COLLISIONLESS SHOCK WAVES
IN HIGH $\beta$ PLASMAS

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ABSTRACT

Compressional large-amplitude waves propagating along the magnetic field direction necessarily destabilize Alfvén waves in high β plasmas by the firehose mechanism. When the electron temperature is much larger than the ion temperature, a closed set of fluid equations, including the turbulent Alfvén wave dissipation in the quasi-linear approximation, may be constructed. This set, investigated here in the limit of weak shocks propagating along the magnetic field, has shock solutions. Microscopic short wavelength turbulence may also be generated in large amplitude shocks, but will probably not affect the Alfvén wave spectrum. The existence of large amplitude and yet-persisting-magnetic field turbulence behind the Earth's bow shock is in qualitative agreement with this model.
1. INTRODUCTION

The possible existence and structure of shock waves in high-temperature collision-free plasmas is a crucial test of the rapidly developing theory of plasma turbulence [Sturrock, (1957); Litvak, (1960); Fishman et al., (1960); Camac et al., (1962); Vedenov et al., (1962); Drummond and Pines, (1962); Galeev and Karpman (1963); Kadomtsev and Petviashvili (1963); Aamodt and Drummond (1964); Galeev et al.; (1965) Kadomtsev (1965); Sagdeev and Galeev, (1966)]. Shock waves, lying as they do at the nexus of turbulence theory, exhibit as great a variety as there are forms of and complications to turbulent collision-free dissipation. In the past, most attention has been concentrated on low Mach number shocks in low β plasmas, where the particle pressure is much smaller than the static magnetic pressure. Roughly speaking, interest has been divided into two camps. One approach has been to construct non-linear coherent wave trains, which, assuming small dissipation, allow irreversible transitions between two asymptotically steady states. Steady wave trains propagating strictly normal to the magnetic field were considered independently by Adlam and Allen (1958), Gardner et al. (1958), and Davis et al., (1958). For moderate magnetic fields, the non-linear waves, which trail the shock front, have short-scale lengths, roughly $C/\omega_p$ where $C$ is the velocity of light and $\omega_p$ is the electron plasma frequency. Sagdeev (1958) considered at the same time the opposite limit of the same problem for very large static magnetic fields and found somewhat longer scale lengths in this limit. Later, this work was generalized to oblique propagation to the magnetic field. (Sagdeev, 1962; Karpman, 1963; Morton, 1964; Petschek, 1965). For angles to the normal direction, $\theta$, greater than $M_+ / M_-$, the electron to ion mass ratio, the wave train has 40 times larger scale lengths, roughly $C/\omega_{p+} \theta$, where $\omega_{p+}$ is the ion plasma frequency. In addition, the wave train propagates upstream. Clearly this regime is the one which will be most often encountered in practice. The analogy with non-linear dispersive waves in shallow water motivated much of this theoretical work [Vedenov et al. (1961); Petschek (1965)]. The $C/\omega_p$ trailing wave train for perpendicular propagation has recently been verified in laboratory experiments (Kurumulaev et al., 1965), and similarly the
leading \( (C/\omega_p)^\theta \) behavior for oblique wave trains (Nesterikhin et al., and S. Segrè, private communications, 1966). Zabusky and Kruskal (1965) have recently extended considerably the theoretical study of general one-dimensional non-linear dispersive systems to the case of non-steady interacting solitary waves. They find that solitary waves preserve themselves in mutual interactions.

Each non-linear wave train has a certain maximum amplitude. Generally speaking, for Mach numbers greater than some critical value (about two, for example for perpendicular shocks) the wave overturns and breaks. Therefore, the other major direction of research has been to construct shocks which were turbulent at the outset, although a low Mach number assumption usually has been necessary. Camac et al. (1962), Galeev and Karpman (1963), Sagdeev (1965) and Kurtmullaev et al. (1966) are representative of such theories. In the completely cold plasma theories of Camac et al. (1962) and Galeev and Karpman (1963), non-linear mode-mode couplings of whistler waves generated by an adiabatic amplification of those waves standing in a shock front propagating normal to the field provide the randomization. Sagdeev (1965) used the instability of the ion sound wave which arises when a strong current is passed by the plasma to generate turbulent dissipation. This mechanism is limited to nearly normal propagation because of the need for short magnetic scale lengths. Both mechanisms yield shock thicknesses of a few times \( C/\omega_p \). Recent experimental results (Kurtmullaev et al., 1966) indicate that ion sound turbulence is present in perpendicular shocks with a shock thickness of \( 10 C/\omega_p \).

All the above theories of turbulence considered perpendicular shocks in low \( \beta \) plasmas. An exception is Moiseev and Sagdeev (1963) who considered parallel propagating shocks in high \( \beta \) plasmas. They constructed the non-linear ion sound solitary wave and wave train. In addition, they considered the dissipation arising from a pitch angle anisotropy instability of electromagnetic waves with wavelengths shorter than an electron gyro-radius. Unfortunately, their method is limited to plasmas with \( \beta \) larger than the ion-to-electron mass ratio, about 2000, and is inappropriate to the solar wind. They suggested that firehose-unstable Alfvén wave turbulence could be important at lower \( \beta \) and, therefore, the solar wind.
In this paper, we propose a turbulent shock theory appropriate to the solar wind for at least part of the time—when $\beta$ is somewhat larger than one, when the electron temperature is larger than the ion temperature, and when the shock has a reasonably large component of propagation along the static magnetic field direction. We hope this theory could be applied to the Earth's bow shock and magnetic storm shockwaves propagating outward from the sun.

Particles play an important role in high $\beta$ shocks. In general, considerable energy is invested in pressure anisotropies created by shock compressions. Thus, in addition to the turbulent resistivity needed to prevent magnetic field gradients from steepening, a "viscosity" is needed to isotropize the pressures. In high $\beta$ as opposed to low $\beta$ plasmas, electromagnetic waves travel slowly relative to the ions and therefore interact strongly. Therefore the pitch angle anisotropy instabilities of electromagnetic waves will be a source of turbulent viscosity in the solar wind. A similar conclusion has been reached by Scarf (1966).

This paper is separated into two distinct parts. In sections 2-6 we attempt to construct a shock model, paying attention to mathematical and physical consistency. Here we are limited by mathematical exigency to weak shocks propagating strictly along the magnetic field. In sections 7-10 we speculate upon the extensions of these ideas to the strong oblique shock case. In section 8 we discuss extremely qualitatively the available relevant satellite data. In section 9 we make some tentative suggestions regarding prospective experiments. The reader should be fully aware of the speculative nature of these sections; our aim is to attempt to establish a philosophical framework about future work. About such efforts there can always be divergences of opinion.

Parker (1958) was the first to discuss the firehose instability in the solar wind context. Following Moiseev and Sargadeev (1963), firehose unstable Alfvén waves provide the shock dissipation. In section 2 we review the firehose instability first in the hydromagnetic limit, in 2.a, where Alfvén waves are always unstable, and in the finite Larmor radius limit, in 2.b, where stabilization
occurs at short wavelengths. Kantrowitz and Petschek (1964) have reviewed the arguments from fluid hydromagnetic theory showing that compression waves steepen to form shocks. In section 3 we show, using kinetic theory, that compressional ion sound waves propagating along the magnetic field also steepen and furthermore, create a firehose unstable ion pressure anisotropy in sufficiently high $\beta$ plasmas. In section 4 we discuss, following Shapiro and Shevchenko (1964), the quasi-linear development of the firehose instability in an idealized spatially homogeneous plasma. The fact that the spatially homogeneous instability saturates suggests that a similar shock dissipation can occur. An interesting fact is that Alfvén wave turbulence is degenerate, being always linear. In section 5 we turn to spatially inhomogeneous turbulence, and show that when the electron temperature is larger than the ion temperature a closed set of fluid equations, including the Alfvén wave dissipation, can be constructed. In section 6 we attempt to solve this set for weak planar shocks. We are able to reduce the set to a non-linear integro-differential equation and demonstrate that, in the weak shock limit transitions between two steady states—shock solutions—exist. Accurate solutions of this integral equation must be the subject of another investigation. (Zabusky et al., to be published). Another interesting sidelight is that, since the turbulence is purely macroscopic, quasi-linear averaged equations can probably also be constructed from finite Larmor radius moment equations [Chew et al., 1959; Roberts and Taylor, 1962; Zaslavskii and Moiseev, 1962] and the quasi-linear and accurate theory compared numerically.

In section 7 we speculate upon the extension of the physical ideas contained in the above mathematically restricted solution to more general situations. A reasonable case can be made that Alfvén turbulence will also play an important role in strong shocks propagating at an angle to the magnetic field and when the ion-electron temperature ratio and $\beta$ are of order one.

In section 8 we discuss briefly the relevant space observations. The closest point of contact between theory and experiment are the observations (Ness et al., 1964) of large amplitude fluctuations in the magnetic field direction which persist for hundreds of Larmor radii behind the shock. This appears to correspond to firehose
amplification of degenerate linear Alfvén wave turbulence.

In section 9 we consider qualitatively the possibility that various forms of microscopic turbulence at short wavelengths, each responsible for selective forms of dissipation, may be present in highly turbulent strong shocks. However, significant long-wavelength Alfvén turbulence will still be generated, since averaging over all the microscopic fluctuations will probably create a macroscopically unstable pressure anisotropy.

Due to limitations of imagination and mathematical technique, it appears difficult to describe all aspects of strong collisionless shocks. However, we would like to suggest that, even if at present it is only possible to formulate weak shock theories, it may be more reasonable in many cases - such as for the solar wind - to construct high $\beta$ shock models, because strong shocks invariably heat the plasma. Since many particles can be trapped and reflected in high $\beta$ waves, wave trains probably all overturn, and turbulent high-$\beta$ models are more likely to be realistic. This philosophy has been our guideline.
2. FIREHOSE INSTABILITY OF ALFVÈN WAVES

Here we review the linear theory of the firehose instability. We begin by limiting our discussion to electromagnetic transverse waves propagating along the magnetic field, and to the fluid-like zero Larmor radius approximation. In 2.a we show that the Alfvén wave grows exponentially in time when the pressure component parallel to the magnetic field is sufficiently larger than the perpendicular pressure component. When the ion β is large, the unstable pressure anisotropy is small. The growth rate increases with decreasing wavelength and exhibits no maximum in this approximation. In 2.b we show that the firehose mechanism stabilizes at wavelengths well below the ion gyroradius for small pressure anisotropies. Thus, firehose Alfvén wave turbulence behaves like a fluid. Waves propagating at an angle to the field are not considered.

2.a) Parallel propagating firehose unstable Alfvén waves in the hydromagnetic limit

The dispersion relation for left-hand circularly polarized electromagnetic waves propagating parallel to a uniform static magnetic field \( B_0 \) in an infinite homogeneous plasma is

\[
\frac{c^2 k^2}{\omega^2} = 1 + \sum \frac{(\omega_p^2)}{N} \int dV_c \int dV_i \frac{\sum \frac{f^+ - f^-}{\omega - \omega_p^+ + i \omega_z}}{k^2} \]

(2.1)

Here \( K \) is the (real) wave number, \( \omega \) the complex wave frequency; all perturbations have a space-time variation proportional to \( e^{i(\kappa z - \omega t)} \) where \( \kappa \) is a space coordinate in the direction of the static magnetic field. Instability occurs if (2.1) has solutions with \( \text{Im} \omega > 0 \). \( \omega_p = \sqrt{4 \pi n e^2 / M_z \omega} \) is the plasma frequency of each species, and \( \Omega_z = \frac{e B_0}{M_z c} \) the gyrofrequency. \( N \) is the number density, \( e \) the electronic charge, \( M_z \) the mass of each species, and \( c \) the velocity of light. Gaussian units are used throughout.

