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These notes were made during the authors' study of the Bethe-Salpeter equation. They may form part of a subsequent publication.

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BOUND STATE EQUATION FOR QUARK-ANTIQUARK SYSTEM

§1 INTRODUCTION

It is now generally accepted that if the spectrum of elementary particles contains traces of higher spin-containing symmetries, such symmetries must have a dynamical origin. A number of quark models, both relativistic as well as non-relativistic, have recently been considered in this connection. In this paper we treat explicitly the relativistic bound state (Bethe-Salpeter) equation for quark-antiquark binding with a specific kernel corresponding to a zero boson-singlet exchange (Fig. 1) with a view to seeing if one can reproduce the meson-multiplet spectrum assumed by the higher symmetry schemes.

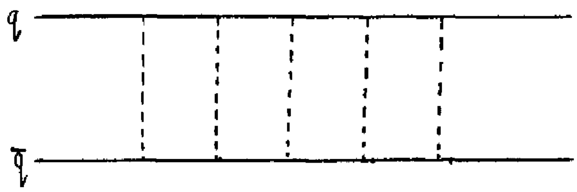

Fig. 1

Fig. 2

Bethe-Salpeter equations have been considered before in this context; in reference 1, for example, a chain diagram of the type shown in Fig. 2 was summed and shown in certain strong-coupling approximations to reproduce the physical multiplet spectrum. A different approximation to the kernel was considered by Bogoliubov [2] and his co-workers; in this approach spin-containing terms were neglected at the outset. We believe the considerations of the present paper are possibly closer to the realistic situation in the following sense: the physically observed bosons are likely to be much lighter than the quarks if they exist, [3] so that it is a reasonable first approximation to consider zero boson mass exchange terms as providing the dominant part of the binding potential.* If one further looks for meson bound states of zero mass (the case we shall explicitly solve) one is considering in effect a reciprocally self-consistent situation - a Bethe-Salpeter bootstrap. The usual objections to retaining only the two-particle Bethe-Salpeter amplitude are, within this context, easy to meet. If mesons are indeed made up from quarks and antiquarks, the meson-pole terms occur equally in the two-particle propagator ($q\bar{q}$ $q\bar{q}$) as in the four-particle ($q\bar{q}$ $q\bar{q}$ $q\bar{q}$ $q\bar{q}$) propagator and so on. To include the four-particle amplitude effects within the two-particle propagator equations, one should include in the Bethe-Salpeter kernel two-particle irreducible iterated terms of the type shown in Fig. 3

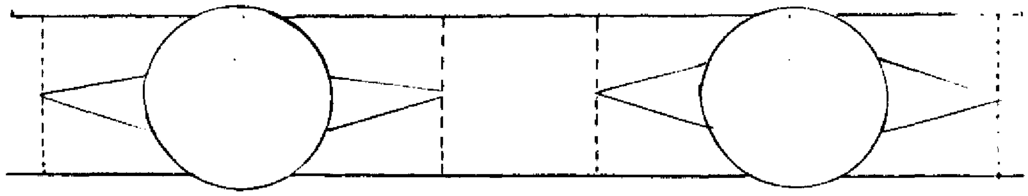

Fig. 3

We believe it is plausible to neglect such terms in comparison with the kernel of Fig. 1 in so far as the "potential" which the inclusion of terms of Fig. 3 will give rise to must inevitably be of a very short range. The plan of these notes is as follows. We first consider in §2 spin-less quarks, interacting through mass-less bosons. A stereographic projection to a 5-dimensional pseudo-sphere, first introduced by CUTKOSKY [4] exhibits the symmetries of the equation fqr the two cases (1) when total energy p_μ of the bound system equals zero (maximal binding) and (2) for the case $p_\mu \neq 0$. The Bethe-Salpeter equation is exactly soluble (§3) in case (1) and reduced to a tractable differential equation [5] in case (2).

* A recent preprint by J. Harte (CERN) contains an analysis along these lines.

In §4 we consider the spinor equation, and its transformation properties, particularly for the case $p = 0$. In §5 an approximate spinor equation exhibiting a 5-dimensional symmetry is solved explicitly. In §6 we make remarks about the problem of defining particle wave functions.

