r} \) smaller than \( \Theta V_{\parallel} \); \( \Omega_L \) is the Larmor radius. We have:

\[
\frac{d\varphi}{dt} = \frac{1}{\Omega_j} \left( \sqrt{\left( \sigma V_{\parallel} + V_{\parallel}^2 \right)^4 + \left( \sigma V_{\parallel} - V_{\parallel} \right)^4 \left( \frac{d\varphi}{dt} - \frac{dV_{\parallel}}{dt} \right)} \sin \varphi \right) \tag{9}
\]

where \( r = r_0 \) when \( \varphi = 0 \), \( \varphi = \varphi_0 \), the magnetic moment \( \mu = \mu B \), \( 2 \chi^2 \) and we have neglected terms containing \( \frac{dV_{\parallel}}{dt} \) and \( \frac{dV_{\perp}}{dt} \).

The trapped particles are reflected at \( \varphi = \varphi_0 (\chi) \) given by \( \cos \varphi_0 = 1 - 2 \chi^2 \). We require \( \chi^2 < 1 \). The period of oscillation \( T \) is

\[
T = 4 \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\left( \Omega_j - \theta V_{\parallel} \right)^2 + \left( \sigma \frac{d\varphi}{dt} - \frac{dV_{\parallel}}{dt} \right)^2}} \tag{10}
\]

\[
= 4 \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{2(\mu B^2 + V_{\parallel}^2) \epsilon} \left( \sqrt{2(\mu B^2 + V_{\parallel}^2) \epsilon} \right)} = 4 \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{2(\mu B^2 + V_{\parallel}^2) \epsilon}} = 4 \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{2(\mu B^2 + V_{\parallel}^2) \epsilon}} = K (\sin \frac{\chi}{\chi_0})
\]

where \( K \) is a complete elliptic function of the first kind, and \( \chi \) is given by \( \cos \chi = 1 - 2 \chi^2 \).
We shall now derive the dispersion relation. We assume that 
\[ \beta = (\partial nT) / B^2 \ll 1, \]
where \( n \) is the density and \( T \) the temperature of the plasma, and we consider only electrostatic perturbations. The equilibrium distribution function \( f^{(0)} = f(r, v, \mu) \) is taken to be a local Maxwellian, where \( v_n \) is the velocity along the line of force. We neglect the dependence on \( \gamma \), since it does not make an essential contribution to the dispersion relation. The linearized equation for the perturbed distribution function in the drift approximation is

\[ \frac{\partial f_j^{(r)}}{\partial t} + v_d \cdot \nabla f_j^{(r)} = \frac{e_j}{m_j} \frac{\partial \Phi}{\partial v} \frac{\partial f_j^{(0)}}{\partial x} + \frac{e_j}{m_j} \nabla \Phi \cdot \frac{\partial f_j^{(0)}}{\partial v} \]  

where \( v_d \) is the velocity of the guiding centre. \( \Phi \) is the potential of the perturbed electric field

\[ \Phi(r', \phi, z, t) = \Phi(r, \phi) e^{-i\omega t} + i \ell \xi \]

where \( \ell \) is an integer. In the usual way we obtain for \( f_j^{(r)} \)

\[ f_j^{(r)} = -\frac{e_j}{m_j} \left[ \Phi(r, \phi) + i \tau \int \left[ i \omega \cdot \xi_j \frac{\partial }{\partial \phi} \Phi(r', \phi') \right] e^{-i\omega(t' - t)} \right] \]

where

\[ \omega_j^{(r)} = \frac{1}{m_j} \frac{1}{\tau_j} + \frac{1}{\tau_j^{(0)}} \frac{\partial f_j^{(r)}}{\partial v} \]

and we have replaced \( \gamma \) by \( \xi = \gamma (\phi - \phi') \) from Eq. (3). The time integration must be taken over the guiding centre trajectory.

We have two groups of particles which we must consider separately:

1) The trapped particles have a periodic oscillatory motion in restricted regions of \( (r, \phi) \)-space. We can expand \( \Phi(r', \phi') \) in a sum over the harmonics of this periodic motion. The integration over time becomes trivial. We obtain a contribution to the dispersion relation which is similar to that obtained when we consider the effect of Larmor orbiting in the magnetic field. The frequency of the periodic motion plays the same role as the Larmor frequency, except that here it would be a function of the velocity of the particles.