\( f^+(V_c, V_i) \) is the equilibrium velocity distribution function for each species, in a cylindrical system of velocity space coordinates.
with the static magnetic field direction as axis of symmetry. \( \perp \) and \( \parallel \) denote components perpendicular and parallel to the static magnetic field respectively. The Larmor phase variable \( \Phi \) has already been integrated in Eq. (2.1) since it is easy to show that \( F^\pm \) is independent of \( \Phi \) in a spatially homogeneous plasma with no electric and only a uniform magnetic field. \( F^\pm \) is normalized so that

\[
2 \pi M^\pm_1 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty dV_\perp dV_1 dV_\parallel F^\pm = \rho = N M^\pm_1
\]  
(2.2)

where \( \rho \) is the mass density of ions. When \( \text{Re}\ \omega > 0 \), (2.1) corresponds to a left-hand circular polarization; the dispersion relation for the opposite right-hand circular polarization is the same as (2.1) except that the sign of \( \Omega^\pm \) is reversed.

We look for solutions to (2.1) where \( \gamma = \text{Im} \ \omega \neq 0 \), so that the velocity space integral is non-singular. For wavelengths much longer than typical ion gyroradii, and frequencies well below the ion gyrofrequency, we may expand the denominator of the integrand in (2.1).

\[
\frac{1}{\Omega^+\Omega^- - \gamma^2} \approx \frac{1}{K_\perp K_\parallel} \left\{ 1 + \frac{K_\perp V_\parallel}{\Omega^+} + \left( \frac{K_\parallel V_\perp}{\Omega^-} \right)^2 + \cdots \right\}
\]  
(2.3)

With the above assumptions, (2.1) reduces to a sum of non-singular moments of the distribution functions. In the coordinate system such that \( \int F^\pm \gamma dV = 0 \) (2.1) becomes approximately, taking the first few terms in (2.3),

\[
\frac{\omega^2}{K_\perp^2 \omega^2} = 1 - \sum_{\pm} \frac{\omega_\parallel^2}{(\omega^2 + \Omega^2)^2} \left\{ 1 + \frac{\omega_\parallel^2 / \Omega^2}{(\omega^2 + \Omega^2)^2} \right\} + \sum_{\pm} \frac{\omega_\parallel^2}{(\omega^2 + \Omega^2)^2} \left( \frac{K_\perp}{\Omega^+ - \Omega^-} \right)^2
\]  
(2.4)

where we have defined the parallel and perpendicular pressure moments of the distribution functions:

\[
P^\parallel = 2 \pi \int_0^\infty \int_{-\infty}^\infty dV_\parallel dV_1 dV_\perp M^\parallel_1 F^\parallel_\perp V_\perp^2
\]  
(2.5a)
The small parameters, \( \omega / \Omega_+ \), the ratio of wave to gyrofrequency, and \( K R_+ \) where \( R_+ = 1 / \Omega_+ \sqrt{\mu / \rho} \), the ion gyroradius, have not yet been ordered with respect to one another in (2.4). In the hydromagnetic zero Larmor radius approximation they are the same order of magnitude, \( \omega / \Omega_+ \sim K R_+ \). Dropping all terms of order \( (\omega / \Omega_+)^2 \sim k^2 R_+^2 \) from (2.4) as well as terms in the mass ratio between species \( M_+ / M_- \), and assuming that the plasma dielectric constant,

\[
\varepsilon \approx \frac{2(\pi \rho \omega_f^2)}{B_0^2} ,
\]

is large, (2.4) reduces to

\[
\omega^2 = K^2 \left( V_A^2 - \frac{P_+ - P_+}{P} - \frac{P_- - P_-}{P} \right)
\]  

\( V_A = B_0 / 4 \pi \rho \) is the Alfvén speed. The dispersion relation for the opposite right-hand polarization is the same as (2.6). Since both right and left-hand waves propagate at the same speed, there is no Faraday rotation effect and the polarization can be considered linear.

In terms of the ratio of parallel particle to magnetic pressure, \( \beta_\parallel = \frac{31 \Pi \mu}{2 B_0} \), and the normalized pressure anisotropy

\[
\left( \frac{\Delta P}{P} \right)_\parallel = \frac{P_\parallel - P_\perp}{P_\parallel} ,
\]

the stability criterion arising from (2.6)

\[
\Delta \equiv \left\{ \left( \frac{\Delta P}{P} \right)_\parallel + \left( \frac{\Delta P}{P} \right)_\perp \frac{\beta_- - \beta_+}{\beta_+} \right\}^2 > 0 \quad \text{(Unstable)}
\]  

When \( \Delta P/P = (\Delta P/P)_\parallel = 0 \), (2.6) describes an ordinary Alfvén wave.

When (2.7) is satisfied, there is a purely growing non-convective aperiodic instability. Of course, the roots come in stable-unstable pairs, but there is one growing made for each direction of \( K \), parallel or antiparallel to \( B_0 \), and for all magnitudes of \( K \).

Notice that when \( \beta_\parallel \) are large, very small pressure anisotropies can be unstable. For the collision-free shock, the electron
pressure anisotropies will be small, since the shock propagates very slowly relative to the electron thermal velocity and can only slightly distort the electron pressure distribution.

The instability criterion 2.7 depends only on fluid behavior of the plasma, and not upon details of the particle distribution, as in resonant instabilities. The turbulence created by the firehose instability will likewise be of the fluid type. A physical picture of the instability has been given by many authors. We follow the treatment of Vedenov et al., (1961b). Alfvén waves distort the lines of force, which can be visualized as "rubber bands". Consider the forces on the "rubber band" when it is curved. For these low frequencies and long wavelengths, particles are attached to the line of force. The motion along the line creates a centrifugal force \( F_c \).

\[
F_c = \int d^3\nu \frac{M\nu^2}{R_c}, \quad R_c = \text{radius of curvature} \quad (2.8)
\]

which tends to increase the line curvature. In addition, the rapid Larmor orbiting, relative to these low frequencies, makes each gyrating particle look like a small current loop whose magnetic moment is conserved. In an inhomogeneous magnetic field there will be a force \( F_M \) on the plasma, due to the magnetization current, \( J_M \), where

\[
J_M = C \nabla \times \int d^3\nu F^+(\mu, \nu) d^3\nu, \quad \text{and} \quad \mu = -\frac{M\nu^2}{\delta B_1} \cdot \frac{B}{|B|}
\]

\[
F_M = \frac{J_M \times B}{C} = \left\{ \nabla \times (\int F_M d^3\nu) \times B \right\} \quad (2.9)
\]

The above magnetization force, together with the "tension" force below in the rubber bands, \( F_T \)

\[
F_T = \frac{J \times B}{C} = \frac{(\nabla \times B)}{4\pi} \times B \quad (2.10)
\]

act as restoring forces. Instability takes place when \( |F_c| > |F_M + F_T| \) or when after substituting \( \nabla = (0, 0, \kappa_c z) \) in (2.8), (2.9), (2.10),

\[
P^0 - P^c > B^2 / 4\pi \quad (2.11)
\]
The small parameters, \( \Omega / \Omega_+ \), the ratio of wave to gyro-frequency, and \( K_R \), where \( R_+ = 1 / \Omega_+ \sqrt{B_0 / \rho} \), the ion gyroradius, have not yet been ordered with respect to one another in (2.4). In the hydromagnetic zero Larmor radius approximation they are the same order of magnitude, \( \Omega / \Omega_+ \sim K_R \). Dropping all terms of order \( (\omega / \Omega_+)^2 \sim k^2 R_+^2 \) from (2.4), as well as terms in the mass ratio between species \( (M_+ / M_0)^2 \), and assuming that the plasma dielectric constant, \( (4 \pi \rho c^2) / \beta_0^2 \), is large, (2.4) reduces to

\[
\omega^2 = K^2 \left( V_A^2 - \frac{P_+ - P_0}{\rho} - \frac{P_- - P_0}{\rho} \right) \tag{2.6}
\]

\[ V_A^2 = \frac{B_0^2}{4 \pi \rho} \] is the Alfvén speed. The dispersion relation for the opposite right-hand polarization is the same as (2.6). Since both right and left-hand waves propagate at the same speed, there is no Faraday rotation effect and the polarization can be considered linear.

In terms of the ratio of parallel particle to magnetic pressure, \( \beta_\parallel = 8 \pi P_\parallel / B_0^2 \), and the normalized pressure anisotropy \( \Delta P = P_+ - P_- \), the stability criterion arising from (2.6)

\[
\Delta = \left\{ \left( \frac{\Delta P}{P_\parallel} \right) + \left( \frac{\Delta P}{P_\perp} \right) \frac{\beta_- - \beta_+}{\beta_+} \right\} > 0 \quad \text{(unstable)} \tag{2.7}
\]

When \( (\Delta P / P_\parallel) = (\Delta P / P_\perp) = 0 \), (2.6) describes an ordinary Alfvén wave.

When (2.7) is satisfied, there is a purely growing non-convective aperiodic instability. Of course, the roots come in stable-unstable pairs, but there is one growing made for each direction of \( K \), parallel or antiparallel to \( B_0 \), and for all magnitudes of \( K \).

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force $F_u$ on the plasma, due to the magnetization current, $J_u$, where

$$J_u = C \nabla \times \int d^3v F^t (u, v) d^3v, \quad \text{and} \quad u = - \frac{M_e^2}{3|B|} \frac{B}{|B|}$$

$$F_u = \frac{J_u \times B}{C} = \left\{ \nabla \times \left( \int d^3v F^t d^3v \right) \times B \right\} \quad (2.9)$$

The above magnetization force, together with the "tension" force
below in the rubber bands, $F_T$

$$F_T = \frac{J \times B}{C} = \left( \frac{\nabla \times B}{4\pi} \right) \times B \quad (2.10)$$

act as restoring forces. Instability takes place when $|F_c| > |F_u + F_T|$ or when after substituting $\nabla = (0, 0, K_u E_z)$ in (2.8), (2.9), (2.10),

$$P_u - P_T > B^2 / 4\pi \quad (2.11)$$

- 10 -
The growth rate, equation (2.6), could be obtained by equating the sum of the above forces to the mass times the acceleration of a given plasma element, \( \rho V = \frac{d}{dt} \left( \frac{CE}{C} \right) \). This instability is similar to that of a rapid flow of water through a firehose, thereby motivating the picturesque name.