§2 THE LADDER APPROXIMATION FOR SCALAR QUARKS AND PROJECTIVE TRANSFORMATIONS

The ladder approximation for spin-less quarks of masses m and m_2 interacting through the exchange of a spin-less boson of mass μ is

$$[p_1^2 - m_1^2] \; [p_2^2 - m_2^2] \; \phi(p_1, p_2) =$$

$$= \frac{ig^2 m^2}{(2\pi)^4} \int \frac{d^4 p_1'}{(2\pi)^4} \; \frac{\phi(p_1' p_2')}{(p_1 - p_1')^2 - \mu^2 + i\epsilon}$$

where

$$(2\pi)^4 \delta^4(p_1 + p_2 - p)\, \phi(p_1, p_2) = \int d^4x_1\, d^4x_2 \, \langle 0 | T[\phi(x_1)\, \phi(x_2)] | p \rangle \, e^{ip_1x_1} e^{ip_2x_2}$$

Introduce total and relative momenta

$$p_1 = \tfrac{1}{2}p + q \quad , \quad p_2 = \tfrac{1}{2}p - q$$

for the equal mass case [6] in terms of which the equation reads

$$\left[(\tfrac{1}{2}p + q)^2 - m^2\right] \; \left[(\tfrac{1}{2}p - q)^2 - m^2\right] =$$

$$= - \frac{ig^2 m^1}{(2\pi)^4} \int d^4q' \; \frac{\phi(p, q')}{(q - q')^2 - \mu^2 + i\epsilon} \tag{1}$$

As shown by CUTKOSKY and SCHWINGER [7], one can make a stereographic projection, mapping the Lorentz space onto a unit hyperboloid in 5 dimensions. Several cases must be distinguished according as p^2 falls in the intervals $(-\infty, 0)$, (0), $(0, 4m^2)$, $(4m^2, \infty)$. Introduce the unit vector $\vec{\eta} = (\eta_\mu, \eta_5)$ such that

$$\eta^2 + \eta_5^2 = 1 \qquad \text{where } p^2 > 4m^2 \quad \text{— (3 + 2) space}$$
$$\eta^2 - \eta_5^2 = -1 \qquad \text{where } p^2 < 4m^2 \quad \text{— (4 + 1) space.} \tag{2}$$

and connect the η to the relative momentum via the relation

$$q_\mu = q_5 \frac{\eta_\mu}{\eta_5 + 1}$$
$$q_5^2 = \pm\left(\tfrac{1}{4}p^2 - m^2\right), \qquad q_5(\text{real}) > 0. \tag{3}$$

The various kinematical factors in the equation reduce to

$$\left[\left(\tfrac{1}{2}p+q\right)^2 - m^2\right]\left[\left(\tfrac{1}{2}p-q\right)^2 - m^2\right] = \frac{4q_5^2}{(\eta_5+1)^2}\left[q_5^2 - \tfrac{1}{4}(p\cdot\eta)^2\right]$$

$$(q-q')^2 = \frac{q_5^2}{(\eta_5+1)(\eta_5'+1)}\left(\vec{\eta} - \vec{\eta}'\right)^2$$

where $\qquad \vec{\eta}\cdot\vec{\eta}' = \eta_\mu \eta_\mu' \pm \eta_5 \eta_5'$

and the integration over relative momentum may also be referred to the pseudo-sphere

$$\det\left(\frac{\partial q_\mu}{\partial \eta_\nu}\right) = \frac{1}{\eta_5}\left(\frac{q_5}{\eta_5+1}\right)^4$$

whence

$$\int d^4q = \int 2\left(\frac{q_5}{\eta_5+1}\right)^4 \delta\left(\vec{\eta}^2 \mp 1\right) d^5\vec{\eta}$$

Combining these results and defining the new wave function

$$\Phi(p,\vec{\eta}) = \left(\frac{1}{\eta_5+1}\right)^3 \phi(p,q) \tag{4}$$

the B-S equation in the new variables reads

$$\left[q_5^{\,2} - \tfrac{1}{4}(p\cdot\eta)^2\right] \Phi\,(p,\vec{\eta}) =$$

$$= \frac{-im^2\lambda}{8\pi^2} \int \frac{d^5\vec{\eta}'\,\delta\,(\vec{\eta}'^{\,2} \mp 1)\;\Phi\,(p,\vec{\eta}')}{(\pm 1 - \vec{\eta}\cdot\vec{\eta}') - \frac{\mu^2 - i\epsilon}{2q_5^{\,2}}\,(\eta_5+1)(\eta_5'+1)} \tag{5}$$

with $\lambda \equiv g^2/8\pi^2$

This equation possesses the following (little-group) symmetries:*

(A) $\mu = 0$ (mass-less boson exchange); $p_\mu = 0$ (maximal binding). The equation is invariant for O(4,1) rotations of $\eta_0, \eta_1, \eta_2, \eta_3, \eta_5$.

