LECTURES
ON THE NON-LINEAR THEORY
OF PLASMA

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AND
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# LIST OF CONTENTS

## PART I: THE WAVE-WAVE INTERACTION IN PLASMA
1. Resonant non-linear coupling between plasma waves .................. 1
2. Interaction of the three waves of finite amplitude .................. 8
3. Many-wave interaction in random phase approximation .......... 20
4. Plasma turbulence in terms of the kinetic equation for waves ............................................. 28
5. Coupling between different modes in a turbulent plasma ...... 35

References ........................................................................... 40

## PART II: PARTICLE-WAVE INTERACTION
1. Particle interaction with a single wave ........................................... 42
2. The many-waves case ................................................................. 51
3. Quasi-linear theory of the electromagnetic modes ..................... 62
4. Quasi-linear theory of the Post-Rosenbluth "loss-cone" instability ......................................................... 70
5. Non-resonant adiabatic wave-particle interaction .................... 81
6. Quasi-linear theory of drift instability ..................................... 90

References ........................................................................... 109

## PART III: WAVE-PARTICLE NON-LINEAR INTERACTION
1. Electron plasma oscillation turbulence .................................. 112
2. Current-driven ion sound turbulence ..................................... 118
3. Non-linear theory of the drift instability ................................. 131
4. The general scheme of weak turbulence theory ......................... 140

References ........................................................................... 145

## PART IV: NON-LINEAR PLASMA DYNAMICS IN COLLISIONLESS SHOCKS
1. Non-linear dispersive wave trains and solitons ...................... 147
2. The onset of instability and turbulence .................................. 150

Simplified schematic table ..................................................... 159

References ........................................................................... 160

Note: Equations and references are numbered separately for each part.
1. Resonant Nonlinear Coupling Between Plasma Waves

We would like to start with the linear theory of plasma waves. In a uniform plasma we can expand the equations of the linear wave theory in series of Fourier harmonics. The main problem of this chapter will be to apply a systematic approach of this kind to nonlinear phenomena.

The simplest approach to nonlinear dynamics is to look first of all for very small amplitude oscillations in fluid-type equations, then to look at finite amplitudes, where non-fluid-type phenomena depending on the Vlasov equation are now included.

If we start from the analogy with nonlinear gas kinetics and assume we have monochromatic waves of frequency \( \omega \), then the main nonlinear effect will be steepening of these waves:

We can describe this process as the generation of higher harmonics.

If the plasma wave satisfies a linear equation, then this can be written as

\[
\hat{L}(\omega, \kappa) \varphi = 0
\]

\( \hat{L} \) = linear operator

Taking into account the nonlinear terms in real plasma oscillations, we get in the simplest approximation a quadratic term (containing \( \varphi^2 \)) with a small operator \( \hat{L}_1 \):

\[
\left\{ \frac{\partial^2}{\partial \xi^2} + L_0 \right\} \varphi = \hat{L}_1 \varphi^2
\]

- 1 -
If we apply perturbation theory to obtain the solution we have

\[ \psi = \psi_0 + \psi_1 \]

where

\[ \psi_0 = |\psi_0| e^{-i\omega_0 t + ik \cdot r} \]

and

\[ \psi_1 = \sum_k \psi_k(t) e^{ik \cdot r} \]

\[ \left\{ \frac{\partial^2}{\partial t^2} + L_0 \right\} \psi_1 = L_0 |\psi_0|^2 e^{-2i\omega_0 t + 2ik \cdot r} \]

If the relation \( \omega^2 = L_0(k) \) has no dispersion \( (\omega/k = \text{constant}) \), then \( 2\omega_0 \) and \( 2k \) also satisfy it. Then the driving term on the r.h.s. resonates with a normal mode of the unperturbed system, causing (secular) growth. If, on the other hand, the relation \( \omega = \omega(k) \) is far from linear (the modes are characterized by large dispersion) then \( \psi_1 \) is driven only to finite amplitude. This is generally the case with plasma waves, in contrast with acoustic phenomena in fluid dynamics.

Qualitatively, we see that if we start with an initially monochromatic wave of frequency \( \omega \), a substantial amount of energy will be transferred to the \( 2\omega \) mode, a small amount to \( 3\omega \), and higher modes will be almost unexcited. This qualitative illustration demonstrates that periodic waves in a plasma, in the region of frequencies where there is an essential deviation from a linear law of dispersion, may propagate without considerable non-linear shape distortion. Moreover, such one-dimensional periodic waves are sometimes the exact steady-state non-linear solutions \([1]\). *

* This problem will be considered in a second set of lectures: "Collision-free Shocks in Plasma"
From these results we may draw the conclusion that nonlinear periodic one-dimensional waves in a plasma can exist for a long time, at least in a case of no damping. However, we must also ascertain if they are stable with regard to various random distortions. If they turn out to be unstable, this implies a transition of their energy to some other type of motion of the plasma, possibly to random turbulent motion.

We will illustrate the physical mechanism of one of these instabilities by means of the example of the simplest nonlinear plasma wave, that is, the flute oscillation (the Alfvén mode in the MHD limit) [2]. For this mode, nonlinear effects never change the wave shape even for the linear dispersion law $\omega = \kappa V_A$, because of transversal polarization.

We will take an initial undisturbed flow as a monochromatic wave described by

$$H_\perp(z, t) = 2 \delta H_\perp \cos (k_0 z - \omega_0 t) \ , \ \frac{\omega_0}{k_0} = V_A$$

$V$, $E$, etc. can be related to $H_\perp$ by the fluid and Maxwell equations, and so are also specified by implication. Of course, any other periodic dependence (e.g., exponential) on $(k_0 z - \omega_0 t)$ could have been taken, but this is convenient to work with.

In order to treat the stability of this wave as usual, we investigate the time dependence of the small perturbations. We write the fluid velocity $V$, field $H$ and number density $N$ in terms of perturbed and unperturbed quantities:

$$V = u + \delta V_\perp \cos (k_0 z - \omega_0 t) + \nu$$

$$H = H_0 + \delta H_\perp \cos (k_0 z - \omega_0 t) + h$$

$$N = N_0 + n$$

- 3 -
where small letters designate perturbed quantities and $U$ is the velocity with which the Alfvén wave propagates in the $z$-direction. For the sake of simplicity we will restrict ourselves to consider one-dimensional perturbations.

First we discuss the equation for perturbed longitudinal motion (i.e., $v^\|_n$). Time derivatives of perturbed quantities are treated by assuming dependence like $\exp(-i\omega t)$. The pressure term on the r.h.s. is rewritten using the adiabatic law connecting pressure and density, and the lowest order magnetic force term is retained:

$$-i\omega v^\|_n + U \frac{dv^\|_n}{dz} = -\frac{C_s^2}{N_0} \frac{dn}{dz} - \frac{1}{4\pi N_0 M} \frac{d}{dz} (H^\perp h^\perp)$$  \hspace{1cm} (1)

Here $C_s$ is the sound velocity, and $h^\perp$ is taken as a first-order quantity.

The linearized continuity equation is

$$-i\omega n + U \frac{dn}{dz} + \frac{d}{dz} (N_0 v^\|_n) = 0$$  \hspace{1cm} (2)

Eqs. (1) and (2) just describe the longitudinal behaviour of the system, and, with the deletion of the last term in (1), yield longitudinal sound waves.

For the third equation, we take the linearized induction equation

$$-i\omega h^\perp - \frac{d}{dz} (H_0 v^\perp_n) = -\frac{d}{dz} (U h^\perp) - v^\perp_n \frac{d}{dz} H^\perp$$  \hspace{1cm} (3)

which gives an equation generating $h^\perp$. The fourth equation describes the change of the transverse linearized velocity, driven by the magnetic force, expanded for small density perturbation

$$-i\omega v^\perp_n + U \frac{dv^\perp_n}{dz} = \frac{H_0}{4\pi \rho_0} \frac{dh^\perp}{dz} - \frac{H_0 n}{4\pi \rho_0 N_0} \frac{dH^\perp}{dz}$$  \hspace{1cm} (4)
Here \( \rho_0 = N_0 M \) is the unperturbed mass density and \( H_0 \) is the uniform magnetic field along which the Alfvén waves propagate. Eqs. (3) and (4) describe the transverse motion and if the last terms (the nonlinearity) are dropped, yield the dispersion relation for Alfvén waves.

Let us make an additional simplifying assumption, namely, \( \beta \equiv \frac{c_s^2}{\gamma - 1} \). We can estimate the effectiveness of the nonlinear terms in (3) and (4) by comparing them with linear terms in their respective equations.

\[
R_1 = \left[ \frac{\rho_0 H_0}{4\pi \rho_0^2} \frac{d}{d\xi} \frac{H_1}{\xi^2} \right] \left[ \frac{H_0}{4\pi \rho_0} \frac{d}{d\xi} \frac{h_1}{\xi^2} \right]^{-1} \sim \frac{\rho_0 H_1}{\rho_0 h_1}
\]

and

\[
R_2 = \left[ \frac{d}{d\xi} \left( \nu_0 H_1 \right) \right] \left[ \frac{d}{d\xi} \left( \nu_0 h_1 \right) \right]^{-1} \sim \frac{\nu_0 H_1}{\nu_0 h_1}
\]

From (1) we can estimate (neglecting the nonlinear term)

\[
\nu_0 \sim \frac{k_s c_s^2 \rho}{\rho_0 \omega_s} , \quad \omega_5 = \omega - k_s \nu_0
\]

\( \beta \ll 1 \) implies \( \nu_0 \gg c_s^2 \), or since \( \omega_s \approx k_s c_s \), \( \nu_0 \gg c_s^2 k_s / \omega_s \). But we see, on substituting for \( \nu_0 \), that this implies \( R_2 \ll R_1 \), thus the nonlinear term of (3) may be dropped, and we are left with a very symmetrical system of equations. They almost split into longitudinal and transverse systems, but do not because of the crossing terms in the equations for \( \nu_0 \) and \( \nu_1 \).

Let us calculate the effect of these crossing terms by means of perturbation theory[2]. As we know from quantum mechanics, the "energy shift" (properly, the frequency shift here) is calculated in first order by means of the diagonal matrix element

\[
\int \psi_0^* \delta V \psi_0 \, d\Omega
\]

If we take for \( \psi_0 \) either of the two eigenmodes of the unperturbed system, the \( z \)-integration gives zero.
because of the periodic behaviour of the perturbed "potential". If, however, we start with a pure state (consisting, in this case, of the Alfvén wave) and include a small amount of the other mode in the perturbation, the result in second order is

\[ \oint \psi^* \delta \psi \, dz \sim \oint e^{-ikz'} e^{\pm ikz} e^{ik \alpha z} \neq 0 \]

provided \( k_{\alpha} \pm k_0 - k_\alpha = 0 \). Thus, if there are waves in the excited spectrum which satisfy this selection rule, the matrix elements can be non-zero, in contrast to the non-degenerate situation.

In the wave frame, the argument of the cosine in \( H \) becomes simply \( k_0 z \), i.e., the frequency \( \omega_0 \) of the wave in its own rest frame vanishes, so that the selection rule

\[ -\omega_S \pm \omega_0 + \omega_A = 0 \]

holds trivially, stating that both modes propagate at the same frequency.

We can now write the \( z \)-dependence of all longitudinal perturbed quantities as \( e^{ik_\alpha z} \) and of all transverse perturbed quantities as \( e^{ik_\alpha z} \) and use the momentum selection rule (i.e., \( k_\alpha = k_0 + k_\alpha \)) to cancel this \( z \)-dependence everywhere. Now (1) to (4) take the form

\[ -i \omega v_n + i k_\alpha u v_n = -i k_\alpha C_n z / N_0 - i k_\alpha S H_0 h_0 \]

\[ -i \omega n + i k_\alpha u n + i k_\alpha N_0 v_n = 0 \]

\[ -i \omega h_\perp - i k_\alpha H_0 v_\perp + i k_\alpha u h_\perp = 0 \]
\[-i\omega \upsilon_1 + i k_A \upsilon_1 \upsilon_1 = \]
\[= \frac{i k_A H_0 \upsilon_1}{4\pi \rho_o} - \frac{nM H_0 i k_0 \delta H}{4\pi \rho_o^2}\]  \hspace{1cm} (8)

Note that in (5) and in (8) only one of the exponentials in \(\cos (k_0 z)\) satisfies the selection rule, and the other may thus be dropped. Combining (5) and (6) we get

\[v_h \left[ \omega - k_s u - \frac{k_s^2 c_s^2}{\omega - k_s u} \right] = \frac{k_s \delta H \cdot h_j}{4\pi \rho_o}\]  \hspace{1cm} (9)

Likewise, combining (7) and (8) and substituting for \(n\) from (6) we find

\[\frac{1}{K_A H_0} \left[ (\omega - k_A u)^2 - \frac{k_A^2 H^2}{4\pi \rho_o} \right] h \parallel = \frac{k_s \upsilon_h}{\omega - k_s u} \cdot \frac{k_0 H_0 \delta H}{4\pi \rho_o}\]  \hspace{1cm} (10)

Eqs. (9) and (10) without the crossing terms on the r.h.s. are the Doppler–shifted dispersion relations for sound and Alfvén waves, respectively. In order to discuss stability we put \(\omega + k_s A u = \omega_s A + i \nu\) \((\omega_s^2 = k_s^2 c_s^2, \omega_s^2 = k_A^2 V_A^2)\) and calculate the determinant of this system of equations. The result is

\[\gamma^2 = \frac{k_s^2 k_A \delta H \cdot \upsilon_A^2}{4\pi \rho_o \omega_s \omega_A}\]  \hspace{1cm} (11)

The selection rules used in Eqs. (5) to (8) are

\[\omega_1 = \omega_o + \omega_2, \quad \omega_o = k_0 u\] \hspace{1cm} (12)
\[k_1 = k_o + k_2\] \hspace{1cm} (13)
Because of the form of the dispersion relations, all terms in the above equations can be written with either plus or minus sign. For convenience, let us specify that $\omega = \kappa_0 V_A > 0$. From the assumption $S \ll 1$, it follows that $C_s \ll V_A$. Now, if $\omega_A = \kappa_A V_A$, (12) and (13) imply that $K_s = 0$; therefore, we must have $\omega_A = -\kappa_A V_A$. Then $K_A \approx -K_0$ and $K_s \approx -2\kappa_0 < 0$. Thus, Eq. (11) implies $\gamma^2 > 0$ (instability) only for $\omega_s < 0$. The energy of a "quantum" of the initial wave must be larger than the energy of a "quantum" of the perturbed waves (i.e., $\omega_e > \omega_A, |\omega_s|^{(4)}$).

2. Interaction of the three waves of finite amplitude

The problem of the stability of Alfvén waves of small amplitude was considered in Section 1. For disturbances in the form of a sum of sound and Alfvén waves, we can write two coupled equations for the perturbed longitudinal velocity and transverse magnetic field. These equations describe the coupling between sound and Alfvén waves. We have found the frequency rule for the decay instability, which corresponds to the usual energy conservation law of quantum mechanics. Therefore, it can be seen more directly if we write the wave equations in the Hamiltonian form using quantum parameters, such as the energy of a wave quantum and the number of waves.

To do this, let us rewrite the wave equations. We will replace $V_{h1}$ and $h_{1l}(t)$ by $[V_{h1}(t), h_{1l}(t)] exp (-i\omega t + ikz)$ where $V_{h1}(t)$ and $h_{1l}(t)$ are slowly varying and $\omega$ and $k$ satisfy the sound dispersion relation in the longitudinal equation and the Alfvén dispersion relation in the transverse equation. If we go to the laboratory coordinate system and label transverse quantities with an index (2) and longitudinal ones with an index (1),

$$\omega_1 = \omega - \kappa_A u_1$$
$$\kappa_A = \kappa_1$$
$$\omega_1^2 = \kappa_1^2 V_A^2$$
$$\omega_2 = \omega - \kappa_A u_2$$
$$\kappa_2 = \kappa_2$$
$$\omega_2^2 = \kappa_2^2 C_s^2$$
$$v_2 = \omega_1$$
$$h_2 = h_{1l}$$
Eqs. (9) and (10) become

\[ i \frac{\partial \psi}{\partial t} = -\frac{k_2 \delta H}{4\pi \rho_0} h_x \, e^{-i(\omega_1 - \omega_2 - \omega_2) t} \quad (14) \]

\[ i \frac{\partial h_x}{\partial t} = \frac{\kappa \kappa_2 \omega_2^2 \delta H}{\omega_1 \omega_2} v_x \, e^{-i(\omega_1 - \omega_1 + \omega_2) t} \quad (15) \]

where we have now taken into account the phase of the finite amplitude Alfvén wave \( \delta H = |\delta H| e^{i\varphi} \). Let us define the number of quanta \( n_k \) as the total energy in a mode divided by the frequency of the mode.

\[ n_k = \omega_k^{-1} \left\{ \frac{\rho_0 v_x^2}{2} + \frac{H_k + E_k^2}{8\pi} + \frac{\rho_2^2 C_x^2}{2 \rho_0^2} \right\} \quad (16) \]

Then we can write Eqs. (14) and (15) in a symmetrical form by introducing probability amplitudes (i.e., \( |C_j|^2 = n_j \)). For the present case these amplitudes are the following:

\[ C_0(t) = \frac{\delta H}{|4\pi k \omega_0|} \quad , \quad C_1(t) = \frac{h_1}{|4\pi k \omega_1|} \quad , \quad C_2(t) = \frac{v_x}{|1 \omega_2| / \rho_0} \quad (17) \]

In terms of these variables, Eqs. (14) and (15) can be written in a form similar to the Schrödinger equation in the interaction representation

\[ i \frac{\partial C}{\partial t} = V_{\kappa_1, \kappa_2} C_0 C_2 \quad (18) \]
\[
\frac{\partial C_i}{\partial t} = V_{k_2, \kappa, k_1} C_i^* C_i
\]

(19)

where

\[
V_{k_2, \kappa, k_1} = V_{k_1, \kappa, k_2} \text{Sign}(\omega_1, \omega_2) \equiv -\sqrt{1/\rho \epsilon} \text{Sign} k_2
\]

\[
\omega_1 = \omega_0 + \omega_2
\]

We see that the "matrix elements" of the interaction operators differ only by a sign for the two modes. This is due to the Hamiltonian form of the hydromagnetic equations. Starting from this point we would like to introduce an important generalization: For any kind of plasma waves (not only for Alfvén ones), if it is possible to represent three-wave interaction in a Hamiltonian form (as was done here for MHD Alfvén waves) the same symmetry rules should exist also for decay-type instability. Actually, this holds for any fluid*approximation. Moreover, in such arbitrary cases one may expect to have for the probability amplitude variables the same type of equations as (18) and (19). Of course, the matrix elements and normalizations should be specified in a different way for the various cases. Besides, in each case it is necessary to fulfill the "decay" conditions (12) and (13).

In Fig. 2 various forms of the spectra are represented for clarity. As it is easy to demonstrate, "decays" of the oscillations may occur for spectra 1 and 4. Oscillations having spectra similar to spectra 2 and 3 are stable with relationship to "decay". However, in the presence of

* Including multi-fluid cases.
several branches in the spectrum of the oscillations, that is, oscillations characterized by the spectra similar to spectra 2, may be unstable with relationship to the "decays" to oscillations out of which at least one does not belong to the given branch. More accurately, "decays" are possible when we may draw a curve similar to either curve 1 or curve 3 through the three points corresponding to oscillations $\omega_0, \omega_4, \omega_2$ (these three points, generally speaking, may lie on different branches). (For different branches of the oscillations, however, "prohibitions" associated with the polarization of the waves may arise.) The fulfilment of "decay" conditions, however, in itself still does not signify instability.

By solving Eqs. (18) and (19) we find an exponential behaviour (i.e., $|\mathbf{C}_j|^2 = \exp[2\gamma t]$). In the general case the growth rate of the perturbing waves is given by

$$\gamma^2 = - |V_{k_4, k_0, k_2}|^2 \text{Sign}(\omega_4, \omega_2) |C_0|^2$$

From this expression we find that the decay-type instability occurs only when the frequencies of the perturbing waves are both smaller than the large amplitude waves\textsuperscript{[4]}, i.e., $|\omega_4|, |\omega_2| < |\omega_0|$. In the case when the resonance conditions for the frequencies and wave vectors cannot be satisfied between three waves only, we may include in our consideration the fourth wave. In order to have the finite growth rate, this fourth wave must obviously be the finite amplitude wave. Therefore, we must, in fact, consider the stability of the second harmonic of finite amplitude waves. The resonance conditions have now the form

$$2\omega(k_0) = \omega(k_4) + \omega(k_2)$$

$$2k_0 = k_4 + k_2$$

Using the decay rule (i.e., $2|\omega(k_0)| > |\omega(k_4)|, |\omega(k_2)|$) and these resonance conditions, it may be shown that in the second order the spectra 3 of Fig. 2 is unstable and spectra 2 is stable. The instability of the gravity waves on the ocean surface and ion sound waves in the second order was considered as an example of this kind of instability\textsuperscript{[5]}. We can expect that the diagram of the unstable regions in the frequency wave amplitude plane qualitatively looks like the same kind of diagram as for the parametric resonance problem (Fig. 3).
The width of the unstable region near the nth harmonic is of the order of the growth rate and proportional to the nth power of the amplitude. Of course, we must satisfy the decay conditions

\[ n \omega(K_0) = \omega(K_1) + \omega(K_2) \]

\[ n K_0 = K_1 + K_2 \]

One of the interesting examples of higher-order decay of this type is the instability of the periodical Alfvén waves with a saw-toothed wave form of a period \( T \) with respect to perturbations with \( \text{max} \{ \omega_1, \omega_2 \} \geq \gamma / \alpha \) frequencies. Another question is the relaxation of the initially steady waves under the influence of the disturbances. To deal with this question we must consider the case of finite amplitudes of the disturbances and the nonlinear terms in the equation for decaying, say, Alfvén waves must be taken into account. When \( C_0, C_1 \), and \( C_2 \) are all of the same order, one would expect to find the following equation:

\[ i \frac{\partial c_0}{\partial t} = \sqrt{K_0, K_1, K_2} C_1 C_2^* \]  \hspace{1cm} (20)

One can derive Eq. (20) from the same MHD equation as we have used to derive Eq. (18).
Let

\[ c_j(t) = a_j(t) e^{i \phi_j(t)} \], \quad \text{Im} a_j = \text{Im} \phi_j = 0 \quad (21) \]

In the case where \( \omega_o > \omega_1 > |\omega_2| \) and \( \omega_2 < 0 \), the symmetry rules of the \( V \)'s imply

\[
V_{k_1, k_0, k_2} = -Jk \\
V_{k_2, k_0, k_1} = V_{k_1, k_0, k_2} \text{sign} (\omega_2, \omega_2) = Jk \\
V_{k_0, k_2, k_1} = V_{k_1, k_2, k_0} \text{sign} (\omega_0, \omega_1) = -Jk
\]

Taking the real and imaginary parts of (18)-(20) and using the variables \( a_j(t) \) and \( \Theta = \phi_1 - \phi_2 - \phi_0 \) gives us the equations \([7]\)

\[
\frac{\partial a_1}{\partial t} = Jk a_0 a_2 \sin \Theta \\
\frac{\partial a_2}{\partial t} = Jk a_0 a_1 \sin \Theta \\
\frac{\partial a_0}{\partial t} = -Jk a_1 a_2 \sin \Theta \\
\frac{\partial \Theta}{\partial t} = Jk \left( \frac{a_0 a_2}{a_1} + \frac{a_0 a_1}{a_2} - \frac{a_1 a_2}{a_0} \right) \cos \Theta = \text{ctg} \Theta \frac{\partial}{\partial t} \left( a_0 a_1 a_2 \right)
\]

By integrating the last of these equations, we find

\[
a_0 a_1 a_2 \cos \Theta = \Gamma = \text{const.} \quad (23)
\]
By using the frequency rule, we can easily find a first integral of the remaining equations

$$a_1^2 \omega_1 + a_2^2 \omega_2 + a_0^2 \omega_0 = \text{const.}$$

(24)

By integrating $a_0 \frac{\partial a_0}{\partial t} + a_1 \frac{\partial a_1}{\partial t}$ we can also find the following constants of the motion:

$$m_1 \equiv n_0 + n_1 = \text{const.}$$

$$m_2 \equiv n_0 + n_2 = \text{const.}$$

$$m_0 \equiv n_1 - n_2 = \text{const.}$$

(25)

They are well known in the theory of the so-called "parametric amplifiers" as the vectorial Manley-Rowe relations taken in the direction of propagation, and may be understood in terms of the diagram for the three-wave decay process

by saying that when a quantum disappears from the $\omega_{k_0}$ mode, quanta appear in each of $\omega_{k_1}$ and $\omega_{k_2}$, so $\Delta n_0 = -1$, $\Delta n_1 = 1 = \Delta n_2$. By combining the relations (23) and (25) we have

$$\frac{d}{dt} n_0 = 2 \delta k [n_0 (m_1 - n_0) (m_2 - n_0) - r^2]^{1/2}$$

(26)

The three roots of

$$n_0 (m_1 - n_0) (m_2 - n_0) - r^2 = 0$$

are labelled as

$$n_c = n_\delta \geq n_a > 0$$
Eq. (26) can be transformed into

\[ H(t-t_o) = \frac{1}{2} \int \frac{dn_o}{\sqrt{(n_o-n_c)(n_o-n_l)(n_o-n_e)}} \frac{n_o(t)}{n_o(t_o)} \]

This can be transformed to an elliptic integral by the change of variables

\[ y(t) = \sqrt{\frac{n_o(t)-n_a}{n_b-n_a}} \quad , \quad \gamma = \sqrt{\frac{n_c-n_b}{n_c-n_a}} \]

If we define the time \( t_o \) in such a way that \( y(t_o) = 0 \), then

\[ H(t-t_o) \sqrt{n_c-n_a} = \int\frac{dy}{(1-y^2)(1-\gamma^2y^2)} \]  

Therefore,

\[ y(t) = sn \left( H \sqrt{n_c-n_a} (t-t_o), \gamma \right) \]

and, by definition of \( y \), we have the general solution

\[ n_o(t) = n_c + (n_b-n_a) sn^2 \left( H \sqrt{n_c-n_a} (t-t_o), \gamma \right) \]

Let us consider two simple cases[7]:

\[ -15 - \]
Case A: At time $t = 0$, $n_0(t) = 0$, $n_1(t) > n_2(t)$. Without loss of generality we can put $t = 0$ in Eq. (26). Then the three roots of Eq. (26) are simply

$$m_1 = n_1(t) = n_c \gg m_2 = n_2(t) = n_b > 0 = n_a$$

We can simplify the solution by neglecting $y^2 y^2$ in Eq. (28), since in this case $y \ll 1$. Thus

$$n_0(t) = n_2(t) \sin^2 \left( \sqrt{\ell t} \sqrt{n_1(t)} \right)$$

$$n_1(t) = n_1(t) - n_2(t) \sin^2 \left( \sqrt{\ell t} \sqrt{n_1(t)} \right)$$

$$n_2(t) = n_2(t) \cos^2 \left( \sqrt{\ell t} \sqrt{n_1(t)} \right)$$

The time variation of the occupation number is shown in Fig. 4.

![Figure 4](image_url)

The situation described by this figure is the case in which the frequency of the finite amplitude wave is smaller than the frequency of the perturbations, and, hence, it is stable to the decay-type instability. The small periodic variation in its amplitude is due to the small amount of
energy initially in the perturbations.

Case B: Let us now consider the decay of finite amplitude waves, when at time $t=0$, $n_z(0) = 0$, $n_o(0) > n_i(0) > 0$. Putting $P = 0$ again, we find the constants

$$n_c = m_4 = n_o(0) + n_i(0) > n_b = m_2 = n_o(0) > n_a = 0.$$ 

Thus from (29) and (25) we obtain

$$n_o(t) = n_o(0) \sin^2 \left[ \mathcal{H} (t-t_0) \sqrt{n_c} , \gamma \right]$$

$$n_i(t) = n_i(0) + n_o(0) \left( 1 - \sin^2 \left[ \mathcal{H} (t-t_0) \sqrt{n_c} , \gamma \right] \right)$$

$$n_2(t) = n_o(0) \left( 1 - \sin^2 \left[ \mathcal{H} (t-t_0) \sqrt{n_c} , \gamma \right] \right)$$ 

where

$$1 - \gamma^2 = \frac{n_i(0)}{n_o(0) + n_i(0)} \ll 1$$

Thus

$$1 - \sin^2 \left[ \mathcal{H} (t-t_0) \sqrt{n_c} , \gamma \right] = 0$$

Therefore, $\mathcal{H} t_0 \sqrt{n_c}$ is $\frac{1}{2}$ of the period of $\sin$, 

$$\mathcal{H} t_0 \sqrt{n_c} = K(\gamma)$$

With $\gamma' = \sqrt{1 - \gamma^2}$, we have
\( K(y) = - \left[ (y'_{2})^2 + O(y^{4}) \right] + \ln \left( \frac{2}{y'} \right) \left[ 1 + (y'_{2})^2 + O(y^{4}) \right] \) (33)

Therefore, from (31)-(33)

\[
t_0 \approx \frac{1}{2 \pi \sqrt{n_c}} \ln \frac{n(\infty)}{n(0)}
\]

(34)

During this time the amplitude of the initial steady wave \( n_0 \) decreases so we can say that it is the time of decay. It differs from the linear growth rate \( \gamma = \omega \sqrt{n_c} \) only by logarithm factor. This factor we need when the amplitudes of disturbances are comparable with the amplitude of the initial wave.

The behaviour of the relative number of wave quanta are shown in Fig. 5.
It turns out that the amplitude of the two perturbing modes for long times drop down to zero again. Thus, the "decay" found in this problem is not really "irreversible".

This type of problem was considered first in 1954 by Pierce [8]. He discussed the use of such resonant three-wave interaction in high frequency electronics. Now it is a highly developed part of radiofrequency electronics where these nonlinear methods are used for the production of mixing frequencies and parametric amplification.

Another illustration of these methods is nonlinear optics. Here the nonlinearity comes usually in terms of the current dependence on electric field [7].

\[ j \sim \alpha E + \beta E^2 + \gamma E^3 + \ldots \]

Also, all optical media have dispersive properties.

In the simplest approximation the dielectric function is given in terms of the resonant frequencies — at least far from these resonances — by

\[ \varepsilon = \sum \alpha_i \left( 1 - \frac{\omega_{pi}^2}{\omega^2 - \omega_i^2} \right) \]

Such a dispersion makes it possible to satisfy the resonance relations \( \omega_0 = \omega_1 + \omega_2 \), \( \kappa_0 = \kappa_1 + \kappa_2 \). Thus, if a ruby laser beam is transmitted through quartz, a small amount of the "second harmonic" is generated. However, if two lasers are passed through quartz at the appropriate angle with respect to one another, in order to satisfy the conditions \( \omega_1 + \omega_2 = \omega_3 \), \( \kappa_1 + \kappa_2 = \kappa_3 \), a considerable amount of energy can be converted into harmonics. In general, in nonlinear optics there is a fairly high variety of different kinds of nonlinear effects. As an example of another sort, one might mention the effects which are seen by reflecting laser beams off the surface of a metal, and so on.
3. Many-wave interaction in random phase approximation

In plasmas, we do not deal very often with highly ordered phenomena since many modes are available to satisfy the conservation relations.

Instead of the Manley-Rowe type of behaviour (Fig. 6a)

there are many modes present, and the recurrent character of the evolution is destroyed (for incommensurable frequencies) (see Fig. 6b)

We can treat this situation using the random phase approximation (RPA) and look at the modulus of the amplitude, averaging over phases. Now there is no way to distinguish the "main wave" from the others, and this is reflected in the notation in the equation for the wave amplitude

\[ i \frac{d\phi(t)}{dt} = \sum_{\kappa'} V_{\kappa, \kappa', \kappa-\kappa'} C_{\kappa'}(t) C_{\kappa-\kappa'}(t) e^{-i(\omega_{\kappa} + \omega_{\kappa'} - \omega_{\kappa'})t} \]

(35)
With a particular choice of normalization the amplitudes $C_k$ play the role of the probability amplitude; so, the mode occupation number is equal to the square of this amplitude:

$$N_k = |C_k|^2$$  \hspace{1cm} (36)$$

Within this normalization the "matrix elements" have the following symmetry properties (compare with the ones for the interaction of the Alfvén and sound waves):

$$V_{\bar{r},r',\bar{r}-\bar{r}'} = V_{\bar{r},\bar{r}-r',r} \quad \text{sign} \quad \omega_{\bar{r}} - \omega_{\bar{r}'}$$  \hspace{1cm} (37)$$

Since we are interested in the time variation of the occupation number $N_k$ only for the eigenoscillations, we can involve only them in this consideration. Then the equation for the amplitudes of the eigenoscillations can be treated with the methods of time-dependent perturbation theory from quantum mechanics \[^4\]. On this basis we expand $C_k(t)$ in powers of the interaction $V_{\bar{r},r',\bar{r}-\bar{r}'}$ as

$$C_k(t) = C_k^{(0)} + C_k^{(1)} + C_k^{(2)} + \cdots$$

and calculate the change in the occupation number after time $\tau$, 

$$|C_k(t)|^2 = |C_k^{(0)}|^2 \quad \text{From (35) we have}$$

$$= 21$$
The quantity $C^{(o)}_{\mathbf{\kappa}}$ is time-independent and corresponds to the solution in the absence of interaction between modes. It can be written as a positive amplitude times a phase factor of the form $e^{i\phi_{\mathbf{\kappa}}}$.