2.b) Parallel propagating firehose waves in the finite Larmor radius approximation

In the zero Larmor radius limit, \( \omega_k = \omega \) has no maximum with increasing \( k \). To find the wave number of maximum growth rate, \( K_m \), we must include higher order terms in \( \gamma R \). Reducing (2.4) in the finite Larmor radius approximation we take \( \omega / \omega_+ \sim (\gamma R)^2 \sim \Delta \ll 1 \), and find for the left-hand mode

\[
\omega^2 - \omega \Omega_+ K^2 \gamma^2 + \Omega_+^2 K^2 \gamma^2 \Delta = 0
\]  

(2.12)

or solving for the real and imaginary frequencies \( \omega_k = \Re \omega, \gamma_k = \Im \omega \),

\[
\omega_k = \pm \Omega_+ \frac{K^2 \gamma^2}{2}
\]  

(2.13)

\[
\gamma_k \frac{K \gamma^2}{\gamma R} = \pm \sqrt{\Delta - \frac{K^2 \gamma^2}{4}}
\]  

(2.14)

When \( K > 0 \), the (+) root grows, and the (-) for \( K < 0 \). Thus the maximum growth rate occurs when \( (K \gamma R)^2 = 2 \Delta \); the maximum growth rate \( \gamma_{K_m} \) equals the wave oscillation frequency at \( K = K_m \)

\[
\gamma_{K_m} = \Re \omega_{K_m} = \Omega_+ \Delta
\]  

(2.15)

Near marginal stability, \( \Delta \sim 0 \), the maximally growing wave has a very low frequency. When the pressure anisotropy is large, \( \Delta \sim 1 \), waves near the ion gyrofrequency will be generated, and the fluid nature of the instability will be lost.

In the finite Larmor radius approximation, firehose unstable Alfvén waves propagate with a group velocity which is, for \( k = K_m \)

\[
\frac{d \omega_{K_m}}{d k} = \sqrt{\frac{2 P^*}{\rho} \Delta}
\]  

(2.16)

Since the instability now propagates, albeit slowly, it is convective.
The critical length $L_c$, over which the amplitude of the maximally growing wave $e$-folds once is

$$L_c = \frac{1}{\Omega_{km}} \frac{\delta \omega_{km}}{\delta \Omega} = R \sqrt{2/\Delta} \quad (2.17)$$

Significant nonlinear effects should occur over lengths a few times $L_c$.

The right-hand wave propagating along $B$ has a dispersion relation similar to eqs. (2.12) and (2.13) but with the signs of $\Omega_+ \Omega_-$ changed. The maximum growth rate remains the same, but the unstable waves propagate in opposite directions for a given sign of $k$. For the opposite sign of $k$ there will again be growing waves propagating in opposite directions with equal magnitude phase and group velocities. In each direction, a right and left-hand wave propagate together, so the polarization remains linear.

As $k$ increases beyond $K_m$ to $K^*$, where $(K^* R) = 4 \Delta$, the growth rate goes to zero, while the oscillation frequency increases. Now, however, there can be resonant particle effects, given by the zeroes of the denominator of the integrand of (2.1), which is now singular since $\chi$ is zero. However, the ions now in resonance have velocities given by

$$V_n = \sqrt{\frac{P_{\parallel}}{P_{\perp}}} \frac{1 - 2\Delta}{2\Delta} \quad (2.18)$$

If these cyclotron resonance ions have more energy in motion parallel than perpendicular to the magnetic field, the right-hand polarization (magnetosonic wave) will be unstable and the left-hand ion cyclotron (Alfvén) wave damped. However, when $\Delta << 1$, the resonant ions have energies well above the mean parallel ion energy. Since the number of resonant ions should therefore be small, the resonant growth of these wave components should be slower than that of the nonresonant firehose instability. Henceforth, we will neglect these resonantly unstable electromagnetic waves, even though fast ions on the tail of the heated distribution behind the shock will in general rush ahead and create pitch angle anisotropies upstream. Because the ensuing instabilities are resonant, their non-linear development will redistribute in pitch angle only the high-energy...
resonant ions. Fast ions may possibly be contained in the shock by this mechanism; however, only firehose turbulence effects both the main body and the tail of the ion distribution.

3. STEEPENING OF ION SOUND WAVES AND CREATION OF ALFVEN WAVES

The ion sound wave propagating parallel to the magnetic field is the fastest propagating linear fluid-like wave in a high $\beta$ plasma. Thus, the fast shock wave will be related to the ion sound wave. In 3a, we review the linear theory of ion sound waves, and show that a compressional pulse steepens to form a shock. In 3b, we suggest that firehose unstable pressure anisotropies are created as the ion sound wave steepens.

3a) Ion sound step waves

The Vlasov equations for ion and electron motion along the lines of force are

$$\frac{dF^+}{dt} + \nabla \cdot \frac{F^+}{\mu^+} = \frac{e}{M^+} \frac{\nabla \phi}{\mu^+} \frac{\partial F^+}{\partial \mu} = 0$$

(3.1)

Here we have assumed the characteristic times of the motion to be sufficiently much longer than the ion gyroperiod that $(V \times B_{\parallel}) \frac{\partial F^+}{\partial \mu} = 0$.

$\Phi$ is the electrostatic potential, and the parallel electric field $E_{\parallel} = -\Phi/\lambda_{\parallel}$. Henceforth, we define $\psi = e\Phi/M_i$. Poisson's equation completes the set of equations

$$\frac{\partial^2 \psi}{\partial x^2} = 4\pi e^2 \int \left( F^+ - F^- \right) d\nu$$

(3.2)

By comparing orders of magnitude in (3.2), we find, normalizing $\psi$ to the ion sound velocity $c_s$ (to be derived), that, for spatial scale lengths much longer than ion sound Debye length, the plasma is quasi-neutral.

$$0 \approx \frac{\lambda_D^2}{\lambda_{\parallel}^2} \frac{1}{2} \left( \psi/c_s^2 \right) = \frac{\int (F^+ - F^-) d\nu}{\int F^+ d\nu}$$

(3.3)
Since one more space derivative on \( \psi \) appears in the Poisson equation (3.2) than in the Vlasov equation (3.1), dispersive effects will occur at scale lengths the order of the ion sound Debye length.

Suppose a piston located at large distances creates a small compressional pulse which then propagates away from the piston along the lines of force with the velocity \( C_s \). If the perturbation extends only over a narrow region, we may integrate in \( Z \) over the perturbation to compute the "jump" in the particle distributions, \( \delta F^\pm \), due to the step function wave, neglecting non-linear terms of \( O(\delta^2) \)

\[
\delta F^+ = \delta \psi \frac{\delta F_0^+}{\delta V_n} \quad ; \quad \delta F^- = - \frac{M^+}{M^-} \delta \psi \frac{\delta F_0^-}{\delta V_n} - C_s
\]  

(3.5)

\( F_0^\pm \) are the velocity distributions ahead of the step. The self-consistent linear wave speed is found by substituting (3.5) into the quasineutrality condition (3.3), and assuming that \( C_s \) is much larger than the mean ion velocity, but much less than the mean electron velocity.

\[
\int d^3V \frac{\delta F_0^+}{\delta V_n} \left( 1 + \frac{V_n}{C_s} + \cdots \right) \approx \frac{M^+}{M^-} \int d^3V \frac{\delta F_0^-}{\delta (V_n^2/2)}
\]

(3.6)

If we define the effective electron temperature, \( T^- \), as follows

\[
\frac{N^- M^-}{T^-} = - \int \frac{\delta F_0^-}{\delta (V_n^2/2)} \ d^3V
\]

(3.7)

where \( N^- \) is the electron number density, then (3.6) yields for \( C_s \)

\[
C_s^2 = \frac{N^+}{N^-} \frac{T^-}{M^+}
\]

(3.8)

In order that our approximations \( T^+/M^+ \ll C_s^2 \ll T^-/M^- \) be consistent, we must have \( T^-/T^+ \gg 1 \). When \( T^-/T^+ \to 1 \), this wave becomes heavily Landau damped, in the sense that many ions travel
with the same velocity as the step function wave. Electron Landau
damping is small, even though the wave lies on the main part of the
electron distribution, because the slope of the electron distribution
is small at small velocities. Since when $\frac{T_e}{T_i} < 1$, the ion
sound wave is heavily damped by ions, a collective fluid-like oscil-
lation will not propagate upstream from the piston in this limit.

If $N_+ = N_-$ ahead of the compressional pulse, behind the
compression pulse, $N_+/N_- > 1$. The electrostatic potential down-
stream from the pulse will be non-zero to compensate for the
difference in ion and electron number densities. Suppose the
piston now creates a second compressional pulse in the wake of the
first. The phase velocity of the second pulse will be larger than
that of the first, and, furthermore, each succeeding pulse will pro-
pagate slightly faster than its predecessors. Since an arbitrary
density profile may be decomposed into a series of small steps, a
compressional ion sound wave will steepen, since the high density
parts travel faster than the low.

The processes which limit the steepening define the shock
structure. There may be a number of such mechanisms. When the scale
lengths become comparable with the ion sound Debye length, the dis-
persive terms hitherto neglected change the linear wave velocities
so that they no longer propagate together with the front. A laminar
solitary wave, a soliton, may be formed by balancing the production
of short wavelength components due to steepening with their dispersive
propagational loss. Such a soliton is discussed in Moiseev and
Sagdeev (1963). Some small dissipation then converts the soliton
into a laminar wave train which then allows a transition between the
asymptotically stationary states. Another possibility is that the
ion sound perturbation destabilizes other plasma modes, and their
turbulent dissipation limits the steepening. This case is treated
here. Finally, a mixed state where both dissipation and dispersion
play important roles could occur. We do not discuss this possibility
in any detail.

3.b) Adiabatic production of Alfvén waves

To test whether a non-linear ion sound amplitude creates
instabilities with wave lengths longer than the ion sound Debye
length, we use the computed distortion in the distribution functions \( \delta F^\pm \), equation (3.5), in the linear dispersion relation for the various oscillatory plasma modes. \( \delta F^\pm \) may be taken as the amplitude of the pulse integrated now over many small steps. For propagation along \( B_0 \) with wavelengths greater than the ion gyroradius, the plasma waves are the ion sound mode and the two polarizations of the Alfvén wave. Since the velocity distortions are such that \( \delta \psi \left( \frac{F_1^+ + \delta F^\pm}{\delta \psi} \right) < 0 \) in general, steepening will not cause an outbreak of unstable ion sound oscillations due to the resonant beam-type instability of this wave. However, Alfvén waves will be firehose unstable. Computing \( \delta P^\pm \) and \( \delta P^- \) we find, in the laboratory frame, for the ions

\[
\delta P^+ = 2 \delta \psi \frac{\rho^+}{C_s^2} \quad \delta P^- \approx \delta \psi \frac{\rho^-}{C_s^2} \quad (3.9)
\]

and similarly for the electrons

\[
\delta P^+ = \delta \psi N \frac{M^+ C_i^2}{C_s^2} \quad \delta P^+ = - \frac{M^+}{M^-} \delta \psi \left( \frac{1}{2} \frac{\delta F^-}{\delta \psi} \right) \approx \frac{\delta \psi}{C_s^2} \quad (3.10)
\]

Computing out the anisotropies, we find

\[
\frac{\delta P^+ - \delta P^-}{\delta \psi} \approx \left( \frac{\delta \psi}{C_s^2} \right) \left( 1 + \frac{\rho^+}{\rho^-} \right) \approx \frac{\delta \psi}{C_s^2} \quad (3.11)
\]

\[
\frac{\delta P^+ - \delta P^-}{\delta \psi} \approx \left( \frac{\delta \psi}{C_s^2} \right) \left( \frac{T^+}{m^+} - \frac{T^-}{m^-} \right) \approx 0 \quad (3.12)
\]

Thus the ion sound wave creates a firehose unstable ion anisotropy but leaves the electron distribution nearly isotropic. Significant electron anisotropies are not expected because the wave travels much slower than the electron mean thermal velocity. Turning to the instability criterion, we expect firehose wave generation when, neglecting electron anisotropy,

\[
\frac{\delta \psi}{C_s^2} > \frac{2}{\beta^+} \quad \text{or} \quad \delta \psi > \frac{T^+}{m^+} \quad V_A^2 \quad (3.13)
\]

For large \( \beta^+ \), the ion sound wave is easily destabilized.
In summary, a compressional longitudinal ion sound wave steepens because the high ion density parts of the wave travel faster than the low ion density regions. Associated with steepening is a distortion of the ion velocity distribution behind the wave. When $\varrho_+$ is large, Alfvèn waves are easily destabilized. This non-linear interaction between ion sound and Alfvèn waves is adiabatic, because the slow distortions in the gross properties of the ion velocity distribution cause the ensuing firehose instability.