(B) $\mu = 0$, $p_\mu \neq 0$, $\frac{p^2}{4} - m^2 < 0$; the symmetry is O(3,1) corresponding to $\eta_1, \eta_2, \eta_3, \eta_5$ in the rest frame $\underset{\sim}{p} = 0$.

For $\mu \neq 0$, the symmetry is reduced to O(3,1) and O(3) in the two cases (A) and (B).

§3 WICK ROTATION AND SOLUTION OF THE SCALAR PROBLEM

Before attempting to solve the equation as it stands, it is convenient to perform a Wick rotation [8] to a complex relative time variable. In practice this consists of replacing q_0 by $q_4 = i q_0$. The Wick boundary condition which allows for this transformation is a consequence of the stability requirements on the bound system and a postulated behaviour at large momenta of the wave function. In group theory terms the transformation has an important consequence in that one can now use the orthogonal set of harmonic functions appropriate to a Euclidean metric instead of the finite-dimensional (non-unitary) representations of the corresponding Lorentz group. [9] It is however to be emphasized that in our view the use of this Wick ansatz is of no fundamental significance; one could either choose to work with the Euclidean harmonics from the beginning and transform back to the Lorentz metric in the Bethe-Salpeter propagator at the end, or one could equally well set up the entire calculation within the Lorentz frame-

* These higher symmetries emerge providing the $i\epsilon$ term is dropped, which is possible after Wick rotation. See Section 3.

work, recognizing explicitly at each stage of the calculation analyticity properties guaranteed by the stability condition and the appropriate boundedness.[10] We shall in the sequel, however, carry out the Wick rotation whenever feasible and work within a Euclidean framework.

(A) Case $p_\mu = 0$

From the evident O(5) symmetry of this limit

$$\Phi(\vec{\eta}) = \frac{\lambda}{8\pi^2}\int d^4\Omega(\vec{\eta}')\,\frac{\Phi(\vec{\eta}')}{1-\vec{\eta}\cdot\vec{\eta}'} \tag{6}$$

an expansion in 5-dimensional hyperspherical co-ordinates is indicated,

$$\Phi(\vec{\eta}) = \sum_{N=1}^{\infty} \Phi_N\, Y_N(\vec{\eta}) \tag{7}$$

Explicit expressions for the relevant harmonics are given in the Appendix. For each value of N, the equation (6) is satisfied only for a particular eigenvalue λ_N. The order of degeneracy of this solution is the number of independent Y_N which equals $\frac{1}{6} N(N+1)(2N+1)$.
Using the expansion[4] (see eq. (A-7) in the Appendix)

$$\frac{1}{1-\vec{\eta}\cdot\vec{\eta}'} = 8\pi^2 \sum_N \frac{Y_N(\vec{\eta})\, Y_N^*(\vec{\eta}')}{N(N+1)} \tag{8}$$

we deduce that $\qquad \lambda_N = N(N+1).$ (9)

(B) $p_\mu \neq 0$

More generally we have only O(4) symmetry. Let $p^2/4m^2 = \epsilon^2$ and orient p along the 0 axis (rest-frame):

$$\left[1-\epsilon^2(1-\eta_4^2)\right]\Phi(\vec{\eta}) = \frac{\lambda}{8\pi^2}\int d^4\Omega(\vec{\eta}')\,\frac{\Phi(\vec{\eta}')}{(1-\vec{\eta}\cdot\vec{\eta}')} \tag{10}$$

There is a degeneracy with respect to rotations in the 1235 subspace implying that the solution is proportional to a characteristic 4-dimensional spherical harmonic. We exploit the fact that these [4] harmonics are contained in

[5] harmonics Y_N, by the formula

$$Y_{Nn}(\vec{\eta}) = (1-\eta_{\parallel}^2)^{\frac{1}{2}(n-1)} \mathcal{C}_{N-n}^{n+\frac{1}{2}}(\eta_{\parallel}) \, Y_n(\eta_{\perp}) \tag{11}$$

where $(\eta_{\perp} = \eta_1 \, \eta_2 \, \eta_3 \, \eta_5)$, $\eta_{\parallel} = \eta_4$.