We postulate that although the $\phi_{\mathbf{\kappa}}$ are fixed by initial conditions at time $t = 0$, the correlation between the phases $C_{\mathbf{\kappa}}$ vanishes completely in a time small compared with the time required for the change in $|C_{\mathbf{\kappa}}|^2$ owing to the non-linear interaction between modes. Therefore, there is no correlation at all between $\phi_{\mathbf{\kappa}}$ and $\phi_{\mathbf{\kappa}'}$ unless they correspond to the same mode, i.e., we can average over an ensemble of states with completely chaotic phases in the wave amplitudes. Generally, application of the random phase average (indicated by a bar over the quantity being averaged) takes the form of an identity such as

$$
\delta_{\mathbf{\kappa}, \mathbf{\kappa}'} = \left\{ \begin{array}{ll}
1 & , \mathbf{\kappa} = \mathbf{\kappa}' \\
0 & , \mathbf{\kappa} \neq \mathbf{\kappa}'
\end{array} \right.
$$

(38)
Thus, the lowest order

\[
\frac{|C_\alpha(t)|^2}{|C_\alpha(t)|^2 - |C_\alpha(o)|^2} = \frac{|C_\alpha(\omega)|^2}{|C_\alpha(t)|^2} + \frac{C_\alpha^{(1)} C_\alpha^{(2)*}}{C_\alpha^{(1)}} + \frac{C_\alpha^{(3)} C_\alpha^{(4)*}}{C_\alpha^{(3)}}
\]  

(40)

Substituting (38) in (39) we obtain

\[
\left| \frac{C_\alpha(t)}{C_\alpha(t) - C_\alpha(o)} \right|^2 = \sum \left\{ \frac{C_\alpha^{(o)} C_\alpha^{(o)} C_\alpha^{(o)*} C_\alpha^{(o)*}}{C_\alpha^{(1)}} \right\} \int_0^t V_{\alpha, \alpha, \alpha} \left( t' \right) dt' \int_0^t V_{\alpha, \alpha, \alpha} \left( t'' \right) dt'' -
\]

\[
- \text{Re} \left[ \frac{C_\alpha^{(o)} C_\alpha^{(o)} C_\alpha^{(o)} C_\alpha^{(o)*}}{C_\alpha^{(1)}} \right] \int_0^t V_{\alpha, \alpha, \alpha} \left( t' \right) dt' \int_0^{t'} V_{\alpha, \alpha, \alpha} \left( t'' \right) dt'' -
\]

\[
- \text{Re} \left[ \frac{C_\alpha^{(o)} C_\alpha^{(o)} C_\alpha^{(o)} C_\alpha^{(o)*}}{C_\alpha^{(1)}} \right] \int_0^t V_{\alpha, \alpha, \alpha} \left( t' \right) dt' \int_0^{t'} V_{\alpha, \alpha, \alpha} \left( t'' \right) dt'' \}
\]

(41)

The product of four $C_\alpha^{(o)}$ on the r.h.s., after averaging over chaotic phases in the wave amplitude, can be reduced to the product of two occupation numbers. The two possibilities of combination of $C_\alpha^{(o)}$ are shown in (41) by dashed and straight lines. In the first term the $C_\alpha^{(o)}$ combine as $|C_\alpha^{(o)}|^2 / |C_\alpha^{(o)}|^2 = N_\alpha / N_\alpha^{(o)}$ or in the last two terms $|C_\alpha^{(o)}|^2 / |C_\alpha^{(o)}|^2$ or $|C_\alpha^{(o)}|^2 / |C_\alpha^{(o)}|^2$.
Application of the symmetry relations among the matrix elements shows that the product of two of them can always be written as \( \sum \mathcal{V}_{\vec{r}', \vec{r}''}(t) dt \) times a sign depending on the signs of \( \omega_{\vec{r}}, \omega_{\vec{r}'}, \omega_{\vec{r}''} \).

For the times longer than a wave period, the time integrals can be evaluated approximately as

\[
\frac{4 \pi r^2}{2} \frac{(\omega_{\vec{r}} - \omega_{\vec{r}'} - \omega_{\vec{r}'}) t}{(\omega_{\vec{r}} - \omega_{\vec{r}'} - \omega_{\vec{r}'})^2} \left| \mathcal{V}_{\vec{r}, \vec{r}', \vec{r}''}(t) \right|^2.
\]

Thus, the change in occupation number is proportional to time; for times shorter than the characteristic time required for \( N_{\vec{r}} \) to change, \( \frac{d N_{\vec{r}}}{dt} \) is approximately. In this manner, we obtain the kinetic equations \([10, 11, 14]\)

\[
\frac{d N_{\vec{r}}}{dt} = 4 \pi \sum_{\vec{r}', \vec{r}''} \left| \mathcal{V}_{\vec{r}, \vec{r}', \vec{r}''}(t) \right|^2 \left\{ N_{\vec{r}', \vec{r}''} N_{\vec{r}''} - \text{sign}(\omega_{\vec{r}} \omega_{\vec{r}'}) N_{\vec{r}} N_{\vec{r}'}, \right. \\
- \left. \text{sign}(\omega_{\vec{r}} \omega_{\vec{r}'}) N_{\vec{r}} N_{\vec{r}''} \right\} \delta \left( \omega_{\vec{r}} - \omega_{\vec{r}'} - \omega_{\vec{r}''} \right) \delta_{\vec{r}, \vec{r}' \vec{r}''}.
\]

(42)

We can go from Eq. (35) directly to this result, written in terms of positive frequencies only, by using time-dependent perturbation theory and the "golden rule". Really, let us consider the part of the nonlinear processes with \( \omega_{\vec{r}} > \omega_{\vec{r}''}, \omega_{\vec{r}''} > 0 \), which corresponds to decay process of the high frequency mode \( \omega_{\vec{r}} \) and the combination process of the low frequency modes \( \omega_{\vec{r}'}, \omega_{\vec{r}''} \). Then we can write the changing of the occupation number \( N_{\vec{r}''} \) due to these processes.
In the classical limit, \( N_R \gg 1 \), we recover the part of the previous result. From this derivation it follows that the "collisional term" for the four-wave interaction is proportional to the third power of the occupation number. In the absence of the resonance interaction between three waves (as, for example, for the electron plasma waves [12] or shallow water waves [13]) the mode coupling then is the third order in the wave energy.

The kinetic equation in the form (43) is used in the physics of solids for the description of the phonon interaction due to lattice irregularities, which can be important (e.g., heat transfer) [9]. But there is a qualitative difference between applications of this equation to the phonon and plasma turbulence. In solids we usually deal with a state not far from thermal equilibrium. Thus, nonlinear phenomena produce small corrections only. On the contrary, in a plasma they are very important. The mean free path for a wave in a turbulent plasma can be quite short and a behaviour like that described in the Rayleigh-Jeans law does not hold for plasma. As to particles, we can follow the familiar treatment of the Boltzmann equation by expanding around a Maxwellian.

In concluding this section we would like to make the following notation. In the previous considerations we used the time-dependent perturbation theory from quantum mechanics, which operates in terms of the eigenoscillations only. But very often it is more convenient to use the Fourier transform for the description of turbulent plasma motion. Then we cannot exclude from our consideration the beats with the frequency different from the eigenfrequency. The equation for the wave disturbance,
where $V^\star_{\gamma, K', K}$ is the operator with respect to the time variable, can be written now in the Fourier representation in a more general form:

\[ \langle \omega - \omega K, -i \gamma K \rangle C_{\gamma, \omega} = \sum_{K', \omega'} V_{\gamma, K', K, K'} (\omega, \omega', \omega - \omega') C_{\gamma, \omega'} C_{\gamma - K', \omega - \omega'} \tag{44} \]

where $\gamma K$ is the growth rate of the eigenoscillations, and $C_{\gamma, \omega}$ are the Fourier transform of $B_{\gamma}(t)$. The application of the Fourier transform is rigorous for the stationary amplitude of the disturbances $\text{Im} \omega = 0$.

We will work within the same normalization of the amplitude $C_{\gamma, \omega}$ as before; so the matrix elements have the same properties of symmetry as (37) if we put all frequencies equal to the eigenfrequencies.

In the first approximation we omit the terms of order $\sqrt{\gamma K} \sim (\gamma K)^{1/2} C_{\gamma, \omega}$ and obtain from Eq. (44) the usual result:

\[ C^{(o)}_{\gamma, \omega} = C_{\gamma, \omega} \delta_{\omega, \omega} \quad , \quad \frac{\langle C^{(o)}_{\gamma, \omega}, C^{(o)}_{\gamma, \omega'} \rangle}{\omega - \omega' + i \gamma} = N_{\gamma} \delta_{\gamma, \gamma'} \delta_{\omega, \omega'} \tag{45} \]

We have again supposed that the phases of $C^{(o)}_{\gamma, \omega}$ are completely chaotic. Then, multiplying Eq. (44) by $C_{\gamma, \omega}$ and averaging the result over the chaotic phases, we obtain that the nonlinear term cancels in the first approximation (45). Therefore, we must take into account the beats with the amplitude

\[ C^{(1)}_{\gamma, \omega} = \sum_{K', \omega'} \frac{V_{\gamma, K', K - K'} (\omega, \omega', \omega - \omega')}{\omega - \omega' + i \gamma} C^{(o)}_{\gamma, \omega} C^{(o)}_{\gamma - K', \omega - \omega'} \tag{46} \]

Here we suppose that the beats damp very fast, so the damping rate is much higher than the growth rate of the eigenoscillations. We need
this condition in the Fourier representation in order to get the same
result as in the time-dependent perturbation theory and we will use it
in the consideration of the wave interaction in the kinetic description
of the collisionless plasma. Substituting (46) in (44) we obtain the
stationary state equation very similar to Eq. (42) [14]

\[ \gamma \zeta N_{k^*} = -2 \sum_{k', \omega} \left| \frac{V_{k^*, k^*, -k^*}}{\omega - \omega_{k^*} - i (\nu - k^*)} \right|^2 N_{k^*} N_{k'^*} \]

Using the definition

\[ \lim_{\nu \to 0} \Im \frac{f}{\omega - \omega_{k^*} + i \nu} = -\pi \delta(\omega - \omega_{k^*}) \text{sign} \nu \]

and the symmetry properties (37) we can reduce this equation to
Eq. (42) only if the beats damp rapidly enough (\( \nu - k^* < 0 \)) [15].
4. Plasma turbulence in terms of the kinetic equation for waves

In the previous section we derived a kinetic equation for waves using the random phase approximation (RPA). Now we will investigate some of the properties of this equation, and how to apply it to several problems.

Investigation of plasma stability in magnetic confinement shows that often, under the influence of various small disturbances, the plasma arrives at a state of disordered motion. In general we must describe this motion by means of all of the plasma characteristics: velocity, temperature, etc. (in the fluid-type description of the plasma) at each point in space and time. If the deviation from steady state is slight (or the total turbulent energy is small), we can represent this turbulent motion by a set of collective oscillations in a linear approximation.

\[ u(\mathbf{r}, t) = \sum \mathbf{\nu}_k e^{-i\omega t + i \mathbf{k} \cdot \mathbf{r}}, \text{ etc.} \]

where the frequencies are determined from the dispersion relation

\[ \omega = \omega(\mathbf{k}) \]

Now the state may be described in terms of the amplitudes of the eigen-modes. The problem is to determine these amplitudes in phase space as a function of \( \mathbf{r} \).

The distribution of energy between different scales of turbulence can be found on the basis of wave-wave interactions. These interactions are easily treated within the framework of the RPA, which is probably valid for the case of many interaction waves. This kind of approach was called "weak turbulence theory".

If we wish, we can write the results of this treatment either in terms of mode energy or the number of waves \( n_k \) in the \( k \)th mode. In the latter representation, the wave kinetic equation becomes

\[
\frac{\partial n_k}{\partial t} = 2 \gamma_k n_k + \text{St} \{ n_k, n_k \}
\]

where the form of the collision term, \( \text{St} \{ n_k, n_k \} \), was derived in the previous lecture. In problems of nonlinear stability, it is
necessary to consider the sources of instability and dissipation. This accounts for the inclusion of the first term on the r.h.s. In steady state, we can drop $\frac{\partial n_e}{\partial t}$ and just consider the r.h.s. of Eq. (49) equated to zero.

Let us begin a preliminary examination of the relationship of weak plasma turbulence to Kolmogoroff's kind of treatment in the conventional hydrodynamic turbulence. It is very difficult to find a rigorous description of the energy transfer among different scales of hydrodynamic turbulence, because we cannot express "strong turbulence" in terms of eigenoscillations (waves) and we have no rigorous statistical equivalent of the equation corresponding to the wave kinetic equation above. In conventional hydrodynamics the most reliable procedure is the use of dimensional arguments.

\[ E(k) \sim k^{-5/3} \]  
\text{(Kolmogoroff-Obukhov law)}

where $E_k$ is the energy per wave number per unit volume.
In making this it was assumed that the turbulence is isotropic and that it may be described in terms of local characteristics. Energy is transferred from small $k$ to large in a resonant manner, passing through each decreasing scale in succession (see, for example, Ref. (16)).

It is interesting to find an analogy with Kolmogoroff's spectrum for a weakly turbulent plasma since in this case we have the equation for the spectral density derived from the previous principles. In the plasma, however, we can imagine a great variety of different waves, so probably there is no universal-type spectrum with some power dependence $\Pi k \sim k^{-5}$. However, for any given type of plasma waves one hopes to have a simple spectrum.

Let us consider an idealized model investigated by ZAKHAROV [17].

We will deal with the nonlinear dispersive (and therefore, plasma-like) media described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} - (\nabla^2 - \epsilon \nabla^2 \nabla^2) u = \nabla^2 u^2 \quad (\epsilon > 0) \quad (50)$$

where $\nabla^2$ is the Laplacian.

When we Fourier-transform, we find

$$\frac{\partial^2 \tilde{u}_{k}}{\partial t^2} - \omega^2_{k} \tilde{u}_{k} = \frac{k^2}{\omega^2} \int d \mathbf{k}' \tilde{u}_{k'} \tilde{u}_{k-k'} \quad (51)$$

with

$$\omega^2_{k} = k^2 + \epsilon k^4 \quad (52)$$

which corresponds to a decay-type spectrum. We can introduce the mode amplitudes $C_{k}$ defined by

$$u_{k}(t) = C_{k}(t) \frac{2 \omega_{k}}{\sqrt{\omega_{k}}} e^{-i \omega_{k} t} \quad (53)$$

In terms of $C_{k}$ we have Eq. (51) in a canonical form:
Let us note here that the wave kinetic equation was derived in Section 3 for the case of a set of discrete nodes. In the continuum limit, \( V \sim \frac{1}{(\kappa^3 c_{\kappa} t)} \) (instead of \( \frac{1}{(c_{\kappa} t)} \)) and the kinetic equation seems to lack a dimensionality factor of \( \kappa^3 \) in the r.h.s. Therefore, in the derivation, instead of \( \delta_{\vec{r}_x} \), we will have \( \delta \)-functions: \( \delta_{\vec{r}_x} \rightarrow \frac{1}{V} \delta(\vec{r} - \vec{r}_x) \), where \( V \) is a normalization volume.

Thus we should really define \( n_{\vec{r}} = \frac{1}{V} |\vec{c}_{\vec{r}}|^2 \), whereupon everything is correct.

We will take \( \omega = \kappa \) for all modes (i.e., \( \epsilon \ll 0 \)) except in evaluating resonant denominators; this imposes a restriction on the magnitude of the amplitudes for which the analysis applies, namely, \( |c_{\vec{r}}| \ll 4 \epsilon \kappa \). All modes would interact strongly if \( \epsilon = 0 \) exactly.

In terms of \( n_{\vec{r}} \) we have

\[
\frac{\partial n_{\vec{r}}}{\partial t} = 2 \chi \kappa n_{\vec{r}} - 4\pi \int \frac{k^2 k'^2 k''^2}{\omega_k \omega_{k'} \omega_{k''}} \left\{ \delta(\omega_k - \omega_{k'} - \omega_{k''}) \cdot \delta(\vec{r} - \vec{r}' - \vec{r}'') \left[ n_{\vec{r}'} n_{\vec{r}''} - 2 n_{\vec{r}} n_{\vec{r}'} \right] + 2 \delta(\omega_k + \omega_{k'} - \omega_{k''}) \cdot \delta(\vec{r} + \vec{r}' - \vec{r}'') \left[ n_{\vec{r}} n_{\vec{r}'} + n_{\vec{r}''} n_{\vec{r}'} - n_{\vec{r}} n_{\vec{r}'} \right] \right\} d^3 \vec{r}' d^3 \vec{r}''
\]

(55)
To avoid difficulty with signs, we have legislated all frequencies to be positive \( \omega_k = k \equiv \mid k \mid \). The term \( 2y \cdot \kappa \) has been added to represent a source and sink of wave energy.

Since we are interested in isotropic solutions, we will average the wave equation over angles. First we integrate out the \( k' \) dependence, replacing \( k \) by \( \pm k - k' \) according to the frequency rule. Then we write

\[
d^3 k' = k'^2 d k' \sin \theta d \theta d \varphi
\]

(\( \theta \) measured from \( \vec{k} \) direction)

\[
\delta(\omega_k - \omega_{k'} - \omega_{k''}) = \delta \left( k - k' - \sqrt{k^2 - 2kk' \cos \theta + k'^2} \right)
\]

(in the first term)

and integrate with respect to \( \cos \theta \) from \(-1\) to \(+1\). As a result the \( \delta \) -function is replaced by

\[
\left[ \frac{K K'}{k - k'} \right]^{-1}
\]

Integration with respect to \( \varphi \) gives a factor of \( 2\pi \). The second term can be treated similarly.

In the first term, the condition \( \omega_k < \omega_k \) means that the integration over the modulus of \( K' \) is only over the interval \( 0 \rightarrow k \). In the second term, however, this restriction is lifted and the upper limit is \( \infty \).

We can put the wave kinetic equation in a more convenient form by introducing the new variable. If we define

\[
N_k = n_k \cdot k^2
\]

the result is

\[
\frac{\partial N_k}{\partial t} = 2y_k N_k + 8\pi^2 k^2 \left\{ \int_0^k dk' N_{k'-k} N_{k'} - 4 N_k \int_0^k dk' N_{k'} d k' + 2 \int_0^\infty N_{k'} N_{k+k'} d k' - 4 \frac{N_k}{k} \int_0^k k'^2 N_{k'} d k' - 8 \frac{N_k}{k} \int_k^\infty N_{k'} d k' \right\}
\]

(57)
One solution in the form of a power law can be found (for $k=0$) by inspection, in the form

$$N_k = \frac{T}{\omega_k} k^2 \approx T k$$

i.e., $N_k \sim \frac{T}{\omega_k}$. This is just the Rayleigh-Jeans law for equipartition of energy among all modes. However, it is clear that for the turbulence problem this law is useless, and some of the integrals taken separately in Eq. (16) diverge at large $k$.

It is clear that we must seek a solution of the form $N_k \sim k^{-\delta}$, $\delta > 2$, to avoid this "ultraviolet catastrophe". The proportionality constant will be determined by requiring that the result be connected with the regions of growth and dissipation.

It might appear that a similar divergence now occurs at $k=0$. That this is not so may be seen by combining terms in the previous equation:

$$\int_0^{K/2} \left[ N_k - 4 N_k N_{k-k'} + 2 N_k N_{k+k'} \right] dk' \approx$$

$$\sim \int_0^{K/2} \left[ 2 N_{k-k'} - 4 N_k + 2 N_{k+k'} \right] N_k' dk' =$$

$$\approx 2 \int_0^{K/2} N_k' k_{k'}^2 \frac{\partial^2 N_k'}{\partial k'^2}$$

Note that the first term diverges in the same way at both $k'=0$ and $k'=k$. The result is convergent if $N_k \sim k^{-\delta}$ where $\delta < 3$.

Now substitute $N_k = \frac{A}{k^{\delta}}$ into the collision term and look for solutions with $2 < \delta < 3$. Using the relation

$$\int_0^K \frac{k^{m-1} (k-k')^{n-1}}{k} dk' = k^{m+n-1} B(m, n)$$

where

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

- 33 -
we find

\[
\begin{align*}
\mathbf{S} \{ N_\kappa \} & \equiv A^2 F(s) = 0 \\
F(s) & = \frac{\Gamma^2(1-s)}{\Gamma(2-2s)} - \frac{4}{1-s} + 2 \frac{\Gamma(1-s)\Gamma(2s-1)}{\Gamma(s)} - \frac{4}{3-s} - \frac{8}{s-2}.
\end{align*}
\]  

(58)

From the behavior of the \( \Gamma \)-functions and the last two terms, it is clear that \( F(s) \rightarrow +\infty \) at \( s=3 \) and \( -\infty \) at \( s=2 \), and so one expects to find a root near the middle of this interval. In actual fact, \( s = 2.5 \) is a solution, as may be verified by substitution.

Let us apply this analysis to the problem in which a source of instability and damping are present. Ordinarily we can estimate \( N_\kappa \) simply by

\[
\frac{\gamma}{4\pi^2 k_o^2} \sim N_\kappa \tag{59}
\]

where \( k_o \) is a characteristic magnitude in \( \mathbf{K} \)-space over which \( \gamma \) varies. This is incorrect if regions of damping and growth are separated by a transparent region. Let the damping be described by

\[
\gamma \sim -\gamma k^d/4\pi^2 \quad (d > \frac{1}{2}) \tag{60}
\]

and the instability by Eq. (59) with \( \gamma = \bar{\gamma} \).

For sufficiently large \( k \), the damping must end. The intermediate region is described (for small enough \( \bar{\gamma} \) ) by

\[
N_\kappa \sim B k^{-2.5} \tag{61}
\]

The boundary of the damping region is given by \( k_\perp \), where
By integration of the steady state equation, we obtain the conservation law

\[ \int_0^\infty k \log k K_k = 0 \]

From this follows

\[ \gamma K_0^2 \frac{B}{k_0^2.5} \sim \gamma B \int_0^k k^{-1.5 + \lambda} dk \]

where the lower bound of the integration on the r.h.s. is taken at some \( k \gg k_0 \). We use Eq. (62) to solve for \( B \)

\[ B \sim \gamma \left[ \frac{\gamma (1 - 0.5) \left(1 - 0.5)(d + 2.5) \right)}{k_0 \nu} \right]^{1/2} \]  

Thus Eq. (61) is approximately valid if the transparent region is sufficiently broad and the damping sets in sufficiently abruptly. Then a spectrum similar in appearance to the Kolmogoroff spectrum holds, although the mechanics involved are quite different. In our case, this spectrum is a result of wave-wave hierarchy; in the Kolmogoroff's case, the modes are the vortices whose statistical interaction we cannot deduce from the previous principles.

5. Coupling between different modes in a turbulent plasma

This simple form of the turbulence spectrum is due to the fact that the kernel of the wave-wave collision integral is very simple for the idealized model described in Eq. (50). In a more realistic situation, this kernel usually has a complicated form. Even the procedure for its derivation is quite cumbersome, especially in the
cases of interaction of different modes. It is, therefore, very important to find some additional simplifications of this problem.

Let us show how it can be simplified if the interacting modes have very different dispersive properties. If we have two modes (characterized by frequencies \( \omega_k \) and \( \Omega_k \)) such that \( \omega_k > \Omega_k \), we may treat the problem adiabatically. For example, let us consider the interaction between high frequency short wave longitudinal plasma oscillations

\[
\omega_k^2 \approx \omega_{pe}^2 + 3 \kappa \frac{T_e}{m}
\]

and low frequency long wave ion acoustic vibrations

\[
\Omega_k^2 = \pm q \cdot C_s^2 \quad , \quad C_s^2 = \frac{T_e}{M}
\]

with \( \kappa >> q \). This problem was first considered by VEDENOV and RUDAKOV [18].

We start from the Liouville equation for the number of electron plasma oscillations (plasmons) in the six-dimensional space of coordinates and wave vectors:

\[
\frac{\partial N_k}{\partial t} + \frac{\partial \omega_k}{\partial r} \cdot \nabla_r N_k - \frac{\partial \omega_k}{\partial r} \cdot \nabla_r N_k = 0
\]

\[
N_k = \frac{|E_k|^2}{4\pi \omega_k}
\]

(64)

In a uniform plasma, changes in the frequency \( \omega \) as a function of position arise only from the ion-acoustic vibration which we describe by the hydro-magnetic equations:

\[
\n M \left[ \frac{\partial \overrightarrow{V}}{\partial t} + (\overrightarrow{V} \cdot \nabla) \overrightarrow{V} \right] = -\frac{\partial p}{\partial r} - \frac{n_e e^2}{2m} \overrightarrow{V} \sum_{k} \frac{|E_k|^2}{\omega_k^2}
\]

(65)

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \overrightarrow{V}) = 0
\]

(66)

\[
p = MC_s^2 n
\]

(67)

where the last term on the r.h.s. of (65) is obtained by averaging the
electron term \(-n_e m \vec{\nabla} \cdot \vec{\nabla} \cdot \vec{\nabla} \psi\) over many plasma oscillation periods, and just represents a gradient of the radiation pressure:

\[
\vec{\nabla} \cdot \vec{\psi} = \frac{e \vec{E}_x}{-i \omega_{pe} m}
\]

\[
- n_e m (\vec{v} \cdot \vec{\nabla}) \vec{\psi} = - n_e m \sum_{\kappa, \kappa'} e^{i \vec{k}_x \cdot \vec{r}} (\vec{v}_{\kappa, \kappa'} \cdot \vec{\nabla}) e^{i \vec{k}'_x \cdot \vec{r}}
\]

\[
= - \frac{n_e m}{2} \sum_{\kappa, \kappa'} \frac{e^2}{m^2 \omega_{pe} \omega_{\kappa, \kappa'}} \left[ i \left( \vec{k}_x \cdot \vec{E}_x \right) E_{\kappa, \kappa'} + i \left( \vec{k}'_x \cdot \vec{E}_{\kappa, \kappa'} \right) E_{\kappa, \kappa'} \right]
\]

\[
e^{i (\vec{k} + \vec{k}') \cdot \vec{r}} \vec{r} = - \frac{n_e e^2}{2 m} \frac{\partial}{\partial r} \sum_{\kappa} \frac{|E_{\kappa, \kappa'}|^2}{\omega_{\kappa, \kappa'}^2}
\]

(67) is just the adiabatic equation of state.

Using the definition of the plasma frequency

\[
\omega_{\kappa, \kappa'}^2 = \frac{4 \pi n_e e^2}{m} \left\{ 1 + 3 \frac{k^2 T_e}{m \omega_{pe}^2} \right\}
\]

and linearizing Eq. (66), we can write Eq. (64) in the form

\[
\frac{\partial N_{\kappa, \kappa'}}{\partial \tau} + \frac{\omega_{\kappa, \kappa'}}{\partial \kappa} \frac{\partial N_{\kappa, \kappa'}}{\partial \kappa} + s \nabla \left( \vec{v} \cdot \vec{E} \right) \cdot \frac{\partial N_{\kappa, \kappa'}}{\partial \kappa} = 0
\]

\[
s = \frac{\omega_{pe}}{2}
\]

Since

\[
- \frac{\partial \omega_{pe}^2}{\partial \kappa} = - \frac{\omega_{pe}}{2} \frac{1}{n_0} \nabla n = s \nabla \left( \vec{v} \cdot \vec{E} \right)
\]

While (65) becomes

\[
- 37 -
\]
In (69) and (70), \( \vec{\xi} \) is the fluid displacement.

In order to see that this set of equations is quite acceptable to work with, let us apply it to some specific problems.

(a) **Damping of an ion acoustic wave in a gas of Langmuir plasmon.**

Let us assume a dependence of \( N^e_\vec{q} \) and \( \vec{v} \) on \( \vec{r} \) and \( t \), of the form \( e^{-i\Omega t + i\vec{q} \cdot \vec{r}} \). Linearizing Eq. (69), we obtain the correction to the plasmon distribution function:

\[
S N^e_{\vec{q}} = \frac{ie}{\Omega - \vec{q} \cdot \frac{\partial \omega}{\partial \vec{K}}} (\vec{q} \cdot \vec{\xi}) (\vec{q} \cdot \frac{\partial N^e_{\vec{q}}}{\partial \vec{K}})
\]

(71)

Substituting in (70) and dotting with \( \vec{q} \) we find the dispersion relation connecting the frequency \( \Omega \) and wave vector \( \vec{q} \) of an ion acoustic wave:

\[
-\Omega^2 + q^2 c_s^2 = q^2 \frac{S^2}{\partial} \sum \frac{d^3 \vec{k}}{\Omega - \vec{q} \cdot \frac{\partial \omega}{\partial \vec{K}}} + i \vec{\xi}
\]

(72)

For \( \Omega \) having a small imaginary part \( \Gamma_{\vec{q}} \ll \Omega \), we find

\[
\Omega = \pm q c_s + i \Gamma_{\vec{q}}
\]

\[
\Gamma_{\vec{q}} = \frac{\pi}{2 q c_s} \sum d^3 \vec{k} \cdot (\vec{q} \cdot \frac{\partial N^e_{\vec{q}}}{\partial \vec{K}}) \delta (\pm q c_s - \vec{q} \cdot \frac{\partial \omega^e_{\vec{q}}}{\partial \vec{K}})
\]

(73)

We can see a direct analogy with the usual Landau damping (growth) of waves on particles. Here, ion sound waves damp on quasi-particles with velocity \( \frac{\partial \omega^e_{\vec{q}}}{\partial \vec{K}} \) and distribution function \( N^e_{\vec{q}} \). And, of course, under the reciprocal influence of the ion sound wave spectrum, this plasmon distribution will relax.

For this relaxation process we have from (69) and (73)
\[
\frac{\partial}{\partial t} \langle N_{\lambda} \rangle = \frac{\partial}{\partial k_{\lambda}} D_{\lambda\beta} \frac{\partial}{\partial k_{\beta}} \langle N_{\lambda} \rangle
\]

\[
D_{\lambda\beta} = \pi \sum_{q} q^{2} q_{\lambda} q_{\beta} q^{\frac{1}{2}} \delta \left( \Omega - q \cdot \frac{\omega}{2} \right)
\]

where \( \langle \rangle \) means an average over the fast (plasma) oscillations.

(b) Instability of a plasmon gas.

Another interesting problem which we can solve within this framework is the instability of a gas of plasmons due to the coupling with the sound waves.

Under suitable conditions, a narrow spectrum of Langmuir oscillations is formed, which we can regard as excitation of a single mode:

\[
N_{\lambda} = \kappa_{0}^{3} N_{0} \delta(\lambda - k_{0})
\]

After substitution in (72) and integration by parts, we obtain

\[
\Omega^{2} = q^{2} C_{s}^{2} \left[ 1 + \frac{3}{4n_{0} m} \frac{q^{2} \omega}{(\Omega - \frac{q^{2} \omega}{2k_{0}})^{2}} \right]
\]

where \( \omega = \sum_{\lambda} N_{\lambda} \delta d \kappa_{0}^{3} = N_{0} k_{0}^{2} \) is the total energy in the spectrum and the dispersion equation (68) has been used again in calculating \( q \cdot \frac{\omega}{2k_{0}} \).

If \( q \cdot \frac{\omega}{2k_{0}} \gg C_{s} \), this equation has a solution

\[
\Omega = \frac{1}{2} q \cdot \frac{\partial \omega}{\partial k_{0}} + \gamma
\]

\[
\left( \frac{1}{2} q \cdot \frac{\partial \omega}{\partial k_{0}} + \gamma \right)^{2} \left( \frac{1}{2} q \cdot \frac{\partial \omega}{\partial k_{0}} - \gamma \right)^{2} \approx \frac{3}{4n_{0} m} q^{2} C_{s}^{2}
\]

\[
\gamma^{2} = \left( \frac{1}{2} q \cdot \frac{\partial \omega}{\partial k_{0}} \right)^{2} + \left( \frac{3}{4} \frac{\omega}{n_{0} \frac{T_{e}}{m}} \cdot M \right)^{\frac{1}{2}} q^{2} C_{s}^{2}
\]

From which it is clear that the criterion for instability is
The growth rate is

\[ \frac{\omega}{n_0T_e} > \frac{3M}{4m} \left( \frac{k}{\lambda_{De}} \right)^4 \left( \frac{\epsilon_0}{|\epsilon|^4} \right)^{1/4} \]

The growth rate is

\[ \gamma \approx qC_s \left( \frac{\omega}{n_0T_e} \frac{M}{m} \right)^{1/4} \] (76)

It can also be shown that the "plasmon gas" is unstable with respect to the decay mode discussed in the first lecture in connection with Alfvén waves. A "steady state" plasma oscillation \((\omega_0, \vec{k}_0)\) decays into another plasmon \((\omega, \vec{k})\) and an ion sound wave \((\Omega, \vec{q})\) \([3, 19]\).

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PART II. PARTICLE-WAVE INTERACTION

1. Particle interaction with a single wave

We have finished consideration of non-linear wave theory in fluid-type media. Let us now start to take into account a second type of interaction, namely that between waves and particles.