4. QUASI-LINEAR DESCRIPTION OF ALFVÈN WAVE TURBULENCE

As a prelude to the inhomogeneous shock problem, we consider the quasi-linear development of the firehose instability in an infinite homogeneous plasma. Because the real and imaginary parts of the wave frequency are comparable, and because the wave-particle interaction is adiabatic, firehose Alfvèn turbulence differs from the more familiar resonant-particle turbulence [Vedenov. et al., (1962); Drummond and Pines, (1962)]. If the wave distribution changes slowly due to non-linear interactions on the time a typical ion crosses one Alfvèn wavelength, $\left( K_0 \sqrt{P/P_r} \right)^{-1} \sim \sqrt{\Delta / \Omega_+}$, one can compute the slow quasi-linear changes in the distribution functions averaged over many wave crossings. For this procedure to be consistent, it is necessary that both $\lambda = \sqrt{\Delta / \Omega_+}$, the ratio of wave magnetic to static magnetic energy be small parameters. In (4.a) we derive the diffusion equation for particles; and in (4.b), the equation for waves; in (4.c), we show that the firehose instability saturates in the non-linear regime.

We denote the right-hand circularly polarized wave magnetic field component by $H^r_k$ and the left-hand by $H^l_k$. The general quasi-linear diffusion equation for a distribution at circularly polarized electromagnetic waves propagating along the magnetic field is
In the finite Larmor radius approximation, we must take \( v_n \sim \frac{\omega_k}{\Omega_p} \sim (\kappa R_a)^2 \). Since \( \gamma_k \sim \omega_k \), the instability is of the "strong" non-resonant type, and the denominators in (4.1) cannot be replaced by a delta function as for resonant wave-particle interactions; however, we can expand them in \( (\kappa R_a) \) and \( (\omega / \Omega_p) \). Expanding up to, but not including \( O \left( \frac{1}{\kappa R_a} \right) \), and collecting all real terms, including those from \( \frac{\omega}{k_n} = \frac{\omega_k + i \gamma_k}{\kappa_n} \), we find

\[
\frac{\partial F}{\partial t} = \sum_{\ell,r} \sum_k \gamma_k^\ell h_k^r \left\{ \frac{\partial^2 F}{\partial V_1} + \frac{1}{\nu} \frac{\partial F}{\partial V_1} - 2 \frac{\partial}{\partial V_1} \left( \frac{V_1 \partial F}{\partial V_1} \right) + \frac{1}{\nu} \left( \frac{V_1 \partial F}{\partial V_1} \right) \right\}
\]

where

\[
h_k^r = \left| \frac{H_k^r}{\omega} \right|^2;\quad h_k^\ell = \left| \frac{H_k^\ell}{\omega} \right|^2
\]

We call the right-hand side of (4.2), \( \delta t \{F^\pm\} \), the quasi-linear "stoss". This stoss was derived by Shapiro and Shevchenko (1964) starting, not from the general quasi-linear stoss (4.1), but directly from the zero Larmor radius approximation. They also derive the diffusion equation for a distribution of waves propagating at an angle to the field. To be strict, their diffusion equation contains the replacement

\[
\frac{V_1^2}{\nu \partial V_1} \frac{\partial F}{\partial V_1} \rightarrow \frac{V_1^2 + \gamma_k^2}{\nu \partial V_1} \frac{\partial F}{\partial V_1} + \frac{\nu \partial F}{\partial V_1}
\]

relative to (4.2). However, since \( \gamma_k / \kappa_n \sim \Delta \ll 1 \), we may drop this term. To include it here in the finite Larmor radius approximation, we would have to go to higher order in \( (\kappa R_a) \) than we did for (4.2).

Since the diffusion coefficient \( \sum_k \gamma_k h_k \) simply multiplies the velocity space operator, each particle diffuses at the same rate,
corresponding to a non-resonant adiabatic wave-particle interaction.
A physical picture for adiabatic diffusion is as follows. Each particle has a small quasi-oscillatory motion due to the waves superposed on its zero order motion. In a steady wave field \( \sum \sum Vh_{\kappa} = 0 \), the particle distribution, averaged over many wave periods, is steady. However, if the wave distribution evolves with time, so does the averaged particle distribution. The particles adjust to the presence of waves by changing their average locations in velocity space.

4. b) Time dependence of wave distribution

The wave distribution \( h_K \) evolves in time due to wave-particle interactions, which to lowest order are described by the linear growth rate \( \gamma_K \), eq. (2.14). In addition, waves can exchange energy through direct mode-mode couplings. The mode coupling rate will be given in order of magnitude by \( \Omega \sum M_{kk} h_k h_k' \), where \( M_{kk} \) represents a matrix element which depends on the specific polarizations and geometric orientations of the waves involved. In general, wave-wave and wave-particle interactions are competitive.

In the fluid hydromagnetic theory, which assumes isotropic particle pressures, the linear Alfvén wave is always a solution of the equations of motion. The propagation speed, polarization and all other properties are independent of the wave amplitude, even for arbitrary \( \beta \).

Since there can be no nonlinear interaction for fluid Alfvén waves, \( M_{kk} \equiv 0 \). The matrix elements in the non-fluid, finite-Larmor-radius, non-isotropic pressure limits can at most be the order of the deviations from ideal fluid theory. Thus

\[
M_{kk} \sim 0 \left\{ \left( \frac{\Delta P}{\rho} \right)_z, \left( k^2 R^2 \right)_+ \right\} \sim 0 \left\{ \left( \frac{\Delta P}{\rho} \right)_+ , \Delta \right\} \tag{4.5}
\]

For \( M_{kk} \) to be small, both \( (\Delta P/\rho)_z \) and \( \Delta \) must be small. In order that the wave-particle interaction dominate the wave-wave, we must assure

\[
\sum h_k \sim \Delta \sim \left( \frac{\Delta P}{\rho} \right)_+ \sim 2/\beta_+ \ll 1 \tag{4.6}
\]

Thus, the mode coupling terms will be small only if \( \beta_+ \ll 1 \). These arguments agree with the results of zero Larmor radius

- 19 -
calculations by Shapiro and Shevchenko (1964). Mode couplings between
Alfvén waves and other modes can also be ignored, since other modes
are not excited to large amplitudes in this model. Thus

\[
\frac{\delta n^r}{\delta \Delta} = 2 \gamma^n \eta^n (4.7)
\]

Using the fact that \( \gamma^n = \gamma^n \), we may write

\[
\sum_k \sum_{n^l} \gamma^n \eta^n \eta^n = \frac{\delta h}{\delta t} (4.8)
\]

where

\[
h = \frac{1}{2} \sum_k \sum_{n^l} \eta^n (4.9)
\]

\( h \) is proportional to the ratio of total wave magnetic energy to
static magnetic energy.

4.c) Quasi-linear stabilization of firehose instability

Using an approximation method similar to the Chapman-Enskog
method appropriate to collision-dominated gases, we assume that the
velocity distributions never deviate much from the time-independent
base distributions \( F_0 \). Writing \( F^\pm(x,t) = F_0^\pm(x) + \Theta^\pm(x,t) \), we
neglect \( \Theta^\pm \) in the stoss term of (4.2), since \( h \) is small. We take
the parallel and perpendicular pressure moments of (4.2) to find the
time evolution of the growth rate. After substituting (4.9) in the
stoss, we find

\[
\frac{\partial P^\pm}{\partial h} = -8 P_\|^{(o)\pm} + 4 P_{1}^{(o)\pm} (4.10)
\]

\[
\frac{\partial P^\pm}{\partial h} = 2 P_{\perp}^{(o)\pm} (4.11)
\]

where \( P_\|^{(o)\pm} \) and \( P_{1}^{(o)\pm} \) are calculated using \( F_0 \)
and \( P_{\perp}^{(o)\pm} \) using \( \Theta \). The solutions are trivial:

\[
P_\|^{(o)\pm}(t) = P_\|^{(o)\pm} + (4 P_{1}^{(o)\pm} - 8 P_\|^{(o)\pm}) h(t) (4.12)
\]

\[
P_{\perp}^{(o)\pm}(t) = P_{\perp}^{(o)\pm} + 2 P_{1}^{(o)\pm} h(t) (4.13)
\]

\( \zeta^\pm(t,0) \) has been assumed zero. Now we compute the growth discrimin-
ant \( \Delta \), e.g. (2.7), as a function of time using (4.12) and (4.13),

\[ -20 - \]
assuming \( \frac{P_{\nu}}{P_{\nu}} = \frac{P_{\nu}}{P_{\nu}} \) and assuming \( h(t) \ll 1 \).

\[
\Delta(t) = \Delta_0 - 12 h(t) = \left( \frac{\Delta P}{P} \right)_{t=0}^+ + \left( \frac{\Delta P}{P} \right)_{t=0}^- - \frac{2}{\beta^4} - 12 h(t) \quad (4.14)
\]

Thus the growth discriminant goes to zero when the ratio of fluctuation to static magnetic energy becomes the order of the initial deviation from marginal stability. The wave pressure then compensates for the initial particle pressure imbalance. Since the firehose waves conserve the first and second adiabatic invariants it can reasonably be asked how pitch angle anisotropies can be altered. Because the second invariant, \( J_2 = \int \psi \, d\ell \), where \( d\ell \) is an element of length along a line of force, is conserved, the increase in average magnetic line length due to the turbulent motion requires that \( \nabla_n \) decrease for each particle, thereby decreasing \( P_{\nu} \). Another point of view is to note that the growth rate involves pressure anisotropies with respect to the average magnetic field. A small random fluctuation in the field direction, conserving particle invariants, clearly will reduce anisotropies with respect to the average field direction.

