SELF-CONSISTENT DESCRIPTION
OF A WARM STATIONARY PLASMA IN A
UNIFORMLY SHEARED MAGNETIC FIELD

A. SESTERO

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SELF-CONSISTENT DESCRIPTION OF A WARM STATIONARY PLASMA
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*On leave of absence from "Laboratorio Gas Ionizzati", EURATOM-CERN, Frascati, Italy.
An exact, self-consistent solution of the system of Vlasov's plus Maxwell's equations is derived, describing a warm stationary plasma in a uniformly sheared magnetic field. It is found that there is no upper limit for the rate of shear, that is, the distance over which the direction of the magnetic field rotates of an angle $2\pi$ can be as small as desired, in particular smaller than the ion or even the electron Larmor radius. Within the assumed model, for any value of plasma density, magnetic field strength and rate of shear a priori assigned, there exists a one-parameter family of solutions, differing from each other essentially for the fraction of the total current that goes into ion or electron current respectively. The two currents can also flow in opposite directions, subtracting thus from each other in producing the net total current; in this case, however, they cannot separately exceed certain limiting values, set by the requirement of integrability for the distribution functions.
1. INTRODUCTION

The continuing interest in fusion research for plasma-magnetic configurations with sheared magnetic flux lines has prompted the writing of the present note, in which the simple situation of a warm plasma in a uniformly sheared magnetic field is investigated. By uniformly sheared magnetic field we mean here a configuration in which the magnetic field vector, while remaining of constant magnitude, rotates with changing $x$ in such a way as to describe a helix of constant pitch with its tip (it is assumed, furthermore, that there is no $x$ component of the magnetic field, and that the system is uniform in the $y$ and $z$ directions).

In the following, first the general procedure is described, which is appropriate for dealing with one-dimensional plasma-magnetic equilibria in presence of shear (Sec. 2); the equations employed are two Vlasov equations for the two species of particles, plus Maxwell's equations. Then, in Sec. 3, the special (force free) situation of the uniform shear is considered, and an explicit, exact solution of the equations is derived for this case.

2. GENERAL FORMULATION OF THE PROBLEM

For a steady-state, one-dimensional case, Maxwell's equations read (in rationalized units, with $\mu$ and $\kappa$ as constants of the vacuum):

$$-\frac{i}{\mu} \frac{dB_x}{dx} = j_y, \quad \frac{1}{\mu} \frac{dB_y}{dx} = j_z, \quad \kappa \frac{dE_x}{dx} = q,$$

(1)

with

$B_x = \text{const.}, \quad E_y = \text{const.}, \quad E_z = \text{const.}$

We choose here $B_x = 0 \quad , \quad E_y = E_z = 0$. In terms of the usual vector and electric potentials, the above equations become:

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The charge and current densities in the above equations can be explicitly computed from the solutions of two Vlasov equations, respectively for electrons and (singly charged) ions:

\[ \frac{dA_y}{dx} = B_z, \quad -\frac{dA_z}{dx} = B_y, \quad -\frac{d\phi}{dx} = E_x. \]  

These are linear homogeneous partial differential equations of first order, and can be integrated with the usual procedure. Any function of the so-called constants of the motion,

\[ p_y = m_e v \cdot e A_y(x), \quad p_z = m_e w \cdot e A_z(x), \quad \epsilon = \frac{1}{2} m_e (u^2 + v^2 + w^2) + e \phi(x), \]

is a solution. In evaluating the charge and current integrals, it is convenient to transform from the variables \( u, v, w \) to the variables \( p_y, p_z, \epsilon \). Correspondingly one obtains:

\[ q_a = \pm 2 e \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\epsilon}^{\epsilon} I_s(p_y, p_z, \epsilon) \, dp_y \, dp_z \, de \, d\epsilon \]

\[-3-\]
The factor "2" in the r.h.s. of Eqs. (6) takes care of the double mapping, due to the appearance of \( u^3 \), but not \( u \), in relations (5). The functions \( f_2(p_y, p_z, \epsilon) \) in Eqs. (6) are in principle arbitrary, and it is

\[
\mathcal{J}_x(p_y, p_z, \epsilon) = \pm v^2 \left[ \frac{m_1}{p_y} (\epsilon - e\phi) - \left( p_y + eA_y \right)^2 - \left( p_z + eA_z \right)^2 \right]^{1/4} .
\]

The distribution functions \( f_2(p_y, p_z, \epsilon) \) are actually only defined in a certain domain, namely, the domain of integration of the r.h.s.'s of Eqs. (6), which in general will change from one value of \( x \) to another (as the limits of integration \( \epsilon_x \) are \( x \)-dependent). This entails the possibility that in some region of the \( (p_y, p_z, \epsilon) \)-space the functions \( f_2 \) may be assigned different values for different values of \( x \) (possible "multivaluedness" of the distribution functions in the space of the constants of the motion).