We shall be interested only in the zero harmonic. It is sufficient therefore to average the periodic variation of the integrand over one period of oscillation.
The co-ordinate $\xi$ makes small fluctuations about an average value which increases monotonically in time. For times longer than the period of one oscillation, the fluctuations are small compared with its average value and can be neglected. We can therefore approximate $\xi(t')$ by

$$\xi(t') \approx \langle \frac{d\xi}{dt} \rangle t' = \nu_{\xi} t'$$  \hspace{1cm} (12)

where $\langle \rangle$ imply averaging over one period of oscillation.

The contribution of the trapped particles to the integral in Eq. (11) becomes

$$\int_{-\infty}^{t} dt' e^{-i(\omega - \frac{\nu_{j}^2}{\kappa_{j}}) (t'-t)} \langle \left[ (i\omega - \Omega_{j}^{*} \frac{\partial}{\partial \varphi}) \Phi(t', \varphi) \right] e^{-ilq_{j}(\varphi_{j} - \varphi)} \rangle f_{j}^{(0)}$$

2) The untrapped particles circulate completely around the minor axis and are characterized by large angular velocities $r(d\varphi/dt)$. Neglecting the drift contributions, we can approximate $\varphi(t')$ by

$$\varphi(t') \approx - \frac{\Theta}{F} \nu_{\varphi} t' + \frac{\nu_{\varphi} t'}{F}$$  \hspace{1cm} (13)

The contribution of the untrapped particles to the integral in Eq. (11) is

$$\int_{-\infty}^{t} dt' e^{-i(\omega - \frac{\nu_{j}^2}{\kappa_{j}}) (t'-t)} \left[ (i\omega - \Omega_{j}^{*} \frac{\partial}{\partial \varphi}) \Phi(t', \varphi) \right] \frac{f_{j}^{(0)}}{f_{j}}$$

Assuming quasi-neutrality, we have for the dispersion relation

$$\left( \frac{1}{\nu_{e}} + \frac{1}{\nu_{i}} \right) n \Phi(t, \varphi)$$

$$= - \sum_{j=1}^{e, i} \int_{-\infty}^{t} d^{2} \nu_{j} \int_{\text{trapped}} dt' e^{-i(\omega - \frac{\nu_{j}^2}{\kappa_{j}}) (t'-t)} \langle \left[ (i\omega - \Omega_{j}^{*} \frac{\partial}{\partial \varphi}) \Phi(t', \varphi) \right] e^{-ilq_{j}(\varphi_{j} - \varphi)} \rangle f_{j}^{(0)} \frac{f_{j}^{(0)}}{f_{j}}$$

$$- \sum_{j=1}^{e, i} \int_{-\infty}^{t} d^{2} \nu_{j} \int_{\text{untrapped}} dt' e^{-i(\omega - \frac{\nu_{j}^2}{\kappa_{j}}) (t'-t)} \left[ (i\omega - \Omega_{j}^{*} \frac{\partial}{\partial \varphi}) \Phi(t', \varphi) \right] \frac{f_{j}^{(0)}}{f_{j}}$$  \hspace{1cm} (14)
This is an integral-differential equation which we must solve to obtain the eigenfrequencies. This is a difficult problem, and we shall content ourselves here by estimating the simplest eigenvalue of the dispersion equation.

We assume that

$$\Phi(t, \varphi) = \sum_{m} \Phi_{m} e^{im\varphi}$$

(15)

where \( m \) is an integer. We shall neglect the radial motion of the wave packet (for further discussion of this point see ref. 3). We obtain the dispersion relation similar to that first obtained by KADOMTSEV and POGUTSE:

\[
\left( \frac{1}{T_e} + \frac{1}{T_i} \right) \Pi \sum_{m} \frac{d\tau}{d^3} \left[ \frac{1}{\mu_e} + \frac{1}{\mu_i} \right] \frac{d\tau}{d^3} \left[ \frac{1}{\mu_e} + \frac{1}{\mu_i} \right] \Phi_{m} e^{im\varphi} \frac{f_{j}(\varphi)}{T_{j}}
\]