Referring to equation (2.14), we may now visualize the evolution of the wave spectrum with time. All those fluctuation components originally present at \( t=0 \) with wave numbers such that \( 4 \Delta_0 > k^2 R^2_i \) grow. However, the wave growth alters the particle distributions and reduces \( \Delta \). When, for a given \( k \), \( 4 \Delta(t) \) becomes equal to \( (k^2) R^2_i \), that wave ceases to grow and becomes a pure oscillation. Since it no longer grows, it no longer contributes to the adiabatic diffusion. However, longer wave length components still grow, and still reduce \( \Delta \). The highest wave number components stabilize first, followed later by the long wavelength part of the spectrum. A complete steady state is reached when \( \Delta = 0 \). There remains a set of oscillating Alfvén waves. Since \( \Delta \) approaches zero in the turbulent steady state, where wave pressure compensates for particle pressure anisotropies, the turbulent plasma is more nearly fluid-like and the mode-coupling matrix elements again become very small. Thus, fully developed Alfvén wave turbulence should persist, even at large amplitudes.
5. QUASI-LINEAR MOMENT EQUATIONS FOR A SPATIALLY INHOMOGENEOUS PLASMA

5.a) Approximate quasi-linear kinetic theory for the wave and particle distributions

For a spatially inhomogeneous plasma, we must assume that the background particle distributions and electromagnetic fields vary only on space scales large relative to typical wavelengths. The ion Vlasov equation averaged over many wave oscillations in time and wavelengths in space is

$$\frac{\partial F^+}{\partial t} + \nabla \cdot F^+ + \frac{q}{m_i} \left\{ E_x + E_L + \frac{v_x B_0}{c} \right\} \cdot \frac{\partial F^+}{\partial v_x} = \delta t \{ F^+ \}$$  \hspace{1cm} (5.1)

where $\delta t \{ F^+ \}$ represents a modified quasi-linear stoss. For flows strictly along $B_0$, we may assume $E_L = 0$. Then, we may write

$$\frac{E_x}{M_i} = -\frac{\nabla \psi}{k^2}.$$  \hspace{1cm} (5.2)

In addition, since the average plasma properties are assumed to vary extremely slowly, we may take $\nabla \times B_0 \cdot \frac{\partial F^+}{\partial v_x} = 0$.

The quasi-linear assumption requires that the average plasma properties vary slowly compared with typical wave fluctuations; i.e., the waves must obey a WKB approximation with respect to the slowly varying background both in space and in time. The wave polarizations, and therefore the forces between waves and particles, are strongly affected only when the waves propagate nearly normal to the magnetic field, i.e., when $\frac{K^2}{K_L^2} \sim \frac{R}{L} \ll 1$, in the so-called drift approximation (Rosenbluth et al., 1962). However, the firehose mechanism destabilizes waves with $K_L = 0$, so that drift effects do not occur, and the wave-particle interaction will have only non-essential and small corrections due to the propagation in a non-uniform medium. These small terms of $O(KL)$ will add space derivatives in the stoss. If these provided dispersion rather than dissipation, there might be some difficulty in dropping these terms, but as it is, they only make a small correction to the dissipation. For high $\beta$ non-linear wave trains propagating normal to the field, the drift dispersive corrections are important. For parallel propagation, they need not be taken into account.
Neglecting small corrections of order \((KL)^2\), and remembering that the stoss term of section 4 was derived for a plasma at rest, we find

\[
S_t\{F^+\} = \sum_k \gamma_k h_k \left[ (\mathbf{V} - \mathbf{U}) \cdot \nabla \mu \right] - 2 \frac{\partial}{\partial \mathbf{V}_\perp} \nabla \cdot \left\{ \nabla \cdot (\mathbf{U} - \mathbf{U}) \right\} + \frac{\partial}{\partial \mathbf{V}_\parallel} \nabla \cdot \left\{ \nabla \cdot \mathbf{F}^+ \right\} \tag{5.2}
\]

where \(\mathbf{V}(z,t)\) is the mean flow velocity parallel to the magnetic field. That this is the correct form for the stoss is suggested by the analogous result of Galeev et al., (1965) for electrostatic turbulence.

The general forms will be similar if we drop resonant interactions and terms of order the wave energy squared \(h_k^2\) from their expressions, and note that, \(\frac{Dh_k}{Dt} \approx 2 \sum \gamma_k h_k\) from (5.2).

The electrons cross the shock extremely rapidly. They feel primarily the change in electrostatic potential and, in their short interaction time with the shock, they are not affected by Alfvén turbulence. From the complete electron equation

\[
\frac{\partial \mathbf{F}^+}{\partial t} + \nabla \cdot \mathbf{F}^+ + \frac{M_e}{M_i} \frac{\partial \mathbf{V}}{\partial t} \mathbf{F}^- = S_t\{F^+\} \tag{5.3}
\]

we compare the order of magnitudes of convective and dissipative terms.

\[
\sqrt{\frac{\partial \mathbf{F}^+}{\partial t}} \approx \frac{\sqrt{\frac{\partial \mathbf{F}^-}{\partial t}}}{\frac{L h_A}{\Omega_p}} \approx \sqrt{\frac{T_e}{T_i}} \frac{1}{\Omega_p} \gg 1 \tag{5.4}
\]

Since in the ion equation (5.1) we took effectively \(R_i/L h_A \sim 1\), the firehose stoss term is very small for electrons. In addition we may neglect \(\partial \mathbf{F}^- / \partial t\), since

\[
\sqrt{\frac{\partial \mathbf{F}^-}{\partial t}} \left[ \frac{\partial \mathbf{F}^-}{\partial t} \right] \sim \frac{C_2}{L \sqrt{\frac{\pi}{\Omega_p}}} \approx \sqrt{\frac{M_e}{M_i}} \ll 1 \tag{5.5}
\]

Therefore we write

\[
\nabla \cdot \mathbf{F}^- + \frac{M_e}{M_i} \frac{\partial \mathbf{V}}{\partial t} \mathbf{F}^- \approx 0 \tag{5.6}
\]

Only the electron density, \(N^-\), appears in Poisson's equation and we
therefore integrate (5.6) above to find the electron density

$$\frac{\partial N}{\partial \psi} = -\frac{M_e}{M_c} \int \frac{\partial E}{\partial W^2} \, dV \equiv + \frac{N_0}{C_s^2}$$

(5.7)

or

$$N(z,t) = N_0 e^{\psi(z,t)/C_s^2}$$

where $N_0$ is an arbitrary constant. Thus, the electrons have a Boltzmann distribution.

The quasi-linear assumptions require that we average over many Alfvén wavelengths $\lambda_A \sim \frac{1}{\sqrt{A} \, R_i}$. Except for extremely low density plasmas where the ion sound Debye length is comparable with the ion gyroradius, i.e., when $\frac{V_A}{c} \sqrt{\frac{e \Theta}{m_i}} > 1$, the quasi-linear theory also averages over many ion Debye lengths. It is therefore only consistent to use the quasi-neutrality condition.

$$\int F^+ dV = N_0 \frac{\psi(z,t)}{C_s^2}$$

(5.8)

Dispersive wave-train effects cannot be described by (5.8).

In section 3, we argued that the electron anisotropy created in the compressional ion sound wave is small. Therefore, the electrons do not play a large role in the shock structure. In addition, there are a number of instabilities (Furth, 1962) which can correct electron pressure anisotropies much more rapidly than can Alfvén turbulence. Before the ions can move, the electrons rapidly reach equilibrium among themselves. Thus, the above choice of the Boltzmann electron distribution is probably a physically more realistic procedure than the retention of the full firehose stoss for electrons. When turbulence in other modes is important for electrons, the effective ion sound velocity will not be determined by the undisturbed electron distribution, but by that electron distribution created by the turbulent interaction with the aforementioned high frequency modes.

The kinetic equation for waves [Camao et al. (1962); Galeev and Karpan (1963); Kadomtsev (1965) and others] describes the evolution of wave energy in the phase space of wave number $k$ and position $x$. 
The terms of $O(h_k^4)$ represent wave-wave couplings. $U$ is the flow velocity in the plasma and $\omega_k / \lambda K$ the group velocity of the $K$th component in the excited spectrum. The term involving $\frac{2}{\lambda K} (\omega_k)$ represents the distortion of a wave packet due to its motion in an inhomogeneous medium. Suppose we compare the order of magnitude of terms in (5.9). These are, in order of magnitude, using the estimates, eqs. (2.13) and (2.16), for $\frac{2}{\lambda K}$ and $\frac{\omega_k}{\lambda K}$ (appropriate to Alfvén waves):

$$\Delta \frac{\rho_T}{\rho} \cdot \frac{h_K}{L} : U \cdot \frac{h_K}{L} : \Delta \frac{\rho_T}{\rho} \cdot \frac{h_K}{L} (5.10)$$

Here we estimated the wave-packet distortion $\nabla(\omega_k)$ by $\omega_k / L$ and assumed that the wave spectrum is moderately broad so that $\frac{\rho_T}{\rho} \sim \frac{h_K}{L}$. If $U$ is roughly the ion sound velocity, $U \sim \sqrt{\frac{\rho}{\rho_T}} \gg \sqrt{\frac{\rho}{\rho_T}}$

then

$$\frac{\partial h_K}{\partial t} + U \frac{\partial h_K}{\partial x} = 2 \omega_k h_K + O(h_k^2, \sqrt{T / T^0} h_K \Delta) (5.11)$$

Since in the shock solution of section 6, $h_K \sim \Delta \sim T / T^0 \ll 1$, the neglect of both the mode coupling terms and of the group velocity and distortion terms is consistent.

5.b) Closure of moment equations

Since when $T / T^0 \gg 1$, the ion sound mode travels much faster than the ion thermal velocity $\sqrt{\rho_T / \rho}$, it cannot feel the fine structure of the ion velocity distribution. The ions therefore behave as a fluid. This suggests a macroscopic moment approach for the ions. Firehose turbulence is also amenable to this treatment, since both the instability and quasi-linear diffusion are both macroscopic. In terms of the following moments:

The ion mass density,

$$f = \rho (+ \int \frac{\partial}{\partial x} \nabla \cdot \mathbf{F}^+ (5.12))$$
the mean flow velocity,
\[ U = \frac{M_i \int F^+ V_i dV}{\rho} \]  
(5.13)

the parallel ion pressure,
\[ P_\parallel = M_i \int (V_i - U)^2 F^+ \]  
(5.14)

and the perpendicular ion pressure,
\[ P_\perp = M_i \int \frac{V_i^2}{2} F^+ \]  
(5.15)

we arrive at the following hierarchy of moment equations, whose first few members are
\[ \frac{\partial P}{\partial t} + \frac{\partial}{\partial z} \left( P U \right) \]  
(5.16)

\[ \frac{\partial}{\partial t} \left( P U^2 + P_\parallel \right) + \frac{\partial}{\partial z} \left( P U^3 + 3 P U \right) + P \frac{\partial U}{\partial z} + \frac{\partial M_1}{\partial z} = 4 \left( P_\perp - 2 P_\parallel \right) \sum \chi_k h_k \]  
(5.17)

\[ \frac{\partial P}{\partial t} + \frac{\partial}{\partial z} \left( U P_\parallel \right) - \frac{\partial M_2}{\partial z} = 2 P_\parallel \sum \chi_k h_k \]  
(5.18)

The moments \( M_1 \) and \( M_2 \) are
\[ M_1 = M_i \int dV F^+ (V_i - U)^3 ; \quad M_2 = M_i \int dV F^+ \frac{V_i^2}{2} (V_i - U) \]  
(5.19)

The hierarchy will close if \( M_1 \) and \( M_2 \) can be neglected.