It is convenient to start with the simplest problem, the interaction between small amplitude monochromatic waves and particles. For simplicity we assume that there is no magnetic field, and we take the wave potential in the form

\[ \varphi = \varphi_0 e^{-i(\omega t - kx)} \]

The main contribution to the interaction arises from the group of resonant particles, those with velocity close to the wave phase velocity \( \omega/\kappa \). In linear theory it gives as a result the so-called Landau damping. The non-linear problem has not been solved in its entirety, but qualitative arguments show that the distribution function is reconstructed for the velocities close to \( \omega/\kappa \). The simplest type of such reshaping is expected to be in the form of a "plateau".

As an indication of such non-linear distortion we would remind you of the Bernstein-Greene-Kruskal (BGK) [1] solution which is an exact non-linear steady-state solution. According to this, there is no Landau damping in steady state. It is tempting to treat such steady state waves as an asymptotic limit \( (t \to \infty) \) - Landau's initial value problem.

Let us see in a very crude qualitative fashion how to approach the BGK-type solution from an initial choice of the distribution function. Consider a wave propagating in a plasma with phase velocity \( \omega/\kappa \). Looking in the frame of reference moving with the wave, we see that there are two groups of particles: if \( |v| > v_{\text{crit}} = |e\varphi_0/m| \), particles pass over the peaks in the periodic potential without trapping; if \( |v| < |e\varphi_0/m| \), they are trapped. If the number of particles moving in one direction is not balanced by the number moving the opposite way, as indicated by the arrows in Fig. 1, the distribution function changes with time.

- 42 -
Let us define energy-angle (similar to action-angle) variables $E$ and $\theta$ and write the distribution function for the trapped particles 

$$ f = f_{tr}(E, \theta) $$

at $t=0$, and for short times, $t>0$, $f$ is anisotropic (depends on $\theta$) in contrast to the steady-state BGK solutions, which depend only on $E$. Every particle has some frequency of oscillation $\omega(E)$. For particles near to the bottom of the potential well, $\omega$ is independent of $E$. For two particles with $E_1 \approx E_2$ and $\theta_1 \approx \theta_2$ we have slightly different frequencies

$$ \omega_1 - \omega_2 = \Delta \omega = \omega' \Delta E $$

After a time $t \gg \tau$, where $\tau \Delta \omega \sim 1$, the particles will be separated by a phase $\Delta \theta \sim \frac{1}{\omega}$. Thus the phases of the particles become randomized, and $f$ ceases to depend on $\theta$. This phase-mixing process is similar to that which affects particles in a non-uniform magnetic field (where the frequency depends on particle energy) and has been treated many times.

We can ask about the cause of the apparent change in entropy as a result of phase mixing (since a phase-independent function has more entropy than the original $f_{tr}(E, \theta)$). This question can also be

* Such an entropy production was discussed in [2,3] as a dissipation mechanism in collisionless plasma shocks.
answered qualitatively for the case of trapped particles in such a monochromatic wave.

The distribution function must conserve the characteristics of the collisionless kinetic equation. These characteristics are the particle orbits. Now, on the basis of period differences we can conclude that the distribution function at any fixed $x$ will be modulated in velocity space (oscillations appear in $f'(v)$), and this modulation will grow with time $[3]$ (see Fig. 2).

![Fig. 2](image)

There is a limit in which one can easily solve the problem of such peculiar behaviour of the distribution function. Suppose $\omega / K$ is so large that the resonant particle energy is small compared with the initial wave energy. Since the Landau damping is small in such an approximation the particle motion occurs in an almost constant periodic field.

$$\Psi = \frac{i}{2} \Psi_0 (t - \cos K x)$$

We can easily find the particle orbits in such a field and immediately write a general solution for the distribution function:

$$f(x, E, t) = f_0 (\omega) + 6 \int f_0' \sqrt{\frac{2}{m}} \frac{\left[ E - \epsilon \Psi_0 \left[ \sin \frac{\epsilon}{2} K x \right] \right]}{m \sqrt{\frac{2}{m}} \left[ E - \epsilon \Psi_0 \left[ \sin \frac{\epsilon}{2} K x \right] \right]}$$ (1)

where $x_0(x, E, t)$ is the point from which $x$ evolved and $\epsilon = \text{sign} U$; at the initial moment $t=0$ we neglected here the correction to the
distribution function due to the presence of a sinusoidal wave in plasma $\sim f^i \sin kx$ and had used the initial condition in the form expanded near the resonance velocity:

$$f(v, t) = f_0(\frac{\omega}{k}) + \frac{\partial f_0(v)}{\partial v} \left|_{v=\omega/k} \right. \cdot (v - \omega/k)$$

The distribution function (1) is normalised to the energy variable $\varepsilon$. Introducing the new variable

$$\sin x = \varepsilon \sin \frac{x}{\varepsilon} \quad , \quad \varepsilon^2 = \frac{eV_0}{E}$$

for trapped particles with $\varepsilon^2 > 1$, we can express the particle trajectory in terms of an elliptic integral of the first kind and $\varepsilon$ takes the sign of $\varepsilon$.

$$F\left(\frac{1}{\varepsilon} , \varepsilon_0\right) = F\left(\frac{1}{\varepsilon} , \varepsilon\right) - \frac{t}{\varepsilon} \quad , \quad \varepsilon = \left(\frac{2m}{eV_0 k}\right)^{\frac{1}{2}}$$

Therefore Eq. (1) for trapped particles takes a form [3]

$$f(x, E, t) = \frac{f_0 + f_0' \delta \sqrt{2E/m} \cn \left[ F\left(\frac{1}{\varepsilon} , \varepsilon\right) - \frac{t}{\varepsilon} ; \frac{1}{\varepsilon}\right]}{\sqrt{2m \left[ E - eV_0 \left(\sin \frac{kx}{2}\right)^2 \right]}}$$

(2a)

Analogously, for the untrapped particles we obtain

$$f(x, E, t) = \frac{f_0 + f_0' \delta \sqrt{2E/m} \dn \left[ F\left(\varepsilon , x\right) - \frac{t}{\varepsilon} ; \frac{1}{\varepsilon}\right]}{\sqrt{2m \left[ E - eV_0 \left(\sin \frac{kx}{2}\right)^2 \right]}}$$

(2b)

If we now go back to the variables $x, u, t$, we can describe quantitatively the modulation of the distribution in the velocity space.

It becomes much easier if we introduce the less realistic but simpler model with a square-tooth wave [3]. (see Fig. 3). It is known that this model permits one to derive linear Landau damping correctly. As with a sinusoidal wave, $\omega = \omega(E)$. Now we can obtain an
analytic solution for the time development of oscillations in \( v \) space, and find that as \( t \to \infty \), the wavelength \( \lambda_v (t) \to 0 \) \( (1/\lambda_v = k_v = \frac{d}{dt} \) and \( f(v) \) is strongly modulated.

It is interesting to compare this with the results obtained by Landau in his original treatment of damping. He found two types of damping: for a monochromatic wave, there are terms in \( f \) like \( e^{i \kappa v t} \) arising from the initial value of \( f \), in addition to the exponential damping. Although Landau did not discuss the point in the paper, he used this result to show how entropy changes. For long enough times, the wavelength of the modulation becomes so small that it is no longer justified to neglect the \( \frac{\partial^2 f}{\partial v^2} \) terms in the collision term; so even for a "collisionless" plasma, collisions will act to smooth out \( f \). The time required for this will be finite and quite insensitive to the collision frequency \( \nu \). The same argument applies here, and thus we get relaxation to one particular steady state trapped particle distribution (out of the entire set of BGK solutions).

We now have a situation similar to that in MHD turbulence theory. There, in the usual co-ordinate space, turbulence first develops on the scale of large wavelengths, then degrades into smaller scales. For sufficiently short lengths, real damping occurs. In the usual picture, the spectrum is independent of viscosity. Here we have a similar situation in \( \nu \)-space: the steady-state distribution is independent of collisions (viscosity).

\[ \text{Now we can calculate the Landau damping as} \]

\[ * \text{ We can now see that this kind of approximation is actually the first term in the series of } \left( \frac{1}{\mathcal{E}} \frac{d \mathcal{E}}{d t} \right). \]
The decrement starts off with the value found by Landau, but after a time $t/\omega(\varepsilon)$ changes sign. This occurs because after one wave period the direction of the imbalance in trapped particle motion is approximately reversed \[3\]. For the exact sinusoidal wave treated by O'NEIL \[4\], the same qualitative behaviour exists.

Assuming the slow change of the wave amplitude O'Neil calculated the time-dependent growth rate of the oscillation. The result is

\[
\gamma(t) = -\gamma_1 \sum_{n=0}^{\infty} \frac{64}{\pi} \int_0^1 dx \left\{ \frac{2 \pi n^2 \sin \left[ \frac{\pi n t}{\lambda} \right]}{x^5 \left( 1 + q^{2n} \right) \left( 1 + q^{-2n} \right)} \right[ \frac{(2n+1)x^{\pi t}}{2 \pi t} \right] \frac{1}{\lambda} \int_0^\infty d\nu \int_0^\infty \nu \int_0^\infty \nu \int_0^\infty d\nu \int_0^\infty d\nu
\]

\[47\]
where \( F = F(\infty, \pi/2); F' = F[(1-x^2)^{1/2}, \pi/2], \phi = e^{\pi F'/F} \)

\[
\gamma_{\text{L}} = \frac{\pi}{2} \omega \frac{\omega^2}{k^2} \frac{\partial f}{\partial u} \frac{1}{\omega_k} \quad \text{is the damping coefficient found by Landau. For the time less than the period of motion of trapped particles and larger than the phase-mixing time} \quad t_\gamma = \frac{L}{k U_{\text{the}}} \text{for untrapped ones* we obtain the usual formula of Landau:}

\[
\gamma(t) \approx \gamma_{\text{L}}, \quad \frac{1}{k U_{\text{the}}} < t < \tau.
\]

In the limit of large time the growth rate goes to zero \( \gamma(\infty) = 0. \)

A more complicated problem, taking the collisions into account, is now solved only for the asymptotic limit of the steady state \([5]\), since finding the distribution function is not so easy as in the case of equations(2).

The kinetic equation for this function in the system of the resting wave may be written down in view of collisions as

\[
u \frac{\partial f}{\partial y} - \phi' (y) \frac{\partial f}{\partial u} = \frac{\partial}{\partial u} \left[ \frac{\partial f}{\partial u} + (\lambda + \nu) f \right]
\]

where all values are reduced to the dimensionless form:

\[
\phi(y) = \frac{\varphi}{T}, \quad y = k x, \quad \phi(x) = \varphi \cos^2 \frac{k x}{2},
\]

\[
\lambda = \frac{V_{\text{ph}}}{(T/m)^{1/2}}, \quad \nu = \frac{3}{2 \kappa U_{\text{the}}}, \quad \tau_D = \frac{m^2 V_{\text{ph}}^3}{8 \pi e^4 n L}
\]

where \( \varphi(x) \) is the wave potential in the system at rest, \( k \) and \( v_{\text{ph}} \) are its wave number and phase velocity respectively, \( \tau_D \) is the effective time of electron collisions whose velocity coincides with the wave phase velocity, \( L \) are Coulomb logarithms and \( \varphi(y), y, \varphi \) and \( \nu \) are the dimensionless potential energy, co-ordinate, phase velocity and collision frequency, respectively.

* As was shown by Landau, a large damping of the wave appears at the initial time \( t < \frac{1}{k U_{\text{the}}} \) and the wave amplitude cannot be treated as a constant. Therefore that formula for this time should not be believed.
Let us introduce one restriction which makes the problem soluble (in practice):

\[ u \ll \varphi_0 \ll 1 \]  

(4)

This corresponds to a wave of finite but small amplitude and low frequency of collisions.

Now it will be more convenient to deal with new independent variables for the distribution functions

\[ u_D y \rightarrow E = \frac{u^2}{2} + \varphi(y), y \]

where \( E \) is the dimensionless total particle energy in the wave field.

Then Eq. (3) takes the form

\[ \frac{\partial f}{\partial y} = \sqrt{\frac{8}{3}} \{ \pm [E - \varphi(y)]^{1/2} \left( f + \frac{\partial f}{\partial E} \right) + cf \}, \]

where the \( \pm \) signs before the root correspond to the different directions of electron velocities (plus sign is taken for particles overtaking the wave).

In solving Eq. (5) we must consider two cases: \( E > \varphi_0 \) (external region) and \( E < \varphi_0 \) (internal region). We find the solution of Eq. (5) as an expansion in series,

\[ f(y, E) = f_0(E) + \sqrt{2} f_1(E, y) + \cdots \]

(6)

By substituting Eq. (6) into Eq. (3) we get

\[ \frac{\partial f}{\partial y} = \frac{\partial f_0}{\partial E} \left[ \pm [E - \varphi(y)]^{1/2} \left( f_0 + \frac{\partial f_0}{\partial E} \right) + cf_0 \right] \]

(7)

where \( f_0(E) \) (zero approximation) is defined (in the external region) by the
condition that $f_1(y)$ is periodic, and by the boundary condition that at $E >> \Psi_0$ the value $f_0(E)$ should asymptotically approach the Maxwell distribution for the plasma moving at the velocity $-c$ relative to the wave. Thus the equation for $f_0(E)$ at $E > \Psi_0$ will be

$$\frac{d}{dE} \left\{ J(E) \left[ f_0 + \frac{df_0}{dE} \right] + cf_0 \right\} = 0 \quad (8)$$

where $J(E) = \pm \frac{1}{2\pi n} \int_{-\pi}^{\pi} \left( E - \Psi_0 \sin \frac{\theta}{2} \right)^{\frac{1}{2}} dy$ are some elliptic integrals.

Now we can find $f_0(E)$ and substitute it in Eq. (7). Finally we must have $f^+_{\text{ext}}$ and $W_{\text{ext}} = -\int_{\omega}^{V_+} \varphi \, du$, which, as one can see, will be expressed in terms of some elliptic integrals.

Let us write this, omitting a calculation and using some expansions of elliptic integrals for the case $\Psi_0 \ll 1$.

$$W_{\text{ext}} = 0.1 \varphi^{\frac{1}{2}} e^{-c^2 \omega \tau n} \quad (9)$$

The same procedure must be used also in the internal region ($E < \Psi_0$). However, the distribution function here is quite symmetrical $f_+(E) = f_-(E)$ where $f_+(E, y)$ and $f_-(E, y)$ correspond to electrons with positive and negative velocities relative to the wave. This means that its contribution to wave damping $W$ is quite small, and we can therefore neglect it.

The major problems arise when we want to find distribution functions inside the singular domain lying between external and internal regions. Fortunately, as shown by ZACHAROV and KARFMAN [5], we do not need to know in detail the behaviour of this function to calculate $W_{\text{sing}}$; it is enough to know its value at both sides of the singular region. In fact,

$$W_{\text{sing}} = V_{\text{ph}} \cdot \varphi \cdot \rho \cdot \omega T \text{ V}_{\text{the}} \left( \frac{\pi}{\sqrt{8 \pi}} \right)^{\frac{1}{2}} \int_{-\pi}^{\pi} \frac{dE}{[E - \varphi(y)]^{\frac{1}{2}}} \rho(y) (f_+ + f_-) =$$

$$= -\omega T \text{ V}_{\text{the}} \left( \frac{\pi}{\sqrt{8 \pi}} \right)^{\frac{1}{2}} \int_{-\pi}^{\pi} \frac{dE}{[E - \varphi(y)]^{\frac{1}{2}}} \rho(y) (f_+ + f_-),$$

- 50 -
where $S$ is the width of the singular region.

Now, by using integration by parts and taking into account only the lowest derivatives of the distribution function inside a singular region, we obtain

$$
\int_{-\pi}^{\pi} dy \left\{ \int_{\gamma_0-\delta}^{\gamma_0+\delta} dE \frac{d}{dy} \left[ f_+(y, E) + f_-(y, E) \right] \right\} =
$$

$$
= \gamma_0 \cos^2 \frac{\omega}{2} \left( \frac{\partial f_0^+}{\partial E} \bigg|_{E=\gamma_0+0} - \frac{\partial f_0^-}{\partial E} \bigg|_{E=\gamma_0-0} \right)
$$

If we substitute here the values of $\frac{\partial f}{\partial E}$, which we find from external ($E = \gamma_0 + 0$) and internal ($E = \gamma_0 - 0$) regions, we finally have

$$
W_{\text{sing}} = 3 \gamma \gamma_0^2 e^{-c^2 c \omega T n}
$$

2. The many-waves case

Let us now turn to a problem with two waves. If the waves have well-separated phase velocities $\left(\frac{\omega}{k}\right)_{1,2}$, then they will not interact and we can just superpose the results for the trapped particles of a single wave. However, if the waves are close together,

$$
\Delta \left(\frac{\omega}{k}\right) \sim \sqrt{\frac{e \gamma}{m}}
$$

where $\gamma$ is the order of magnitude of the amplitude, then the situation becomes completely different. We expect to find overlapping or "collectivization of the trapped particles."
We can continue by looking for solutions with 3, 4, etc., waves present, but the analysis is hopelessly complicated and only rough qualitative conclusions can be drawn. However, with a very large number of waves present, we can introduce the random phase approximation and the statistical approach used before in these lectures.

Suppose that there is a domain in velocity space

\[ \left( \frac{\omega}{K} \right)_{\text{min}} < \mathcal{U} < \left( \frac{\omega}{K} \right)_{\text{max}} \]

with waves present having phase velocities throughout this interval, such that between any two neighbouring waves there is collectivization of the resonant particles. If the phases of these waves are random, particles undergo Brownian motion in velocity space and we derive the familiar quasi-linear theory \([6,7]\). We will not go into the derivation in detail here, as it is well known to most people and discussed critically in a number of books (e.g., that of Timman and Montgomery \([8]\)). The most rigorous way of obtaining the finite quasi-linear equations, as well as the criterion of overlapping between neighbouring monochromatic waves, was given by Al'tshul' and Karman \([9]\).
We content ourselves here with a simple non-rigorous derivation. Starting from the Vlasov equation

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} \nabla \psi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0
\]  

(10)

in one dimension, we write the distribution function as the sum of a slowly varying and rapidly varying part

\[ f = \bar{f} + f_\sim \]

For the rapidly varying part, we have just the usual linearized form of (10):

\[
\frac{\partial f_\sim}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\sim}{\partial \mathbf{x}} - \frac{e}{m} \nabla \psi \cdot \frac{\partial \bar{f}}{\partial \mathbf{v}} = 0
\]

During a single oscillation period, \( \bar{f} \) changes only slightly, and we can use a WKB approximation in time to solve for \( f_\sim \). For \( \bar{f} \) we have

\[
\frac{\partial \bar{f}}{\partial t} + \mathbf{v} \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} - \frac{e}{m} \nabla \psi \cdot \frac{\partial f_\sim}{\partial \mathbf{v}} = 0
\]  

(11)

The only non-linear effect of which quasi-linear theory takes account is the average of the product of two rapidly oscillating functions, i.e., the last term in (11).

The analysis is mathematically similar to that of the familiar van der Pol method, where we have two kinds of motion, one fast and one slow. The same kind of averaging in time is used there. The analogy with the van der Pol method can be useful in justifying quasi-linear theory for many waves. Here we have a partial differential equation in \( (x, \mathbf{v} \cdot t) \) instead of an ordinary differential equation. If just one monochromatic wave were present, we could transform the frequency to zero by going to a frame of reference moving with the phase velocity of the wave, so that an average over the fast oscillations becomes meaningless. But if many waves are present, no such co-ordinate system exists, and the average can be
employed. We take the fast time \( T_1 \approx 1/\Delta t \), where \( \Delta t = \omega_{\text{max}} - \omega_{\text{min}} \), and compare it with the time \( T_2 \) on which \( f \) changes; to get the criterion for the applicability of the quasi-linear theory, \( T_1 \ll T_2 \).

This criterion does not depend on the group velocity.

Although the discussion has concerned itself only with the one-dimensional problem, we will write down the results for electrostatic waves in three dimensions:

\[
\frac{\partial f}{\partial t} = \frac{e^2}{m^2 \partial v_\alpha} \mathcal{D}_{\alpha \beta} \frac{\partial f}{\partial v_\beta}; \quad \alpha, \beta = x, y, z \tag{12}
\]

\[
\mathcal{D}_{\alpha \beta} = \sum_k \left| V_k \right|^2 k_\alpha k_\beta \frac{\gamma_k}{(\omega_k - Kn)^2 + \gamma_k^2} \tag{13}
\]

where \( \gamma_k \) is related to \( f \) by Landau's formula.

Usually, Eq. (13) is written with \( \delta(\omega - Kn) \) instead of the last factor; this is done because the condition that the wave amplitude be small requires \( \gamma_k \ll \omega_k \). However, then \( \mathcal{D}_{\alpha \beta} \) contains only the interaction with the resonant particles; the form written above also contains the adiabatic interaction with the non-resonant particles. The resonant part of (13) gives the reconstructions of \( f \) in the resonant region; the adiabatic part describes the ordered motion of the plasma fluid under the influence of the wave field.

In the one-dimensional case we get a steady state solution after some time with a plateau for a restricted region in \( U \) space. We can estimate the time \( T_R \) required for relaxation from the initial velocity distribution to this plateau from Eq. (12):

\[
\frac{1}{T_R} \sim \frac{e^2}{m^2 \left( \Delta U \right)^2} \frac{\kappa^2 \varphi^2}{\Delta \omega}
\]

\[
T_R \sim \frac{m^2}{e^2} \frac{\left( \Delta U \right)^2 \Delta \omega}{\kappa^2 \varphi^2}
\]
where $\Delta \omega = (\omega)_{\text{max}} - (\omega)_{\text{min}}$, the width of the wave packet in velocity space, and $\Delta \omega \sim K \Delta \omega$. Then the condition $\tau_R \geq \frac{1}{\Delta \omega}$ just yields

$$\Delta \omega \gg \sqrt{\frac{e \Phi}{m}}$$

or, equivalently, $1/\Delta \omega$ is much smaller than the trapping time. It is the case opposite to the condition which we obtained before for two neighbouring monochromatic waves to overlap, $\Delta \omega < \sqrt{e \Phi/m}$.

The main result we find is that the plasma rapidly evolves to a state with a plateau on the distribution function and a finite amplitude wave packet which does not interact with the plasma. Of course, this is not really a steady-state solution: we know already from considering fluid-type phenomena that the wave spectrum will interact with itself and change in time. However, for small amplitude waves we may hope that the time required for establishing the plateau is much shorter than that for wave-wave interactions:

$$\tau_R \ll \tau_{WW}. \quad (14)$$

But it turns out that both $\tau_R$ and $\tau_{WW}$ are inversely proportional to the first power of the wave energy or wave number for three-wave interactions. In order to justify the ordering we need to take into account the other parameters of the system, $\tau_R$ is very short for narrow wave packets, so we make a new assumption,

$$\left(\frac{\Delta \omega}{\omega}\right)^2 \ll 1 \quad (15)$$

where $\omega$ is a typical phase velocity.

We can invoke a second argument. The criterion (14) comes from wave-wave interactions which obey the decay or resonance conditions (see Part I); for Langmuir oscillations we cannot conserve quasi-momentum and energy in a three-wave interaction. For a four-wave interaction, however, we find that the typical interaction times go inversely as the square of the wave energy. So, one way or the other, we can sometimes justify neglect of non-linear wave-wave interaction on the scale $\tau_R$. 

- 55 -
Now our task is to find some examples of situations where quasi-linear theory is applicable, just to see what kinds of phenomena can arise.

For Langmuir plasma oscillations we have the system of Eq. (12) and (for the wave energy)

$$\frac{\partial \mathcal{E}_\mathbf{k}}{\partial t} = 2\chi_\mathbf{k} \mathcal{E}_\mathbf{k}$$

(16)

together comprising the quasi-linear approximation. The first problem considered was that of the relaxation of an unstable distribution with a gentle bump, where it is possible to find the wave energy $\mathcal{E}(\mathbf{r})$ (where $\mathcal{E}(0) = 0$) as a function of the bump parameters and of time for $t \sim 0$ and $t \to \infty$. This solution is, of course, well known from many papers [6,7], so we shall not discuss it again here. The second problem treated using (12) and (16) was that of the quasi-linear Landau damping of a wave packet $\omega/\mathbf{k} \leq \omega/\mathbf{k}_\text{max}$ with $\mathcal{E}(0) \neq 0$ inside this interval [6,7].

These are two typical examples of one-dimensional quasi-linear problems. In two or three dimensions, it is easy to find complications and additional properties of the theory, because the one-dimensional problem is in a sense degenerate: the resonant particles occupy only a restricted region in velocity space. In higher-dimensional cases, even for a wave packet which is localized in $\mathbf{k}$-space, we find a broadening of the resonance region.

Let us consider a two-dimensional problem. In Fig. 6 are plotted the curves of the initially constant $f$, which are, say, circles centered on the origin. Let us introduce a fairly narrow wave packet propagating in the $\mathbf{v}_x$-direction. Because of the formation of a quasi-linear plateau, there will be established inside this narrow band, $\mathbf{v}_x \sim \omega/\mathbf{k}$, a new system of equipotentials (characteristics) parallel to $\mathbf{v}_x$. These, of course, connect with the circles in the part of velocity space outside the resonant band. It is easy to see that only a finite amount of energy is needed to reconstruct $f$ in this way.

Now suppose we have many such wave packets present propagating in different directions. In Fig. 6 each will be traversed by its own
set of equipotentials. Since these equipotentials will intersect some domain extending to infinity, an infinite amount of energy is now needed to make $\mathcal{f}$ be constant along all these equipotentials*. If, for example, all directions of propagation are present, this domain fills all of velocity space outside of the circle

$$v_x^2 + v_y^2 = \left(\frac{\omega}{k}\right)^2$$

because every part of this region is common to at least two different resonance bands. Since $\mathcal{f}$ is constant out to infinity, it is obvious that an infinite amount of energy is involved. This means that any steady state, corresponding to finite wave packet energy, is impossible, and the wave spectrum must damp to zero [10].

* Since the quasi-linear plateau is no longer in exactly steady state in the two-dimensional case (even for quite narrow wave packets) it is necessary to understand its meaning by considering it as an approximation, i.e., a "quasi-plateau".
To see what will actually happen to the distribution function, let us suppose that we have a two-dimensional wave packet which has cylindrical symmetry in $K$-space. Then $\bar{f}$ will be isotropic

$$\bar{f} = \frac{1}{V} \left( \bar{u}_x^2 + \bar{u}_y^2, t \right)$$

and on substitution in (12) we find

$$\frac{\partial \bar{f}}{\partial t} = \frac{e^2}{m^2} \frac{1}{|\psi|^2} \omega^2 \frac{1}{u} \frac{\partial}{\partial \psi} \frac{1}{\sqrt{u^2 - (\omega/K)^2}} \frac{1}{\sqrt{u^2 - (\omega/K)^2}} \frac{\partial \bar{f}}{\partial \psi}$$

(17)

\[ \bar{u}^2 = \bar{u}_x^2 + \bar{u}_y^2 \]

To get this, we replace the summation over $K$ by an integral, and take a very narrow spectrum, $|\psi|^2 = |\psi|^2 \delta(\kappa - \kappa_0)$. (Quasi-linear theory is still valid because this distribution gets smeared out when the spread in angles is taken into account.) For $\omega/K > \bar{u}$ we have

$$\frac{\partial \bar{f}}{\partial t} = 0$$

Together with (15), Eq. (17) has an exact solution if we make one additional simplification. If initially there is a loss of energy in the wave packet, then we obtain a reconstruction of $\bar{f}$ in the form of an outward expansion, so that finally for most of the distribution function we can neglect $\omega/K$ in comparison with $\bar{u}$. Now it is quite easy to solve the equations. As an example of a case where it is valid to neglect $\omega/K$ without restrictions, we can point to the interaction between electrons and ion sound waves, since

$$\frac{\omega}{K} \sim \sqrt{\frac{T_e}{M}} \ll \sqrt{\frac{T_e}{m}} \sim \bar{u}$$

For Langmuir oscillations this will be valid only after a considerable time, when the distribution has spread a long way.

Now let us introduce a new variable in place of time
\[ \zeta = \frac{25 e^2}{4 m^2} \int_0^t \omega^2 |\varphi^2(t')| \, dt' = \int_0^t D(t') \, dt' \]

We have
\[ \frac{\partial f}{\partial \tau} = \frac{4}{25} \frac{\partial}{\partial \nu^2} \left( \frac{1}{\nu} \frac{\partial f}{\partial \nu} \right) \]  \hspace{1cm} (18)

where \( \zeta \) is a constant.

To solve (18) for a given initial distribution \( f_0 \), we can Laplace-transform and obtain the solution in terms of the Green's function for \( f \) using modified Bessel functions of order \( -\frac{3}{2} \).

This is not a very convenient solution to work with, but we can go to \( t = \infty \), where \( f \) no longer depends on \( f_0 \), and write the similarity solution of (18) as [11]

\[ f = C \exp \left\{ -\frac{\nu^5}{\int_0^t D(t') \, dt'} \right\} \left[ \int_0^t D(t') \, dt' \right]^{-\frac{2}{5}} \hspace{1cm} (19) \]

where
\[ C = \frac{5}{\Gamma\left(\frac{3}{2}\right)} \int_0^\infty \omega \left( f_0(\nu) \nu \, d\nu \right), \quad \nu > \frac{\omega}{K} \]

To find \( \gamma \), we have
\[ \gamma_x = \frac{\pi}{2} \omega \frac{\omega^2}{K^2} \int_K \frac{\partial f}{\partial x} \delta \left( \omega - K \nu \right) \, d^2 \nu \]

For the asymptotic form (19), we calculate [11]
\[ \gamma_x = -\frac{\beta_x}{\int_0^t D(t') \, dt'}^{3/5}, \quad \beta_x = C \frac{\omega^2 \omega_{pe}^2 \Gamma\left(\frac{1}{2}\right)}{10 K^3} \]
\[
\frac{d|\psi|^2}{dt} = -\frac{2\beta s |\psi|^2}{\left[\int_0^t D(t)dt\right]^{3/2}}
\]  
(20)

Note that \(\omega/\kappa\) drops out of the expression (20) for \(\gamma\) because the main contribution to \(\gamma\) comes from \(\nu \gg \omega/\kappa\).

The original system of equations has now been reduced to (20) alone, which can easily be transformed to a second order non-linear differential equation, which can be solved by a not quite convenient quadrature, though this is a cumbersome procedure; the qualitative behaviour is obvious. Starting with some initial value, the energy will damp and \(\gamma\) will go to zero.

Initially we have Landau damping; after some finite time, \(E(t) \to 0\). There is no energy available for further reconstruction of \(f\), and \(\gamma\) tends to a constant value. The results are clearly different from the one-dimensional case in which a plateau is formed.

The picture we have obtained has obvious applications to so-called turbulent heating. Suppose we have a system in which turbulent heating is being employed; usually this means that the plasma is carrying a current which drives some instability, so we have relative motion of the electrons and ions. If this drift velocity is slightly greater than \(\omega/\kappa\) ion sound waves are unstable*. As mentioned above, \(\omega/\kappa\) drops out of the problem and the results can be applied direct. So even without knowing the wave spectrum we know that, after heating, the electron spectrum will not be Maxwellian but will have the form just seen. The actual form of the tail of the distribution will come not from quasi-linear theory but from other considerations, perhaps wave-wave interactions.

However, it must be remarked that in the calculation just completed, we assumed an isotropic wave spectrum; for ion sound turbulent heating this is certainly not satisfied. Suppose a current to be directed perpendicular to the magnetic field (the usual situation):

* Non-linear theory of this kind of instability will be discussed later (see Part III).
is the \(x\) -direction, \(H\) is in the \(z\) -direction.

For a small but finite value of the magnetic field, this problem is exactly the one just considered, since particles gyrate about the direction of the field, so that after a time of the order of a few Larmor periods, particles are mixed in the \(\mathcal{U}_x-\mathcal{U}_y\) plane. We can regard this as a rotation of the wave packet instead of a rotation of particles, and the distribution function will depend in \(\mathcal{U}_x, \mathcal{U}_y\) through \(\mathcal{U}_x^2 + \mathcal{U}_y^2\) only, even for a one-dimensional wave packet.

We thus find that there is also a difference between the one-dimensional case with and without a magnetic field. Of course, \(H\) must not be too large, because the simple picture of plasma dynamics used here then gets drastically modified. \(H\) is used only to contain the particles. We can neglect it in considering longitudinal wave properties because \(\omega_{pe}^2 > \omega_\|^2\); this is always the case in turbulent heating situations.
3. **Quasi-linear theory of the electromagnetic modes.**

In this section we will apply quasi-linear theory to electromagnetic modes propagating through a plasma immersed in a uniform magnetic field \( \vec{H}_0 \). We will consider the simplest cases of these modes, namely, ion and electron whistlers propagating parallel to the field \( \vec{H}_0 \). The limitation to parallel propagation simplifies the algebra but does not significantly change the results.