Define the amplitude

$$\Phi_{n\ell m}(\eta_{\parallel}) = \int \Phi(\eta) \, Y^{*}_{n\ell m}(\eta_{\perp}) \, d^3\Omega(\eta_{\perp}) \tag{12}$$

Integrating (10) over the 4-dimensional subspace, and using (12), we get an integral equation for this amplitude:

$$\left[1-\epsilon^2(1-\eta_{\parallel}^2)\right] \Phi_n(\eta_{\parallel}) = \sum_N \frac{\lambda}{N(N+1)} (1-\eta_{\parallel}^2)^{\frac{1}{2}(n-1)} C_{N-n}^{n+\frac{1}{2}}(\eta_{\parallel})$$

$$\cdot \int (1-\eta_{\parallel}'^2)^{\frac{1}{2}(n+1)} C_{N-n}^{n+\frac{1}{2}}(\eta_{\parallel}') \, \Phi_n(\eta_{\parallel}') \, d\eta_{\parallel} \tag{13}$$

We have suppressed the ℓ, m dependence of $\Phi_{n\ell m}$ above. Define the function

$$g_n(z) = (1-\epsilon^2+\epsilon^2 z^2) \, \Phi_n(z) \, (1-z^2)^{\frac{1}{2}(n+1)} \tag{14}$$

which satisfies $g_n(1) = g_n(-1) = 0$ for regular solutions. Eq. (13) may be rewritten as

$$g_n(z) = \sum_N \frac{\lambda}{N(N+1)} (1-z^2)^n \, C_{N-n}^{n+\frac{1}{2}}(z) \int dz' \, C_{N-n}^{n+\frac{1}{2}}(z') \, \frac{g_n(z')}{1-\epsilon^2+\epsilon^2 z'^2}$$

This integral equation can be recast into a differential form for g if we notice that

$$f_{Nn}(z) = (1-z^2)^n \, C_{N-n}^{n+\frac{1}{2}}(z)$$

satisfies the differential equation,

$$(1-z^2) \, f''(z) + 2(n-1) \, z \, f'(z) + \left[N(N+1) - n(n-1)\right] f(z) = 0 \, .$$

Thus

$$(1-z^2)\, g_n''(z) + 2(n-1)\, z\, g_n'(z) - n(n-1)\, g_n(z) =$$
$$= -\sum_N \lambda\, (1-z^2)^n\, C_{N-n}^{n+\frac{1}{2}}(z) \int dz'\, C_{N-n}^{n+\frac{1}{2}}(z')\, \frac{g_n(z')}{1-\epsilon^2+\epsilon^2 z'^2}$$

Integrating w.r.t. $C_{N-n}^{n+\frac{1}{2}}$ (2) and using the completeness property of the C functions in the interval (-1,1) we finally obtain the differential equation

$$(1-z^2)\, g_n''(z) + 2(n-1)\, z\, g_n'(z) + \left[\frac{\lambda}{1-\epsilon^2+\epsilon^2 z^2} - n(n-1)\right] g_n(z) = 0 \tag{15}$$

This equation was obtained by CUTKOSKY [4] by a different (and perhaps more cumbersome) method. The solutions are characterized by an additional number N where $N-n \geqslant 0$ denotes the number of zeros of g_n within the interval (-1,1). Since Wick and Cutkosky have solved the equation in various limits no further elaboration is needed. In the rest frame then, apart from a normalization factor,

$$\phi(\epsilon, \vec{\eta}) = \frac{Y_{n\ell m}^{(\eta_1)}}{(\eta_5+1)^3} \quad \frac{g_{Nn}(\eta_{\shortparallel})}{(1-\epsilon^2+\epsilon^2\eta_{\shortparallel}^2)\,(1-\eta_{\shortparallel}^2)^{\frac{1}{2}(n+1)}} \tag{16}$$

Since $g_{Nn} = (1-z^2)^n\, C_{N-n}^{n+\frac{1}{2}}(Z)$ in the limit $\epsilon \to 0$, it is easy to verify that $\phi(0, \vec{\eta})$ equals the 5-dimensional harmonic $Y_{Nn\ell m}(\vec{\eta})$ when λ takes the appropriate value N (N + 1).