Notice that in eqs. (5.18) and (5.19)
\[ \frac{M_1}{U P_\parallel} \sim \frac{M_2}{U^2 P_\parallel} \sim \sqrt{\frac{P}{P U^2}} \sim \sqrt{\frac{T}{T^*}} \ll 1 \]  
(5.20)
and that the other terms on the r.h.s. of (5.18) and (5.19) are even larger. Not only should $M_1$ and $M_2$ be small initially, but the dynamics must preserve their small magnitude. Taking the $M_1$ and $M_2$ moments of (5.1) and (5.2) assuming that $M_1 = M_2 = 0$ at $t = 0$ and neglecting all fourth order moments, we find

$$\frac{\Delta M_1}{\Delta t} = -2 \frac{P}{\rho} \frac{\Delta \psi}{\Delta z} ; \frac{\Delta M_2}{\Delta t} = -3 \frac{P}{\rho} \frac{\Delta \psi}{\Delta z} \quad (5.22)$$

The flow time scale is roughly $L/U$, so that the changes in $M_1$ and $M_2$, in the shock are

$$\delta M_1 \sim \frac{P \Delta \psi}{U} ; \quad \delta M_2 \sim \frac{P \Delta \psi}{U}$$

where $\Delta \psi$ is the change in potential. Thus,

$$\frac{\delta M_1}{\rho \Delta z} \sim \frac{\Delta \psi}{U} ; \quad \frac{\delta M_2}{\rho \Delta z} \sim \frac{\Delta \psi}{U} \quad (5.23)$$

Since $U \sim C_s$, we may drop the third order moments on this basis if $\frac{\Delta \psi}{C_s^2} \ll 1$. This will clearly be true for weak shocks. In a steady flow, we must compute the evolution of $M_1$ and $M_2$ in space. This third condition illustrates in physical terms the difficulties involved when $T^-/T^+$ approaches one. Suppose we compute

$$\frac{\Delta M_1}{\Delta z} = \frac{\Delta \psi}{\Delta z} \left\langle \frac{M_1 (V_\eta - U)^3}{V_\eta} \frac{\Delta F^+}{\Delta V_\eta} \right\rangle +$$

$$+ U \frac{\Delta h}{\Delta z} \left\langle \frac{M_1 (V_\eta - U)^3}{V_\eta} \left[ \frac{1}{V_\eta} \left( \frac{V_\eta - U}{U} \right)^2 \frac{\Delta F^+}{\Delta V_\eta} - 2 \frac{\Delta \eta}{V_\eta} \left( \frac{V_\eta - U}{U} \right) \frac{\Delta F^+}{\Delta V_\eta} \right] \right\rangle \quad (5.24)$$

where $\langle \rangle$ denotes velocity space integration. When $U \sim C_s \gg (V_\eta - U)$ we may approximate the "resonant" denominator in (5.24) by

$$\frac{1}{V_\eta} \sim \frac{1}{U} \left\{ 1 - \frac{(V_\eta - U)}{U} - \frac{(V_\eta - U)^2}{U^2} - \cdots \right\} \quad (5.25)$$

so that the moments in (5.25) are non-singular, and (5.24) becomes

$$\frac{\Delta M_1}{\Delta z} \sim - \frac{2P}{U} \frac{\Delta \psi}{\Delta z} + 6 \frac{\Delta h}{\Delta z} (M_1 - 2M_2) \quad (5.26)$$

- 27 -
and similarly for $M_2$. $S M_1$ and $S M_2$ can then be neglected if

$$\frac{S_{\psi}}{C_s^2} \ll 1 \quad \text{and} \quad S h \ll 1 \quad (5.27)$$

When $T^*/T^* \rightarrow 1$, $U \rightarrow \sqrt{\beta^*/\rho^*}$, the ion thermal velocity, and

the $1/\nu_i$ singularity in $\partial M_i/\partial \mathcal{Z}$ and $\partial M_2/\partial \mathcal{Z}$ emphasizes a region of velocity space near the maximum of the ion distribution creating a large moment. Thus, the non-closure of the moment hierarchy is associated with a Landau-damping-type of resonance involving low velocity ions which would stand in a steady shock front. Here the full kinetic equation (5.1) must be solved and the resonance explicitly accounted for.

Consider again the steady state with $\partial F/\partial t = 0$. Let us compare the smallest terms on the r.h.s. with the stoss term of (5.18)

$$\frac{4(P - 2P_o)}{3 \sum_k \frac{h_k}{h_s}} \sim - \left(1 + \frac{\Delta P}{P_s} \right) h \Delta \quad (5.28)$$

Since $h \Delta$ is $\ll 1$, the term $(\Delta P/P_s) h \Delta$ is higher order and may be neglected. Thus, the moment equations are (5.16), (5.17), (5.18), (5.19) together with (5.11) and the quasi-neutrality condition (5.9). Summarizing the approximations (5.11), (5.23), (5.27) and (5.28)

$$\frac{\nu_i}{n_i} \sim \Delta \sim h \sim \frac{T^*}{C_s^2} \sim \frac{T^*}{\beta^*} \quad (5.29)$$

6. COLLISION-FREE SHOCK STRUCTURE

Consider a steady shock, moving with velocity $U_o$ into an undisturbed medium with density $\rho_o$, and isotropic pressure, $P_o$. We set all time derivatives zero and consider only variations with $Z$. The conservation of mass equation (5.16) has an immediate integral

$$F V = P_o \quad U_o \quad (6.1)$$
If we substitute the quasi-neutrality condition into the momentum equation (5.17), we find another integral using (6.1)

\[ \rho U_0 (U - U_0) + P_u - P_o + \rho \kappa^2 (e \gamma / c_s^2 - 1) \]  

(6.2)

Here we assumed that the electrostatic potential ahead of the shock, \( \psi_0 \), is zero. Noting that \( e \gamma / c_s^2 = \frac{f}{\rho} = \frac{U}{U_0} \), we rewrite (6.2)

\[ P_u = P_o + \frac{U - U_0}{U_0} \left\{ \frac{U_0}{U} \rho \kappa^2 - \rho \kappa^2 U_0^2 \right\} \]  

(6.3)

The various terms in the third order moment equation (5.18) scale roughly in order of magnitude as

\[ \rho \kappa^2 : 3P_o : 2\rho \kappa^2 : 4\rho \kappa^2 \]  

(6.4)

Thus \( h \sim T^+, T^- \). The r.h.s. is doubly small, \( O(T^+/T^-)^2 \), so that it is a good approximation to write, as follows

\[ \frac{1}{\rho \kappa^2} \left( P_u U^2 + 3U P_u \right) + 2U \frac{e \gamma}{\kappa} \frac{\Delta \psi}{\Delta x} = -4P_o U_0 \frac{\Delta h}{\Delta x} \]  

(6.5)

Neglecting all perturbations of the flow and pressure fields in the term containing the fluctuation field thus leads to another integral:

\[ \rho \kappa^2 (U' - U_0') + 3U P_o - 3U_0 P_o + 2U_0 \psi = -4P_o U_0 (h - h_0) \]  

(6.6)

where \( h_0 \) is the integrated fluctuation field ahead of the wave. Substituting (6.3) for \( P_u \) into (6.4), we arrive at

\[ U' - U_0' + 3(U - U_0)\left\{ \frac{U_0}{U} - \frac{U_o}{U_o} \right\} - \frac{3P_o}{P_o} + \frac{2U_0}{U_0} U = -4P_o U_0 (h - h_0) \]  

(6.7)

Eqs. (6.1), (6.3) and (6.5) and the quasi-neutrality condition, Eq. (5.6), determine the variables \( f_0, U_0, \psi \), and \( P_o \) in terms of the initial values \( f, U_0, P_o \), and \( \psi_0 = 0 \), and the fluctuation field \( h - h_0 \). The integro-differential equation for the wave field completes the set of equations:

\[ U_0 \frac{\Delta h}{\Delta x^2} = 2 \sum_K \psi_K h_K, \quad h = \sum_K h_K \]  

(6.8)
The expression relating \( P_\perp \) to \( h-h_0 \) is, under the same assumptions as those leading to Eq. (6.7), after integrating (5.14)

\[
P_\perp = \frac{2v_p}{v} P_0 (h-h_0) + \frac{P_0}{U} \frac{V_0}{U}
\]

Combining with (6.3), we find

\[
\frac{P_\perp - P_0}{P_0} = \frac{1}{P_0} \left( \frac{V-U_0}{V_0} \right) \left\{ -P_0 U_0^2 + \frac{U_0}{U} (\frac{P C_s^2}{P_0} + P)_0 \right\} - 2(h-h_0)
\]

Eq. (6.8) together with (6.11) and (6.1), (6.3), (6.8) and the quasi-neutrality condition constitute a nonlinear integro-differential equation for the shock structure.

In general, the nonlinear equation (6.8) is difficult to solve analytically. In addition, there is no unique solution for the wave spectrum because the shock dissipation is due to an instability. Only those wave components present ahead of the shock can be amplified and contribute to the shock structure. Since there is always a thermal level of wave excitation, there is always a wave source at each wave number. On the other hand, if the plasma ahead of the shock is unsteady and turbulent, as, for instance, is likely in the case of the solar wind, those components with large non-thermal amplitudes upstream will contribute the dominant dissipative effect. Thus, the shock thickness will also be a function of the initial conditions.

The limit of weak shocks is probably the only regime where the quasi-linear theory is truly valid. Some progress can be made here.

Treating \( \Phi \equiv \left( \frac{U}{U_0} - 1 \right) \) and \( \frac{P_\perp - P_0}{P_0} \) as small parameters, and dropping all terms in third order, eq. (6.7) becomes

\[
\Phi \left\{ -\frac{U_0^2}{P_0} + \frac{C_s^2}{P_0} \left( \frac{3P}{P_0} \right) \right\} + \Phi^2 \left\{ \frac{C_s^2}{P_0} - 2U_0^2 \right\} = -\frac{P_\perp}{P_0} (h-h_0)
\]
Notice that by dropping terms of order \( \phi^4 C_s^2 \) and \( \frac{P}{\rho^2} (h - h_o) \) in (6.12) we recover the linear result for the wave propagation speed,
\[
V = C_s^2 + \frac{3P}{\rho C_s^2}.
\]
Furthermore, when \( \phi = 0 \), the wave spectrum is undisturbed, since \( h - h_o \) is necessarily zero.

Henceforth, we will use all the small parameters at our disposal, arranged as follows. First we define the Mach number in terms of the linear ion sound velocity:
\[
M^2 V^2 = \frac{U_o^2}{\rho_o^2} \tag{6.13}
\]
and then order small parameters,
\[
M^2 - 1 \sim \phi \sim \frac{T^+}{T^-} \sim h - h_o \tag{6.14}
\]
To second order in the above parameters, eq. (6.7) becomes
\[
\phi (\dot{M}^2 - 1) + \phi^2 \sim 4 \frac{T^+}{T^-} (h - h_o) \tag{6.15}
\]
The pressure anisotropy, (6.11), becomes in the same ordering of parameters
\[
\frac{P_h - P_i}{P_o} \sim - \frac{T^-}{T^+} \phi \left\{ \frac{3}{2} (\dot{M}^2 - 1) + \frac{T^+}{T^-} + \phi^2 \right\} \tag{6.16}
\]
or
\[
\frac{P_h - P_i}{P_o} \sim - 2 \left\{ \phi - 3 (h - h_o) \right\} \tag{6.17}
\]
Both these forms indicate that pressure anisotropies created by compression pulses \( \phi < 0 \) are diminished in the nonlinear limit, since the \( \phi^2 \) term in (6.13) diminishes \( \frac{P_h - P_i}{P_o} \), and \( h - h_o > 0 \) diminishes \( \frac{P_h - P_i}{P_o} \) in (6.17). Notice that the pressure anisotropy is strictly zero when there is no disturbance, \( \phi = 0 \).