The basic equations for the problem are the kinetic equation

\[
\frac{\partial \bar{f}_i}{\partial t} + \frac{e_i}{m_j c} \vec{v}_i \cdot \nabla \vec{H}_0 \cdot \frac{\partial \bar{f}_i}{\partial \vec{v}} = \frac{e_i}{m_j} \left[ \vec{E}^2 + \frac{1}{\epsilon} \left[ \nabla \vec{H}_0 \right] \right] \cdot \frac{\partial \bar{f}_i}{\partial \vec{v}} = 0
\]

and Maxwell's equations. If we divide the distribution into a slowly varying and rapidly varying part, the equation for the slowly varying part takes the form

\[
\frac{\partial \bar{f}_i}{\partial t} = \left( \frac{e_i}{m_j} \right)^2 \sum_k \left\{ -\frac{k v_i}{\omega_k} \frac{\partial}{\partial v_z} + \left( 1 + \frac{k v_i}{\omega_k} \right) \frac{1}{v_i} \frac{\partial}{\partial v_i} v_i \right\} \left( \frac{|E_k|^2}{\omega_k \pm \omega_{nj} + ik v_z} \right) \left\{ (1- \frac{k v_z}{\omega_k}) \frac{v_i}{v_d} \frac{\partial}{\partial v_d} v_i + \frac{k v_i}{\omega_k} \frac{\partial}{\partial v_z} v_z \right\}
\]

where we have introduced cylindrical co-ordinates in velocity space, and the \( \pm \) sign in the resonant denominator refers to right-hand and left-hand circularly polarized waves. In the resonant region this equation becomes

\[
\frac{\partial \bar{f}_i}{\partial t} \cong \frac{e_i^2}{m_j} \sum_k \left\{ (1- \frac{k v_z}{\omega_k}) \frac{1}{v_d} \frac{\partial}{\partial v_d} v_i + \frac{k v_i}{\omega_k} \frac{\partial}{\partial v_z} v_z \right\} |E_k|^2 \left( \omega_k - k v_z \pm \omega_{nj} \right) \left\{ (1- \frac{k v_z}{\omega_k}) \frac{\partial}{\partial v_d} v_i + \frac{k v_i}{\omega_k} \frac{\partial}{\partial v_z} v_z \right\} \left( \bar{f}_i \right)
\]
The resonance condition now picks out particles with velocity
\[ \mathbf{v}_e = \frac{\omega \pm \omega_{\text{gj}}}{K} \]
The frequency of the field as seen in a coordinate system moving with this velocity has been Doppler-shifted to the gyrofrequency. Consequently, the resonant particles rotate around the field $\mathbf{H}_0$ at the same rate as the electric vector $\mathbf{E}_1$, and they are accelerated very effectively by this field.

As a first application of Eq. (21), we assume a large amplitude wave packet is impressed on a Maxwellian plasma. The resonant region for the wave packet will be as shown in Fig. 1. It lies in the left-hand plane because $\omega - \omega_{\text{gj}}$ is negative for the whistler modes. The solid circles in this figure are the level curves for the Maxwellian.
As in the quasi-linear theory of Langmuir oscillations, the resonant particles diffuse in velocity space until a quasi-static state is reached. From Eq. (21) we can easily see that this state will be such that

\[ \left\{ \left(1 - \frac{k v_x}{\omega_k} \right) \frac{\partial}{\partial v_x} + \frac{k v_x}{\omega_k} \frac{\partial}{\partial v_z} \right\} \overline{f_j} = 0 \quad (22) \]

This condition is completely equivalent to the condition that \( \frac{\partial f}{\partial v} = 0 \) in the quasi-linear theory of Langmuir oscillations. The solution of Eq. (22) is

\[ \overline{f} = \overline{f} \left( \frac{v_x^2}{2} + \frac{v_z^2}{2} - \int \frac{\omega}{K} d v_z \right) \quad (23) \]

as can be checked by substituting solution (23) in Eq. (22). Thus, it follows that the level curves of the quasi-static distribution are given by

\[ \frac{v_x^2}{2} + \frac{v_z^2}{2} - \int \frac{\omega}{K} d v_z = \text{const.} \]

(For a sufficiently narrow wave packet \( \Delta \frac{\omega}{K} \ll \frac{\omega}{K} \) it will appear \( \frac{v_x^2}{2} + \frac{v_z^2}{2} - \frac{\omega}{K} v_z = \text{const.} \)). Therefore, these curves are almost circles but their origin is displaced to the right by \( \frac{\omega}{K} \). Consequently, they will look like the dashed curves in Fig. 1. Since the marginal stability condition for the whistler mode is just Eq. (22) integrated over \( v_\perp \),

\[ \int dv_\perp v_\perp^2 \left\{ \left(1 - \frac{k v_x}{\omega} \right) \frac{\partial \overline{f}}{\partial v_\perp} + \frac{k v_x}{\omega} \frac{\partial \overline{f}}{\partial v_z} \right\} = 0 \quad (24) \left| v_z = \frac{\omega - \omega_\perp}{K} \right. \]

we find that the damping coefficient goes to zero in the time-asymptotic limit, just as it did in the analogous case for Langmuir oscillations.
In order to continue this analogy, one can reduce the two-dimensional quasi-linear diffusion operator in Eq. (21) to the one-dimensional form.

Indeed, if we introduce

$$\mathcal{W} = \frac{v_1^2}{\omega} + \frac{v_2^2}{\omega} - \int \frac{\omega}{\kappa} d\Omega_2$$

as one of the new variables, the derivatives will cancel, and Eq. (21) with the new variables \(\mathcal{W}\) and, say, \(U = U_e\) will have the form

$$\frac{\partial \tilde{f}}{\partial t} = \sum \left[ \left( 1 - \frac{kU}{\omega} \right) \frac{1}{v_1} + \frac{kU}{\omega} \frac{\partial}{\partial v} \right] \frac{e^2}{m^2} \frac{|E_x|^2}{\kappa} \pi \delta \left( \Omega - \epsilon U \pm \omega_n \right) \frac{kU}{\omega} \frac{\partial \tilde{f}}{\partial v}$$

$$= \frac{e^2}{m^2} \frac{\partial}{\partial v} \left[ v_1^2(\omega, v) \frac{|H_x|^2(v)}{|v - \frac{d\omega}{d\kappa}|} \frac{\partial \tilde{f}}{\partial v} \right]$$

also the integral (24), which determines the imaginary part of the frequency, can easily be written in a one-dimensional form

$$\text{Im} \omega \sim \int dv_1 \frac{v_1^2}{\omega} \left\{ \left( 1 - \frac{kU_2}{\omega} \right) \frac{\partial \tilde{f}}{\partial v_1} + \frac{kU_1}{\omega} \frac{\partial \tilde{f}}{\partial v_2} \right\} \left| v_2 = \frac{\omega - \omega_n}{\kappa} \right|$$

$$= \int v_1^2(\omega, v) \frac{\partial \tilde{f}}{\partial v} \, dv \quad \text{[12]}$$

* In the original paper[6] the one-dimensional form of the quasi-linear equation for whistlers was written inaccurately. It was corrected in Ref. [12]. But the one-dimensional whistler case is also degenerate in some respect; there is no exact plateau in a non-one-dimensional case. So actually, we can have only a quasi-plateau [13].
Now we already have the full analogy with longitudinal oscillations.

The integral (27) might be negative, giving an instability, in plasma with non-isotropic distribution of particle velocities. As one of the most interesting examples of this distribution one should mention the distribution with the so-called "loss cone". This situation will have some bearing on laboratory plasmas and on space plasmas. The loss-cone distribution is of the form

$$f = f_0 \left( v_{\perp}^2 + v_{\parallel}^2 \right) \eta \left( \lambda v_{\perp}^2 - v_{\parallel}^2 \right)$$

where

$$\lambda = \frac{H_{\text{max}} - H}{H}$$

and

For a given $\lambda$, the level curves appear as follows

For a given $\xi$ the level curves appear as follows
Substituting this distribution into the stability criterion for whistlers gives

\[ \int_0^\infty \mathcal{E}_l \left[ \left(1 - \frac{k v_z^2}{\omega^2} \right) \frac{\partial^2}{\partial \mathcal{E}_l^2} (f_0 \eta) + \frac{k}{\omega^2} \frac{\partial}{\partial (m v_z^2)} (f_0 \eta) \right] d\mathcal{E}_l < 0 \]  

(29)

where \( \mathcal{E}_l \equiv \frac{m v_z^2}{\omega} \).

Integrating the first term by parts and carrying out the differentiation on the other two terms gives

\[ - \int_0^\infty f_0 \eta d\mathcal{E}_l - \frac{2 k v_z^2}{\omega m} \int_0^\infty \mathcal{E}_l \eta \left[ \frac{\partial f_0}{\partial (v_z^2)} - \frac{\partial f_0}{\partial v_z^2} \right] d\mathcal{E}_l - \]

\[ \frac{k v_z^2}{\omega} (d+1) \int_0^\infty \mathcal{E}_l \eta' f_0 \]  

\[ < 0 \]

(30)

The second term in this equation vanishes identically, and, provided the range over which \( \eta' \) changes is small compared to \( T \), we can replace \( \eta'(x) \) by \( S(x)^2 \) in the third term. Consequently we find \([13]\)

\[ - \int_0^\infty \mathcal{E}_l \eta d\mathcal{E}_l < 0 \]

(30)

where \( \mathcal{E}_l = \frac{\omega - \omega_l^{1/2}}{k} \). Since \( \mathcal{E}_l \) is negative, this criterion gives instability for large enough \( |\mathcal{E}_l| \) or small enough \( \lambda \).

At the end of a mirror machine, \( \lambda \) goes to zero so there is always a small unstable region.
However, the violation of the local stability criterion in a small region $\Delta z$ does not mean that the plasma is unstable as a whole. This would require the wave to grow through many $\varepsilon$-foldings before it left the unstable region

$$\int \frac{\Im \omega}{\partial \omega / \partial k} dz \gg 1.$$

Carrying out this integration gives the following non-local criterion for instability:

$$\Delta z \gg \frac{\varepsilon}{\omega_{pj}} F(\beta_j) \quad (31)$$

where $\beta_j = \frac{8\pi n_j T_j}{\Omega_0^2}$ and $F(\beta_j)$ goes to infinity as $\beta_j$ goes to zero. In present machines this condition cannot be satisfied by ion whistlers, but it can be satisfied by electron whistlers, if $\beta_e$ is not too small.

According to Eq. (30) the plasma will also be unstable when the resonance velocity is much larger than the thermal velocity. In any thermalized laboratory plasma there will, of course, be very few particles so far out on the tail of the distribution, and the instability will be slow. So the point is that particles are swept into the loss cone (Fig. 10) when the plasma tries to form the quasi-linear plateau, but very slowly. C.F. KENNEL and H.E. PEPSCHEK [14] successfully applied this basic diffusion into the loss-cone mechanism to the plasma in the magnetosphere where the necessary high-energy particles exist in large enough numbers.
to make this instability an important mechanism for particle untrapping.

The importance of whistler-type loss-cone instability is often decreased by the competition of another instability due to the loss cone: namely, the Post-Rosenbluth one. This instability is a pure electrostatic mode with a more complicated kind of polarization. Since this is very important we will discuss it in detail.

Fig. 10
4. Quasi-linear theory of the Post-Rosenbluth "loss-cone" instability

We should like to consider in more detail the Post-Rosenbluth instability \[14\] in the case where the volume of the "loss cone" in the velocity space is very small (for example, it may correspond to the large mirror ratio). In this case, we will use expansion on the small parameter (the relative volume of the loss cone). Further, in real traps of finite length \( L \), the time of escape through the magnetic mirrors is finite and is of order \( L/V_z \)

where \( V_z \) is the ion velocity along magnetic field lines. Here we will consider the highly idealized situation where this time is much greater than the time of the quasi-linear diffusion into the loss cone \( \tau_D \)

\[
L > \tau_D V_z
\]

If we neglect the particle escape through the mirrors, the loss cone will be filled during the time \( \tau_D \) and then instability will disappear. The relaxation of the ion distribution function can be described in the frame of the quasi-linear theory. In the case of the large mirror ratio it turns out that the level of the turbulence is sufficiently small to be able to neglect mode-mode coupling.

As in Ref.\[15\] we assume that:

(a) unstable oscillations are electrostatic;

(b) perturbation scale is sufficiently small for the plasma to be regarded as homogeneous, so that the electric field potential \( \varphi \) can be expanded in a sum over the fields of the individual harmonic oscillations:

\[
\varphi = \sum \varphi_{n,\omega} \exp \left( -i \omega t + i k_z z + i \vec{k}_\perp \cdot \vec{r} \right)
\]

where \( k_z \) and \( \vec{k}_\perp \) are the components of the wave vector, respectively, along and transverse to the unperturbed magnetic field \( \mathbf{H}_0 = (0,0,H_z) \);

(c) electrons are cold;

(d) the frequency and wavelength of the oscillations are within the interval,
\[ \omega_{He} \gg \Im \omega \gg \omega_{Hi}, \quad k_{\perp} r_{H} \gg 1 \gg k_{\perp} r_{He} \] (33)

where \( \omega_{hi} \) is the gyrofrequency of the species, \( r_{H} = \frac{v_{thj}}{\omega_{Hi}} \) is the Larmor radius of the particle with thermal velocity \( v_{thj} \). Development of the instability within this interval of frequency is possible only in a dense plasma.

\[ \sqrt{\frac{4 \pi e^2 n}{m_i}} = \omega_{pi} \gg \omega_{Hi} \] (34)

Under the assumptions (33) we neglect the influence of the magnetic field on the ion motion and use the drift approximation for the description of the electron motion. Then the kinetic equations for the ions and electrons respectively can be written in the form

\[ \left\{ \begin{array}{l}
\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla - \frac{e}{m_i} \mathbf{v} \mathbf{\nabla} \phi \left( \mathbf{r}, t \right) \frac{\partial}{\partial \mathbf{v}} \int f_i \left( \mathbf{r}, \mathbf{v}, t \right) = 0 \ , \\
\frac{\partial}{\partial t} + \mathbf{v}_e \frac{\partial}{\partial \mathbf{v}_e} - \frac{e}{m_e} \frac{\mathbf{\nabla} \phi \times \mathbf{H}_e}{\mathbf{H}_e} \cdot \frac{\partial}{\partial \mathbf{r}} - \frac{e}{m_e} \frac{\mathbf{\nabla} \phi}{\mathbf{v}_e} \cdot \frac{\partial}{\partial \mathbf{v}_e} \int f_e \left( \mathbf{r}, \mathbf{v}, t \right) = 0 \end{array} \right. \] (35)

where \( \mathbf{M} \) is the magnetic moment of the electron (i.e., \( \mathbf{M} = m_e v^2 / 2 \mathbf{H}_e = \text{const} \)). The complete system of equations contains, besides Eqs. (35) and (36), the equation for the electric field potential

\[ \Delta \phi = -4 \pi \sum_{\mathbf{r}} \delta \left( \mathbf{r} - \mathbf{r}_j \right) \epsilon_j \int f_j \left( \mathbf{v} \right) d^3 \mathbf{v} \] (37)
In the linear approximation we reduce Eqs. (35)-(37) to the dispersion equation
\[ \mathcal{E}(\omega, \kappa) = 1 + \frac{\omega_{pe}^2}{\omega_{he}^2} + \frac{\omega_{pe}^2}{\kappa^2} \int_{-\infty}^{\infty} \frac{d\nu_x}{\omega - \kappa \nu_z + i \epsilon} \]
\[ \left. \frac{\partial f}{\partial \nu_z} \right|_{\nu_z = 0} = 0 \]

In a very dense plasma \( \omega_{pe} \gg \omega_{he} \) we must take into account in Eq. (36) the inertial drift of the electrons which corresponds to the additional term \( \omega_{pe}^2 / \omega_{he}^2 \) in Eq. (38). We omit this term in the following calculations because it changes only the definition of the plasma frequencies:
\[ \omega_{pi} \rightarrow \omega_{pi} \left( 1 + \frac{\omega_{pe}^2}{\omega_{he}^2} \right)^{-1/2} \]

Insofar as \( k_z \ll k_L \), we neglect the ion motion along magnetic field lines. Then the unperturbed ion distribution does not depend on the azimuthal angle in the velocity space, and we can easily integrate over this angle:
\[ \psi(w) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\phi}{\omega - k_z \nu_z \cos \phi} f_i(\nu_x, \nu_z) = \frac{1}{\nu_{thi}^2} \int_{-\infty}^{\infty} \frac{d\nu_x}{\nu_x^2} f_i(\nu_x, \nu_z) \]
\[ F(y) \equiv 2 \int_{d\nu} \frac{d\psi/d\nu}{\sqrt{1 - w/y^2}} \]

Here \( \psi(x) \) is the ion distribution with respect to the dimensionless
velocities \( w = \frac{v_0^2}{v_{th}^2} \), and it satisfies the normalization condition. The root of the integrand of \( F(y) \) corresponds to the taking of the integrals in (39) for \( \omega \) in the upper half plane; so that for \( y = y_r + i\xi \)

\[(1 - w/y^2)^{-\frac{1}{2}} = -iy_r (w - y_r^2)^{-\frac{1}{2}}, \quad w > y_r^2. \quad (40)\]

Finally, we neglect the electron thermal motion according to the third assumption and rewrite the dispersion equation in the form:

\[\mathcal{E}(\omega, \xi) = 1 - \frac{\omega_p^2}{\omega_r^2} \frac{k^2}{\omega_r^2} \left[ \psi(\omega) + F \left( \frac{\omega}{k v_{th}} \right) \right] = 0. \quad (41)\]

For the plasma near the marginally stable state one may expand Eq. (41) over \( \text{Im} \omega \equiv y \ll \omega_r \):

\[y = -\frac{\omega_r \text{Im} \mathcal{F} \left( \frac{\omega}{k v_{th}} \right)}{2 \left[ \frac{k^2 \lambda^2_{Di}}{\omega_r} + \psi(\omega) + F_r \left( \frac{\omega_r}{k v_{th}} \right) \right]}; \quad (42)\]

\[\omega_r = \pm \omega_p e \frac{k^2}{\omega_r} \left\{ 1 + \left[ \psi(\omega) + F_r \left( \frac{\omega}{k v_{th}} \right) \right] \right\}^{-\frac{1}{2}} \frac{k^2 \lambda^2_{Di}}{\omega_r} \]

The solution with \( \text{Im} \omega > 0 \), corresponding to growing disturbances, appears only if

\[\int_0^{\infty} dw \frac{\partial \psi}{\partial w} w^{-\frac{1}{2}} > 0 \quad (43)\]

The particle escape from the loss cone through the magnetic mirrors leaves this condition unfulfilled.
Averaging the kinetic equation over the period of the rapid oscillations in the co-ordinate space we derive the quasi-linear equation for the averaged distribution function \( \rho \): \[
\frac{\partial \rho}{\partial t} = -\frac{e}{m_i} \sum_{\mathbf{k},\omega,\omega'} \mathbf{v}_{\mathbf{k}} \frac{\partial}{\partial \mathbf{v}} \rho_{\mathbf{k},\omega,\omega'}(\mathbf{v})
\]

Under our first assumption \( L > T_c T_D \) (\( T_D \) is the diffusion time into the loss cone) no loss of the particles from the trap appears during the process of the relaxation of the ion distribution. Then, due to the small volume of the loss cone (our second assumption) the wave energy is small. Hence, we can neglect mode-mode coupling and take the rapidly oscillating part of the distribution function \( \rho_{\mathbf{k},\omega} \) and the electric field potential \( \mathbf{E}_d \) in the linear approximation

\[
\rho_{\mathbf{k},\omega} = \rho_{\mathbf{k},\omega} \delta_{\omega,\omega^*}
\]

where \( \omega^* \) is the frequency of the eigenoscillation with wave vector \( \mathbf{k} \) and amplitude \( \rho_{\mathbf{k},\omega} \) ;

\[
\delta_{\omega,\omega^*} = \left\{ \begin{array}{ll} 1 & , \omega = \omega^* \\ 0 & , \omega \neq \omega^* \end{array} \right.
\]

Within this approximation we reduce Eq. (44) to the "quasi-linear equation" in the usual form to be valid now even for strong instability \( \gamma \sim \omega^* \) if \( \gamma \ll k V_{th} \).
We used here the axial symmetry of the distribution function in the velocity space to average Eq. (44) over the azimuthal and longitudinal components of the ion velocity. We can do this even for the wave spectrum without axial symmetry in the $k$-space since the gyration of the particles in the strong magnetic field symmetrizes the velocity distribution.

The quasi-linear kinetic equation (45) and the equation

$$\frac{\partial}{\partial t} \langle \psi(z) \rangle = 2 \int_0^\infty \left( \frac{\partial}{\partial z} \int_0^\infty \frac{d\psi(z)}{d\psi(z)} \right) \, dz$$

where $\langle \psi(z) \rangle$ is determined in terms of $\int_0^\infty \langle \psi(z) \rangle$ by the expression (42), are the basic system, which we have now to solve to find the time evolution of plasma with the loss cone for $t = 0$. This is actually a very complicated system. However, we can simplify Eq. (45) in the same way as we did once in Section 1 of Part II, with Eq. (12). This procedure is right if the main contribution in the integral (43)

$$\int_0^\infty \frac{d\psi(z)}{d\psi(z)} \, dz$$

comes from that domain of the velocity space where $V | V_0 \approx \omega/k_1$. If we take the initial distribution function in the form $a)$ shown qualitatively in Fig. 11,
the condition $\nu \gg \frac{\omega}{k}$ would mean that $\frac{\omega}{k} \ll \nu_p$, where $\nu_p$ is the plateau size. Actually, any kind of initial distribution, even, say, of the type $b)$ very soon will have a form $a)$, since the quasilinear diffusion is very high for smaller $\nu$.

Thus, Eq. (15) might be rewritten in the approximate form [17]

$$\frac{\partial \psi}{\partial D} = \frac{2}{25} \frac{\partial}{\partial w} w^{-\frac{1}{2}} \frac{\partial \psi}{\partial w}$$

(47)

where

$$D = \frac{25 \Sigma}{k} \int_0^1 y_x w_x \frac{e^2 |\psi_x|^2(t)}{m_i^2 \nu_{thi}^4} dt.$$

This equation can be easily solved using Laplace transform. The result is [17]

$$\psi(w, D) = \int_0^\infty dw' \psi_0(w') G(w, w'; D)$$

(48)

where $\psi_0(w)$ is the initial ion distribution,

$$G(w, w'; D) = \frac{5}{2} \alpha e^{-\frac{1}{4} \left(\frac{5}{2} + w + w' \frac{5}{2}\right)} \int_{\frac{3}{5}}^\infty \left[2d w \frac{5}{4} w' \frac{5}{4}\right] w \frac{3}{4} \frac{3}{4}.$$

is the Green function of Eq. (46); $\alpha = D^{-1}$.

Now we can find the growth rate $\nu_2$ in terms of $\psi(w, D)$, and substitute it in Eq. (46):

$$\frac{d|\psi_2|^2}{dt} = \frac{\nu_2 \omega_x}{\kappa^2 \lambda_{\nu}^2 + \nu_0} \int_0^\infty dw \frac{d\psi(w, D)}{dw} w^{-\frac{1}{2}} \times |\psi|^2$$

(49)

- 76 -
Then we will have \[17]\]

\[
\frac{dD}{dt} = 5 \sum_k \frac{y_k^2 \omega_k^2}{\psi(0) + k^2 \lambda_{D1}^2} \frac{e^2 / \psi_k^2}{m_i^2 \nu_{ri}^{-4}} \frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{4}{5})} \int_0^\infty ds \left( \frac{d}{s} \right)^{\frac{1}{5}}
\]

\[
\psi_0 \left( \frac{s^{\frac{1}{5}}}{\omega^{\frac{1}{5}}} \right) e^{-\frac{s^{\frac{1}{5}}}{2}} \left\{ s^{-\frac{1}{10}} M_{-\frac{1}{10}, \frac{3}{5}} (s) - \frac{2}{5} s^{-\frac{3}{5}} M_{\frac{3}{5}, \frac{3}{10}} (s) \right\}
\]

(50)

where \(M_{\mu, \nu}\) is the degenerate hypergeometric function [18].

Let us consider the idealized ion distribution for which we can take the integrals of Eqs. (48) and (50) exactly:

\[
\psi_0 (w) = A (1 - \Delta) e^{-\frac{w^{\frac{5}{2}}}{\epsilon^{\frac{5}{2}}}} e^{-w^{\frac{5}{2}}}
\]

(51)

where \(A \equiv \frac{\Gamma \left( \frac{1}{5} \right)}{2 \Gamma \left( \frac{4}{5} \right) \epsilon^{\frac{5}{2}} \left( 1 + \epsilon^{\frac{5}{2}} \right)^{\frac{2}{5}} \Delta \epsilon} \) is the normalization constant;

\(1 \geq \Delta \geq 0\) (\(\Delta = 1\) corresponds to the case of empty loss cone: \(\psi_0 (0) = 0\)).

Then the solution for the distribution function and the equation for wave amplitude can be written in the form

\[
\psi (w, D) = A e^{-\frac{w^{\frac{5}{2}}}{1 + D}} \left\{ (1 + D)^{-\frac{2}{5}} - \Delta (1 + D + D/\epsilon^{\frac{5}{2}})^{\frac{2}{5}} \right\}
\]

(52)

\[
\times \exp \left[ -\frac{w^{\frac{5}{2}}}{\epsilon^{\frac{5}{2}} (1 + D) (1 + D + D/\epsilon^{\frac{5}{2}})} \right]
\]

\[
\frac{d|\psi_k|^2}{dt} = \omega_{pi} \frac{y_k^2 K^2 \lambda_{Di}^2}{\psi(0) + \psi(D)} \left\{ \psi_k^2 \right\} A \Gamma \left( \frac{4}{5} \right) \left\{ \frac{\Delta \epsilon^{-\frac{5}{2}} \left[ 1 + D/\epsilon^{\frac{5}{2}} \right]^{\frac{2}{5}}}{(1 + D)^{\frac{5}{2}}} - \frac{1}{(1 + D)^{\frac{5}{2}}} \right\}
\]

(53)
From the last expression we see that, if initially $\Delta > \sqrt{\epsilon}$, the growing oscillations cause the turbulent diffusion of ions into the loss cone.

As we can see from expression (52), for the particular choice of the initial ion distribution (51) the quasi-linear relaxation of the turbulent spectrum in time can be described by changing in time the parameters $\Delta(t)$, $\epsilon(t)$ and the effective "temperature" of the main body of the ion distribution only. This change can be represented by using the formula (42):

$$\Delta(t) = \Delta \sqrt{1 + D + D/\epsilon^{5/2}}$$

$$\epsilon(t) = \epsilon [1 + D + D/\epsilon^{5/2}]^{2/5}$$

$$T_i(t) = T_i [1 + D]^{2/5} \quad (54)$$

Now it is easy to describe the solution of Eq. (53) qualitatively. At some moment $t_0$ we reach the marginally stable point $\Delta(t_0) = \sqrt{\epsilon(t_0)}$ and the oscillations stop growing. However, the ions continue to diffuse into the loss cone (see Eq. (47)) and the oscillations become damping. It is obvious that the relaxation process stops only when the oscillation amplitude reaches the zero level, when $t \to \infty$.

The final ion distribution has a margin of stability with respect to the considered perturbations (i.e., $\chi < 0$ for arbitrary wave vector $k$). The parameters of this distribution depend on $D_\infty$ only and can be found from the energy conservation law:

$$\int_0^\infty \psi_0(w) \omega d\omega = \int_0^\infty \psi(w, D_\infty) \omega d\omega =$$

$$= D^{3/5} \frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{2}{5})} \int_0^\infty d\omega \psi_0(\omega) \omega^{-3/2} \epsilon^{-\frac{W^{5/2}}{2D_\infty}} M_{\frac{3}{5}, -\frac{3}{10}} \left(\frac{W^{5/2}}{2D_\infty}\right) \quad (55)$$

where $D_\infty = \lim_{t \to \infty} D(t)$

* In Ref. [17] it was proposed to use for this aim the equation $\chi(D_\infty) = 0$ because after the first state of relaxation to the quasi-steady state (55) the particle loss through the mirrors supports the instability on a very low level. But in that case we must add at the r.h.s. of Eq. (47) the term describing this particle loss and change the expression for the growth rate (50) (This was not done in Ref. [17].)
From (52) and (55) we immediately obtain

\[
1 + \frac{\Delta \varepsilon^2}{(1 + \varepsilon^{5/2})^{4/5}} = \frac{\Delta \varepsilon^2}{(1 + D_{\infty} + D_{\infty}/\varepsilon^{5/2})^{5/3} (1 + D_{\infty}^{5/2})^{5/2}}
\]

In the limit of large mirror ratio the parameter \( \varepsilon \) is small and the amplitudes of the oscillations remain small during the relaxation process (i.e., \( D \ll 1 \)). On the other hand, by reducing the quasi-linear equation (45) to the form (47) we suppose that the width of the sink on the distribution function essentially increases due to the quasi-linear diffusion. Hence, we can apply the result (56) only to the case of initially strong instability

\[
\Delta \gg \sqrt{\varepsilon}
\]

Expanding (56) on small parameters \( \varepsilon \), \( D_{\infty} \), \( \sqrt{\varepsilon}/\Delta \ll 1 \) we obtain

\[
\frac{D_{\infty}}{\varepsilon^{5/2}} = \left( \frac{5 \Delta}{2 \sqrt{\varepsilon}} \right)^{5/3} \gg 1
\]

The ion distribution in the large time limit, as it follows from Eqs. (52) and (58), is equal to

\[
\psi(w, D_{\infty}) \approx \frac{5}{2 \Gamma(1/2)} e^{-w^{5/2}} \left\{ 1 - \left( \frac{5 \Delta}{2} \right)^{5/3} \left[ \frac{1}{2} + w^{5/2} \right] \right\} - \left( \frac{2 \sqrt{\varepsilon}}{5 \Delta} \right)^{5/3} e^{\frac{w^{5/2}}{(2/5 \varepsilon \Delta)^{5/3}}}
\]
Here we can see directly that the sink on the ion distribution under condition (57) becomes wider and less deep after relaxation, and we can justify the approximation (47). The main part of the distribution function is only slightly disturbed by "quasi-linear diffusion".

In order to write the equation for the waves in terms of one variable \( D \) only, we need to suppose, as usual in the quasi-linear theory, that the wave spectrum has a sharp maximum at the point \( \mathbf{k} = \mathbf{k}^* \), 
\[ \gamma(\mathbf{k}^*) > \gamma(\mathbf{k}^*) \] . Now we neglect the change of frequency for \( D(t) \gg \varepsilon^{5/2} \) and integrate Eq. (53) up to the quadrature with conditions 
\[ \frac{dD}{dt} \Big|_{t=\infty} = 0 \quad , \quad D(\infty) = D_\infty \]

\[
Bt = \int \frac{d\mathbf{x}}{5\Delta \varepsilon} \frac{(1 + x)^{3/2} - x}{D_\infty / \varepsilon^{5/2}}
\]

(60)

where
\[
B \approx \frac{5 \omega_p}{2} \mathbf{y}^2 \mathbf{y}^2 \frac{k^* \lambda_p}{k^* \lambda_p + \gamma(0, D_\infty)} \frac{\Gamma(4/5)}{\Gamma(2/5)}
\]

For the distribution (51) the maximal growth rate is achieved at point

\[
\mathbf{y}^* \sim \varepsilon^{1/2} \quad , \quad k^* \lambda_p \sim \sqrt{F_p(y^*)} \sim \Delta^{1/2}
\]

(61)

Parameter \( D_\infty \) in the saturation regime can be easily found from Eq. (60) as well as from Eq. (55)

\[
\left[ \frac{D(t)}{\varepsilon^{5/2}} \right]^{3/5} - \frac{5 \Delta}{2 \sqrt{\varepsilon}} \approx e^{-\frac{3}{5} B^* t}, \quad t \to \infty
\]

(62)
By doing this, we introduced a lot of idealizations in order to get soluble equations. Of course, in a realistic situation, the development and non-linear relaxation of the loss cone instability have a more complicated nature, due, for example, to the continuous loss of particles through the mirrors, which we did not take into account, and so on. In these circumstances, the non-linear mode-mode coupling becomes very important, as was shown in Ref. [17].

5. Non-resonant adiabatic wave-particle interaction

Up till now we have considered only the resonant interaction of particles and waves. For their non-resonant (or adiabatic) interaction we can no longer approximate the resonance term in the diffusion equation by a $\delta$-function. For example, in the case of Langmuir oscillations, the diffusion equation takes the form

\[
\frac{\partial \tilde{f}}{\partial t} = \frac{e^2}{m^2} \frac{\partial}{\partial v} \sum_{\kappa} |E_\kappa|^2 \frac{t}{i(\kappa v - \omega_\kappa)} \frac{\partial \tilde{f}}{\partial v} = 
\]

\[
= \left( \frac{e}{m} \right)^2 \frac{\partial}{\partial v} \sum_{\kappa} |E_\kappa|^2 \frac{J_{m \omega_\kappa}}{\omega_\kappa - \kappa v^2} \frac{\partial \tilde{f}}{\partial v} = 
\]

\[
\approx \left( \frac{e}{m} \right)^2 \frac{\partial}{\partial v} \sum_{\kappa} J_{m \omega_\kappa} \frac{1}{\omega_p^2} \frac{E_\kappa^2}{\omega_\kappa^2} \frac{\partial \tilde{f}}{\partial v}
\]

where we have approximated $1/\omega_\kappa - \kappa v^2$ by $\omega_p^2 = \omega_\kappa^2$ in the last step. Using the equation for wave growth, we can rewrite this as

\[
\frac{\partial \tilde{f}}{\partial t} = \frac{i}{2} \frac{e}{m} \omega_p^2 \frac{d}{dt} \left( \sum_{\kappa} |E_\kappa|^2 / 4\pi n \right) \frac{\partial \tilde{f}}{\partial v^2} = 
\]

\[
= \frac{i}{2m} \frac{d}{dt} \left( \sum_{\kappa} |E_\kappa|^2 / 4\pi n \right) \frac{\partial^2 \tilde{f}}{\partial v^2} .
\]
Multiplying both sides of Eq. (63) to $\frac{m v^2}{2}$ and integrating over all $v$, we have

$$\frac{d}{dt} \frac{m}{2} \int f v^2 dv = \frac{d}{dt} \sum \frac{|E_k|^2}{8 \pi}$$

(64)

From Eq. (64) it is obvious that non-resonant interaction in quasi-linear theory has a trivial meaning: it describes the participation of the main body of the plasma distribution in the oscillations. Therefore, such interaction should be taken into account in order to have an energy conservation law, and so on.