The level structure of the system can be described by using representations of non-compact groups.

(A) <u>$p_\mu = 0$ (maximal binding)</u>

Since to each level (N) is associated a degeneracy $\frac{1}{6}$ N(N+1)(2N+1), this corresponds to a component of a fully-symmetric traceless tensor $\Phi_{\mu_1\mu_2\ldots\ldots}$ ($\mu_1, \mu_2, \ldots = 1, \ldots 5$), which provides an elegant alternative description of a general spherical harmonic. When arranged in a tower it is easily seen that such symmetric tensors correspond to a <u>single</u> irreducible representation of the non-compact group O(5,1) with the maximal compact subgroup O(5).

(B) As shown above, each O(5) level is split and labelled by two quantum numbers n and κ ($N = n + \kappa$). It is easy to see that the levels may now be arranged according to the following irreducible representations of O(4,1).

$$\begin{array}{lll} \kappa = 0: & (0,0) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (1,1) \oplus \ldots & \\ \kappa = 1: & (\tfrac{1}{2},\tfrac{1}{2}) \oplus (1,1) \oplus \ldots & \text{etc.} \\ \kappa = 2: & (1,1) \oplus \ldots & \text{etc.} \end{array}$$

i.e., the original $p = 0$, O(5,1) representation splits as $O(5,1) = \Sigma \oplus O(4,1)$. The O(4) levels in the notation above are characterized by quantum numbers of the groups $U(2) \times U(2) \approx O(4)$.

§4 SPINOR QUARKS

The relevant equation for mass-less singlet exchange is

$$\left[\tfrac{1}{2}\not{p} + \not{q} - m\right]^{\gamma}_{\alpha} \phi^{\delta}_{\gamma}(p,q) \left[\tfrac{1}{2}\not{p} - \not{q} + m\right]^{\beta}_{\delta} = - \frac{ig^2}{(2\pi)^4} \int d^4q' \, \frac{\phi^{\beta}_{\alpha}(p,q')}{(q-q')^2} \tag{17}$$

(γ_0 hermitian, $\underline{\gamma}$ anti-hermitian).

Let R represent little group transformations which leave p invariant ($p = Rp$). In general we have

$$\phi(p,Rq) = S(R)\,\left[O(R)\,\phi(p,q)\right]\,S^{-1}(R) \tag{18}$$

where S(R) acts on spin indices and O(R) is an orthogonal transformation appropriate to the representations of the little group. We can always write

$$\phi^{\beta}_{\alpha}(p,q) = \chi^{\beta}_{\alpha}(p,q)\;Y(q) \tag{19}$$

and there is a special class of solutions to the equation where $\chi^{\beta}_{\alpha}(p,q)$ is scalar under the combined spin and co-ordinate transformations $q \to Rq$, $p \to Rp = p$, and $Y(q)$ is a scalar harmonic of the little group of some particular dimensionality. There may be other classes of solutions which we have not investigated.

We exemplify these remarks with reference to the case $p = 0$, when the little group is O(3,1) or, after Wick rotation, O(4). For this case the special class of solutions mentioned above consists of $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$ representations; the functions χ^{α}_{β} are of the form $\not{q}$, $\frac{\partial}{\partial \not{q}}$ etc., and $\phi(0,q)$ can be written as

$$\phi(0,q) = \left[\begin{array}{l} S_n(q^2) + V_n^{(1)}(q^2)\not{q} + V_n^{(2)}(q^2)\not{\partial} \\ + T_n^{(1)}(q^2)\sigma^{\mu\nu} q_\mu \partial_\nu + T_n^{(2)}(q^2)\sigma^{\mu\nu} q_\mu \partial_\nu \gamma_5 \\ + A_n^{(1)}(q^2)\, i\not{q}\gamma_5 + A_n^{(2)}(q^2)\, i\not{\partial}\gamma_5 + P_n(q^2)\gamma_5 \end{array} \right] Y_{n\ell m}(\hat{q}) \qquad (20)$$

At this point it becomes convenient to pass to the γ-matrix basis

$$K_{\alpha}^{\beta} = \tfrac{1}{4}(\gamma_R)_{\alpha}^{\beta}\, K_R$$

which is the procedure followed by MANDELSTAM and KUMMER.[11] The equivalent form of eq. (17) is

$$D_{RS}(0,q)\,\phi_S(0,q) = -\frac{ig^2}{(2\pi)^4}\int d^4q'\,\frac{\phi_R(p,q')}{(q-q')^2} \qquad (21)$$

where

$$D_{RS}(p,q) = \tfrac{1}{4}\,\mathrm{Tr}\left[\gamma_R\left(\tfrac{1}{2}\not{p} + \not{q} - m\right)\gamma_S\left(\tfrac{1}{2}\not{p} - \not{q} + m\right)\right]$$

(The detailed expression for D is given in the Appendix.)