This point has been thoroughly discussed in Ref. 1, Sec.II, (actually, only for the case of two-dimensional distribution functions, i.e., independent of \( p_z \); but the appropriate extension to the present three-dimensional case is easily inferred). Here, however, we shall restrict ourselves to considering only proper, single valued distribution functions, which we assume to be defined by their analytic form in the whole of the \( (p_y, p_z, \epsilon) \)-space, irrespective of the boundaries of the actual \( (x \)-dependent) domain of integration of the charge and current integrals (6).

Also, we shall further restrict ourselves to the class of solutions that can be obtained from distribution functions of the form
\[ f(p_x, p_z, \epsilon) = N \left( \frac{m_+}{2\pi kT} \right)^{3/2} \exp \left( -\frac{\epsilon}{kT} \right) S_{y+}\left( p_y \right) S_{z+}\left( p_z \right), \quad (8) \]

where \( S_{y+} \left( p_y \right) , S_{y-} \left( p_y \right) , S_{z+} \left( p_z \right) , S_{z-} \left( p_z \right) \) are in principle arbitrary (dimensionless) functions of their arguments. The form (8) of the distribution functions includes of course the case of the thermodynamic equilibrium, which corresponds to the simple choice \( S_{y+} \left( p_y \right) \equiv 1 , S_{z-} \left( p_z \right) \equiv 1 \).

It is convenient at this point to reduce all the equations to non-dimensional form. Paralleling closely the procedure of Ref. 1, we define

\[ \gamma = \left( m_- / m_+ \right)^{1/2} , \quad \gamma_+ = \gamma , \quad \gamma_- = 1 , \]

and introduce \( x^* = e \left( \frac{\mu N}{m_-} \right)^{1/2} \) together with

\[ \epsilon^* = \epsilon / kT , \quad p_y^* = \pm p_y / \left( 2m_+ kT \right)^{1/2} , \quad p_z^* = \pm p_z / \left( 2m_+ kT \right)^{1/2} , \]

\[ \phi^* = e \phi / kT , \quad A_y^* = \epsilon A_y / \left( 2m_+ kT \right)^{1/2} , \quad A_z^* = \epsilon A_z / \left( 2m_+ kT \right)^{1/2} , \]

\[ q_{\pm}^* = q_{\pm} / e N , \quad j_{y+}^* = \frac{j_{y+}}{e N} \left( \frac{m_-}{2kT} \right)^{1/2} , \quad j_{z+}^* = \frac{j_{z+}}{e N} \left( \frac{m_-}{2kT} \right)^{1/2} . \]

In terms of these non-dimensional quantities Eqs. (2) take the following form (dropping, for simplicity, all the asterisks from now on):

\[ \frac{d^3 A_y}{dx^3} = - \left( j_{y+} + j_{y-} \right) , \]

\[ \frac{d^3 A_z}{dx^3} = - \left( j_{z+} + j_{z-} \right) , \quad (9) \]

\[ \frac{\kappa m kT}{m_-} \frac{d^2 \phi}{dx^2} = - \left( q_+ + q_- \right) , \]
whereby the right-hand sides of the equations are given by the following expressions:

\[ q_s = \exp(\pi \phi) q_{y_s}(A_y) q_{z_s}(A_z), \]

\[ j_{y_s} = y_s \exp(\pi \phi) h_{y_s}(A_y) q_{z_s}(A_z), \]

\[ j_{z_s} = y_s \exp(\pi \phi) q_{y_s}(A_y) q_{z_s}(A_z), \]

with

\[ q_{y_s}(A_y) = \pi^{-1/2} \int_{-\infty}^{\infty} dp_y s_{y_s}(p_{y_s}) \exp\left[-\left(p_{y_s} - y_s A_y\right)^2\right], \]

\[ h_{y_s}(A_y) = \pi^{-1/2} \int_{-\infty}^{\infty} dp_y s_{y_s}(p_{y_s}) \exp\left[-\left(p_{y_s} - y_s A_y\right)^2\right] \left(p_{y_s} - y_s A_y\right), \]

\[ q_{z_s}(A_z) = \pi^{-1/2} \int_{-\infty}^{\infty} dp_z s_{z_s}(p_{z_s}) \exp\left[-\left(p_{z_s} - y_z A_z\right)^2\right], \]

\[ h_{z_s}(A_z) = \pi^{-1/2} \int_{-\infty}^{\infty} dp_z s_{z_s}(p_{z_s}) \exp\left[-\left(p_{z_s} - y_z A_z\right)^2\right] \left(p_{z_s} - y_z A_z\right). \]