Let us come back to Eq. (63). Changing co-ordinates from $t$ to $\tau = \sum \frac{|E_k(t)|^2}{4 \pi n}$ gives

$$\frac{df}{d\tau} = \frac{1}{2m} \frac{\partial^2 f}{\partial v^2}$$

(65)

If we choose the initial conditions $\bar{f}(v) = \delta(v)$ and $\tau = 0$, then Eq. (65) has the solution

$$\bar{f}(v) = \sqrt{\frac{m}{2 \pi \tau}} e^{-\frac{m v^2}{2 \tau}}$$

(66)

Consequently, the non-resonant interaction heats the bulk of the plasma to an effective temperature $\tau = \sum \frac{|E_k|^2}{4 \pi n}$.

We should note in passing that the Gaussian shape of the final distribution is another demonstration that quasi-linear theory cannot be applied to a single wave. For a single monochromatic wave generated in an initially cold plasma, the distribution would be of the form

$$f(v) = \frac{1}{\pi \frac{2 e E}{m \kappa} - v^2}$$

(67)

A lot of instabilities in plasma are due to the non-resonant interaction of plasma and waves. It is obvious that the quasi-linear relaxation in such cases should be described in terms of adiabatic
interaction. Let us start with the firehose instability [19, 20] which gives the simplest example of adiabatic wave–particle interaction [21]. In this instability the $R_e(\omega)$ is zero and the $I_m(\omega)$ is given by

$$I_m(\omega) \approx K V_k h_i \sqrt{\frac{\eta - T_e}{T_e}}$$

when $\beta \gg 1$.

If the plasma is not far from marginal stability (i.e., $\frac{\Delta T}{T} \ll 1$) then $I_m(\omega)$ will be much less than $K V_k h_i$, and we can apply quasi-linear theory. Note that quasi-linear theory does not always require $I_m(\omega) \ll R_e(\omega)$; as mentioned above, the $R_e(\omega)$ is in fact zero for this instability. Since $K V_k h_j \ll \omega h_j$ for the firehose instability, there will be no resonant particles; the instability itself is algebraic in nature and does not require resonant particles.

To get the diffusion equation, we replace $\omega K$ in Eq. (20a) by $i I_m(\omega_K)$ and use the reality condition $I_m(\omega_K) = I_m(\omega_D)$. This procedure gives

$$\frac{\partial \bar{f}_j}{\partial t} = \frac{e_j^2}{2 m_j c^2 \omega_{h_j}^2} \left\{ V_k^2 \frac{\partial^2 \bar{f}_j}{\partial V_k^2} + V_e^2 \frac{1}{V_k} \frac{\partial}{\partial V_k} \frac{\partial \bar{f}_j}{\partial V_k} \right\} \left( \sum_{k} \frac{H_k^2}{H_k} \right)$$

(69)

where we have used

$$|E_k|^2 = \frac{\omega_k^2}{k^2 c^2} \frac{1}{H_k}$$

(70)

We can solve Eq. (69) by the following approximate procedure. Since the plasma is near marginal stability (i.e., $\frac{\Delta T}{T} \ll 1$), the distribution can be expressed as

$$f \approx f_M + \frac{\Delta T}{T} f_1$$

where $f_M$ is a Maxwellian and $f_1$ is a corrective term producing the
anisotropy. If we neglect this small corrective term when evaluating
the r.h.s. of Eq. (69) we obtain

$$\frac{\partial f_i}{\partial t} = \frac{e_i^2}{2 m_j^2 c^2 \omega_{\nu j}} \left( \frac{u_{\nu j}^2 - 2 u_{\lambda j}^2}{u_{\nu j}^2} \right) f_{2m} \frac{d}{dt} \sum_K |H_k|^2 \quad (71)$$

The solution of this equation is

$$f_i = f_0 + \frac{e_i^2}{2 m_i^2 c^2 \omega_{\nu i}} \left( \frac{u_{\nu i}^2 - 2 u_{\lambda i}^2}{u_{\nu i}^2} \right) f_{2m} \sum_K |H_k|^2 \quad (72)$$

This function can now be used to calculate the growth rate

$$\text{Im}(\omega_k) = \kappa \sqrt{\int d\nu \text{Im}(\nu \omega_k)} \approx$$

$$\approx \kappa \frac{(\Delta T)}{T} \sqrt{ \left( \frac{\Delta T}{T} \right)_0 - 3 \frac{e_i^2}{m_i^2 c^2 \omega_{\nu i}^2} \sum_K |H_k|^2 } \quad (73)$$

The maximum growth rate in the linear theory is found for $\kappa \approx \frac{1}{T} \sqrt{\omega_{\nu i}} \quad [22]$. If we are willing to use this value of $\kappa$ for the coefficient in Eq. (73), we obtain a tractable non-linear equation for the growth of the waves

$$\frac{d}{dt} \sum_K |H_k|^2 = \sum_K |H_k|^2 \omega_{\nu i} \left[ \left( \frac{\Delta T}{T} \right)_0 - 3 \frac{\sum_K |H_k|^2}{H_0^2} \right] \quad (74)$$

The solution to this equation is of the form

$$\sum_K |H_k|^2 = \left( \frac{\Delta T}{T} \right)_0 \frac{h_0 e^{(\Delta T)/T_0 \omega_{\nu i} t}}{1 + h_0 e^{(\Delta T)/T_0 \omega_{\nu i} t}}$$

where $h_0 = \left. \sum_K |H_k|^2 \right|_{t=0}$.
One can give a simple physical interpretation for the removal of the temperature anisotropy. The instability is low frequency so the integral

$$J = \sum_{k} |\mathbf{H}_{k}|^2$$

should be constant. At first the field lines are straight, but, as the amplitude of the wave builds up, the field lines become very wiggly.

Since the path of integration in Eq. (75) becomes longer $V_{\parallel}$ must become smaller, and this presumably reduces $T_{\parallel}$.

Although the adiabatic quasi-linear diffusion describes the wave interaction with all particles, nevertheless the strength of this interaction might be quite different for different domains in velocity.
space. In such situations the quasi-linear distribution function might, in the process of relaxation, have a quite peculiar form.

An example of such quasi-linear theory is the case of the "whistler" mode with no (initial) magnetic field [23,24]. If we have a non-isotropic distribution function, it is usually unstable against whistler type perturbations.

![Fig. 25](image)

Let us consider such an anisotropic distribution \( f(\mathbf{v}_x^2, \mathbf{v}_y^2) \). As indicated in Fig. 25, the effective temperature in the x-direction is greater than that in the y-direction. We can easily show that a pure transverse perturbation (\( \mathbf{E} \perp \mathbf{k} \)) will be unstable for any arbitrarily small anisotropy. To do this, we write the linearized Vlasov equation, where the first order quantities contain the factor \( e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \):

\[
-i(\omega - \mathbf{k} \cdot \mathbf{v}_e) f_j + \frac{\mathbf{E} \cdot \mathbf{x}}{m_j} \frac{\partial f_{0j}}{\partial \mathbf{v}_x} + \frac{\mathbf{E} \cdot \mathbf{y}}{m_j c} \mathbf{v}_x \frac{\partial f_{0j}}{\partial \mathbf{v}_x} - \frac{e}{m_j c} \mathbf{H} \cdot \mathbf{v}_x \frac{\partial f_{0j}}{\partial \mathbf{v}_x} = 0, \quad j = i, e
\]

and
So we have four equations for $f_i, f_e, H_y, E_x$. We get (expressing the fields in terms of $H_y$)

$$ i \mathbf{k} \cdot E_x = \frac{i\omega}{c} H_y \quad (78) $$

Substituting in the expression (77) for the current, we get

$$ f_j = \frac{1}{i(\omega - \mathbf{k} \cdot \mathbf{u}_e)} \left\{ \frac{eH_y^j}{m_j c} \left[ \mathbf{v}_e \frac{\partial f_j}{\partial \mathbf{v}_e} - \mathbf{v}_e \frac{\partial f_j}{\partial \mathbf{v}_e} \right] - \frac{e}{m_j} \frac{\omega}{\mathbf{k} \cdot \mathbf{u}_e} H_y \frac{\partial f_j}{\partial \mathbf{v}_e} \right\} \quad (79) $$

Suppose now that the ion distribution is anisotropic. (If $f_{0j}$ were isotropic, the magnetic field terms in the Lorentz force would cancel.) Substituting in the expression (77) for the current, we get

$$ -i \mathbf{k} \cdot H_y = -\frac{4\pi e}{c} \left\{ \frac{eH_y}{m_i c} \int \frac{d^3 \mathbf{v}_i}{\omega - k \cdot u_e \text{r}_i} \left\{ \mathbf{v}_e f_i + \mathbf{v}_i \frac{2}{\omega} \frac{\partial f_i}{\partial \mathbf{v}_i} \right\} \right\} \quad (80) $$

from the current in the x-direction.

The dispersion relation now has a simple form

$$ \omega^2 \mathbf{k}_x = \sum_j \frac{4\pi e^2}{m_j c^2} \left\{ \int d^3 \mathbf{v}_j \frac{K_{2j} V_{x,j}^2}{\omega - K_{2j} V_{x,j}^{\text{r}_j \text{r}_j} \mathbf{v}_j} \frac{\partial f_j}{\partial \mathbf{v}_j} + 1 \right\} \quad (81) $$

If the plasma is anisotropic, this becomes the usual dispersion relation for waves in a plasma in the absence of a magnetic field.

When we have an anisotropic distribution, we can look for a new root...
of this equation. It arises as very low frequency mode. Let us suppose that \( f \sim e^{-\frac{m v_x^2 - m v_z^2}{2T_e}} \). Taking \( \omega < k_z v_{th} \), we can derive from (81)

\[
\omega = i \frac{c^2}{\tau_0^2} \omega_{pe}^2 \left\{ \sum_j \left( \frac{T_x}{T_{e,j}} - 1 \right) - k_z^2 \right\} \frac{T_e}{T_x} |k_z| \sqrt{\psi_{t\theta \ell}} \tag{82}
\]

Now we can recognize that this wave is unstable for sufficiently small \( k_z^2 \) if \( 1 - \frac{T_x}{T_{e,j}} < 0 \) (In terms of a more general distribution function \( f(v_x^2, v_z^2) \), it would be \( \int \frac{v_x^2}{v_z^2} \frac{df}{v_z} dv_z + 1 < 0 \). There is no real part in \( \omega \), and therefore it describes an aperiodic instability. The imaginary part is very small if the electron distribution is anisotropic.

Suppose we have the opposite situation, \( T_x > T_{e,j} \). Then instability results if the direction of propagation is rotated (i.e., by interchanging the roles of \( x \) and \( z \)) by 90 degrees. So this situation is absolutely unstable, even for very small anisotropy.

If we look at the imaginary part of \( \omega \), we see that it increases with increasing wave number; but we cannot take \( k_z \) very large because the additional term becomes significant and gives stabilization. Thus there exists a critical wave number below which waves are unstable:

\[
k_z^2 < \frac{\Delta T}{T} \frac{\omega_{pe}^2}{c^2}.
\]

Let us see how quasi-linear theory may be applied to such a wave. To find it we consider the quasi-linear theory for this kind of perturbation. The equation for \( f \) has the form

\[
\frac{\partial f}{\partial t} + \left[ \frac{eE_z}{m} + \frac{e}{mc} \frac{H_z}{z^2} \right] \cdot \frac{\partial f}{\partial \nu} = 0 \tag{83}
\]

where \( \sim \) denotes rapidly varying functions. Next we use the expression found in the linear theory for the rapidly varying functions, as is customary. The quadratic expression is averaged over many cycles, and we obtain the final result.
\[
\frac{\partial f}{\partial t} = J m \frac{e^2}{m^2 c^2} \left( \frac{v}{c} \right) \frac{\partial}{\partial v} \left( \frac{v}{\omega + k \cdot v} \right)
\]

\[
\left\{ \frac{\omega}{k \cdot c} E + v \times \mathcal{H} \right\} \frac{\partial f}{\partial v}
\]

where the bar indicates the average.

We can represent \( \frac{i}{\omega + k \cdot v} \) as \( \frac{d/\partial t}{i/\omega^2 + k^2 v^2} \) and neglect \( \omega^2 \) in the denominator since \( \omega^1 \ll k^2 v^2 \).

Now we can easily go through the same sort of calculation as that just completed for the firehose instability. Finally, the quasi-linear equation becomes

\[
\frac{\partial f}{\partial t} = \frac{e^2}{m^2 c^2} \sum_k \frac{1}{k^2} \left( \frac{\partial}{\partial t} \langle \mathcal{H}_k \rangle \right) \left( \frac{\partial}{\partial v_x} \frac{\partial}{\partial v_y} \right) \frac{\partial f}{\partial v_x} \frac{\partial f}{\partial v_y} (84)
\]

The physical meaning of this quasi-linear diffusion in the velocity space is clear. The instability creates magnetic flutes, producing a chaotic magnetic field. This chaotic field influences the motion of the particles. Thus we have scattering of particles due to small-scale fluctuations in the magnetic field.

Although this case is an adiabatic interaction of modes with all particles, we can see from Eq. (84) that the quasi-linear diffusion for the particles with small \( v_2 \) is much greater than for the main body of the distribution.

Therefore one can expect the considerable reconstruction of particle distribution to occur mostly for very small regions, where \( \mathcal{V}_2 \ll \mathcal{V}_{th} \).

- 89 -
6. Quasi-linear theory of drift instability

A. Linear theory of drift waves

One of the most important types of plasma instability is the drift instability of non-uniform plasma, which leads to increasing heat and particle loss due to the turbulent transport process in the plasma. Therefore, more careful attention should be paid to the influence of this type of instability on plasma confinement in traps and to the non-linear stage of their development. Let us consider here in detail the low-frequency (i.e., $\omega \ll \omega_{\pi i}$) "universal drift instability" of a plane slab of low $\beta$ plasma in a strong magnetic field $H=\{0,0,H\}$ [25-28]

$$\frac{m_i}{m_e} < \frac{4\pi n_0 \left( T_i + T_e \right)}{H^2} \equiv \beta << 1$$  \hspace{1cm} (85)

We take the plasma to be inhomogeneous in the $x$-direction, with $\nabla T_{i,\perp} = 0$ and small density gradient

$$\Gamma_{wi} \frac{v_n}{n_i} < \sqrt{\frac{m_e}{m_i}}$$  \hspace{1cm} (86)

For simplicity we put $T_i = T_e$ and $\omega_{pi}^2 \gg \omega_{\pi i}^2$, so the perturbation can be considered as quasi-neutral. Then the plasma is unstable in respect to the electrostatic perturbation with the phase velocity within the interval

$$U_{bi} \quad < \quad \frac{\omega}{K_z} \quad < \quad U_A = \frac{H_0}{4\pi n_i M}$$  \hspace{1cm} (87)

The electric field potential $\varphi$ is written here as the sum of propagating plane waves

$$\varphi = \sum_{\kappa, \omega} \varphi_{\kappa, \omega} \ e^{-i(\omega t - \kappa \cdot r)}$$  \hspace{1cm} (88)

Generally speaking, the equation for the $x$-dependence of the wave disturbance has a form of the integro-differential equation [29-31] which has in the WKB approximation a solution of the usual type:
Here $k_\infty$ is the complex function, but with small imaginary part for the instability with small growth rate. Then the square of this function 

$$-k_\infty^2(x, \omega, k) \equiv \omega_{\perp} \xi(k) + i V_{\omega, \xi}(x)$$

plays a role analogous to the complex potential of the Shrödinger equation. The turning points of the real part of " $-k_\infty^2$ " restricts the region of coordinate space, where the wave packet in the form (88) can move. The description of the unstable perturbation by wave packets (88) is justified if they grow in the time between reflection from the turning points up to the level at which we cannot neglect the non-linear mode coupling [32].

Therefore, by choosing perturbation in the form (4) we suppose that [17]

$$\delta_\infty \Delta x \frac{\partial \omega}{\partial k_\infty} > \Lambda = \ln \left| \frac{\varphi^{(a)}}{\varphi^{(b)}} \right|$$

(90)

where $\varphi^{(a)}$ is the amplitude of the wave in the quasi-stationary turbulent state and $\varphi^{(b)}$ is the amplitude of the electric field fluctuation at time $t = 0$ (for the quiet plasma it is the amplitude of the thermal fluctuation).

We start by deriving the linear dispersion relation for the drift instability with the help of the integration of the Boltzmann equation over the unperturbed particle trajectory [34]. The equilibrium distribution function depends on the constants of motion $\mathcal{V}_1^2, \mathcal{V}_1, \mathcal{X}_1 = x + \frac{\xi}{\mathcal{V}_1}$, only,

$$f_{\mathcal{X}_1} = f_{\mathcal{X}_1} (\mathcal{V}_1^2, \mathcal{V}_1, \mathcal{X}_1)$$

(91)

* Convection of the wave packets in the non-uniform plasma was considered in [33]. The instability condition found there is similar to (90).
In the linear approximation we can write the Boltzmann equation with an electrostatic potential perturbation in the form
\[
\left\{ \frac{\partial}{\partial t} + \frac{e_j}{m_j c} \left[ \nabla \times H \right] \cdot \frac{\partial}{\partial v} \right\} \delta f_j^{(t)} = - \frac{e_j}{m_j} \nabla \varphi \left( \frac{\partial \delta f_j}{\partial v} \right) \tag{92}
\]

On the l.h.s. we have a derivative taken along the unperturbed trajectory of particles in the magnetic field
\[
x_j(t') = x_j(t) + \frac{V_m}{\omega_{nj}} \left[ \sin \left( \theta_j(t) - \omega_{nj}(t' - t) \right) - \sin \theta_j(t) \right]
\]
\[
y_j(t') = y_j(t) + \frac{V_m}{\omega_{nj}} \left[ \cos \left( \theta_j(t) - \omega_{nj}(t' - t) \right) - \cos \theta_j(t) \right] \tag{93}
\]
\[
z_j(t') = z_j(t) + \frac{V_m}{\omega_{nj}} (t' - t)
\]

where $V_m$ and $\omega_{nj}$ are the velocities of the particle across and along the magnetic field, and $\theta_j$ is the angle the velocity vector makes with the $U^\parallel$ axis at the instant $t' = t$.

We will write these in vector form using the unit vector $\hat{h} = \frac{h}{H}$.

\[
\hat{r} = - \frac{[\nabla \times \hat{h}]}{\omega_{nj}}
\]

Carrying out the integration over trajectories in (92) using (91) and (93), we obtain the first order correction to $f_j$ in the form
\[
\delta f_j^{(t)} = \frac{i e_j}{m_j} \sum_k \varphi_k \left[ e^{i (\omega_k (t - k \cdot r))} \right] \left[ \frac{k \cdot \hat{u}}{\omega_{nj}} \right] \cdot (t' - t) - i \left[ \frac{k \cdot \hat{u}(t')} {\omega_{nj}} \right] \cdot \hat{h}
\]

where all quantities which do not depend on the time $t'$ following the
trajectory have been carried outside the integral.

Now we use the fact that the total time derivative along the particle trajectory is

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial z} = -i \omega_z + i k \cdot \mathbf{v}(t) \quad (95)$$

and use the Bessel function expansion

$$e^{-\frac{i k \cdot \mathbf{v}}{\omega_{j}}} = \sum_{l=-\infty}^{+\infty} \mathcal{J}_l \left( \frac{k_x v_x}{\omega_j} \right) e^{i \left( \frac{\pi}{2} + \theta_j - \omega_j t - \psi_k \right)} \quad (96)$$

where

$$\kappa = \left\{ -k_x \sin \psi_k, k_x \cos \psi_k, k_z \right\}$$

Then we can do the time integration in (94) and obtain the Fourier component of the perturbed distribution function

$$f_{ij}^{(n)} = \frac{e^{i \psi_j}}{m_j v_j} \left\{ \frac{\partial}{\partial v_j} - \sum_{l=-\infty}^{+\infty} \omega \frac{\partial}{\partial v_j} + \kappa_x v_z \left( \frac{\partial}{\partial v_j} - \frac{\partial}{\partial v_j} \right) + \frac{k_y}{\omega_j} \frac{\partial}{\partial X_j} \right\}$$

$$\cdot \mathcal{J}_l \left( \frac{k_x v_x}{\omega_j} \right) e^{i \left( \frac{\pi}{2} + \theta_j - \omega_j t - \psi_k \right)} \left\{ f_{ij}(v_j^x, v_j^z, X_j) e^{-i \omega t - \kappa \cdot t} \right\} \quad (97)$$

Here the small additional term $i \in$ has been inserted to make the integral convergent, and corresponds to the adiabatic switching on of the perturbation at $t = -\infty$.

Let us now consider the stability of the local Maxwellian distribution

$$f_{ij} = n_{ij}(X_j) \left( \frac{m_j}{2 \pi T_j} \right)^{3/2} \exp \left\{ - \frac{m_j v^2}{2 T_j} \right\}$$
where \( n_{ij}, T_j \) are the density and temperature of the \( j \)th species. For low frequency waves (\( \omega < \omega_{Hi} \)), we neglect in Eq. (97) all terms with \( \ell \neq 0 \). Also, by (87) we may drop the \( k_z T_i \) term for ions, which describes resonant wave-ion interaction. Then on integrating with respect to velocity we find the perturbed ion density

\[
\dot{n}_i = -\frac{\mathcal{E}_{\parallel}}{e} n_o \left( 1 - \frac{\omega - k_y \nu_{di}^i}{\omega} \right) \Gamma_i \left( \frac{l_i \nu_{di}^i}{n_o} \right) \tag{98}
\]

Here the function

\[
\Gamma_i (\lambda) = I_o \left( \lambda^{3/2} \right) e^{-\lambda^2/2} , \quad \lambda = k_z \nu_{di}^i
\]

describes the weakening of the effective electric field experienced by the ions, averaged over their circular Larmor orbits. In the limits of small and large Larmor radius, respectively,

\[
\Gamma_l \approx 1 - \frac{1}{2} \lambda^2 + 0 (\lambda^4) , \quad \lambda \ll 1
\]

\[
\Gamma_l \approx \frac{1}{\nu_{Hi} \lambda} \left[ 1 + 0 (\lambda^{-2}) \right] , \quad \lambda \gg 1
\]

The Larmor radius of the electrons is very small and for them Eq. (97) corresponds to the usual drift approximation. Expanding the electron term in \( \omega / k_z \nu_{te} \) we obtain

\[
\dot{n}_e = \frac{\mathcal{E}_{\parallel}}{e} n_o \left\{ 1 + i \frac{\nu_{Te}}{\nu_{Hi} \nu_{te}} \left( \frac{\omega - k_y \nu_{de}^e}{|k_z| \nu_{te}} \right) \right\} \tag{100}
\]

In the limit of quasi-neutrality (\( k^2 \lambda_D^2 \ll 1 , \lambda_D^2 = T / 4 \pi n_e e^2 \)) the frequency and the growth rate are given by

\[
\omega_k + i \gamma_k = \frac{k_y \nu_{de}^e \Gamma_{ei}^i}{\lambda_i^2 \Gamma_{ei}^i + i \pi \nu_{Hi} \frac{k_y \nu_{de}^e (\Gamma_{ei}^i - 1)}{|k_z| \nu_{te}}}
\]

- 94 -
We see that the instability arises as a result of finite ion Larmor radius effect and that the growth rate goes to zero simultaneously with the ion Larmor radius. In order to see more clearly what can happen near this marginally stable case we will write the expression for the growth rate through the local characteristics of the distribution of the resonant electrons:

\[ \gamma = - \frac{\omega}{2} \frac{m_e}{k_z} \left( k_z \frac{\partial}{\partial v_z} + \frac{k_z}{\omega_{ci}} \frac{\partial}{\partial x} \right) f_e(v_x, x) \left/ \frac{v_x}{\omega_{ci} k_z} \right. \]

(102)

In the approximation of the small ion Larmor radius, two terms on the r.h.s. almost cancel accidentally for the Maxwellian electron distribution at one point \( v_x = \omega / k_z \), and instability becomes very weak. (This cancellation takes place also for the arbitrary temperatures of the species \( T_i \neq T_e \).) Therefore, even small deviations from the linear theory may strongly change the stability in respect of the finite amplitude waves. There are two effects of this type:

1) If the amplitude of the drift wave is large enough, the resonance region will be broadened by a considerable amount (i.e., \( \Delta v_x \sim \sqrt{2 e \Psi / m_e} \)). We will now take this broadening into account and show that a wave of finite amplitude can be stable in the non-linear regime.

2) The relaxation of the electron distribution under the influence of the oscillations of finite amplitude, as we know already, also leads to stabilization of the instability.

Let us first consider case 1) which takes place in the stability problem of the monochromatic drift wave.

B. Non-linear stability of the monochromatic drift wave

We choose the following form for the electric potential:

\[ \psi(y, z, t) = - \psi_0 \left[ \cos(k_y y + k_z z - \omega t) + O(\frac{e\psi_0}{T}) \right] \]

(103)

where we have set \( k_z = 0 \) for simplicity. If we work in a co-ordinate
system moving with the wave, the rate of increase in resonant electron kinetic energy, which must equal the rate of decrease in wave energy, can be written as

\[
\frac{dT}{dt} = \frac{r_{\text{in}}^2 m_e}{2} \int_{-\infty}^{+\infty} \frac{d\xi}{\lambda_z} \left[ d\nu_\xi \left( \nu_\xi + \omega K_\xi \right) \right]^2 \frac{\partial f_\xi}{\partial t}
\]

In the drift approximation, the general solution for the distribution function can be written as

\[
f_\xi [\xi, \nu_\xi, t] = f_\xi [\nu_\xi (\xi, \nu_\xi, t), \nu_\xi (\xi, \nu_\xi, t), 0],
\]

where \(f_\xi [\xi, \nu_\xi, 0]\) is the initial distribution and \((\xi_0, \nu_\xi_0)\) is the initial position of the particle. We can divide the distribution into two parts

\[
f_\xi [\xi, \nu_\xi, t] = f_0 [x_0, \nu_\xi_0] + f_4 [x_0, \nu_\xi_0, 0] \cos (k_y y_0 + k_z z_0),
\]

where the first part is the local Maxwellian and the second part is a perturbation due to the wave. The second part only makes a contribution to the harmonic generation and we can neglect it while we evaluate the rate of change of kinetic energy. Consequently, we find

\[
\frac{\partial f}{\partial t} = \frac{\partial f_0 [x_0, \nu_\xi_0, 0]}{\partial x_0} \frac{\partial x_0}{\partial t} + \frac{\partial f_4 [x_0, \nu_\xi_0, 0]}{\partial \nu_\xi_0} \frac{\partial \nu_\xi_0}{\partial t} =
\]

\[
= \frac{e}{m_e} \nu_0 \sin (k_y y_0 + k_z z_0) (k_z \frac{\partial}{\partial \nu_\xi_0} + \frac{k_y}{\omega H_0} \frac{\partial}{\partial x_0}) f_0 \nu_0
\]

The particle equations of motion in the drift approximation,

\[
\begin{align*}
\dot{y} &= 0 \\
\dot{x} &= \frac{k_x \nu_0}{H_0} \sin (k_y y + k_z z) \\
\dot{z} &= -\frac{e k_z \nu_0}{m_e} \sin (k_y y + k_z z)
\end{align*}
\]
have a solution in the form of elliptic integrals. First we introduce a new variable, $k_y y + k_z z = 2 \xi$, and rewrite the equation for conservation of energy,

$$\frac{m \dot{z}^2}{2} - e \psi_0 \cos (k_y y + k_z z) = W$$

in the form

$$\frac{\ddot{\xi}}{\dot{\xi}^2} = \frac{1}{\alpha^2 \tau^2} \left[ 1 - \alpha^2 \sin^2 \xi \right], \quad (108)$$

where $\alpha^2 = \frac{2e\psi_0}{W + e\psi_0}$ and $\tau \equiv \left( \frac{m_e/e\psi_0 \kappa_z^2}{\alpha^2} \right)^{1/2}$.

For the case where $\alpha^2 < 1$, we can write the solution of Eq. (108) as

$$F(\xi, \xi_0) = F(\alpha, \xi) - \frac{t}{\alpha \tau}. \quad (109)$$

When $\alpha^2 > 1$, it is convenient to introduce the new variable $\xi'$, defined as

$$\alpha \sin \xi' = \sin \xi, \quad \xi'^2 = \frac{1}{\alpha^2} \left[ 1 - \frac{1}{\alpha^2} \cdot \sin^2 \xi \right]$$

This equation has the solution

$$F\left( \frac{1}{\alpha} \xi_0 \right) = F\left( \frac{1}{\alpha} \xi, \xi' \right) - \frac{t}{\tau}. \quad (110)$$

To get the time dependence of the growth rate we need only put these solutions into Eqs. (104) and (106). This problem was solved by O'NEIL [5] in the approximation which reduces to the linear growth rate at $t = 0$. We would now like to take into account non-linear effects that are present initially, so we must include more terms than O'Neill did. After substituting Eq. (106) into Eq. (104), we obtain

$$\frac{dT}{dt} = \frac{16 n e \psi_0}{\kappa_z^2} \int_0^{\frac{3a}{2}} d\xi \int_0^{\xi} d\xi' \left\{ \left. \left[ k_z \frac{\partial \sigma_{ee}}{\partial \xi} + k_y \frac{\partial \sigma_{ee}}{\partial x_0} \right] \right|_{x_0 = x, \eta_0 = 0} \right\}.$$
\[
+ \frac{\partial^2}{\partial \xi^2} \left[ \frac{\partial f_{ce}}{\partial \xi} + \frac{\partial f_{re}}{\partial \xi} \right] \bigg|_{\xi=0} \frac{\xi^2}{4 \xi^2} \sin(2\xi) \right) , \quad (111)
\]

where we have dropped even functions of $\xi$ by symmetry arguments and have neglected the time dependence of $\xi$.

Carrying out the integration in Eq. (111) gives us the following expression for the growth rate:

\[
\gamma_\delta(t) = \frac{T_e}{n_e e^2} \left[ \frac{d\gamma}{dt} \right] = \]

\[
= \frac{n_e}{k_e^2 \omega_{ce}} \int_0^1 d\xi \sum_{n=0}^\infty \left\{ \left( \frac{\partial^2 \gamma_{re}}{\partial \xi^2} + \frac{8}{k_e^2 \omega_{te}^2} \right) \left( \frac{2}{\xi^2} + \frac{4}{3} \left( \frac{\pi^2 n^2}{4 \xi^2} - 1 \right) - \frac{\xi^2}{3} \right) \right\} \frac{2 \pi \xi^2 \sin \left[ \frac{\pi n t}{\xi^2} \right]}{\xi^5 \xi^2 (1 + q^{2n}) (1 + q^{-2n})} \]

\[
\left( \frac{2n+1}{2n} \right) \frac{\pi \xi^2 \sin \left[ \frac{(2n+1)\xi t}{2n} \right]}{F^2 (1 + q^{2n+1}) (1 + q^{-2n-1})} \right\} e^{-\frac{\xi^2}{T_e}} \quad (112)
\]

where \( F(\xi, \xi/2) = F \), \( q = \exp \left[ \frac{\pi F'}{F} \right] \), and \( F' = F \left[ (1 - \xi^2)^{1/2}, \xi/2 \right] \).

In the limit where \( \xi / T_e \ll 1 \), the main contribution to the growth rate comes from the \( n = 1 \) term in the first sum; so we find
Consequently, a wave with finite amplitude can be stable in the non-linear regime if the linear growth rate is small enough. Here we took into account only finite Larmor radius effects, but the same idea can also be applied to the current- or gravity-driven drift instabilities in an inhomogeneous plasma. Therefore, we can use this expression to estimate the amplitude of the separate drift waves observed by N.S. Buchelniloova in a potassium plasma.