The following qualitative analysis indicates that the shock integral equation (6.8) allows transitions between two asymptotically steady states. For instance, consider the case where all the excited Alfvén waves upstream obey the MHD limit.
where \( \left( \frac{p_n - p_i}{\bar{p}_0} \right)^* \) is a typical pressure anisotropy in the shock. Then to lowest order in the shock perturbation, the growth rate is, dropping terms of order \( k^3 \frac{(p_n - p_i)}{\beta n \bar{p}_0} \)

\[
\gamma_k \sim k \sqrt{\frac{p_0}{\bar{p}_0}} \left( \frac{p_n - p_i}{\bar{p}_0} - \frac{2}{\beta_n} \right)^{1/2}
\] (6.19)

There is of course a damped mode with \( k \) negative, but its amplitude will be small. Henceforth, we treat only the growing modes. Thus, since \( k \) is positive, \( n \) is constant only if \( \frac{p_n - p_i}{\bar{p}_0} - \frac{2}{\beta_n} = 0 \). Substituting from (6.15), we find two solutions corresponding to a constant growth rate, one for the flow upstream,

\[
\phi \sim \frac{1^+}{T^{-2(M-1)/3^+}} \approx 0
\] (6.20)

and one for the flow behind the shock.

\[
-\phi \sim \frac{3}{2} (M-1) + \frac{1^+}{T}
\] (6.21)

The shock thickness may be estimated using (6.9), (6.19) and (6.23). Since

\[
U_0 \frac{\delta h}{\delta z} = 2 \sqrt{\frac{p_0}{\bar{p}_0}} \sum_k k \sqrt{\frac{p_n - p_i}{\bar{p}_0}} h_k
\] (6.22)

we find, by substituting order of magnitude estimates

\[
\frac{U h}{L_s} \sim 2 \sqrt{\frac{p_0}{\bar{p}_0}} \sqrt{\frac{p_n - p_i}{\bar{p}_0}} \ k^* h
\] (6.23)

where \( L_s \) is the shock thickness and \( k^* \) a typical wave number in the excited spectrum upstream. Then using (6.16) and (6.21), we find
Thus, for weak shocks $M^2 - 1$, $K^* L_s \gg 1$ and the WKB approximation is appropriate for the Alfvén waves. Using (6.10) and estimating the final value of $\phi$ by (6.21), we find the approximate total wave amplitude downstream

\[ h - h_0 \approx \left\{ \frac{1}{2} \frac{T}{T^*} (M^2 - 1)^2 + \frac{1}{8} (M^2 - 1)^5 + \frac{T^*}{T} \right\} \approx (M^2 - 1) \quad (6.25) \]

The excited wave spectrum ahead of the shock will in general have high frequency components which include the wave number of maximum growth rate, $K_m$. Here we argue as follows: If there is stabilization for the maximally growing mode at any location in space, all other modes will be damping and their wave energy diminishing. Near $\gamma_{K_m} = 0$, the total wave energy $\sum K_h$ should be dominated by $h_{K_m}$. Thus, the entire plasma should again be in steady state when $\gamma_{K_m} \approx 0$. For the maximally growing mode, $\gamma_{K_m}$ is

\[ \gamma_{K_m} \sim \Omega_+ \frac{P_u - P}{P_0} \quad (6.26) \]

so that, once again, the two steady solutions are given by (6.7) and (6.18). Of course, $K_m$ varies through the shock with $P_u - P / P_0$, so that each location within the shock front will have a different maximally growing wave component. Since the waves propagate very slowly, there should therefore be spatial structure to the wave spectrum corresponding to the variation of $P_u - P / P_0$.

The shock thickness may be estimated using the same procedures as above if we also assume

\[ \sum K_h \sim \gamma_{K_m} h \quad (6.27) \]

Then, we find for $L_s$
which by taking \( \sqrt{\tau / \tau^*} \ll 1 \) but \( (\tau^* / \tau^*) \ll M^2 - 1 \) reduces to the limit originally estimated by Moiseev and Sagdeev (1962)

\[
L_5 \sim \frac{R_+ \sqrt{\tau / \tau^*}}{4 \left\{ \frac{3}{2} (M^2 - 1) + 1 \right\}}
\]

Thus, when \( M^2 - 1 \ll 1 \), the WKB approximation will again be valid, since the wavelength of the turbulent waves will be shorter than the shock thickness.

Clearly, the validity of these arguments is limited to weak shocks, \( M \approx 1 \). Nevertheless, the fact that the above arguments are self-consistent suggests that firehose Alfvén waves are a realistic mechanism for shock dissipation for finite shock amplitudes.

7. DISCUSSION OF MORE GENERAL SHOCK WAVES

In this section we speculate briefly on the possibility of relaxing the assumptions and approximations which were necessary to generate even a partially tractable theory mathematically. We take these points up individually, but not necessarily in order of their importance.

7.a) Extension to \( M > 1 \) and \( \beta_+ \approx 1 \)

When \( \beta_+ \approx 1 \), \( (\Delta p / p)_+ \) must also be of order one, in order that Alfvén waves be firehose unstable. Thus, since \( (\Delta p / p) \sim (M - 1)^2 \) Alfvén turbulence will only be generated in shock waves with Mach numbers greater than, say, 2 when \( \beta_+ \approx 1 \). In this case, an ion sound compression wave, rather than becoming firehose unstable immediately, may steepen until ion sound Debye length dispersive effects create, with some small dissipation, the non-linear ion sound wave train discussed by Moiseev and Sagdeev (1963). Above \( M \approx 1.6 \), the largest amplitude oscillation "breaks". The large electrostatic potential of the wave reflects many ions. In the reflection region,
there is a two-phase flow, with ion beams propagating in opposite
directions, which may well be unstable to incoherent ion sound
turbulence, as suggested also by Fredricks et al. (1965). There would
then be a high level of electrostatic turbulence in the overturning
region, a region which probably defies mathematical description.
The physical result, that ordered ion motion parallel to the field
be converted to parallel pressure, seems clear. Thus, $\beta_+$, which
is calculated using $P^+_{\parallel}$, will increase, lowering the threshold
for firehose instability. The quasi-linear theory, by averaging
over many Alfvén wavelengths, excludes this fine structure at the
ion Debye length. For strong shocks, there could be an electrostatic
wave train ahead, ion heating in the overturning region, and firehose
turbulence pressure isotropication. Since turbulent ion heating may
occur in strong shocks, it is reasonable to assume high $\beta_+$ at the
outset for the description of the long wavelength shock structure.
In general, the stronger the shock, the more ion heating, and the
more reasonable the high $\beta_+$ assumption.

7.b) Oblique shock waves

There is probably a finite cone of shock propagation angles
to the magnetic field for which $(\frac{\Delta P}{P^+}) > 0$, determined by the
competition of the compressional motion along the field, and the
betatron increase in $P_\perp$ due to the increasing magnetic field
strength in oblique shocks. Within this critical angle, firehose
dissipation will be part of the shock structure.

For normal shocks, drift dispersive effects will probably
play a role. Furthermore, $P^\perp > P^+$, and the "mirror" instability
[Sagdeev et al. (1958); Stix (1962); Scarf (1966)] will probably
destabilize magnetic sound waves. The mirror instability might
also occur in non-linear rarefaction waves propagating along the
magnetic field. Since non-linear wave-wave couplings may occur,
the decay length of the turbulence behind perpendicular shocks may
be shorter than that of the Alfvén turbulence behind parallel shocks.

7.c) Ratio of electron to ion temperature

$T^e/T^i > 1$ is clearly the primary physical assumption used
here. When $T^e/T^i > 1$, a collective, coupled ion-electron motion,
with a velocity larger than the ion thermal velocity, is permitted
along the lines of force. Because the ion motion is fluid-like, the moment equation hierarchy of section 5 has a closure. This greatly simplifies the mathematics. However, even when $T^+/T^+ \leq 1$, similar collisionless dissipation may occur, even if its mathematical description is not so felicitous. A compression pulse still enhances $P_n^+$. In addition, when $T^+/T^+ \leq 1$, thermal ions are not contained in the front. Many such ions, heated behind the shock, can now escape upstream and mix with the oncoming flow, again creating a pressure anisotropy, Alfvén magnetic turbulence, and possibly electrostatic turbulence as well.

If, as suggested in 7.a, there is considerable electrostatic turbulence present in the overturning region due to a two-ion beam ion sound instability, the electrons could be heated, so that behind the overturning region $T^+/T^+$ would be increased [Fredricks et al. (1965); Galeev et al. (1966)]. An interesting experimental test would be the measurement of the electron velocity distribution, which for ion sound turbulence has the form

$$ F^+ = C_1 \frac{-v^+}{e} $$

where $C_1$ and $C_2$ are constants (Galeev et al., 1966). Note that $F^+$ is nearly constant to $\sqrt{C_2}$ and falls rapidly thereafter.

7.d) **Possibility of unsteady shock motions**

The need for mathematical simplicity has restricted consideration to steady shocks. It would be interesting to ask whether shock fronts become unstable to the "resonant" ions which stand in the shock front. Such an effect could be particularly pronounced when $T^+/T^+ \leq 1$, since then the fluid description breaks down due precisely to the presence of too many resonant ions. Nevertheless, unstable pressure anisotropies should also occur in unsteady non-linear motions.
In 8.a, we estimate the relevant shock parameters for the solar wind. There may be a small overlap between the present restricted theory and observations but, in general, strong oblique shocks will be the rule. In 8.b, we compare briefly with observations. The strong low frequency magnetic field fluctuations observed to persist without decay throughout the transition region behind the Earth's bow shock appear to be the closest point of overlap between theory and observation.

8.a) Space plasma

The magnetic field strength is typically $5 \times 10^5$ Gauss ($\equiv 5'$ [Coleman et al. (1962); Hess et al. (1964); Coleman et al. (1966)]) in the solar wind. An ion number density of $\sim 5/cm^3$ is probably typical [Snyder et al. 1963; Wolfe et al. 1966]. The ion temperature seems to vary between 1 and 100 eV, with high temperatures occurring at disturbed times. [Bonetti et al. (1963); Bridge et al. (1964); Snyder and Neugebauer, 1964; Strong et al. (1966)]. $\beta_+$ therefore ranges from $1/10$ to 10, with one a typical value (M. Neugebauer, private communication, 1966).

Using a two-fluid plasma model, Sturrock and Hartle (1966) calculated $T^-/T^+$ in the solar wind near the Earth. Because the electron-electron collision mean free path, and therefore electron heat conductivity, is larger than that of ions, $T^-/T^+$ increases rapidly in this model with increasing distance from the sun. $T_-$ and $T_+^+$ and $T^-/T^+$ should be rather sensitive to the coronal temperatures since the heat conductivities are proportional to $(T^+_+)^{1/2}$. Assuming $T^- \approx T^+_+ \approx 200 eV$ at the corona, Sturrock and Hartle found $T^- \approx 50eV$, $T^+_+ \sim 1 eV$ and $T^-/T^+ \approx 50$ near the Earth.