(16)

where

$$\omega_{j}^* = \frac{T_{j}}{\Omega_{j} m_{j}} \left( \frac{1}{\mu_{e}} + \frac{1}{\mu_{i}} \right) \frac{\partial f_{j}(\varphi)}{\partial \varphi} \frac{f_{j}(\varphi)}{T_{j}}$$

(17)

IV. STABILITY ANALYSIS

1) Zero static electric field

So far the value of \((m - \lambda q)\) has not been specified. We consider the case where
We assume the ion and electron temperatures to be equal \( T_e = T_i \), and static electric fields to be zero. We neglect the contribution of the untrapped particles. The dispersion relation becomes

\[
| \frac{(m - q) u_r}{q R_0} | > | \omega^* | > | \omega - \frac{q V_e}{r} | > | \ell V_e | \quad (18)
\]

\[
2n \Phi(\Phi) e^{-i \frac{q}{r} \varphi}
\]

\[
= \sum \frac{1}{\omega^*} \int d^3 \nu \left[ -\omega^* - \omega^* \ell v_f \right] \left\langle \Phi \ e^{i(m - q) \varphi'} \right\rangle f_i^{(0)}
\]

\[
+ \sum \frac{1}{\omega^*} \int d^3 \nu \left[ \omega^* - \omega^* \ell v_f \right] \left\langle \Phi \ e^{i(m - q) \varphi'} \right\rangle f_e^{(0)}
\]

(19)

where

\[
\omega^* = \omega^*_i = - \omega^*_e
\]

and

\[
V_{s_i} = V_{s_e} = - V_{s_i}
\]

\[
= \frac{1}{\tau} \int_0^\infty \frac{d x}{d t} d t = \frac{1}{\sqrt{2} \ K} \int_0 \frac{d \varphi}{\sqrt{2 x^2 - 1 + \cos \varphi}}
\]

\[
= \frac{-q}{q'} \left[ \left\{ \frac{E}{K} - \frac{1}{2} \right\} + \frac{2 q' x}{q} \left\{ \frac{E}{K} - 1 + x^2 \right\} \right]
\]

\[
= \frac{-q}{q'} \left\{ \sin \left( \frac{x}{2} \right) \right\}
\]

(21)

\( E \) is a complete elliptic function of the second kind with argument \( \sin \pi/2 \).
The trapped electrons and ions are in the main part of the distribution function, and we may assume their densities to be equal. The large terms containing $\omega \omega^*$ in Eq. (19) cancel and we obtain

$$n \tilde{\Phi}(q) e^{-i l q \psi} = -\sum_m 4\pi \omega^2 \int_{0}^{\infty} \frac{d\kappa}{\sqrt{2\kappa^2 - 1 + \cos \psi}} \omega^* \lambda \frac{1}{\kappa} \left( \Phi_m e^{i (m-l)\lambda^*} \right)$$

We have approximated $\int d^3 v$ by

$$\int_{\text{trapped}} d^3 v \approx 4\pi j e \int_{0}^{\infty} \frac{d\kappa^2}{\sqrt{2\kappa^2 - 1 + \cos \psi}}$$

since $v_t \sim v$ for the trapped particles.

Now we can write Eq. (22) in the form

$$\lambda_1 \tilde{\Phi}_m = \sum_{m'} H_{mm'} \tilde{\Phi}_{m'} \cdot \frac{m'}{m}$$

where

$$\lambda_1 = \frac{2\pi \omega^2 \gamma}{2 - \kappa^2 \frac{\lambda}{\kappa} \frac{q_f}{q_f} \omega^*}$$

$$H_{mm'} = \int_{0}^{\infty} d\kappa^2 \frac{G}{K} P_{m-l_q + \frac{1}{2}} P_{m'-l_q + \frac{1}{2}}$$

$$\int_{\text{trapped}} \frac{\cos (m-l_q) \Psi}{\sqrt{2\kappa^2 - 1 + \cos \psi}} = P_{m-l_q + \frac{1}{2}}$$

is the Legendre function.

Eq. (23) must now be solved to obtain a value for $\lambda_1$. KADOMTSEV and POGUTSE derived this equation and solved it numerically. They in fact considered the limit where $(m-l_q) \ll 1$, and also neglected the contribution of the untrapped particles. They found an aperiodic instability with growth rate $\gamma = \text{Im} \omega$ given by $\lambda$.