Using the identity: $\qquad \partial_q^2\left[1/(q-q')^2\right] = -4\pi^2\delta^4(q-q')$

we end up with the set of simultaneous differential equations

$$\partial_q^2\left[D_{RS}(q)\,\phi_S(q)\right] = -\lambda\,\phi_R(q) \qquad (22)$$

S and V are coupled together; so are A and T, while P remains disjoint (the GOLDSTEIN[12] particle). We consider the various sectors in turn after passing to Euclidean space:

P - P sector

Since $P(q) = P_n(q^2)\, Y_{n\ell m}(\hat{q})$ we obtain simply

$$\Box_n\left[(m^2+q^2)\; P_n(q^2)\right] = \lambda P_n(q^2) \tag{23}$$

where

$$\Box_n = \frac{d^2}{dq^2} + \frac{3}{q}\,\frac{d}{dq} - \frac{n^2-1}{q^2} \tag{24}$$

S - V sector

Here we must use the solutions

$$\begin{aligned} S(q) &= S_n(q^2)\; Y_{n\ell m}(\hat{q}) \\ V_\mu(q) &= V_n^{(1)}(q^2)\, Y^{(1)}_{\mu,n\ell m}(\hat{q}) + V_n^{(2)}(q^2)\, Y^{(2)}_{\mu,n\ell m}(\hat{q}) \end{aligned} \tag{25}$$

The properties of $Y^{(1)}$, $Y^{(2)}$ and other harmonics are given in the Appendix. The resulting equations are

$$\Box_n\left[(q^2-m^2)\; S_n - 2mq\; V_n^{(1)}\right] = \lambda S_n \tag{26}$$

$$\begin{pmatrix} \left(\Box_n - \frac{3}{q^2}\right) & , & \frac{2\sqrt{n^2-1}}{q^2} \\ \frac{2\sqrt{n^2-1}}{q^2} & , & \left(\Box_n + \frac{1}{q^2}\right) \end{pmatrix} \begin{pmatrix} 2mq\, S_n + (q^2-m^2)\, V_n^{(1)} \\ -(q^2+m^2)\; V_n^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} V_n^{(1)} \\ V_n^{(2)} \end{pmatrix} \tag{27}$$

T - A sector

$$A_\mu(q) = A_n^{(1)}(q^2)\, Y^{(1)}_{\mu, n\ell m}(\hat{q}) + A_n^{(2)}(q^2)\, Y^{(2)}_{\mu, n\ell m}(\hat{q})$$

$$T_{\mu\nu}(q) = T_n^{(1)}(q^2)\, Y^{(1)}_{\mu\nu, n\ell m}(\hat{q}) + T_n^{(2)}(q^2)\, Y^{(2)}_{\mu\nu, n\ell m}(\hat{q}) \tag{28}$$

Introducing this expansion into (21) we arrive at

$$\Box_n \left[(q^2+m^2)\, T_n^{(1)}\right] = -\lambda T_n^{(1)} \tag{29}$$

$$\Box_n \left[(q^2-m^2)\, T_n^{(2)} - 2mq\, A_n^{(2)}\right] = \lambda T_n^{(2)} \tag{30}$$

$$\begin{pmatrix} \Box_n - \frac{3}{q^2}\,, & + \frac{2\sqrt{n^2-1}}{q^2} \\ \frac{2\sqrt{n^2-1}}{q^2}\,, & + \left(\Box_n + \frac{1}{q^2}\right) \end{pmatrix} \begin{pmatrix} (q^2+m^2)\, A_n^{(1)} \\ (q^2-m^2)\, A_n^{(2)} + 2mq\, T_n^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} A_n^{(1)} \\ A_n^{(2)} \end{pmatrix} \tag{31}$$

Notice the close similarity with the S - V sector.