Eqs. (9), together with (10) and (11), can be looked upon from two different points of view. On one hand, assigned the \( s_{y_s}(p_{y_s}) \) and \( s_{z_s}(p_{z_s}) \), the above equations constitute a system of three (coupled, non-linear) ordinary differential equations for \( A_y, A_z \) and \( \phi \) as functions of \( x \); this is the viewpoint that is usually taken, in analogous problems \(^1\), \(^2\). Alternatively, one may think of assigning \( A_y, A_z \) and \( \phi \) as functions of \( x \), thus obtaining a set of integral equations for the \( s_{y_s}(p_{y_s}) \), \( s_{z_s}(p_{z_s}) \) as unknown functions. In the next section, an example will be given of this second procedure, in which, moreover, the equations will turn out to be simple enough to allow an explicit solution.

3. UNIFORM SHEAR

As anticipated in the Introduction, we shall endeavour in this section to obtain the form of the ion and electron distributions that
corresponds to the following prescriptions for the macroscopic quantities (non-dimensionalized as in the previous section):

\[ q_+(x) = 1, \quad q_-(x) = 1, \quad (12) \]

with \( E_x = 0, \quad \phi = 0 \), and

\[ A_y(x) = A_0 \cos(kx), \quad A_z(x) = A_0 \sin(kx), \quad (13) \]

or also

\[ A_y(x) = A_0 \cos(kx), \quad A_z(x) = -A_0 \sin(kx). \quad (13') \]

In other words, as \( x \) changes, the tip of the magnetic field vector describes a helix of constant pitch \( 2\pi/k \), whose two possible orientations correspond respectively to (13) or (13'). The magnitude \( kA_0 \) of the magnetic field is constant, so is the (common) density of ions and electrons, and there is no electric field. This is perhaps the simplest situation involving a sheared magnetic field that can be envisaged.

Within the framework of the preceding section, the conditions (12), because of (10), are equivalent to

\[ q_+ [A_y(x)] q_- [A_z(x)] = 1, \quad q_+ [A_+(x)] q_- [A_z(x)] = 1, \quad (14) \]

while, inserting (13) or (13') into the first two of Eqs.(9), one obtains, taking also into account (14):

\[ k^2 A_y = \gamma h_y+ (A_y)/q_y+ (A_y) + h_y- (A_y)/q_y- (A_y), \]

\[ k^2 A_z = \gamma h_z+ (A_z)/q_z+ (A_z) + h_z- (A_z)/q_z- (A_z). \]
The above equations are certainly verified if the following is true:

\[ \frac{h_{y+}(A_y)}{g_{y+}(A_y)} = \alpha_+ k^3 A_y, \quad \frac{h_{y-}(A_y)}{g_{y-}(A_y)} = \alpha_- k^3 A_y, \]

(15)

with \( \alpha_+ \) and \( \alpha_- \) such that

\[ y \alpha_+ + \alpha_- = 1. \]  

(16)

Eqs. (15) are all of the form (dropping all subscripts, for simplicity):

\[ h(A) = \alpha k^3 A g(A), \]

or, because of (11):

\[ \int_{-\infty}^{\infty} dp \ s(p) \exp[-(p - yA)^2] (p - yA - \alpha k^3 A) = 0. \]  

(17)

This is a homogeneous Fredholm integral equation of the first type for the unknown \( s(p) \). We need not study it in detail here; we shall content ourselves with obtaining a single, specific solution, which can be shown to exist (and be explicitly derived) through the following procedure.

By simple inspection of Eq. (17), we observe, first of all, that a possible way of satisfying it would be to find a function \( s(p) \) such that the product \( s(p) \exp[-(p - yA)^2] \) were a function symmetric with respect to the point \( p = yA + \alpha k^3 A \), and this identically for varying values of \( A \) (though the function \( s(p) \) must not itself depend on \( A \), by its own meaning; see formula (8)).