Since the amplitude we obtained from Eq. (113),

\[
\frac{e \psi_0}{T_e} \approx \frac{4}{\pi} \frac{\kappa_1^2 \gamma_{hi}^2}{K_e} \frac{1}{|K_e| \Sigma_{the}} \left[ \kappa_1 \gamma_{hi}^2 \right]
\]

is very small, we can obtain the level of harmonic generation by expanding the Vlasov equation in the small parameter \( e \psi_0 / T_e \). For example, the amplitude of the second harmonic is given by

\[
\frac{e \psi_{2K}^{(2)}}{T_e} = \left[ \frac{e \psi_0}{T_e} \right]^2 \left\{ 1 - \frac{\omega - \kappa_1 \gamma_{hi}^2}{\omega} \int_0^\infty \! \! d \gamma_{hi} \left[ 1 - \frac{2 \kappa_1 \gamma_{hi}^2}{\omega} \right] \right\}
\]

where we have chosen \( T_e = T_1 \). In the limit of small Larmor radius this can be written as

\[
\frac{e \psi_{2K}^{(2)}}{T_e} \approx \frac{1}{2} \left[ \frac{e \psi_0}{T_e} \right]^2
\]

One can expect that the amplitude of the nth harmonic will be proportional to the nth power of the main harmonic.
Consequently, we find that the amplitude of the harmonics decreases exponentially as a function of the frequency.

The picture drawn here is valid only for the narrow wave packet in which the phase velocities of the different waves are very close,

\[ \Delta \omega / \kappa z < \sqrt{\frac{e \varphi}{m_e c}} \]. But during experiment the instability can arise in a wide region of the phase space and this condition is violated. Then, the main stabilization effect comes from relaxation of the electron distribution.

C. Quasi-linear relaxation of the particle distribution and transport processes \[35, 36\]

As usual, we express the distribution as the sum of a slowly and rapidly varying part (i.e., \( f_j = \overline{f_j} + \delta f_j \)) and obtain an equation for the slowly varying part by averaging the Vlasov equation over the rapid oscillations

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \varphi + \frac{e_j}{m_j c} \left[ \mathbf{v} \times \mathbf{B} \right] \frac{\partial}{\partial \mathbf{v}} \varphi \right) \overline{f_j} = \overline{\frac{e_j}{m_j c} \varphi \frac{\partial \delta f_j}{\partial x}} \equiv S \int \frac{f_j}{f_j} \quad (114)
\]

Expanding \( \varphi \) and \( \delta f_j \) in Fourier components allows us to express the r.h.s. of Eq. (114) as

\[
S \int \frac{f_j}{f_j} = \sum_k \overline{\frac{e_j}{m_j} \left( \frac{d \varphi_k^*}{d x} \frac{\partial}{\partial x} - i \varphi_k^* \frac{\partial}{\partial x} \right) \delta f_k \left( x, y \right)} \quad (115)
\]

where the cross terms have vanished in the averaging process and we have explicitly made allowance for a slow \( x \)-dependence in \( \varphi_k \left( x \right) \) and \( \delta f_k \left( x, y \right) \).

In accord with the philosophy of quasi-linear theory, we assume that the distribution can be written in the form

\[
\frac{e \varphi_n e}{T} \sim \left[ \frac{e \varphi e}{T} \right]^n
\]
where the dependence on the last two variables is slow enough to be ignored while evaluating $\delta f_{xj}$. With this point in mind, we can easily generalize the expression (97) for $\delta f_{xj}$.

$$\delta f_{xj}(x, v) = \frac{e_j}{m_j} \varphi_k(x + \frac{v_y}{\omega_{lj}}) \left\{ \frac{i}{v_x} \frac{\partial}{\partial v_x} + e^{-i \frac{K_x v_x}{\omega_{lj}}} \int_0^1 \left( \frac{k_y v_y}{\omega_{lj}} \right) \right\}$$

Substituting this expression into Eq. (115) and averaging the result over the azimuthal angle in velocity space, we find

$$\varphi_k \left[ x(t) \right] = \varphi_k (x + \frac{v_y}{\omega_{lj}}) + \frac{v_y(t)}{\omega_{lj}} \int \frac{d\varphi_k(x + \frac{v_y}{\omega_{lj}})}{d\varphi_k}$$

We neglect here the contribution to the integral from the second term of the expansion on slow co-ordinate.

Substituting this expression into Eq. (115) and averaging the result over the azimuthal angle in velocity space, we find

$$S t \{ f_j \} = Re \left( \frac{e_j}{m_j} \right)^2 \sum_k \left\{ k_x \frac{\partial}{\partial v_x} + k_y \frac{\partial}{\partial x} \right\}$$

$$\int \frac{\varphi_k(x)^2}{i \left( k_x v_x - \omega_k \right)} \left\{ k_x \frac{\partial}{\partial v_x} + k_y \frac{\partial}{\partial x} \right\} f_j$$

Since $k_x v_x < \omega < k_x v_y$, the interaction of the waves with the ions will be primarily adiabatic and the interaction with the electrons primarily resonant. Also, we may drop finite Larmor radius effects in the electron equation but not in the ion equation. Consequently, the electron and ion equations become
Of course, the electron equation could have been derived more simply from the drift approximation.

Next we investigate the formation of the quasi-linear "plateau". For this purpose it will be convenient to evaluate the \( K \) dependence of the electron diffusion coefficient at that value of \( K \) which is only unstable (i.e., \( K = K^u \), where \( K^u \sim \omega \Lambda \) and \( K^u \) can be found by taking into account the competition between quasi-linear plateau formation and collisions, as will be shown later. The waves with \( K_2 < K^u \) are damping.) If we introduce the new variables

\[
\gamma = \frac{v_y^2}{2}, \quad \xi = \frac{v_y^2}{2} + \frac{\omega \omega_n}{K_2} \chi
\]

we can reduce the differential operators in Eq. (117) as
\[
\frac{\partial}{\partial \xi} = \left[ \frac{1}{\nu_\perp^2} \frac{\partial}{\partial \nu_\perp^2} + \frac{k_y}{\omega_e \omega_m} \frac{\partial}{\partial \xi} \right]
\]

Consequently, the level curves of the "plateau" are described by the equations

\[
x - \frac{k_y \nu_\perp^2}{2 \omega_e \omega_m} = \xi = \text{const.}
\]

Since \( \nu_\perp \ll \nu_{the} \) in the resonant region, the level curves of the original Maxwellian can be described by the equations

\[
x - \frac{k_y \nu_\perp^2}{2 \omega_e \omega_m k_y \nu_d e} = \text{const.}
\]

where \( \nu_d e = -\frac{c T_e}{e H_o} \frac{n_o}{n_o} \). Since \( \omega_m = k_y \nu_d e \frac{\Gamma_o}{2 - \Gamma_o} \), it follows that the two sets of curves differ only by finite Larmor radius effects (see Fig.26). The energy gain which we receive after relaxation of the electron distribution, for this reason is also small for the small ion Larmor radius.

![Diagram](image)
If we plot the distribution as a function of $U_z$ (see Fig. 27), we find that the slope in the resonant region must become very steep, so that the enhanced Landau damping can stabilize the waves. During this process the electrons lose the energy of the motion along field lines. Therefore, we can say that the energy goes to the unstable waves from the longitudinal motion of the electrons.

![Diagram](image)

**Fig. 27**

We can also use the constancy of $\xi$ to estimate the change in an electron's position \[35\]

$$\xi \propto \frac{k_d}{\omega_z \omega_\mu} \frac{\delta U_z^2}{2}$$

Since $\delta U_z^2 < U_A^{-2}$, the displacement of the resonance electrons is much smaller than plasma radius \( \rho = \frac{n}{n'} \).

$$\xi \propto \frac{k_d}{\omega_z \omega_\mu} \frac{U_A^{-2}}{2} \sim \frac{n}{n'} \frac{U_A^{-2}}{v_{th,e}}$$

Thus the instability inhibits itself rapidly and the change in electron density is small.
One can easily see that the electrons and ions diffuse across the magnetic field at the same rate. Integrating Eq. (117) over velocity co-ordinates gives the diffusion equation for the electrons

$$\frac{\partial n_e}{\partial t} = \left( \frac{e}{m_e} \right)^2 \sum_k \frac{k_y}{\omega_{ne}} \frac{\partial}{\partial x} f_k \left[ \omega_x - k_x v_x \right] \int d\omega \delta \left( \omega_x - k_x v_x \right) \left[ \frac{k_x^2}{\omega_{ne}^2} \omega_{ne} \frac{\partial}{\partial x} \right] f_k$$

which we have introduced the growth rate $\chi$ and multiplied and divided by $\frac{\partial n_e}{\partial x}$. Using Eq. (118), we find a similar equation for the ions

$$\frac{\partial n_i}{\partial t} = \left( \frac{e}{m_i} \right)^2 \sum_k \frac{k_y}{\omega_{ni}} \frac{\partial}{\partial x} f_k \left[ \frac{k_x}{\omega_{ni}} \right] \int d\omega \delta \left( \omega_x - k_x v_x \right) \left[ \frac{\chi}{\omega_x^2} \right] \frac{\partial n_i}{\partial x}$$

One can easily see from the dispersion relation (i.e., $\omega_\perp = \frac{n_e}{n_i} \left( \frac{\omega_{ni}}{\omega_{ne}} \right) k_y \Gamma_0 \left[ 2 - \Gamma_0 \right]$) that the diffusion coefficients in these two equations are identical.

Since the collisionless theory gave only negligible diffusion across the magnetic field, we will modify the previous work by adding a small collision term to the r.h.s. of Eq. (114). The collisions will try to make the distribution take the form of a local Maxwellian and will thereby prohibit the formation of the quasi-linear "plateau". Consequently, we should find enhanced diffusion in this case. The electron equation will now be of the form

$$\frac{\partial f_e}{\partial t} = S_t \left\{ f_e \right\} + S_t \left\{ \text{coll} \left\{ f_e \right\} \right.$$
\[ S_{\text{coll}} \{ f_e \} = \nu_e \nu_{he}^2 \frac{\partial}{\partial V_e^2} (f_e - f_{me}) \]

Since we are treating the case where \( S_{\text{coll}} \{ f_e \} \ll S_{QL} \{ f_e \} \), we can solve Eq. (117) by successive approximations \[36\]. We express the distribution as \( f_e = f_e^{(0)} + f_e^{(n)} \) and then demand that

\[ S_{QL} \{ f_e^{(0)} \} = 0 \quad (123) \]

and that

\[ S_{QL} \{ f_e^{(n)} \} + S_{\text{coll}} \{ f_e^{(0)} \} = 0 \quad (124) \]

Eq. (123) has the solution

\[ \frac{1}{V_e} \frac{\partial f_e^{(0)}}{\partial V_e} = -\frac{k_y}{\omega_k \omega_{ne}} \frac{\partial f_e^{(0)}}{\partial x} \quad (125) \]

To solve Eq. (124), we integrate it with respect to \( V_e \)

\[ \int_0^{V_e} \left( \frac{e}{\nu_{me}} \right)^2 \sum \left[ k_e \frac{\partial}{\partial V_e^2} + \frac{k_y}{\omega_{ne}} \frac{\partial}{\partial x} \right] \Psi_e \, \frac{\partial}{\partial x} \delta (\omega_e - k_e V_e) \, dV_e. \]

\[ \times \left[ k_e \frac{\partial}{\partial V_e^2} + \frac{k_y}{\omega_{ne}} \frac{\partial}{\partial x} \right] f_e^{(1)} = \nu_e \nu_{he}^2 \frac{\partial}{\partial V_e} [f_{me} - f_e^{(0)}] \quad (126) \]

Since \( f_e^{(1)} \) will be more strongly dependent on \( V_e \) than on \( x \), we can neglect the derivative with respect to \( x \) in the first bracket in Eq. (126). If we also use Eq. (125) to evaluate the second term on the r.h.s. of Eq. (126), we find
Integrating this expression over \( \mathcal{U}_\xi \) gives

\[
\left( \frac{e}{m_e} \right)^2 \sum_k |\Psi_k|^2 \kappa_k \mathcal{F} \delta(\omega_k - \mathcal{U}_\xi) \left[ \kappa_k \frac{\partial}{\partial \mathcal{U}_\xi} + \frac{k_y}{\omega_{\text{he}}} \frac{\partial}{\partial x} \right] e^{(i)} = \int \mathcal{U}_\xi^2 \frac{\partial}{\partial \mathcal{U}_\xi} \left[ \frac{1}{\mathcal{U}_\xi^2} \frac{\partial}{\partial \mathcal{U}_\xi^2} + \frac{k_y}{\omega_{\text{he}}} \frac{\partial}{\partial x} \right] f_{\text{me}} \, (127)
\]

where we have introduced the actual growth rate, \( \mathcal{P}^{(i)} \), and the growth rate for a Maxwellian plasma, \( \mathcal{P}^{(m)} \) (see (102)).

Although this expression cannot be used to evaluate either the growth rate or the field amplitude separately, it can be used to evaluate the diffusion coefficient, which depends on the product of these two quantities. From Eq. (121) we can see that the diffusion coefficient is given by

\[
D_\perp \approx - \frac{1}{\mathcal{H}_m n_e} \sum_k \kappa_k \frac{\mathcal{P}^{(i)}}{\omega_{\text{he}}} |\Psi_k|^2 \, (129)
\]

By using Eq. (128) we can express this as

\[
D_\perp \approx \int \frac{\mathcal{U}_\xi \mathcal{P}^{(m)} \kappa_y n_e}{\omega_{\text{he}} \kappa_k \mathcal{U}_\xi n_e'} \left( \frac{\omega_{\mathcal{P}^{(m)}} - k_y \mathcal{U}_\xi}{\kappa_k \mathcal{U}_\xi n_e} \right) \left( \frac{\kappa_k \mathcal{U}_\xi}{n_e} \right) \, (130)
\]

where

\[
\mathcal{P}^{(m)} = \mathcal{P} \omega_{\mathcal{P}} \left( \frac{\omega_{\mathcal{P}}}{k_y \mathcal{U}_\xi} \right) \left( \frac{1}{\kappa_k \mathcal{U}_\xi n_e} \right)
\]

Evaluating this expression at \( k_k \approx \omega_{\mathcal{P}} / \mathcal{U}_\xi \) and \( k_k \mathcal{U}_\xi \approx 1 \), gives [35,36]
\[ D_L = \gamma_e \left( \frac{m_e}{m_\beta} \right)^{3/2} \left[ n / \frac{\partial n}{\partial x} \right]^2. \] (131)

Of course, this formula is only valid when \( \text{St. qu. } \{ f_{\text{me}} \} \gg \text{St. coll. } \{ f_{\text{e}}^{(3)} \}. \)

In a later lecture we will investigate the wave amplitude and rewrite this condition in a concrete way.
REFERENCES


PART III. WAVE-PARTICLE NON-LINEAR INTERACTION

1. Electron plasma oscillation turbulence.

Let us now turn to the last principal mechanism for non-linear interaction between the waves and the plasma, in the case where there is no magnetic field. If we impress two waves on a plasma, then these waves will beat with the mixed frequency \((\omega_1 \pm \omega_2)\) and mixed wavelength \((K_1 \pm K_2)\). We have already considered the resonance of this beating with a third wave \((\omega_3, K_3)\) in the decay-type interaction. But, in analogy with the linear theory, this beating can also resonate with particles moving at the velocity

\[
(K_1 \pm K_2) \cdot \mathbf{u} = (\omega_1 \pm \omega_2)
\]

This type of process has been included for the first time in the theory of weak turbulence by W. DRUMMOND and D. PINES [1] for a one-dimensional wave packet, and by KADOMTSEV and PETVIASHVILI [2] for the general case (see also [3]). In our derivation we will follow the article [4].

Of course, the rate of energy-increase (or decrease) due to this process will be proportional to the product of the energy in the two primary waves. Consequently, it will be a higher order correction in the expansion on wave amplitude. But very often the linear growth rate is small, because only a few particles can resonate with the wave, and the non-linear correction to the damping coefficient can be important.

To illustrate this effect, we consider a wave packet of random-phased Langmuir oscillations. The frequency for Langmuir oscillations is essentially constant (i.e., \(\omega^2 = \omega_{pe}^2 \left(1 + \frac{3}{2} K^2 \lambda_D^2\right)\) where \(K^2 \lambda_D^2 \ll 1\)). Consequently, it takes at least four waves to satisfy the frequency resonance condition

\[
\omega_{K_1} + \omega_{K_2} = \omega_{K_3} + \omega_{K_4}
\]

and wave-wave scattering cannot enter the problem until third order in the wave energy. On the other hand, in second order in the wave energy, we can satisfy the resonance condition for non-linear interaction with the particles
As usual, we expand the distribution in powers of the wave amplitude by using the iteration formula

\[
\mathcal{F}_j^{(n)}(\kappa, \omega) = \sum_{\substack{\kappa' + \kappa'' = \kappa \\ \omega' + \omega'' = \omega}} i \mathcal{E}_j \int d\kappa' \varphi(\kappa', \omega') \kappa' \frac{2}{\omega'} \mathcal{F}_i(\kappa'', \omega'', \gamma) e^{-i(\omega t - \kappa r)}
\]

Substituting this expression into Poisson's equation gives the dynamic equation for the waves

\[
\mathcal{E}^{(1)}(\omega) \varphi(\kappa, \omega) + \sum_{\substack{\kappa' + \kappa'' = \kappa \\ \omega' + \omega'' = \omega}} \mathcal{E}^{(2)} \varphi(\kappa', \omega') \varphi(\kappa'', \omega'') + \sum_{\substack{\kappa' + \kappa'' + \kappa''' = \kappa \\ \omega' + \omega'' + \omega''' = \omega}} \mathcal{E}^{(3)} \varphi(\kappa', \omega') \varphi(\kappa'', \omega'') \varphi(\kappa''', \omega''') + \ldots = 0
\]

where

\[
\mathcal{E}^{(1)}(\omega) = 1 + \sum_j \frac{\omega_j^2}{\kappa^2} \int d\gamma \frac{\kappa \frac{\partial f_j}{\partial \gamma}}{\omega - \kappa r + i \epsilon}
\]

\[
\mathcal{E}^{(2)}(\kappa', \kappa'', \omega') = - \frac{1}{2} \sum_j \frac{\omega_j^2}{\kappa^2} \mathcal{E}_j \int d\gamma \frac{1}{\omega' + \omega'' - (\kappa' + \kappa'') r + i \epsilon}
\]

\[
\int \kappa' \frac{\partial}{\partial \gamma} \frac{1}{\omega' - \kappa' r + i \epsilon} \kappa'' \frac{\partial}{\partial \gamma} \frac{1}{\omega' - \kappa'' r + i \epsilon} \kappa' \frac{\partial}{\partial \gamma} f_j
\]

etc.
We solve Eq. (3) by expanding in powers of the field amplitude. The lowest order solution just defines the field eigenfunctions

$$\varphi^{(i)}(k, \omega) = \varphi^{(i)}_k \delta(\omega - \omega(k))$$

(5)

where $\omega(k)$ is the solution of $R_k \left[ \mathcal{E}^{(i)}_k(\omega) \right] = 0$. One can easily see that the second order solution is

$$\varphi^{(2)}(k, \omega) = \sum_{k' + k'' = k} \mathcal{E}^{(2)}_{k', k''}(\omega') \varphi^{(i)}_{k'} \varphi^{(i)}_{k''} \delta(\omega - \omega(k') - \omega(k''))$$

(6)

To derive the wave kinetic equation, we first multiply Eq. (3) by $\varphi^{*}(k, \tilde{\omega}) e^{i(\tilde{\omega} - \omega)t}$ and integrate over $d\omega d\tilde{\omega}$. The first term in the resulting expression will be of the form

$$\int d\omega \int d\tilde{\omega} \mathcal{E}^{(i)}_k(\omega) \varphi(k, \omega) \varphi^{*}(k, \tilde{\omega}) e^{i(\tilde{\omega} - \omega)t}$$

(7)

Since $\varphi(k, \omega)$ is peaked around $\omega_k$, we can rewrite the imaginary part of this term as

$$\text{Im} \left\{ \int d\omega \int d\tilde{\omega} \mathcal{E}^{(i)}_k(\omega) \varphi(k, \omega) \varphi^{*}(k, \tilde{\omega}) e^{i(\tilde{\omega} - \omega)t} \right\} \approx\frac{1}{2} \frac{\partial \mathcal{E}^{(i)}_k}{\partial \omega_k} \frac{d |\varphi_k(t)|^2}{dt} + \mathcal{E}^{(i)}_k(\omega_k) |\varphi_k|^2$$

where

$$\mathcal{E}^{(i)}_k(\omega) = \mathcal{E}^{(i)}_k(\omega) + i \frac{\mathcal{E}^{(i)}_k(\omega)}{\omega}$$

(8)

If we use Eqs. (5) and (6) to evaluate the remaining two terms and then average over random initial phases (i.e., $< \varphi^{(i)}_k \varphi^{(i)\ast}_k > = \int |\varphi^{(i)}_k|^2 \delta(k, k')$), we find the well-known kinetic wave equation[2,4,5]...
\[
\frac{1}{2} \frac{\partial \varepsilon^{(4)}_{\psi}}{\partial \Omega_{\xi}} - \frac{\partial \psi_{\xi}}{\partial t} = - \varepsilon^{(4)}_{\psi'(\omega_{\xi})} \left| \psi_{\xi} \right|^2 + 
\]
\[
+ \text{Im} \sum_{\kappa' + \kappa'' = \kappa} \frac{2}{\varepsilon^{(2)}_{\psi(\omega_{\kappa'}, \omega_{\kappa''})}} \left| \psi_{\kappa'} \right|^2 \left| \psi_{\kappa''} \right|^2 + 
\]
\[
+ \text{Im} \sum_{\kappa' \neq \kappa''} \left\{ \frac{4 \varepsilon^{(2)}_{\psi(\omega_{\kappa'}, \omega_{\kappa''}), -\omega_{\kappa'}} \varepsilon^{(2)}_{\psi(\omega_{\kappa'}, -\omega_{\kappa'})}}{\varepsilon^{(4)}_{\psi(\omega_{\kappa'}, -\omega_{\kappa'})}} \right\} \left| \psi_{\kappa'} \right|^2 \left| \psi_{\kappa''} \right|^2 
\]
\[
- \text{Im} \sum_{\kappa' \neq \kappa''} \left\{ \frac{\varepsilon^{(3)}_{\psi(\omega_{\kappa'}, -\omega_{\kappa''})}}{\varepsilon^{(4)}_{\psi(\omega_{\kappa'}, -\omega_{\kappa'})}} \right\} \left| \psi_{\kappa'} \right|^2 \left| \psi_{\kappa''} \right|^2 
\]
(9)

where we have dropped terms which are higher than fourth order in the field amplitude. The first term on the r.h.s. of this equation is just the quasi-linear growth rate. The second term gives the rate at which the modes \( \psi_{\kappa'} \) and \( \psi_{\kappa''} \) decay into \( \psi_{\kappa} \). The third term gives the rate at which \( \psi_{\kappa} \) and \( \psi_{\kappa'} \) decay into \( \psi_{\kappa''} \), but it also gives a contribution due to the non-linear interaction of the waves and particles. The third term is also due to the non-linear wave-particle interaction.

For Langmuir oscillations the decay-type interactions make no contribution, since we cannot satisfy the frequency condition

\( \omega_{\kappa' + \kappa''} = \omega_{\kappa} + \omega_{\kappa'} \). If we restrict our considerations to the case of a narrow spherical wave packet with width \( (\Delta k) / \lambda_{\text{De}} \ll (u_{\text{in}} / k \lambda_{\text{De}})^{2/3} \)
then we can also drop the non-linear interaction of the waves with the electrons. With these points in mind, one can easily see that the main contribution to the r.h.s. of Eq. (9) will come from the third term and that in this term \( \varepsilon^{(3)} \) will be determined by the electrons and \( \varepsilon^{(4)} \) by the ions. To evaluate \( \varepsilon^{(3)}_{\psi(\omega_{\kappa'}, -\omega_{\kappa''})} \) we expand the integral in Eq. (4) in terms of the small parameter

\[
\left\{ \left( \omega_{\kappa} - \omega_{\kappa'} \right) / \left( k \kappa' \right) \right\} u_{\text{De}}
\]

\text{Singualrities of type } 1/\varepsilon^{(4)}(\omega, \kappa) \text{ near } \omega = \omega(\kappa) \text{ must be treated in Eq. (9) according to the rule (See Part I)}
\[ \frac{1}{\varepsilon^{(4)}(\omega, \kappa)} = \frac{1}{\varepsilon^{(4)}(\omega, \kappa) + \nu} \]

- 115 -
In a similar way, we find that

$$\mathcal{E}_{k',k-k'}^{(2)} \approx \frac{\mathcal{E}_{k,k-k'}^{(1)}}{(k-k')^2} \cdot \int \frac{\left(k-k'\right) \cdot \frac{\partial f_e}{\partial \nu}}{\omega_k - \omega_k' - (k-k') \cdot \nu} \, d^3\nu \, \frac{k \cdot k'}{\omega_k^2}$$

(10)

In a similar way, we find that

$$\mathcal{E}_{k',k-k'}^{(2)} \approx \frac{e}{T_e} \cdot \frac{k \cdot k'}{(k-k')^2} \cdot \int \frac{\left(k-k'\right) \cdot \frac{\partial f_e}{\partial \nu}}{\omega_k - \omega_k' - (k-k') \cdot \nu} \, d^3\nu$$

(11)

and that

$$\mathcal{E}_{k,k-k'}^{(1)} \approx 1 + \frac{1}{(k-k')^2} \cdot \frac{\omega_k - \omega_k'}{(k-k')^2} \cdot \frac{\omega_k - \omega_k'}{(k-k')^2} \cdot W \left( \frac{\omega_k - \omega_k'}{1/(k-k')^2} \right)$$

where

$$W \left( z \right) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{z - t + i e} \, dt$$

(12)

Consequently, Eq. (9) becomes

$$\frac{\partial}{\partial t} \left[ \frac{\mathcal{E}_{k}^{(1)}(k^2 | \Psi_{k} |^2) \omega_k}{\omega_k} \right] = \sum_{k'} \frac{e^2}{n \tau_i} \frac{|\Psi_{k'}|^2}{(k-k')^2} \cdot \frac{1}{\omega_k - \omega_k'} \cdot W \left( \frac{\omega_k - \omega_k'}{1/(k-k')^2} \right)$$

(13)
We can see from Eq. (13) that the number of waves is conserved in this process:

\[
\frac{\partial}{\partial t} \sum_{\kappa} n_{\kappa} = \frac{\partial}{\partial t} \sum_{\kappa} \frac{1}{\omega_\kappa} \frac{\partial}{\partial \omega_\kappa} \left[ \mathcal{I}_{\kappa}^{(0)}(\omega_\kappa) \omega_\kappa \right] \frac{k^2 |\psi_{\kappa}|^2}{\gamma T_i} = 0
\]

This is easy to understand if we look at the resonance condition as an equation determining the energy a particle receives when wave \( \psi_{\kappa_1} \) scatters from it into wave \( \psi_{\kappa_2} \):

\[
\Delta E = \frac{\partial E}{\partial \kappa_2} \Delta \kappa_2 \equiv (k_1 - k_2) \cdot \psi = \omega_{\kappa_1} - \omega_{\kappa_2}
\]

Since this quantity is much smaller than the energy in either wave (i.e., \( (k_1 - k_2) \cdot \psi = \omega_1 - \omega_2 \ll \omega_1 \omega_2 \)), the number of waves must be conserved. Let us note that in the limit of large phase velocity of waves the wave number conservation law can be found for arbitrary waves using the symmetry properties of the coefficients \( \mathcal{C}^{(2)} \) and \( \mathcal{C}^{(3)} \) in Eq. (9) \[6\].

Eq. (13) can easily be solved in the case where the spread of the wave packet is much larger than the ion thermal velocity

\[
\frac{\Delta \kappa}{\kappa} \gg \left( \frac{m_e}{m_i} \right)^{1/2}
\]

In this limit, Eq. (13) can be reduced to the following differential equation:

\[
\frac{\partial N_\kappa}{\partial \tau} - N_\kappa \frac{\partial N_\kappa}{\partial \chi} = -6 \zeta^2 N_\kappa^2
\]

where

\[
N_\kappa = \frac{4 \pi}{3} \frac{k^3 |\psi_\kappa|^2}{\omega_\kappa} , \quad \zeta = \frac{\pi m_e \omega_K}{8 \pi^2 m_i (T_e + T_i)^2}
\]
The r.h.s. of this equation describes decrease in energy due to the scattering of waves to longer wavelengths. The l.h.s. of the equation describes the steepening of the wave packet in k-space. The general solution of the equation was given in [4],

\[ N_k = e^{\beta x} \left\{ \beta^{-1} \left[ 1 - e^{-\beta x} \left( 1 - \tau \beta N_k \right) \right] \right\} \]

where \( \beta = 6 \omega^2 \), \( \left\{ \beta^{-1} \left[ 1 - e^{-\beta x} \right] \right\} \) is the initial energy distribution in the wave packet.

In the time asymptotic limit the solution can be written in the compact form [7]

\[ N_k = \frac{N_0(\xi)}{1 + \beta N_0(\xi) \tau}, \quad \xi = x + \frac{1}{\beta} \log (1 + \beta N_0 \tau) \]

2. Current-driven ion sound turbulence

Now we will discuss current-driven ion sound turbulence in the limit where \( T_e \gg T_i \) and \( H_o = 0 \). Because the instability is electrostatic in nature, we can use the wave kinetic equation derived in the previous section (i.e., Eq. (9)).

For waves with phase velocities between the ion and electron thermal velocities the linear dispersion relation takes the form

\[ 0 = E_{(n)}^{(i)}(\kappa, \omega) \equiv 1 - \frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2}{k^2 C_s^2} \left\{ 1 + i \sqrt{\frac{n m_i}{2 m_i}} \left[ \frac{\omega}{\kappa |\xi| C_s} - \frac{u \cos \theta}{C_s} \right] \right\} \]
where \( C_s = \sqrt{T_i/m_i} \) is the sound velocity, \( \mathcal{U} \) is the electron drift velocity, and \( \Theta \) is the angle between \( \mathcal{U} \) and \( \mathbf{k} \). In the long wavelength limit, the phase velocity of the oscillation is equal to the ion sound velocity \( \omega = |\mathbf{k}| \cdot C_s \), and in the short wavelength limit the frequency is equal to the ion plasma frequency (see Fig. 1).

\[ \omega = \frac{\omega_{pi}}{C_s} \]

**Fig. 1**

We need

\[ \mathcal{U} > C_s \]  \hspace{1cm} (17)

for the existence of the instability in respect of the long wavelength perturbation. For the short waves the critical current is less for small ion temperature \( T_i \ll T_e \).

Since the ion sound instability is the resonance type weak instability, we may use the quasi-linear theory for the description of the current relaxation in the collisionless plasma. For this aim we consider the particle distribution in a plasma with a current.
Fig. 2 depicts the equipotentials (characteristics) of the electron and ion velocity distribution functions. The two species are displaced by the electron drift velocity $U$.

Let us take a narrow wave packet propagating parallel to $U$ as shown in Fig. 2. The interval between the resonant region and the origin is of the same order as the ion sound velocity, because the ion sound velocity is calculated in the ion rest frame. We can expect a plateau to arise in the resonant region with electron equipotentials (Fig. 2). If we look for waves with $k$ not parallel to the current velocity $U$, then for $U \gg C_S$ we can expect these waves also to be unstable. So we will have a resonant region in the form of a cone as shown in Fig. 2 by dashed lines, inside which all waves are unstable.

The angle of the vertex of this cone of course depends on the ratio of the ordered velocity to the critical velocity. If the ordered velocity is much larger than $C_S$, almost all angles of propagation will be unstable; only waves within a very small angle with respect to $U_y$ will be stable, and this angle $\varphi$ will approximately satisfy

$$\varphi \sim \frac{C_S}{U}.$$
So only electrons moving within a small angle \( \sim \varphi \) of \( \hat{\mathbf{U}}_d \) are not in resonance with the wave spectrum. Some small part of the electrons are always out of resonance with the wave spectrum because the projections of their velocities in the direction of the current are much larger than \( C_s \).

If we now have a magnetic field in the \( \hat{\mathbf{U}}_z \) direction, then, as was mentioned in Part II, it will play the role of a mixer; all electrons will rotate around the \( \hat{\mathbf{U}}_z \) axis, and in this case the small cone of stable electrons has no important effect. So we can consider this problem as if all electrons are resonant with waves.

But if we have no magnetic field of this sort (i.e., if we have \( B \parallel \hat{\mathbf{U}}_y \)), then the electrons in the stable cone will not be in resonance. The only possibility is that they can resonate with waves (ion sound or other) arising through wave-wave interactions.

If we now look for the simplest experiment, we have a magnetic field in the \( \mathbf{y} \)-direction and apply an electric field parallel to \( \mathbf{H} \), producing a current velocity which satisfies (17). We will find a quasi-linear interaction between the ion sound waves and the electrons outside the stable cone. In other words, all electrons outside the stable cone take energy from the electric field, accelerate and then give up energy to the ion sound waves due to the quasi-linear interaction. So we will have some kind of balance between the electric field acceleration and the retarding force due to radiation of ion sound phonons.

A fraction of the electrons will be accelerated just as in the usual runaway processes when this happens. But if \( \varphi \ll \hat{\mathbf{U}}_y \) (i.e., the electric field is much above the critical value) only a small number of electrons will run away. The simplest experimental evidence of such phenomena can be expected to arise as follows: first, if the number of electrons in the stable cone is small, then for such high electric fields we will find that the electrical conductivity is much lower than classical values. Secondly, runaway electrons will be observed. Both of these effects have been observed already in discharges with high electric fields.