Finally we may remark that in the limit $m \to 0$, only $V_1\ V_2$ and $A_1\ A_2$ remain coupled and these equations reduce to the cases considered by MANDELSTAM.[11] Since the equations of the P sector are identical with those considered by GOLDSTEIN,[12] it would be very surprising if the S V A T solutions also do not possess a continuous λ spectrum.

For the case $p_o \neq 0$, the little group transformations are simply pure rotations and we may only use the following expansion for $\phi(p,q)$

$$\left[S + S'\gamma_0 + V_1 \underline{\gamma}\cdot\underline{q} + V_2 \underline{\gamma}\cdot\underline{\partial} + \dots\right] Y_{jm}(\underline{q})$$

where S, S', V, V' ... are functions of E, q^2 and q_o. In the limit $E \to 0$ (with the corresponding eigenvalue change) the functions should merge into the previous 4-dimensional solutions.

§6 A SPINOR MODEL EXHIBITING O(5) SYMMETRY WITH PSEUDO-SCALAR EXCHANGE

The discussion of §4 was framed entirely in terms of relative momentum variables. However, in the limit $p \to 0$ it was found that the equation almost possessed an O(5) symmetry in the sense that the $S - V$ sector equations closely resembled those in the $A - T$ sector. This quasi-symmetry can be made more evident if we use the η variables of §2, when we have

$$\left[\not{\eta} - (\eta_5 + 1)\right] \Phi(\vec{\eta}) \left[\not{\eta} - (\eta_5 + 1)\right] = \lambda \int d\Omega(\vec{\eta}') \frac{\gamma_5 \Phi(\vec{\eta}') \gamma_5}{1 - \vec{\eta}\cdot\vec{\eta}'} \tag{32}$$

for a model where we consider pseudo-scalar meson exchange.
Write

$$i\left[\not{\eta} + \eta_5 + 1\right]\gamma_5 = \left[\eta_K \Gamma_K + i\gamma_5\right] \tag{33}$$

where $\eta_K = (\eta_4, \underline{\eta}, \eta_5)$

and $\Gamma_K = (-\gamma_0\gamma_5, i\underline{\gamma}\gamma_5, i\gamma_5)$ are hermitian

Thus $\left\{\Gamma_K, \Gamma_L\right\} = 2\delta_{KL}$.

As we have seen, the cases $p = 0$ are reproduced from the solutions of the case $p \neq 0$. We expect therefore that the solutions of (32) are related to the solutions of the model equation

$$\left(\eta_K \Gamma_K\right) \hat{\Phi} \left(\eta_K \Gamma_K\right) = -\frac{\lambda}{8\pi^2} \int \frac{d\Omega(\vec{\eta}')\, \hat{\Phi}(\eta')}{1 - \eta\cdot\eta'} \tag{34}$$

where we drop the $i\gamma_5$ term, and $\hat{\Phi} \equiv i\gamma_5 \Phi$

Change to the basis

$$\Gamma_R = 1 \quad , \quad \Gamma_K \quad , \quad \Sigma_{KL} = \frac{i}{2}\left[\Gamma_K , \Gamma_L\right] .$$
$$\hat{\Phi}_R = S \quad , \quad V_K \quad , \quad T_{KL} .$$

In these terms,

$$S(\vec{\eta}) = -\frac{\lambda}{8\pi^2}\int \frac{S(\vec{\eta}')}{1-\vec{\eta}\cdot\vec{\eta}'}\, d\Omega(\vec{\eta}')$$

$$-2\eta_L\eta_K V_L(\vec{\eta}) + V_K(\vec{\eta}) = \frac{\lambda}{8\pi^2}\int \frac{V_K(\vec{\eta}')}{1-\vec{\eta}\cdot\vec{\eta}'}\, d\Omega(\vec{\eta}') \tag{35}$$

$$T_{KL}(\eta) + 2\left(\eta_K\eta_M T_{LM}(\eta) - \eta_L\eta_M T_{KM}(\vec{\eta})\right) = -\frac{\lambda}{8\pi^2}\int \frac{T_{KL}(\vec{\eta}')}{1-\vec{\eta}\cdot\vec{\eta}'}\, d\Omega(\vec{\eta}')$$