This means writing for \( s(p) \) the functional equation

\[ s(yA + \alpha k^3 A - \xi) \exp[-(\alpha k^3 A - \xi)^2] = s(yA + \alpha k^3 A + \xi) \exp[-(\alpha k^3 A + \xi)^2], \]

to be satisfied identically in \( A \) (though \( A \) itself must not enter in the expression for \( s(p) \)). Taking the logarithm of both numbers,
and introducing \( r(p) = \log s(p) \), one obtains

\[
\begin{align*}
  r'((\gamma A + \alpha k^2 A + \xi)) - r'((\gamma A + \alpha k^2 A - \xi)) &= 4\alpha k^2 A \\
  r'((\gamma A + \alpha k^2 A + \xi)) + r'((\gamma A + \alpha k^2 A - \xi)) &= 4\alpha k^2 A \\
  r'(\gamma A + \alpha k^2 A + \xi) - r'(\gamma A + \alpha k^2 A - \xi) &= 4\alpha k^2 \frac{\xi}{(\gamma + \alpha k^2)},
\end{align*}
\]

Deriving alternatively with respect to \( \xi \) and with respect to \( A \), one has

\[
\begin{align*}
  r'((\gamma A + \alpha k^2 A + \xi)) &= 2\alpha k^2 \left[ A + \xi/(\gamma + \alpha k^2) \right], \\
  r'((\gamma A + \alpha k^2 A - \xi)) &= 2\alpha k^2 \left[ A - \xi/(\gamma + \alpha k^2) \right],
\end{align*}
\]

which obviously define the same function \( r'(p) \):

\[
r'(p) = 2\alpha k^2 p/(\gamma + \alpha k^2).
\]

This is actually independent of \( A \), as desired. From it one obtains

\[
r'(p) = \alpha k^2 p/(\gamma + \alpha k^2) + \log C,
\]

(\text{where} \( C \) \text{is an arbitrary, positive constant}), and thus

\[
s(p) = C \exp\left(\frac{\alpha k^2}{\gamma + \alpha k^2} p^2\right).
\]

Upon re-introduction of the appropriate subscripts, the latter actually gives origin to the four relations

\[
\begin{align*}
  s_{\gamma_+}(p_{\gamma_+}) &= C_{\gamma_+} \exp(\eta_+ p_{\gamma_+}^2), \\
  s_{\gamma_-}(p_{\gamma_-}) &= C_{\gamma_-} \exp(\eta_- p_{\gamma_-}^2), \\
  s_{z_+}(p_{z_+}) &= C_{z_+} \exp(\eta_+ p_{z_+}^2), \\
  s_{z_-}(p_{z_-}) &= C_{z_-} \exp(\eta_- p_{z_-}^2),
\end{align*}
\]

(18)
where the parameters $\eta_i$ are defined in terms of the $\alpha_i$ by

$$\eta_i = \frac{\alpha_i k^2}{(\eta_i^2 + \alpha_i k^2)}.$$  

(19)

Inserting the above expressions into the first and third of Eqs. (11) one obtains

$$q_{\gamma \iota}(A_{\gamma}) = C_{\gamma \iota}(1-\eta_i)^{-\frac{1}{2}} \exp \left[ \eta_i (1-\eta_i)^{-1} \frac{1}{2} \eta_i^3 A_{\gamma} \right],$$  

and since it is (from (13) or (13'))

$$A_{\gamma}^2 + A_{z}^2 = A_o^2,$$  

one sees that Eqs. (14) can also be satisfied, by simply taking the constants of integrations $C_{\gamma \iota}, C_{\gamma \iota}, C_{z \iota}, C_{z \iota}$ such as to verify the two conditions

$$C_{\gamma \iota} C_{z \iota} = (1-\eta_i) \exp \left[ -\eta_i (1-\eta_i)^{-1} \frac{1}{2} \eta_i^3 A_o^2 \right].$$  

(20)

Notice that only the products $C_{\gamma \iota} C_{z \iota}, C_{\gamma \iota} C_{z \iota}$ have meaning (and not the four constants of integration separately), since only such products appear in the final expression for the distribution functions (see (21) below).