In the opposite case, if we have a magnetic field perpendicular to \( \hat{\mathbf{U}}_y \), all the particles rotate and we will not have runaways. Now
we expect to observe only the first effect above, i.e., an anomalously low $\sigma$. This also has been confirmed by experiment. In this case, the anomalous conductivity should be even smaller, because when $E$ was parallel to $H$, some current was carried by the runaway electrons. Every electron is resonant at some phase of its rotation about the magnetic field. This kind of geometry has been produced in the fast $\Theta$-pinch experiments at Novosibirsk.

![Diagram showing electrical conductivities](image)

**Fig. 3**

Electrical conductivities are found up to five orders of magnitude lower than the classical conductivity.

Let us look again at the first case, where $H \approx 0$. For strong electric fields we can consider almost all electrons resonant, and we find runaway phenomena which are closely analogous to the runaways for the case of a Lorentz gas in which electron-ion interactions are retained. For a Lorentz gas, the collision integral depends dimensionally on velocity as the inverse fifth power for large velocities. We know that if a Lorentz gas is subjected to an electron field, the acceleration due to the field becomes greater than the collisional retarding force for sufficiently high velocities. This problem was solved by Kruskal and Bernstein. (The Lorentz gas is an exactly soluble model for the runaway problem in a collisional plasma.)

We now recall that the quasi-linear collision term which was
written down in Part II has the form

$$\frac{\partial f_e}{\partial t} = \chi \frac{\partial}{\partial v} \left( \frac{1}{v^2} \frac{\partial f_e}{\partial v} \right)$$

and has the same power dependence on velocity as the Lorentz collisional term. Thus the rather unrealistic Lorentz gas model of Kruskal and Bernstein can now be given the physical interpretation of collisions between electrons and electrostatic waves. (This analogy was suggested by Rudakov [8], who applied the results of Kruskal and Bernstein to this problem and found that for sufficiently large electric fields, even neglecting the cone of stable electrons, runaways result; in other words, over a sufficiently long time, the ion sound instability cannot prevent the runaway process from occurring.) We may inquire what happens after these electrons run away. There exist other instabilities, stronger than this one; for example, two-stream instabilities with a critical velocity on the order of the electron thermal velocity may play a role.

Thus this is the situation in two- or three-dimensional quasi-linear theory. To calculate any kind of concrete expressions (for example, electrical conductivity) we can use the quasi-linear collision term, following the usual procedure starting with the transport equation. To do this, we need to know the quasi-linear diffusion coefficient which is proportional to the square of the electron field. Sometimes this can be done within the framework of pure quasi-linear theory, as in the example of a spherically symmetric problem considered in Part II; but for the ion wave instability, it is necessary to consider higher order processes like wave-wave scattering, etc. So we will now turn to the consideration of mode coupling for this problem. In Part I, it was shown that the dispersion relation of the type drawn in Fig. 1 (i.e., concave downward) cannot satisfy the resonance condition for the decay-type interaction. Consequently, the main contribution to the mode coupling comes from the non-linear wave-particle interaction, which is described by the last two terms in Eq. (9). The electron contribution to these terms is negligible, because only a few electrons can satisfy the resonance condition (i.e., \( \Delta n \sim n_e (\omega + \omega')/|K-K'|v_{th} \ll n_e \)). Since the wave phase velocity is much larger than the ion thermal velocity, the waves are only scattered by the ions, and the number of waves will be conserved.
If we set \( \omega'' = \omega_K - \omega_{K'} \sim k U h \) and \( \kappa'' = \kappa - \kappa' \sim \kappa \), then to lowest order in the small parameter \( \omega''/\omega \) we find

\[
\mathcal{E}^{(1)}(\kappa'', \omega'') \approx \frac{\omega_{pi}^2}{\kappa''^2} \int \frac{\kappa'' \cdot \frac{\partial f_i}{\partial U} \, d^3U}{\omega'' - \kappa'' \cdot U} \]

However, if we substitute these expressions into the wave kinetic equation we see that the non-linear terms cancel. Consequently, we must take into account terms of higher order in the small parameter \( \omega''/\omega \sim \frac{\kappa'' U}{\omega} \). For \( \mathcal{E}^{(2)} \) we find

\[
\mathcal{E}^{(2)}(\kappa', \kappa - \kappa') = \kappa''^2 \mathcal{E}^{(2)}(\kappa', \kappa', \kappa''') = \frac{k^2 m_i \omega_{K} \omega_{K'}}{\mathcal{E}(\kappa' \cdot K')} \tag{18}
\]

\[
\mathcal{E}^{(3)}(\kappa', \kappa, -\kappa') = \frac{e \kappa' \cdot \kappa}{m_i \omega_{K} \omega_{K'}} \mathcal{E}^{(1)}(\omega'', \kappa''') \tag{19}
\]

The value of the first part is obvious from the argument of the delta function and the second part is independent of the delta function. Consequently, we find

\[
\kappa' \cdot \nabla' = \frac{(\kappa'' \cdot \nabla')(\kappa' \cdot \kappa'')}{\kappa''^2} + \frac{[K' \cdot \nabla'] \cdot [K'' \cdot \kappa]}{K' \cdot \kappa''} \tag{20}
\]
In an analogous way we find

\[ \kappa^2 \mathcal{E}_{\kappa', \kappa''} = \kappa'' \mathcal{E}_{\kappa, \kappa'} = \frac{e}{m_i} \left( \frac{\kappa \cdot \kappa'}{\omega} \right) \left\{ 1 + \frac{2 \kappa \cdot \kappa''}{\kappa''^2 \omega} \right\} + \frac{3 (\kappa \cdot \kappa'') \omega''^2}{\kappa''^2 \omega_S^2} + \frac{3 (\kappa \cdot \kappa'')^2 \omega''^2}{\kappa''^2 \omega^2} \right\} \left\{ \frac{\kappa''}{\omega''} \right\} \right\} \}

(22)

By using Eqs. (21) and (22) we can write the kinetic equation in the form \[ (9, 10) \]

\[ \frac{\omega_{pl}^2}{\omega_k^2} \frac{\partial |\psi_k|^2}{\partial (\omega_k t)} = \frac{\omega_{pl}^2}{\kappa^2 C_s^2} \sqrt{\frac{n m_e}{2 m_i}} \left( \frac{u}{c_s} \cos \theta - \frac{\omega}{\omega_S} \right) |\psi_k|^2 + \frac{16 \pi^2 e^4 n}{m_i^3 k^2} \sum_{\kappa'} \frac{(\kappa \cdot \kappa')^2}{\omega_{pl}^2 \omega_{k'}^2} \mathcal{E}_{\kappa, \kappa'} \left\{ \frac{\kappa \cdot \kappa'}{\omega_{pl}^2} \right\} \left\{ \frac{\kappa \cdot \kappa'}{\omega_{pl}^2} \right\}

\[ \kappa', \frac{\partial f_{\text{foi}}}{\partial \omega} |\psi_k|^2 |\psi_{k'}|^2 \]

(23)
As expected, the mode coupling term conserves the number of waves

\[ \sum \omega \xi = \sum \frac{\omega \xi^2}{\omega \xi} \frac{k^2 |\psi|}{4 \pi} \]

Consequently, the mode coupling cannot stop the increase in the number of waves produced by the linear instability, and we need to take into account some additional process if we want to find a stationary spectrum. The energy flows from the high-frequency short-wavelength oscillations to the low-frequency long-wavelength oscillations; so we assume, as KADOMTSIV did /10/, that ion-ion collisions produce a turbulence sink in the long-wavelength region. So we can cut off the spectrum in this region and construct a stationary solution in the rest of phase space by balancing the linear growth rate with the nonlinear flow of energy to the long wavelength.

As an example of this kind of solution, we first assume that the turbulence is limited to two lines in \( K \)-space and consider the long-wavelength turbulence

\[ \left| \psi \xi \right|^2 = I(k) \delta(\psi) \delta(\cos \Theta - \cos \Theta_o), \quad \kappa \lambda \ll 1 \]

where \((k, \Theta, \psi)\) are spherical coordinates in \( K \)-space with the polar axis along the direction of the current.

When \( U \) is only slightly larger than \( C_s \), the linear instability produces waves propagating at small angles with respect to the current (i.e., \( \Theta_o \ll 1 \)). In this case, we can reduce Eq.(23) to the form

\[ \sqrt{\frac{\pi m_e}{2 m_i}} \left( \frac{U}{C_s} \cos \Theta_o - 1 \right) I(k) \approx - \frac{e^2}{T_e} \frac{\lambda}{\pi} I(k) k \frac{\partial}{\partial k} \left[ k^3 I(k) \right] \]

where we put

\[ \left( \kappa'' \frac{\partial}{\partial \psi} \delta(\omega \xi - \omega \xi') - \kappa' \frac{\partial}{\partial \xi} \right) = - \kappa'' \frac{\partial}{\partial \omega \xi} \delta(\omega \xi - \omega \xi') \]

-126-
and replaced the summation over $\kappa'$ by

$$\sum_{\kappa'} = \int_{-\infty}^{\infty} d\kappa' \sum_{0}^{\frac{2\pi}{\kappa}} \sin \theta d\theta \int_{0}^{\frac{2\pi}{\kappa}} \frac{k^2 dk}{(2\pi)^3}$$

The general solution of Eq. (24) is

$$\frac{e^2}{\pi^2 T_e} I(k) = -\sqrt{\frac{\pi m_e}{2 m_i}} \left( \frac{u}{c_s} \cos \theta_o - 1 \right) \frac{T_e}{T_i} \frac{\theta_o^2}{k^3} \ln(kD) \quad (25)$$

where we have demanded that $I(k) = 0$ at some very long wavelength $D$.

This solution is, of course, not unique. Moreover, the solution is unstable, because any wave propagating along the direction of the current has a larger linear growth rate and smaller mode coupling term than the wave propagating at the angle $\Theta_o$. From this argument we conclude that the spectrum tends to collapse to a line parallel to the current.

I.A. AKHIEZER has shown that it is possible to construct a self-similar oscillatory solution in the long-wavelength region (i.e., $k \lambda_D << 1$). Here we only follow his arguments qualitatively. For an axially symmetric solution (i.e., $\phi = \Theta$), we can write Eq. (23) in the form

$$\frac{1}{k c_s} \frac{\partial I(k, \theta, t)}{\partial t} + \frac{\sqrt{\pi m_e}}{2 m_i} \left( 1 - \frac{u}{c_s} \cos \theta \right) I(k, \theta, t) =$$

$$= \frac{e^2 T_e}{\pi T_i} I(k, \theta, t) \frac{\partial^2}{\partial k^2} \left[ 1 - \cos^2 \Theta \cos^2 \Theta' \right] \int \frac{1}{c_s / u} k^3 I(k, \theta', t) d\cos \Theta' \quad (26)$$

One can see that this equation is invariant under the transformation

$$I \rightarrow \lambda^{-3} I, \quad k \rightarrow \lambda \frac{k}{\lambda}, \quad t \rightarrow \lambda^{-1} t \quad (27)$$
where $\lambda$ is an arbitrary parameter. Therefore, we can reduce the number of variables by introducing $I k^3$ and $x = (\frac{n}{2m_i})^2 \omega_{pi} t_k$, which are invariant under the above transformations. In the linear stage, the function $I k^3$ depends only on $x$ like $\exp \{ \frac{2}{x} + \}$ where $y^+ \sim x$. Consequently, $I k^3$ should depend only on $x$ in the later stages. With regard to the angular dependence of the turbulent spectrum, Akhiezer found that the energy oscillates between the Cherenkov cone (i.e., $I(\theta) \sim 0 (\cos \theta - C_i / \omega)$) and the current line (i.e., $I(\theta) \sim 0 (1 - \cos \theta)$) with a period of oscillation proportional to the energy in the spectrum. Consequently, most of the time the energy will be distributed throughout the Cherenkov cone (see Fig. 4).

Fig. 4

In the short-wavelength limit (i.e., $k \lambda_D \gg 1$) again we can reduce Eq. (23) for the wave amplitude to the integro-differential equation of type (26)

$$\frac{\partial I(k, \theta, t)}{\partial \omega_{pi} t} = \frac{\omega_{pi}^2}{k^2 C_s^2} \sqrt{\frac{n m_e}{2 m_i}} \left( \frac{\nu}{C_s} \cos \theta - \frac{\omega_{pi}}{k C_s} \right) I(k, \theta, t) +$$

$$+ \frac{\omega_{pi}^2}{k^2 C_s^2} \frac{e^2 I(k, \theta, t)}{4 T_e^2} \frac{T_i}{T_e} \lambda_{10} \frac{k}{\lambda_D} \frac{d}{dk} \int_0^{2 \pi} \int_0^{2 \pi} d \cos \theta' \frac{2 \pi}{1} \left[ \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi') \right]$$

$\left[ 1 - \left[ \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi') \right] \right]$. 

-128-
There is no rigorous solution of type (27) for this equation. However, we can find, as we did for the wave-wave interaction in Part I, some power-type solution, at least for the k-dependence. The natural sink of wave in the short-wavelength region is due to the linear ion Landau damping. From Eq. (28) we find that the spectral energy of waves very rapidly decreases to the short scales

\[
\frac{e^2 I(k)}{4 T_e^2} \sim \frac{1}{5} \sqrt{\frac{2 m_i}{\pi}} \left( \frac{u_c \cos \theta - 5 \omega_i}{6 \kappa c_s k} \right) \frac{T_e}{\rho_0^2 T_i}.
\]  

(29)

The power of this solution is completely different from that found in Part II. Therefore there is no universal power solution for the plasma turbulence like the Kolmogorov spectrum in hydrodynamic turbulence.

Now with the help of the expression found for the spectral energy of waves in the limits of both large (25) and short (28) scales, we estimate the turbulent resistivity. Multiplying the quasi-linear equation for the electrons

\[
\frac{\partial f_e^{(o)}}{\partial t} = \frac{e^2}{m_e^2} \int \kappa \frac{\partial}{\partial \mathbf{v}} \frac{\partial f_e}{\partial \mathbf{v}} \frac{\omega^2}{(\omega - \kappa \mathbf{v})^2 + \mathbf{v}^2} - \frac{\kappa}{\omega} \frac{\partial f_e}{\partial \mathbf{v}} \frac{d^3 \mathbf{k}}{(2\pi)^3}
\]

by \( m_e \mathbf{v} \) and integrating the result over the velocity space, we obtain the electron momentum loss due to the radiation of the ion sound waves

\[
\mathbf{p}_{\text{rad}} = -\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{e^2 |\mathbf{P}_e|^2}{m_e} \int \frac{d^3 \mathbf{v}}{(2\pi)^3} \kappa \delta (\omega - \kappa \mathbf{v}) \frac{\partial f_e}{\partial \mathbf{v}}
\]

Introducing the effective electron-ion collision frequency as

\[
m_e n_0 v_{\text{eff}} \mathbf{u}_e(t) = \frac{d}{dt} \left( m_e n_0 \mathbf{u}_e(t) \right),
\]

with the help of Eqs. (16) we obtain from this expression

\[
m_e n_0 v_{\text{eff}} \mathbf{u}_e(t) = -\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\partial \mathbf{E}_{\kappa}^{(i)}(\mathbf{w}_\kappa)}{\partial \mathbf{w}_\kappa} \frac{k^2 |\mathbf{P}_e|^2}{8\pi}.
\]  

(30)
On the r.h.s. of Eq. (30) is the wave momentum density radiated by electrons per unit time (i.e., \( -\mathbf{P}_{\text{rad}} \)). By Eq. (30) we supposed that all the waves radiated by electrons are absorbed by ions. The main contribution to the integral on the r.h.s. of Eq. (30) corresponds to the wavelength of order Debye length \( \Lambda_D \sim \lambda \).

For the case of large current velocity \( U \gg C_s \), we find from Eqs. (25) and (29)

\[
\gamma_{\text{eff}} \sim \omega_{pi} U C_s \frac{T_e}{T_i}.
\]

Therefore the effective collision frequency is larger than the growth rate.

As in the usual Joule heating, the turbulent heating changes mostly the electron temperature [12] if \( U \) is greater than the ion thermal velocity. The heating rate of the ions is always much smaller than the heating rate of the electrons. Really, the rate of the energy changes due to the scattering of the electrons by ion sound waves is equal to (see Eq. (30))

\[
n_e \frac{\dot{T}_e}{T_e} = \gamma_{\text{eff}} m_e n_e U^2 = \int \frac{d^3k}{(2\pi)^3} \left( \frac{k^2}{\omega} \right)^2 \left( \frac{k \cdot U}{\omega} \right) \frac{E_k}{\omega} \frac{k \cdot U}{\omega}.
\]

where

\[
E_k = \omega_k \frac{\partial}{\partial \omega_k} E_k(\mathbf{k}, \omega) \frac{k^2 |\psi_k|^2}{2 \pi} \text{ is the wave energy.}
\]

Then, even if we suppose that the total energy of the waves is absorbed by ions, the ratio of the heating rate for both species is limited by the inequality (12)

\[
\frac{\dot{T}_i}{T_i} \lesssim \frac{\int \frac{d^3k}{(2\pi)^3} \left( \frac{k \cdot U}{\omega} \right) \frac{E_k}{\omega}}{\int \frac{d^3k}{(2\pi)^3} \left( \frac{k \cdot U}{\omega} \right) \frac{E_k}{\omega}} \sim \frac{C_s}{U} \quad (32)
\]

This ratio is small if the current velocity is not so close to the critical one.
3. Non-linear theory of the drift instability

In order to complete consideration of the anomalous transport processes in non-uniform plasma, we will try to describe in detail the non-linear stage of the low-frequency "universal drift instability" [3-14]. The non-linear equation for the spectral energy of the electrostatic waves is written in terms of the expansion coefficient of the particle distribution in powers of the wave amplitude. These coefficients are defined by the iteration formula

$$
\delta \sum_{\mathbf{k}'', \mathbf{k}'', \cdots, \mathbf{k}^{(n)}} (\omega', \omega'', \cdots, \omega, \omega')^j \ell \cdot \nu(t) =
$$

$$
= i \frac{e_z}{m_j} \int_{-\infty}^{t} d \tau \left( \varphi_{k', \omega'} \left( \ell'' \cdot \nu(t) \right) \frac{k_z}{\omega_{k_z}} \delta \sum_{\mathbf{k}'', \mathbf{k}'', \cdots, \mathbf{k}^{(n)}} (\omega', \omega'', \cdots, \omega, \omega')^j \ell \cdot \nu(t) \right)
$$

We can describe the mode coupling of drift waves in a non-uniform plasma with the help of this equation so long as

$$
\lambda \propto \frac{d}{dx} \left( \kappa \nu_a \right) \ll \gamma
$$

This inequality just says that the difference between the eigenfrequencies in a non-uniform plasma is much less than the growth rate. In other words, the difference between the energy levels is much smaller than the energy of interaction.

The dispersion relation in the linear approximation was found in Part II. Now let us proceed to the next step: the derivation of the non-linear equation for the wave energy. Substituting in (33) the linear approximation (II - 97), we get

$$
\delta \sum_{\mathbf{k}'', \mathbf{k}'', \cdots, \mathbf{k}^{(n)}} (\omega', \omega'', \cdots, \omega, \omega')^j \ell \cdot \nu(t) = - \frac{e_z^2}{m_j \ell} \int_{-\infty}^{t} d \tau \left( \varphi_{k', \omega'} \left( \ell'' \cdot \nu(t) \right) - i \frac{\kappa z u_{\nu(t)}}{\omega_{k_z}} \right)
$$

$$
\times \left( \kappa z \frac{d}{d \nu(t)} \left[ 1 - \frac{\omega'' - \kappa z u_{\nu(t)}}{\omega'' - \kappa z u_{\nu(t)} + i \epsilon} \right] \frac{\kappa z u_{\nu(t)}}{\omega_{k_z}} \right) e^{i \frac{\kappa z u_{\nu(t)}}{\omega_{k_z}}} \int f_{\nu(t)}. \right)
$$

(35)
In (3) the velocity derivative gives several terms; however, we recall that in the first order calculation, the contribution from differentiation of the argument of the Maxwellian cancelled up to a small term, and the same thing happens here; also, if we assume \( \kappa_+ \gg \eta \), the term arising from differentiation of the density in \( \delta f_j^{(\ell)} \) is negligible. So the main contribution comes from the rapid velocity dependence of the exponential \( \exp \left( \frac{\kappa' \kappa''}{\omega m} \right) \) in the second term of \( \delta f_j^{(\ell)} \). Carrying out the time integration and symmetrizing with respect to \( \kappa \) and \( \kappa'' \), we have

\[
\delta f_j^{(\ell)} = -i \frac{e_j}{T} \sum_{\kappa_0} \frac{\kappa' \kappa''}{\omega m} \frac{h}{\omega'_m} \int_0^\infty \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right) \frac{J_0}{\omega m} \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right).
\]

Using this expression in the same approximations, we obtain the third order correction to the distribution function.

\[
\delta f_{j,\omega}^{(3)} = -i \frac{e_j}{T} \sum_{\kappa_0} \frac{c^2}{H^2} \frac{\kappa''_m}{\kappa = \kappa'_m + \kappa''_m} \int_0^\infty \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right) \frac{h}{\omega m} \int_0^\infty \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right)
\]

\[
\times \left[ \omega - \kappa'_m \kappa''_m / \omega m \right] \left[ \varphi_{\kappa'} \varphi_{\kappa''} \varphi_{\kappa''} \right] \left( \omega + \omega'' \right) \left( \omega - \kappa'_m \kappa''_m / \omega m \right) \left( \omega + \omega'' \right).
\]

\[
\int \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right) \frac{J_0}{\omega m} \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right) e^{-i \left( \omega + \omega'' \right) t} \left[ \varphi_{\kappa'} \varphi_{\kappa''} \varphi_{\kappa''} \right] \left( \omega + \omega'' \right) \left( \omega - \kappa'_m \kappa''_m / \omega m \right) \left( \omega + \omega'' \right)
\]

\[
\int \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right) \frac{J_0}{\omega m} \left( \frac{\kappa'_m \kappa''_m}{\omega m} \right) e^{-i \left( \omega + \omega'' \right) t} \left[ \varphi_{\kappa'} \varphi_{\kappa''} \varphi_{\kappa''} \right] \left( \omega + \omega'' \right) \left( \omega - \kappa'_m \kappa''_m / \omega m \right) \left( \omega + \omega'' \right)
\]
In Eq. (9) for the wave amplitude, the main non-linear contribution comes from the ion terms, until \( \kappa_x^2 \gamma_{ii} < \sqrt{\frac{m_i \beta}{m_e}} \). We also use the fact that the phase velocity along the lines is much larger than the thermal velocity for most unstable waves, so the dielectric constant for the beats can be written in a form:

\[
(k - k')^2 \chi_d^2 \varepsilon_{ij} \left( \omega_x - \omega_{k'} \right) \approx \gamma_i \left( (k - k') \gamma_{ii} \right) \int_{\Sigma} \frac{\left( \omega_x - \omega_{k'} \right) \delta \left( \omega_x - \omega_{k'} \right) \delta \left( \omega_x - \omega_{k''} \right)}{\omega_x - \omega_{k'} - (k_x - k'_x) \Sigma_x} \, d\Sigma_x \delta (k_x - k_x') \delta (k'_x - k_x') \, d\Sigma_x \, d\Sigma_x \tag{38}
\]

If we keep only the ion contribution to \( \varepsilon^{(2)} \) and \( \varepsilon^{(3)} \), Eq. (9) becomes:

\[
\frac{\partial |\psi_x|^2}{\partial t} + \frac{\partial \omega}{\partial k_x} \frac{\partial |\psi_x|^2}{\partial k_x} + \frac{\partial \omega}{\partial k_x} \frac{\partial |\psi_x|^2}{\partial \omega} = - \frac{V_2}{K_x} \frac{\omega_x (\omega_x - k_x \Sigma_x)}{1} |\psi_x|^4 + \frac{C^2 |\psi_x|^2}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right] \left( \frac{\Sigma_x (k_x - k'_x)}{\omega_x} \right)
\]

\[
\left[ \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right] \left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right]
\]

\[
\frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right] \left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right]
\]

\[
\left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right] \left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right]
\]

\[
\left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right] \left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right]
\]

\[
\left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right] \left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right]
\]

\[
\left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right] \left[ \frac{\omega_x}{H^2} \int_{(2\pi)^3} d^3 k \left[ \frac{\delta (\omega_x - \omega_{k'}) (k_x - k'_x)}{\omega_x} \right]^2 \right]
\]

* The approximation (38) breaks down if \( \kappa_x^2 \gamma_{ii} < \sqrt{\beta} \). But for such a long wave the interaction through the beats is very small and we can neglect it.
On the l.h.s. we take into account a weak dependence on \( t \), \( \kappa \) and \( k_x \) in \( \psi_\kappa \), in addition to the oscillatory factor already obtained in the linear approximation. The way to do this is very similar to what we did to obtain the dependence on \( t \) in \( \psi_\kappa \) (see Eq. (8)). But we may obtain the l.h.s. of Eq. (39) by a more formal method if we expand the frequency, \( x \)-component of wave vector and \( x \)-co-ordinate near the real position of the wave packet in the configuration space \( (\omega_\kappa, k_x, x) \).

In accordance with Eq. (8) we find

\[
\begin{align*}
\frac{\partial \psi_\kappa}{\partial \kappa_x} &= \frac{\partial \psi_\kappa}{\partial \kappa_x} \\
&= \frac{\partial \psi_\kappa}{\partial \kappa_x} \\
&= \frac{\partial \psi_\kappa}{\partial \kappa_x}
\end{align*}
\]

This we easily reduce to the substantial derivative in configuration space by the help of relations

\[
\frac{\partial \omega_\kappa}{\partial \kappa_x} = - \frac{\partial \psi_\kappa}{\partial \kappa_x} \left/ \frac{\partial \psi_\kappa}{\partial \omega_\kappa} \right., \quad \frac{\partial \omega_\kappa}{\partial k_x} = - \frac{\partial \psi_\kappa}{\partial k_x} \left/ \frac{\partial \psi_\kappa}{\partial \omega_\kappa} \right..
\]

Three terms on the r.h.s. of Eq. (39) describe respectively the linear growth of instability, mode coupling due to wave scattering by ions and decay type interaction, which can be written in symmetrical form with the particular choice of normalization (See Part I).

Let us first consider the long-wavelength drift turbulence (i.e., \( k_x^2 \ll \kappa^-2 \ll 1 \)). In this limit we neglect the wave scattering by ions, which is small (as \( k_x^2 \ll \kappa^-2 \ll 1 \)) in comparison with the decay type interaction.

As we go to the limit of zero ion Larmor radius, the phase velocity of the wave becomes very close to the electron drift velocity

\[
\omega_\kappa \approx \omega_\kappa - \kappa \nu \ll \omega_\kappa
\]

Therefore, the waves are propagating almost in phase with each other for a long time \( A_t \sim \omega_\kappa^{-1} \), and interaction becomes stronger.
This is reflected by the fact that the integrand of the non-linear decay type term goes to zero as \(\delta \omega_{k}^{2}\). But the argument of the \(\delta\) -function in (39) vanishes also. Thus

\[
\Delta \omega = \omega_{k} - \omega_{k'} - \omega_{k-k'} = \delta \omega_{k} - \delta \omega_{k'} - \delta \omega_{k-k'}.
\]

Then the non-linear term is of order

\[
\frac{c^{2} |\psi_{k}|^{2}}{H^{2}} \int \frac{d^{2}k'}{(2\pi)^{2}} |\psi_{k'}|^{2} (k \times k')^{2} \frac{\delta \omega_{k}}{\omega_{k}} \delta (\omega) \sim \delta \omega_{k} |\psi_{k}|^{2} \left( \frac{d^{2}k'}{(2\pi)^{3}} \frac{c^{2} |k \times k'|^{2}}{H^{2} \omega_{k}^{2}} |\psi_{k}|^{2} \right).
\]

By balancing growth rate and the rate of energy output from the mode due to the decay interaction we obtain [13 - 14]

\[
\frac{c^{2}}{H^{2}} \int \frac{k_{x}^{2} |\psi_{k}|^{2}}{(2\pi)^{3}} \frac{d^{2}k_{z}}{\omega_{k}} \sim \frac{\lambda_{z}}{\delta \omega_{k}} \left( \frac{(\omega_{d})^{2}}{c} \right)^{2} \quad (41)
\]

For the truncation of the wave kinetic equation (39) in the lowest non-linear order to be valid, we need \(\lambda_{z} / \delta \omega_{k} \ll 1\); thus (41) implies that the \(E \times H\) drift is always smaller than the initial velocity due to the pressure gradient, since \(\lambda_{z} / \delta \omega_{k} \ll 1\) is needed for the derivation of (39). In a plasma with \(\beta > m_{e} / m_{i}\), as was shown in Part II, the electrostatic waves satisfy

\[
\frac{\lambda_{z}}{\delta \omega_{k}} \sim \frac{\omega_{x}}{|k_{z}| \nu_{th}^{2}} < \sqrt{\frac{m_{e}}{m_{i}} \beta} \quad (42)
\]

For some instabilities, the perturbation theory leading to (39) breaks down because \(\lambda_{z} > \delta \omega_{k}\). For example, let us consider bad curvature effects on the long-wavelength drift instability [25]. The expressions (II-98) and (II-100) hold in the drift frames of the respective species. In the co-ordinate frames moving with the respective gravitational drift velocities, we have instead

\[
h_{i} = -\frac{e \psi}{T_{i}} n_{o} \frac{k_{y} \nu_{d}}{\omega - \omega_{ni}^{-1} [k \times \hat{r}_{o}] \cdot \hat{r}_{ni}^{2} / \nu_{th}^{2}}
\]

-135-
\[ n_e \approx \frac{e \nu}{T_e} n_0 \left( 1 + \frac{1}{\pi^2} \right) \left( \frac{\omega}{\omega_{th}} \right) \left( 1 - i \frac{1}{\pi} \right) \left( \frac{\nu_{th}}{\omega_{th}} \right) \left( \frac{R}{\omega_{th}} \right) - k_y V_d^e \]

where \( \hat{n}_o \) is the principal normal to the curved field lines and \( R \) is the radius of curvature. Here we neglect the ion Larmor radius in comparison with the wavelength.

Now the dispersion relation yields*

\[ \omega = \frac{[\kappa \cdot \hat{n}^o] h}{\omega_{th} R} + k_y V_d^e \left\{ 1 - i \frac{1}{\pi} \left[ \frac{[\kappa \cdot \hat{n}^o] h}{\omega_{th} R} \left( 1 + \frac{T_e}{\pi} \right) \right] \right\} \]

If we go to the co-ordinate system moving with velocity

\[ U = V_d^e + \frac{\hat{n}_o \cdot \hat{h}}{\omega_{th}} \frac{\nu_{th}^2}{R} \]

the real part of the frequency vanishes and we have pure growing modes, i.e., a strong instability.

This is related to the fact that the non-linear term in (39) vanishes for \( \delta \omega_x \rightarrow 0 \). Up to now we have assumed that the electrons (which can move freely along the lines of force) are established in a Boltzmann distribution. However, this is not true of the resonant electrons. As ion non-linear effects become small, it is necessary to consider electron terms which were not included in the foregoing treatment. When \( \gamma \sim \delta \omega_x \), (41) implies that the electron \( E \times H \) drift is as large as the drift arising from the pressure gradient. Moreover, sometimes we can treat the drift wave as monochromatic in the zero Larmor radius limit. In that case the amplitude limitation due to the non-linear motion of the resonance electrons is more restrictive (see Part II).

Up to now we have considered only long-wavelength drift oscillation. But if we start from the thermal level of fluctuation, the wave grows more rapidly with the largest growth rate.

* Let us note that in the trap with \( q_y \neq 0 \) the wave packets with \( k_x \gg k_y \) can be growing even in the field with minimum \( \beta \)-geometry \( \beta = 0 \) (the plasma stability in the general case \( \frac{dV}{dy} = q_y \neq 0 \) discussed in detail in [17] and [18 - 19] respectively).
The linear growth rate for the drift wave (sec. II-101) increases as a function of $K_{\perp} r_{ni}$, until $K_{\perp} r_{ni} \sim \frac{1}{m_i \beta m_e}$ or until finite plasma pressure effects come into play (16). The finite plasma pressure effects add a term due to the ion diamagnetic drift,

$$\nu = \frac{v_{i2}}{\omega_{ni}} \frac{v_{iH}}{H} i_y = - \frac{B}{\omega_{ni}} \frac{v_{i2}}{n} i_y .$$

Putting this drift into the linear dispersion relation gives us the following marginal stability condition:

$$\pi \int (\omega - k_y v_{rd}) \int \left[ (\frac{k_y v_{iH}}{\omega_{ni}}) \delta (\omega - k_y \beta \frac{v_{i2}}{\omega_{ni}} \frac{v_{n}}{n}) \right] d^3 v +$$

$$+ \pi \frac{\omega - k_y v_{rd}}{1 \frac{k_y \omega_{the}}{\nu_{the}}} = 0 .$$

Consequently, finite plasma pressure stabilizes the drift wave when

$$K_{\perp_{max}} r_{ni} \approx \left( \frac{8 \pi \beta^2}{1} \right)^{1/2} \gg 1 . \quad (43)$$

Thus, the maximum linear growth rate will occur when $K_{\perp} r_{ni}$ is just below the smaller of $(8 \pi \beta^2)^{-1/2}$ or $\sqrt{m_i \beta m_e}$.