Asbridge et al. (1966) found that occasionally some 100 eV electrons were observed ahead of the shock with Velas 2 and 3. Behind the shock, isotropic thermalized electrons with 200 eV mean energies were found. A preliminary estimate suggests that $T^- < 40 eV$ in the solar wind [S. Olbert, private communication; Rossi et al. (1966)]. Clearly, more information is needed to determine $T^-/T^+$. The existing evidence suggests that $T^-/T^+ > 1$, at least occasionally.
Since the ion sound velocity $C_s$ depends, not on the mean electron energy $T_e$, but on the moment $\int \frac{\rho}{\omega} \langle v_Z^2 \rangle$, eq. (3.7), a high energy electron tail, such as that found by Freeman et al., (1963), could enhance the ion sound velocity.

For an electron temperature of 25 eV, the ion sound speed is $5 \times 10^6$ cm/sec, while the solar wind velocity varies between $3 \times 10^7$ and $7.5 \times 10^7$ cm/sec, corresponding to ion sound Mach numbers of the subsolar point of $6 - 15$. These are comparable to the Alfvén Mach numbers, since $\beta_\perp \approx 1$; when $\beta_\perp > 1$, the ion sound wave Mach number is lower than the Alfvén. Away from the subsolar stagnation point, the shock strength decreases, and there may even be a weak shock region where $M - 1 < 1$.

There probably can be only a small subset of all measurements for which $\beta_\perp > \beta_i > 1$, at large distances from the stagnation point (so that $M \approx 1$) and with the proper geometrical location (so that the shock propagates approximately along the lines of force) and which therefore obeys the strict mathematical assumptions of the present theory. In general, however, the plasma requires consideration of the mathematically much more formidable $\beta_\perp \sim \beta_\parallel \sim 1$, $M \gg 1$ oblique propagation case. However, firehose turbulence will probably still occur, perhaps as one part of a turbulent hierarchy.

8.b) Discussion of observations

According to this theory, the shock thickness should be somewhat larger than an ion gyroradius. When $T_e \approx 25$ eV and $B \approx 5$, the ion Larmor radius is roughly 100 kilometers and the gyrofrequency $10^6$ cycle per second. A disturbance a few Larmor radii thick propagating at the solar wind velocity would pass directly over a stationary detector in about one second. An oblique passage of a long, thin object would take somewhat longer.

The Earth's bow shock seems often to be in motion (Heppner [1965]; Holzer et al. (1966); Wolfe et al. (1966)). If the shock velocity were comparable with the solar wind velocity, the shock transition would last a few seconds on a stationary satellite. Heppner (1965), Holzer et al. (1966) and Kaufmann (1966), all report such time scales. However, accurate spatially correlated
measurements are needed to determine the velocity, if the shock moves, or for that matter, to distinguish between a geometrically wrinkled stationary front containing interlocking filaments of solar wind and shock heated plasma and a truly moving front. If the shock velocity is $\frac{1}{10}$ the solar wind velocity as Holzer et al. (1966) estimate, the space scales become much shorter, and the above agreement is fortuitous.

The strong low frequency magnetic turbulence, initiated at the shock front and persisting well behind it, which was observed by Ness, et al. (1964), could well be firehose turbulence. The pressure anisotropy in the shock, $\left(\frac{\Delta p}{p}\right)_f \sim 1$ for strong shocks, should be comparable with the wave turbulence $\mathcal{h} \sim \frac{\Sigma_{1}^{1} \Omega_{1}^{1} B_{0}^{1}}{B_{0}^{1}}$ behind the shock. Ness, et al. (1964) found that $\mathcal{h} \ll 1$ ahead of the shock, and $\mathcal{h} \sim \frac{1}{4} - 1$ behind. Similarly, Ness, et al. (1966) found that a low noise level in the solar wind was amplified to $\mathcal{h} \sim \frac{3}{10} - \frac{1}{4}$ in the frequency range below $1/60$ cps.

The frequency corresponding to the maximum wave amplitude is a complicated function of the initial spectrum ahead and the maximum growth rate within the shock, and so may vary from observation to observation. Since only torsional Alfvén waves, which do not increase the magnetic field strength, are created by the firehose mechanism, the fluctuations in magnetic field direction should be larger than those in its magnitude. There may be some preliminary evidence that this might be so (Leverett Davis, private communication).

The persistence, together with the large amplitude, of the magnetic turbulence observed behind the shock, strongly suggests Alfvén waves. Large amplitude noise fills the transition region, and even penetrates into the magnetosphere, many hundreds of ion gyro-radii behind the shock. For one regime, the non-linear coupling between waves may conceivably even be zero, even for the large observed amplitudes. For instance, it is known that Alfvén waves can decay into an ion sound and an Alfvén wave [Kadomtsev (1953); Sagdeev and Galeev (1966)] or into a pair of Alfvén and magnetosonic waves (Kadomtsev, 1965). However, if $\frac{7}{10} < 1$ well behind the shock, the ion sound wave will be heavily damped. When $3 < 1$ magnetosonic waves propagating at large angles to the magnetic field are heavily damped, while at small angles they are
identical to Alfvén waves. Thus, there would be nothing but other Alfvén waves to couple with, and these are nearly linear because of the virtual pressure isotropy of the fluctuating medium. Here, Alfvén turbulence would be completely "degenerate".

Clearly, accurate calculations of the mode coupling coefficients are needed to estimate the decay length of Alfvén turbulence behind the shock, and the distribution in spectrum and oscillation mode of the turbulent wave energy at large distances from the shock. Moreover, this discussion emphasizes the necessity of making fluctuation spectra measurements as close as possible to the shock front for an unambiguous comparison with theory.

TURBULENCE IN OTHER WAVE MODES

Here we discuss briefly and very incompletely other possible forms of turbulence which may be present in large amplitude shocks. We limited discussion previously to parallel propagating Alfvén waves because these have the largest growth rate. However, the firehose mechanism will destabilize in reality hydromagnetic waves propagating within a certain cone of angles about the magnetic field direction. These could be accounted for, using the method of Shapiro and Shevchenko (1964). However, the basic physics will be the same and the mathematics more complex. As these waves are really part of the basic dissipation mechanism, we turn to the other modes.

9.a) Short-wavelength ion anisotropy instability

For parallel wavelengths less than an ion gyroradius, there is an electromagnetic wave instability with frequency (Kennel and Wong, 1966)

\[ \omega \sim \Omega^+ \left\{ \frac{\Delta P}{P} + \frac{2}{\beta_+} \left( 2 - \beta_+ \frac{\Delta P}{P} \right) \right\} \]  \hspace{1cm} (9.1)

and growth rate

\[ \gamma = \sqrt{\frac{\pi}{2}} \frac{\Omega^+}{\kappa R_+} \left\{ \left( \frac{\Delta P}{P} \right)^2 - \frac{\kappa R^2}{\beta_+} \left( 2 - \beta_+ \frac{\Delta P}{P} \right) \right\} \]  \hspace{1cm} (9.2)
Since this is a resonant instability, not all the ions contribute to the growth rate, and the non-linear diffusion will not reach the total ion distribution. The anisotropy of very low velocity ions, which stand in the shock, is involved in the instability and non-linear diffusion. The instability criterion is \((\Delta P/\rho)^2 \beta^2 \approx \), while for the firehose wave it is \(\Delta P/\rho > \beta^2 \). Since \((\Delta P/\rho) - (m-1)^2\), this instability is competitive with the firehose wave only for strong shocks.

Alfvén waves are unstable to pressure anisotropies measured with respect to the average magnetic field direction, whereas the short-wave length instability above depends upon local anisotropies. As Alfvén waves conserve the ion magnetic moment, relaxation of the ion magnetic moment distribution requires another non-adiabatic short wavelength instability. Each instability may appear with its own function, firehose waves determining the macroscopic shock structure while the resonant instability slowly relaxes the magnetic moment distribution downstream. Since both the Alfvén and the short wavelength wave above are circularly polarized electromagnetic waves propagating along the magnetic field with roughly the same frequency, only a detector sensitive to spatial correlations can discriminate between the two modes of turbulence.

9.b) Whistler and ion sound turbulence

The ion sound wave, which has \(\lambda_{10} \text{ km}\) wavelengths and 300 cps frequencies in the solar wind, was discussed in 7.a. For oblique shocks, both coherent and turbulent whistler structure could also occur [Camac et al. (1962); Sagdeev (1962); Karpman (1963); Morton (1964)]. This would have 10-20 km wavelengths, and frequencies near 1 cps. According to the discussions in the literature, these components arise in wave trains, or have significant mode coupling, and would therefore probably occur primarily in and near the shock front. Finally, there should be some turbulence near and above the electron gyrofrequency associated with the relaxation of the electron distribution, as mentioned briefly in section 5.

Both Heppner (1966) and Holzer et al. (1966) have observed enhanced magnetic fluctuations at roughly...
1 cps behind the shock. \(\rho\), according to Holzer et al., was about \(10^{-2}\) in the frequency range 0.2 - 2 cps, probably insufficient to account for all the shock dissipation. The high frequency enhanced noise cut off at 300 cps, roughly the ion plasma frequency. It is tempting to associate this component with ion sound turbulence, since the ion sound wave propagating at an angle to the field in a high \(\beta\) plasma is not purely electrostatic.

10. DISCUSSION AND SUMMARY

There may be a hierarchy of turbulent modes excited in the Earth's bow shock. To distinguish between them requires at least low frequency detectors for Alfvén waves (<\(1/10\) cps), whistler detectors \(\approx 1\) cps and very fast detectors, preferably electrostatic, at 300 cps. Careful polarization measurements would be needed to distinguish between ion sound noise at 300 cps and electron cyclotron noise, associated with pure electron turbulence, at the electron cyclotron frequency, also roughly 300 cps. Since the turbulent scale lengths differ greatly among the various likely oscillations, each detector will see a different "shock". The overall shock structure is the totality of non-linear effects which contribute to dissipation.

These comments are summarized schematically in fig. 1.

We summarize our arguments as follows. Compressional large amplitude shock waves propagating in the direction of the magnetic field necessarily create pressure anisotropies which in high \(\beta\) plasmas are unstable to the firehose growth of Alfvén waves. The non-linear development creates a large-amplitude wave distribution, with \(\Delta B/\rho_0 \sim 1\). Fully developed Alfvén turbulence should persist well behind the shock front. Other short wavelength modes of turbulence may also exist within the front, but their averaged effect will still probably be to generate Alfvén waves. Considerable low-frequency large-amplitude magnetic turbulence, probably of the type discussed, exists behind the Earth's bow shock.

In addition to reasonable spectral resolution of the fluctuating electromagnetic and electrostatic fields, good knowledge
of static parameters, such as the ratio of ion pressure to magnetic pressure, the ratio of electron to ion temperature, and the angle of shock propagation relative to the magnetic field direction is crucial for comparison of theory and experiment.

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Shown here is a pictorial summary of some of the various scales of turbulence which could form a part of strong shock structure. Ahead could be a whistler wave train with a scale length roughly $c/\omega_{p+}$, while trailing behind might be an ion sound wave train at the Debye length. Shown is a possible overturning region where strong turbulence may heat electrons and ions. Finally there is the Alfvén wave turbulence discussed here, with wavelengths less than the ion gyro-radius. This diagram has been deliberately not drawn to scale, either for amplitudes or wavelengths, because strong shock analyses have not been performed. We hope only to illustrate that many self-consistent turbulent dissipation mechanisms have been proposed in the weak shock limit; if they all appear in strong shocks, the composite picture could be quite complex.
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