For the convenience of the reader, we shall give here the final result in terms of the original dimensional variables. For the $f_{\iota}(p_{\gamma}, p_z, \epsilon)$ one obtains the expression:

$$f_{\iota}(p_{\gamma}, p_z, \epsilon) = \frac{N^2 \mu^2}{2\pi kT}  C_{\gamma \iota} C_{z \iota} \exp \left( \eta_i \frac{p_{\gamma}^2 + p_z^2}{2m_i kT} - \frac{\epsilon}{kT} \right),$$  

(21)
(with the products \( C_{y1} C_{z1} \) given by (20)), which corresponds to the following form of the more familiar \( F_z(u, v, w, x) \):

\[
F_z(u, v, w, x) = N \left( \frac{m_i}{2 \pi kT} \right)^{3/2} C_{yi} C_{zi} \exp \left( \frac{\gamma_i}{2} \frac{e^2 A^2}{m_i kT} \right) \cdot \exp \left\{ -\frac{m_i}{2kT} \left[ u^2 (1 - \gamma_i) v^2 + (1 - \gamma_i) w^2 \right] \right\} \exp \left\{ \frac{\gamma_i e}{kT} \left[ v A_y(x) + w A_z(x) \right] \right\},
\]

where the \( A_y(x) \) and \( A_z(x) \) are to be taken as given by (13) or (13').

It may be of interest to evaluate also the (total) kinetic energy densities of the two species of particles, which are given by the triple integrals:

\[
\mathcal{E}_i(x) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw \ \frac{1}{2} m_i (u^2 + v^2 + w^2) F_i(u, v, w, x).
\]

Using (22), one obtains a result which is independent of \( x \) (as it was clearly to be expected):

\[
\mathcal{E}_i = N kT \left[ \frac{1}{4} \left( 1 - \gamma_i \right)^{-\frac{3}{2}} + 2 \gamma_i \left( 1 - \gamma_i \right)^{-\frac{5}{2}} \frac{e^2 A^2}{2 \pi m_i kT} \right].
\]

For \( k = 0 \) (hence, \( \gamma_i = 0 \)), this gives the appropriate result for a Maxwellian plasma, \( \mathcal{E}_i = \frac{3}{2} N kT \), whereas in the opposite limit, that is, for \( k \gg (\mu N/m_i)^{1/2} \), by making use of (19), one obtains the following asymptotic expression for the energy:

\[
\mathcal{E}_i \sim \gamma_i^{3/2} \frac{k^2}{e^2 \mu N/m_i} \left[ \frac{B_o^2}{2 \mu} + \frac{N kT}{\gamma_i A^2} \right]/
\]

where \( B_o = k A_0 \) is the (constant) magnitude of the magnetic field vector. Notice that this differs from the corresponding cold plasma result – easily obtainable – just because of the presence of the second term in the square brackets, which then represent the "disordered" or
thermal component of the kinetic energy.

In conclusion, we can say that the following results have been established. For any given value of the plasma density $N$, of the magnetic field strength $B_0$, and of the helix pitch $2\pi/k$, a one-parameter family of solutions has been obtained. The free parameter in the solutions is related to the free choice of $\alpha_+$ and $\alpha_-$, subject to the condition $\gamma\alpha_+ + \alpha_- = 1$. This partition of the unity corresponds in fact to the partition of the total current into ion current and electron current. Notice that the $\alpha_\pm$ are also permitted negative values, that is, the ion and electron currents may also subtract from each other, rather than add. However, the condition of integrability of the distribution functions restricts the range of values that are permitted for the $\alpha_\pm$. The integrability condition requires in fact
$$\gamma \alpha_+ < 1,$$
from which one obtains
$$-\frac{k^2}{\gamma} < \alpha_- < 1 + \frac{2}{\gamma k^2},$$
and correspondingly
$$1 + \frac{k^2}{\gamma} > \alpha_+ > -\frac{2}{\gamma k^2}.$$

We shall also point out that there appears to be no upper limit for the values of $k$ that can be prescribed; in particular $k$ may be larger than the reciprocal ion Larmor radius or even electron Larmor radius (though the energy densities (24) become very large in these cases). This is to be contrasted with other solutions, obtained elsewhere, connecting two asymptotically constant states, where the electron Larmor radius or ion Larmor radius were found to give a lower bound for the width of the transition regions.

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3) The condition of integrability for the distribution functions poses further conditions, actually, for the parameters $\alpha_+$ and $\alpha_-$ to be admissible. This point will be commented upon further on in this section.

4) To be sure, the conditions $\eta_1 < 1$ also imply the existence of all the moments of the distribution functions.
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