$\lambda$) Kadomtsev has shown, using numerical analyses, that $P_{mm'}$ decreases rapidly with $|m-m'|$. Therefore it is reasonable to derive the approximate dispersion relation taking simply $\lambda_1 = H_{mm'}$. We will use the Kadomtsev-type approximation later on.
2) Non-zero static electric field

We wish to point out, however, that Eq. (22) was obtained on the assumption of equal densities of trapped ions and electrons. If radial electric fields develop in the plasma (see ref. 2)), the plasma rotates about the minor axis of the toroid. The effect of this is not only to produce a frequency shift. At the same time the trapped particles are pushed out to the tail of the distribution function, since their parallel velocity \( v_n \sim v_E/\theta \) (see Eq. (7)). For equal ion and electron temperatures, the density of trapped ions decreases more rapidly than that of the electrons, and exact cancellation of the large terms in Eq. (19) do not occur. The instability will therefore be eventually quenched\(^*\) and we shall be left with a purely periodic mode with frequency

\[
\omega \sim \varepsilon^{\frac{1}{2}} \omega^* \]

On the other hand, when the density of trapped ions has become small, a new instability arises. The untrapped ions now play a significant role. Assuming that Eq. (18) is satisfied, we neglect in Eq. (16) the untrapped electrons, and retain the imaginary contribution of the untrapped ions coming from the half-residue in the velocity integration. The dispersion relation is

\[
2 \Phi (\phi) e^{-i \ell q \phi}
\]

\[
E \rightleftharpoons \int \frac{d\omega}{\omega^*} \int \frac{d\omega^*}{\omega^*} \frac{\omega^*}{\omega} \left\langle \Phi_m \left( \frac{i (m-\ell q) \eta}{\ell q} \right) \right\rangle \frac{f^{(0)}(\phi)}{f^{(0)}}
\]

\[
- \frac{q R_c}{(m-\ell q)} \left( \frac{\omega^*}{\omega} \right) \int d^3 \vec{v} \frac{\delta(v_n)}{\omega} \Phi^{(0)}(\phi) e^{-i \ell q \phi}
\]

\[
(28)
\]

\(^*\) The stability condition can be written in a form

\[
v_E^2/\theta^2 v_{thi}^2 > \epsilon^2
\]
where we again consider $T_i = T_e$.

If we neglect the effect of the radial electric fields on the trapped electrons, we can write Eq. (28) in the form

$$\sum_{m'} \frac{F_{mm'}}{m'} \frac{\omega^*}{\omega}$$ (29)

where

$$\lambda_2 = \frac{2 \pi \left( \omega - \frac{q R_0}{m} \right)}{2 \sqrt{2 \varepsilon \omega^*}} \left[ 1 + i \pi \frac{q R_0}{m} \omega^* \frac{i}{2 \pi} \int d^3 \Omega \int \frac{d \sigma}{\sigma} \mathcal{E}(\beta) \right]$$ (30)

and

$$F_{mm'} = \frac{\partial^2 \mathcal{E}}{\partial \mathcal{E}_{m'q+\frac{1}{2}}} \mathcal{E}_{m-q+\frac{1}{2}}$$ (31)

We must solve Eq. (29) to obtain a value for $\lambda_2$. The dispersion relation represented by Eq. (30) can be written

$$(\omega - \frac{q R_0}{m}) \approx \frac{2 \sqrt{2 \varepsilon \omega^*}}{2 \pi} \int d^3 \Omega \int \frac{d \sigma}{\sigma} \mathcal{E}(\beta)$$ (32)

This equation is similar to that derived for the ion impurity density gradient instability, the trapped electrons playing the part of the ion impurities.

The worst case results when the second term in the denominator of Eq. (3) is of order unity, and we have

$$\Im (\omega) = \frac{2 \sqrt{2 \varepsilon} \omega^*}{2 \pi} F_{mm} (-\omega^*)$$ (33)

To satisfy Eq. (18) we require

$$\left| \frac{(m-q \frac{1}{2}) \frac{1}{m} \frac{q R_0}{m}}{\varepsilon} \right| > 1 \Rightarrow \left| \int \mathcal{E} F_{mm} \right| > \left| \frac{\lambda \frac{1}{2}}{m \frac{1}{2}} \right|$$ (34)

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