From Eq. (39) we can see that the rate of the wave scattering by ions is the rapidly increasing function of $K_{\perp}$ and that this process makes the main contribution to Eq. (39) for short wavelengths. As usual, the short waves scatter by ions to the long-wavelength region of $K_{\perp}$-space and very soon limit themselves to the level (41) (of course, we now estimate $\gamma_{vd}$, $\omega_{vd}$ in the short-wavelength limit).

From Eq. (41) it also follows that the boundary of the turbulent region moves from small wavelengths to large wavelengths. However, this movement must stop when $K_{\perp} r_{ni}$ becomes of order unity, because the scattering process becomes less effective there and decay processes...
start to return energy to the short wavelengths. In the time-asymptotic limit, the spectral energy of waves must be equal to zero for \( k_{\perp} \gamma_{\nu} > \frac{1}{\lambda} \).

If we use the estimate for \( |\psi_k|^2 \) given in Eq. (41), we find the following value for the diffusion coefficient (see Part II).

\[
D_{\perp} \approx \frac{\frac{1}{2} \chi_{\perp}^2}{\delta \omega_{\perp} \gamma_0 (k_{\perp} \gamma_{\nu})} \tag{44}
\]

As was shown, in the time-asymptotic limit, we maximise \( D_{\perp} \) with respect to the long wavelength \( k_{\perp} \gamma_{\nu} < 1 \).

In this expression \( \chi_{\perp} \) appears as the basic unit in the "random walk". It is tempting to choose it as large as the radius of the plasma; this would correspond to a quasi-mode. But Eq. (41) shows that the amplitude of such a quasi-mode would be enormous, and, consequently, its stability is questionable. Since it is difficult to check the stability of such a mode, we give another argument against its existence. Since the linear growth rate is independent of \( k_{\perp} \), all the normal modes up to \( k_{\perp} \approx k_{\parallel} \) would have to grow to a large amplitude with the quasi-mode, and these modes would suppress the amplitude of the quasi-mode. Therefore, we put \( \chi_{\perp} \approx \chi_{\parallel} \approx \gamma_{\nu} \) and find [13-14]

\[
D_{\perp}^{(0)} \approx \frac{m_e}{m_i \beta} \frac{\gamma_{\nu} \nu n}{\gamma} \left( \frac{e}{\gamma_{\nu}} \right) \tag{45}
\]

When collisions are not strong enough to maintain the Maxwellian form of the electron distribution, quasi-linear relaxation will reduce the growth rate of the oscillations (see Part II). If \( \nu_c \) is the collision frequency and \( |\psi_{\perp}|^2 \) is the amplitude of the turbulence, then the growth rate will be reduced when

\[
\nu_c \leq \pi \sum_{\kappa} \frac{e^2 k_{\perp}^2 |\psi_{\perp}|^2}{m_e^2 \omega_{\perp}^2 v_{\text{th}}^2} \gamma_0 (k_{\perp} \gamma_{\nu})
\]

-138-
Using Eq. (41) to estimate the level of turbulence shows that the oscillations with \( K_{\perp} r_{ni} \sim 1 \) will be suppressed when \( \psi_e \lesssim \Omega_\ast \left( \frac{m_e n}{m_i} \right)^{3/2} \). But this does not mean that the instability will be completely suppressed, because the rate of energy flow from the short-wavelength region to the long-wavelength region will also be reduced. Consequently, the level of turbulence in the short-wavelength region will be given by Eq. (41) with \( K_{\perp} r_{ni} \gg 1 \) and the diffusion coefficient by Eqs. (44) and (45).

The diffusion rate is decreased when the relaxation of the distribution affects the short wavelengths (i.e., \( K_{\perp} r_{ni} \sim (8 \pi \beta^2)^{-1/2} \)). Then the diffusion coefficient can be estimated with the help of Part II [147].

\[
\mathcal{D}^{(1)}_\perp = \psi_e \left( \frac{n}{n_i} \right)^{3/2} \frac{m_e}{m_i} \left( \frac{e^2}{8 \pi m_i \beta^2} \right)^{1/2} \mathcal{D}^{(0)}_\perp \quad (46)
\]

where the condition \( \mathcal{D}^{(1)}_\perp < \mathcal{D}^{(0)}_\perp \) is the condition for the applicability of Eq. (46).
4. The general scheme of weak turbulence theory

In order to establish the relationship between the different parts of weak turbulence approach, it is useful now to summarize what we did in the form of a symbolic tree:

- **Vlasov equation** (for each species of charged particles) + **Maxwell equation**
  - Using statistical approach
    - **Particle kinetic equation** + **Quasi-particle (wave) kinetic equation** (for each "species" of plasma waves)

These kinetic equations have the form:

\[
\frac{df(v)}{dt} = St\{f(v)\} + \frac{dN(k)}{dt} = St\{N(k)\}
\]

where the Stoss-terms represent:

**in the 1st approximation**

\[
St\{f\} = QLA \quad \text{and} \quad St\{N(k)\} = 2Jm(\omega_x)\{f(v)\}N(k)
\]

This is a quasi-linear approximation (QLA, Part II of these lectures). \(Im(\omega_x)\{f(v)\}\) means the imaginary part of \(\omega_x\) as a functional of \(f(v)\) (from the linear dispersion relation).

This approximation takes into account only cross-interaction particles + waves, according to the resonance condition.
\[ \omega_k + \kappa \cdot \psi = 0 \]

in the 2nd approximation (a)

Wave + wave; (3-waves) interaction (part I),
according to the "decay" conditions

\[
\begin{aligned}
\omega_1 + \omega_2 &= \omega_3 \\
\kappa_1 + \kappa_2 &= \kappa_3
\end{aligned}
\]

The collisional term in the particle's kinetic equation describes the adiabatic interaction of particles with waves, since the particles participate in the oscillations.

\[ S^\dagger N \dagger = \hat{N} \cdot \hat{N} \]
Symbolic notation
in order to stress the quadratic character of 3-wave interaction Stoss (part I)

in the 2nd approximation (b)

Wave – particle – wave interaction (part III),

\[
\omega_4 - (\kappa_3 - \kappa_2) \cdot \psi = \omega_2
\]

The contribution of this process to Stoss-term is too cumbersome and we did not consider it in the lectures (see §4, 10)

\[ \hat{N} \cdot \hat{N} f \]
in order to show that the particles also participate, see the details in Part III.

in the 3rd approximation

\[
\begin{aligned}
\omega_1 + \omega_2 &= \omega_3 + \omega_4 \\
\kappa_1 + \kappa_2 &= \kappa_3 + \kappa_4
\end{aligned}
\]

4-wave interaction

-141-
Again only adiabatic interaction of particles and waves.

Since this Stoss-term is cubic, it can be important only in the undecayable cases. We did not consider it in the lectures.

In the case of strong turbulence such approximate procedure breaks down, since higher numbers of wave interactions have the same order of magnitude. In other words, in the strong coupling limit there is no closure. Here we do not discuss the various attempts to make non-linear estimations in the strong turbulence cases \( \text{\cite{10, 20, 21}} \).

In the case of weak turbulence, the statement of concrete problems in terms of, generally speaking, quite complicated non-linear integro-differential kinetic equations for particles and waves does not lead, of course, to easy interpretation. But, at least, there exists a finite number of equations. In the different limiting cases one could solve it using various realistic and idealized approximations, as we were trying to demonstrate in these lectures.

Weak plasma turbulence theory permits a simple generalization in order to take into account the spontaneous (thermal) fluctuations in plasma. One can expect these fluctuations to be the additional source of wave generation in the kinetic equation for the waves and corresponding to it, the recoil effect (friction term) in the particle kinetic equations \( \text{\cite{10, 22}} \).

Let us treat the fluctuating part of the distribution function $
abla f \left( \mathbf{x}, \mathbf{r}, t \right)$, which satisfies the ideal gas correlation law in a plasma without magnetic field

\[
\langle \delta f \left( \mathbf{x}, \mathbf{r}, t \right) \delta f \left( \mathbf{x}', \mathbf{r}', t' \right) \rangle = \delta \left( \mathbf{x} - \mathbf{x}' \right) \delta \left( \mathbf{r} - \mathbf{r}' - \mathbf{v} \left( t - t' \right) \right) f \left( \mathbf{v} \right)
\] (46)
We will take $\delta f$ into account for simplicity in the case of the longitudinal oscillations

$$\mathcal{E}(\kappa, \omega) \psi_{\kappa, \omega} = -\frac{4\pi e}{k^2} \int \delta f_{\kappa, \omega} d^3\Sigma$$

$$\delta f_{\kappa, \omega} = \int \delta f(\xi, \xi, t) \exp \left\{ i \omega t - i \kappa \cdot \xi \right\} dt d^3\Sigma$$

$$\mathcal{E}(\kappa, \omega) \equiv 1 + \sum \frac{i\pi e^2}{k^2} \int \frac{\kappa \cdot \xi}{\omega - \kappa \cdot \xi + i\varepsilon} d^3\Sigma$$

(47)

Since $\delta f$ represents an additional source in the equation for the electric field.

But we are interested in the quadratic terms $|\psi_{\kappa, \omega}|^2$.

Multiplying Eq. (47) by $\psi_{\kappa, \omega}^*$, we obtain

$$\mathcal{E}(\kappa, \omega) |\psi_{\kappa, \omega}|^2 = -\frac{4\pi e}{k^2} \psi_{\kappa, \omega}^* \int \delta f_{\kappa, \omega} d^3\Sigma$$

(48)

or

$$\mathcal{E}(\kappa, \omega) |\psi_{\kappa, \omega}|^2 = \frac{(4\pi e)^2}{k^4 \mathcal{E}^*(\kappa, \omega)} \int \delta f_{\kappa, \omega} d^3\Sigma \int \delta f_{\kappa, \omega}^* d^3\Sigma$$

(49)

Using the same symbolic approximation as we did in Part III, $\mathcal{E}(\kappa, \omega) \equiv \mathcal{E}(\kappa, \omega + \frac{2}{\omega \frac{\partial}{\partial t}})$, and averaging $\delta f \cdot \delta f^*$ according to the correlation law (46), we easily derive

$$\frac{\partial \mathcal{E}(\kappa, \omega)}{\partial \omega} \frac{\partial |\psi|^2}{\partial t} = \text{Im} \left\{ \frac{16 \pi^2 e^2}{k^4} \int f(\xi) \frac{d^3\Sigma}{\mathcal{E}^*(\kappa, \kappa, \xi)} \right\}$$

(50)

* As in most of our lectures, we consider here the discrete set of waves in the finite system of volume equal to unity.
This is an additional term, representing the spontaneous generation of the plasma waves, which we should now add to the r.h.s. of the wave kinetic equation.

The recoil effect in the quasi-linear equation for the distribution function is derived in the following way:

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{fluct.}} = \frac{e \nabla \delta \psi}{m} \cdot \frac{\partial \delta f}{\partial \nu}
\]

\[ (51) \]

where \( \delta \psi \) is the contribution to the electric field from the spontaneous fluctuations. Averaging \( \delta \psi \delta f \), using Eqs. (46) and (47), we derive this additional term in the kinetic equation:

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{fluct.}} = \frac{16 \pi^4 e^4}{m^2} \sum_k \int \frac{\delta \left( \kappa (\nu - \nu') \right)}{\kappa^4 \left| \kappa (\kappa, \kappa, \nu) \right|^2} f(\nu') \frac{\partial f(\nu')}{\partial \nu'} d\nu'
\]

\[ (52) \]

These additional terms (50) (in the wave kinetic equation) and (52) (in the quasi-linear equation) are not significant in the case of unstable plasma. In stable plasma the competition between the term (50) and "Landau damping" determines the equilibrium level of the thermal field fluctuation. This equilibrium background of plasma waves, after the substitution into the quasi-linear Stoss, will give

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{QLA}} \approx -\frac{16 \pi^4 e^4}{m^2} \sum_k \int \frac{\delta \left( \kappa (\nu - \nu') \right)}{\kappa^4 \left| \kappa (\kappa, \kappa, \nu) \right|^2} f(\nu') \frac{\partial f(\nu')}{\partial \nu'} d\nu'
\]

\[ (53) \]

Combining the terms (52) and (53) we get the Lenard-Balescu equation.
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1. Non-linear dispersive wave trains and solitons

One of the most interesting fields where one can apply the whole arsenal of the plasma non-linear theory methods is the theory of collisionless shocks.

Actually, what we know about the wide variety of these problems are some limiting cases where the plasma behaviour can be described almost in terms of moments approximation (as we did in Part I). Such approximations can be applied to collisionless plasma only in some degenerate cases, for example, when we have guiding centre motion across a magnetic field. Another relatively well developed case is the low $\beta$ ($\beta = \frac{nT}{4\pi m}$) limit, where the moments approximation appears simply as the equations of motion for each species, since the thermal spread is almost negligible.

The cases when the fluid-like description breaks down are not yet understood satisfactorily; therefore we have not too much to say about it.

The logical way to find the features of the shock in collisionless plasma is to follow the time evolution of an initial Riemannian-type disturbance. In a fluid-like approximation the development of such profiles can also be described in terms of Riemann's solution (unless dispersion does not play a role) where any point of profile $x$ is moving with velocity

$$\frac{dx}{d\tau} = C(u) + u$$

(1)

where $C(u)$ is a local value of the sound speed and $u$ is the local velocity amplitude. According to solution (1), the non-linear steepening will create sufficiently strong gradients in the front of such a

* This part of the lecture notes represents only a short review. For the omitted details we will refer to the reviews $\{1, 2, 3\}$ (See also $\{3\}$).
wave. Finally, the dispersion phenomena become very important.

They are represented in a fluid-like plasma model in terms of the highest derivatives. Unfortunately nobody has yet found the generalization of a Riemannian-type solution for the case with dispersion. But for a simple analysis one can expect the formula

\[ \frac{dx}{dt} = C(\nu) + \nu + \Delta \nu_{\text{disp}} \]  \hspace{1cm} (2)

to be qualitatively right. Very often the linear dispersion law has a behaviour similar to that of the two different cases shown in Fig. 1.

So one can write it down as

\[ \omega^2 = C^2 K^2 (1 + \lambda K^2 + \ldots) \]  \hspace{1cm} (3)

where \( \lambda > 0 \) for the case (a) and \( \lambda < 0 \) for the case (b).

\[ \Delta \nu_{\text{disp}} \approx C \lambda K^2 \implies -C \lambda \frac{d^2}{dx^2} \]  \hspace{1cm} (4)

Now using (2) and (4) one can easily see that the competition between non-linear and dispersive effects will lead qualitatively to the time evolution of non-linear wave profile, shown in Fig. 2.
In other words, the initially monotonic profile becomes intensively modulated, reminding us finally of a non-linear wave train. Obviously, in the case (a) this wave train is going far ahead and in the case (b) it is going backwards.

The next step in order to see the asymptotic time behaviour \(( t \to \infty )\) of such wave trains is to look for the steady-state solutions \( \frac{\partial}{\partial t} = 0 \) (or the solutions depending on \( x-ut \)). One can expect these solutions to be the asymptotic form of any initial one-dimensional profile, as happens in conventional fluid-dynamics, where \( (x-ut) \)-solutions represent the shock structure. Since in our case dispersion plays a determinant role, these steady-state non-linear solutions are reversible (in the case of no dissipation). They appear to be non-linear periodic waves. In order to find their form in a moments approximation, one must solve the system of non-linear ordinary differential equations. There is no difficulty in doing it, since the coefficients do not depend on the argument \( (x-ut) \). The characteristic space scale for these steady-state waves is the order of characteristic dispersion length \( \sim \sqrt{d} \) in the case of dispersion relation of the type (3). As a degenerate limiting case of these non-linear solutions, the solitary waves (solitons) appear. For sufficiently high amplitudes such steady-state waves do not exist. This happens when the dispersion, at last, cannot compete with non-linear steepening. The same thing happens to shallow water solitary waves, as is well known. This breaking of non-linear steady state waves means the breakdown of the moments approximation: the plasma motion becomes multistreaming and very likely unstable \([2,4]\). But, as we know, some types of instabilities in non-linear waves can arise even for smaller amplitudes (much below breaking conditions).

Before considering the instability problems, we should discuss briefly the influence of specific collisionless damping mechanism on these steady-state waves. The "resonant" particles (with velocities close to the wave velocity) are contributing to the damping of this kind of wave. They are also responsible for the entropy production mechanism similar to what we discussed in Part II, when considering the non-linear Landau damping. Magnetic field plays a special role in this case \([5]\). Also the thermal motion of the particles...
with velocity much less than "resonant" gives rise to an additional randomization, since magnetic moments of ions in such waves are no longer the adiabatic invariants.

2. The onset of instability and turbulence

However, the most effective mechanisms of "collisionless" dissipation are the various types of plasma instability which lead to the transformation of the energy of regular non-linear motion into the energy of chaotic oscillations (and finally to randomization). We start by discussion of the decay instability (Part I) of the almost periodic wave trains in laminar shock waves. Let us take the wave train propagating exactly across the magnetic field in a low $\beta$ case. The dispersion law $\omega(k)$ for this wave is shown in Fig. 1 (case 5) and belongs to the non-decayable type (see Part I). Therefore this wave can be unstable only in respect of the perturbation which includes some other type of wave.

Analysis of stability shows that the non-linear oscillations with frequency

$$\omega_0 = \mathcal{M} k_0 v_A$$

are unstable with respect to decay to a fast magnetohydrodynamic wave and fast whistler-type oscillation with the following frequencies:

$$\omega_1 = k_1 v_A \left(1 + \frac{k_2^2 v_A^2}{2 \omega_{ni}^2}\right), \quad \omega_2 = \frac{|k_2| k_1 v_A^2}{\omega_{ni}}$$

The growth rate of the instability is of order

$$\gamma \sim \omega_0 \sqrt{\mu - 1} \ll \omega_0$$

As a consequence of decay instability, non-linear regular oscillations are damped much earlier than according to the theory of laminar shocks, since their energy is transferred to the energy of a whole spectrum of noises. The length of damping obtained in such a manner may be
identified with the thickness of the front of a shock wave. From the dimensional point of view we may expect that $\Delta$ must be of an order of several lengths of oscillation (the rough estimate for $\Delta$ was given in reference [1]). But if the amplitude of the shock is large enough a number of faster instabilities come into play and are much more important than those of decay-type.

The most obvious type of instability for non-linear waves in a magnetic field is "beam" instability, when the mean regular velocity of the electrons relative to the ions exceeds the mean thermal velocity $(v_y > \sqrt{m_i/m_e})$. Using some results of soliton theory to find the current velocity, we observe that this condition begins to be fulfilled for waves with a Mach number exceeding

$$M^* \approx 1 + \frac{3}{8} \left( \frac{8\pi n T_e}{\mu^2} \right)^{1/3}$$

Physically, this instability signifies that the electrons moving relative to the ions are losing momentum and energy not only because of ordinary collisions, but also because of some new effective friction of a collective nature - coherent "radiation" of plasma oscillation as a consequence of instability. Therefore, upon fulfillment of the condition of instability $v_y > \sqrt{T_e/m_e}$ an anomalous electric resistance should appear, leading to an anomalous dissipation. We expect, of course, that the main part of the ordered energy of the electrons due to this collective friction between the electrons and ions goes into the disordered energy of the electrons (as in usual joule heating by high density electron current). Therefore the effective "temperature" of the electrons increases in comparison with that of the ions.

$$T_e \gg T_i$$ (8)

Starting from this moment the "beam" instability becomes less important because the ion sound wave instability comes into play. Let us show now that the electron current in the turbulent shock front becomes very low since the anomalous electric resistance in the shock front is very large. In order to describe rigorously the shock front structure and compute this current we must at least be able to solve the equations for
the moments of quasi-linear distribution function $f(x, \lambda, t)$
(taking into account the effective electric resistivity) together with
the Maxwell equations for the electric and magnetic fields.

For the steady-state case these equations in a moments approxima-
tion will have a form

$$\rho \nabla = \text{const}$$

$$J = \rho \nabla^2 + q + \frac{H^2}{8\pi} = \text{const}$$  \(I\)

$$A = \frac{8}{\delta-1} \frac{\rho_0}{\sigma} + \frac{\nabla^2}{2} + \frac{H \cdot H_0}{4\pi \rho_0} = \text{const}$$

$$\nabla H = \frac{c^2}{4\pi \delta} \frac{dH}{dx} + \text{const}$$

In such an approach* the only difference from conventional magnetogas-
dynamics lies in a very complicated form of $\sigma$ which is now some
functional of the distribution function, and so on (See Part III, where
we discussed the electric conductivity in the presence of the ion sound
instability).

The system (I) can be easily reduced to the one equation for $H(x)$

$$\frac{c^2}{4\pi \sigma} \frac{dH}{dx} = \left[ \frac{\chi}{(y+1) \rho_0 u_0} \left( \frac{\nabla^2}{8\pi} \right) + \left\{ \frac{\chi}{(y+1) \rho_0 u_0} \left( \frac{\nabla^2}{8\pi} \right) \right\}^2 - \frac{2(y-1)}{(y+1)} \left( A - \frac{H \cdot H_0}{4\pi \rho_0} \right) \right]^{1/2} H - u_0 H_0$$ \(9\)

But since $\sigma$ is now very complicated, there is no reason to try to
solve Eq. (9) exactly. We will give an estimate of the shock thick-
ness using the effective collision frequency between ions and elec-
trons due to the radiation of ion sound waves by electrons, estimated in Part III:

* It is possible to justify this approach, at least, for low Mach numbers
$\mathcal{M} - 1 < 1$, using weak turbulence theory.
Substituting Eq. (10) into the well-known expression for the thickness of the shock wave in the magnetohydrodynamics of a plasma for a case of a weak wave being propagated across a magnetic field, we obtain the thickness of the turbulent shock front

\[ \Delta \approx \frac{c^2}{4\pi |g_{eff}| v_A (\mu - 1) \omega_{pi}} \quad \gamma \quad \kappa_{eff} = \frac{ne^2}{m_e \gamma_{eff}} \]  

We estimate temperature ratio in this expression assuming that the ion sound wave energy is absorbed by ions. Using the results of Part III, we have

\[ \frac{T_e}{T_i} \approx \frac{\mathcal{U}}{C_S} \]  

Finally, we express the current velocity \( \mathcal{U} \) through the thickness of the shock front using a Maxwell equation for the magnetic field:

\[ \mathcal{U} = \frac{c}{4\pi ne} \cdot \frac{H(\mu - 1)}{\Delta} \]  

From Eqs. (12) - (14), we immediately obtain

\[ \Delta = \frac{c}{\omega_{pe}} \left[ (\mu - 1) \frac{C_S}{c^2} \right]^{4/3} \]  

* It can also be found from Eq. (9).
We assume that the turbulent heating in the shock front is sufficiently
effective that

\[ n \frac{T_e}{T} \sim \frac{H^2}{8\pi} \cdot (\mu - 1)^2 \]

Then from (15) we find the final result,

\[ \Delta \sim \frac{C}{\omega_{pe}^2 \sqrt{\mu - 1}} \left( \frac{C}{V_A} \right)^{1/3} \]  \hspace{1cm} (16)

This estimate is close to the experimental data \[9, 10\] in order of magnitude in a fairly wide range of density.

Since the model used is quite similar to what we have in
conventional gas dynamics, we expect the existence of so-called iso-
magnetic jump when the Mach number becomes greater than critical.
This critical Mach number is quite independent of conductivity. It can
be found from the left-hand side of Eq. (9) if we take into account that
the condition of the singularity is that the expression under the square
root be zero at \( H = H_2 \) (magnetic field behind shock).

\[ \left[ \frac{\chi}{(\gamma + 1)\rho_0 U_o} \left( \gamma - \frac{H^2}{8\pi} \right) \right]^2 - \frac{2(\chi - 1)}{(\gamma + 1)} \left( A - \frac{HH_0}{4\pi \rho_0} \right) = 0 \]  \hspace{1cm} (17)

\[ \frac{\chi}{(\gamma + 1)\rho_0 U_o} \left( \gamma - \frac{H^2}{8\pi} \right) H - U_o H_0 = 0 \]  \hspace{1cm} (18)

For the case of \( \chi = \frac{5}{3} \), this critical Mach number coming from
Eqs. (17) and (18) is close to \( \mu_c \approx 2.8 \cdot \sqrt{8} \).

In our case an isomagnetic jump means the breaking up of the
moments approximation for the ions. It probably leads to the multi-
streaming picture of ion motion and to spreading the effective shock
front.

Let us discuss now the case of the shock wave propagating
not exactly across a magnetic field (dispersion law type 1- (a)).
If the angle $\Theta$ between the plane of the shock front and the magnetic field lines is much larger than

$$\Theta \gg \sqrt{\frac{m_e}{m_i}}$$

the turbulent spectrum can be completely different from the case of the perpendicular propagation. The oscillatory structure of the shock (non-linear wave train) is still unstable, in respect of the decay to two Alfvén waves, with growth rate $i\gamma$

$$\gamma \sim \omega_{Hi} \left[ \frac{(\mu-1)/2}{t_\Theta} \right]^{3/2}, \quad t_\Theta \sim 1 \quad (19)$$

But development of the beam instability can now be impossible if the angle of propagation $\Theta$ is large enough*.

Therefore we may expect that in the initially isothermal plasma the ion sound instability cannot also arise. But in a strong magnetic field even small electron current $u \ll U_{th_e}$ leads to the development of some other instability discovered by O. BUNEMAN. The growing oscillations have the frequency and wavelength in the intervals:

$$\omega_{Hi} \ll \omega \ll \omega_{He}, \quad K \gg \frac{\Gamma_{Hi}}{\Gamma_{He}} \quad \Rightarrow \quad K \gg \frac{\Gamma_{He}}{\Gamma_{He}} \quad (21)$$

(Generally speaking, higher cyclotron harmonics can also exist).

The dispersion relation for this kind of wave was found in Part II, where we discussed the "loss-cone" instability.

$$1 + \frac{\omega_{pe}^2}{\omega_{he}^2} = \frac{\omega_{pi}^2}{K^2} \int \frac{d\nu}{\nu} \frac{\nu_{ei}}{\nu - K \nu + \nu_{ei}} \exp \left[ \frac{\omega_{pe}^2}{K^2} \right] \left[ \frac{d\nu E_{\omega_{pe}}}{\nu - K \nu + \nu_{ei}} \right]_0^{+\infty} \quad (22)$$

* For $\mu \sim 1$ this is the case $\Theta > \sqrt{m_e/m_i}$. 

-155-
We expand the integrals in the case of phase velocities within the intervals:

\[
\omega \gg v_{th}, \quad \omega - k \cdot u \gg v_{th}
\]  \tag{23}

and take a plasma with the particle distribution in a form

\[
f_0 \approx n_0 \left(\frac{m_i}{2\pi k T_i}\right)^{3/2} e^{-\frac{m_i (v_i - u)^2}{2 k T_i}}, \quad u_i = 0, u_e = \pm u. \tag{24}
\]

Then from Eq. (22) we obtain

\[
1 + \frac{\omega_{pe}^2}{\omega_{He}^2} - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2 k_z^2 / k_i^2}{(\omega - k \cdot u)^2 - k_z^2 v_{th}^2} = 0 \tag{25}
\]

Further, for simplicity, we consider the case of a dense plasma:

\[
\omega_{pe}^2 \gg \omega_{He}^2 \tag{26}
\]

In this case the growth rate of the instability is given by the equation

\[
\chi \approx K_i \cdot U \left(\frac{k_z^2}{k_i^2} \frac{m_i}{m_e} - \frac{k_i^2}{\omega_{He}^2} \frac{U_{He}}{U_{Hi}}\right)^{1/2} > K_i v_{th} \tag{27}
\]

\[
k_z / k_i < U / v_{th} < 1 \quad \Rightarrow \quad k_i v_{He} < 1
\]

and has a maximum

\[
\chi_{max} \sim \sqrt{\frac{U_{Hi}}{U_{He}}} \tag{28}
\]

-156-
It is difficult to describe rigorously the non-linear stage of the instability, but we can expect the instability to be saturated when the effect of turbulent motion of the electrons* in the random electric field stabilizes the instability

\[ \mathcal{L} \sum_{j} c^2 \frac{q_j^2 |\psi_j|'^2}{H_z^2} \sim (k_z u_n)^2 \]  

(29)

The turbulent Ohm's law can easily be found from the quasi-linear equation for electron distribution and has the form

\[ \frac{m_e}{e^2} \left( \frac{\partial u_n}{\partial t} + (u_n \cdot \nabla) u_n \right) = -e n \left\{ E_0 + \frac{1}{e} \left[ u_n \times H_z \right] \right\} + \]

\[ + \frac{e^2}{m_e} \sum_{k} k \cdot |\psi_k|^2 \cdot k \cdot \frac{\partial f_{te}}{\partial \mathbf{v}_e} \cdot \frac{\gamma_{k'} \cdot \mathbf{v}_e}{(\omega_k - k \cdot u_n - k^2 e^2 + y_e^2)} \]  

(30)

It follows from Eqs. (29) and (30) that

\[ \gamma_{\text{eff}} = \omega_{he} \frac{u_n}{v_{th e}} \]  

(31)

Using this estimation of the effective frequency of collisions one can find the order of magnitude of the shock thickness in the usual manner.

A completely different class of instabilities is becoming very important in the collisionless shock waves in high $\beta$-plasmas. These are mostly the pitch-angle instabilities. For detailed consideration we will refer to the preprints 13, 14.

* Turbulent motion of ions can be neglected because of

\[ \gamma_{\text{i}}^2 = \sum_{j} \frac{e^2 q_j^2 |\psi_j|'^2}{m_i^2 \omega_i^2} \]

< $\gamma_1^2 / q_1^2$. 

-157-
Since the whole problem of plasma collisionless shock waves is divided into a wide variety of separate pieces, it is useful to represent the entire picture of the different existing limiting cases in the form of a simplified schematic table. (See p. 159)

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<table>
<thead>
<tr>
<th>Case</th>
<th>Linear dispersion curve</th>
<th>Dispersion length</th>
<th>Form of growth</th>
<th>Critical number for breaking</th>
<th>Onset of instability and turbulent shock thickness</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{mc}^2 \ll \frac{H^2}{\Delta n} \ll n_{mc}^2$</td>
<td>$\omega_p \omega$</td>
<td>$\frac{H^2}{\omega_p \Delta n m_{mc}^2}$</td>
<td>$\nabla$</td>
<td>$2$</td>
<td>Beam instability $M &gt; 1 + \left( \frac{\gamma m_{H} T}{H} \right)^{2}$</td>
<td>1958 [17]</td>
</tr>
<tr>
<td>$\frac{H^2}{\Delta n} \ll n_{mc}^2$</td>
<td>$\omega = \frac{c}{\omega_p}$</td>
<td>$\frac{c}{\omega_p}$</td>
<td>$\nabla$</td>
<td>$2$</td>
<td>due to ion sound instability $\Delta \sim \frac{c}{\omega_p} \left( \frac{c}{v_A} \right)^{2}$</td>
<td>1958 [15,16,18]</td>
</tr>
<tr>
<td>Two species of ions $e_1, M_1, n_1$</td>
<td>$\omega = \frac{c}{\omega_p}$</td>
<td>$\frac{c}{\omega_p}$</td>
<td>$\nabla$</td>
<td>$2$</td>
<td>Decay instability, mirror instability</td>
<td>14</td>
</tr>
<tr>
<td>$\beta_i \geq 1$</td>
<td>$\omega = \omega_{hi}$</td>
<td>$\omega_{hi}$</td>
<td>( \sim n_{hi} )</td>
<td>( \sim n_{hi} )</td>
<td>( \beta_i \geq 1 )</td>
<td>( \sim n_{hi} )</td>
</tr>
<tr>
<td>$\frac{U}{H} &gt; \sqrt{\frac{m_i}{m_H}}$</td>
<td>$\omega = \frac{c}{\omega_p}$</td>
<td>$\frac{c}{\omega_p}$</td>
<td>$\nabla$</td>
<td>( \sim n_{hi} )</td>
<td>Decay instability, Buneman instability</td>
<td>19</td>
</tr>
<tr>
<td>$H = 0$</td>
<td>$\omega = \omega_{hi}$</td>
<td>$\omega_{hi}$</td>
<td>$\omega_{hi}$</td>
<td>$\sim \omega_{hi}$</td>
<td>$\theta \sim 1, 6$</td>
<td>( \theta \sim 1, 6 )</td>
</tr>
<tr>
<td>$T_e \gg T_i$</td>
<td>$\omega = \omega_{hi}$</td>
<td>$\omega_{hi}$</td>
<td>$\omega_{hi}$</td>
<td>$\sim n_{hi}$</td>
<td>$\sim n_{hi}$</td>
<td>$\sim n_{hi}$</td>
</tr>
<tr>
<td>$g \gg 1$</td>
<td>$\omega_p \omega$</td>
<td>$\omega_p$</td>
<td>$\nabla$</td>
<td>$2$</td>
<td>Fine-hose instability $\Delta \sim \frac{c}{\omega_p}$</td>
<td>13</td>
</tr>
</tbody>
</table>
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160-