We shall consider the solutions which transform as representations of the pure rotation group and therefore use the harmonics below (see the Appendix for definition and properties)

$$S(\vec{\eta}) = S_N Y_N(\vec{\eta})$$
$$V_K(\vec{\eta}) = V_{1N} Y^{(1)}_{K,N}(\vec{\eta}) + V_{2N} Y^{(2)}_{K,N}(\vec{\eta}) \tag{36}$$
$$T_{KL}(\vec{\eta}) = T_N Y_{KL,N}(\vec{\eta})$$

Algebraic relations among the coefficients which give the eigenvalues λ then follow easily:

$$S_N\left(1 + \frac{\lambda_N}{N(N+1)}\right) = T_N\left(1 - \frac{\lambda_N}{N(n+1)}\right) = 0 \tag{37}$$

$$V_{11}\left(1+\frac{\lambda}{2}\right)=0 \qquad , \qquad V_{21} = 0$$

$$\begin{pmatrix} N(N+1)+\lambda & \frac{+2\lambda}{\sqrt{(N-1)\ (N+2)}} \\ \frac{+2\lambda}{\sqrt{(N-1)\ (N+2)}} & -N(N+1)+4\lambda \end{pmatrix} \begin{pmatrix} V_{1N} \\ V_{2N} \end{pmatrix} = 0, \qquad N > 0 \ . \tag{38}$$

The equation

$$4\lambda^2\left[1-\frac{1}{(N-1)\ (N+2)}\right] + 3\lambda N(N+1) - \left[N\ (N+1)\right]^2 = 0$$

has always two real positive roots so there are always two vector states.

As $\quad N \to \infty \qquad \lambda_1 \longrightarrow -N^2$

$$\lambda_2 \longrightarrow +\tfrac{1}{4} N^2$$

for the corresponding eigenvalues.

When the symmetry breaking is switched on (for example $p \neq 0$) these algebraic equations become coupled differential equations that must be solved with appropriate boundary conditions.

§6 PARTICLE ASPECTS

We have so far concentrated on solving the bound state equation; the amplitude $\phi(p, q)$ or $\phi(p, \eta)$ is a function of external as well as internal variables. The next problem is to decide how to define particle wave functions, and for the spinor case to find criteria for deciding how many physical particles the amplitude $\phi_\alpha^\beta(p, q)$ represents.

For the scalar case, the extraction of a particle wave-function from $\phi(p, \eta)$ seems reasonably unique; define the wave-function

$$\phi_{\kappa n \ell m}(p) = \int \phi_\kappa(p, \eta)\ Y^*_{n\ell m}(\eta_\perp)\ d\Omega(\eta) \tag{39}$$

The indices $\kappa n \ell m$ give the internal symmetry characteristics. For the spinor case one may likewise define the particle wave-function

$$\phi_J(p) = \int \left(\overline{\chi Y}\right)^{\beta}_{J\alpha} \; \phi^{\alpha}_{\beta}(p,q) \; d^4q \tag{40}$$

There is however in this case the extra complication arising from redundant components in the spinors. To see this, note that the structure of the B-S equation resembles the BARGMANN-WIGNER equations.[13] Writing the equation in the form

$$\left(\tfrac{1}{2}\not{p} - m\right) \; \Phi(p,q) \; \left(\tfrac{1}{2}\not{p} + m\right) = \lambda \int \frac{\Phi \, dq'}{(q-q')^2} + \left[\not{q} \, \Phi \left(\frac{\not{p}}{2} + m\right) + \left(\frac{\not{p}}{2} - m\right) \Phi \, \not{q} \right], \tag{41}$$

it is clear that a quantity like $\Phi^{\alpha}_{\ell m \beta} = \int \Phi^{\alpha}_{\beta}(p, q) \, Y_{\ell m}(\hat{q}) d^4q$ (which one may reasonably call the orbital projection of the B-S amplitude) satisfies subsidiary equations closely resembling (but more complicated than) those arising from Bargmann-Wigner formalism. The extra complications come from the terms on the right-hand side of (41). The first term $\left(\lambda \int \frac{\Phi \, dq'}{(q-q')^2}\right)$ on the right contributes a mass-correction; the other relate S, T, P, A, V projections among each other. The situation is analogous to the one encountered in field theory where, for example, for a theory with non-conserved currents, $\frac{\partial A_\mu}{\partial x_\mu} \neq 0$ and one has the problem of deciding whether the scalar components of A_μ (with their indefinite metric characteristics) make their appearance as physical particles. Our feeling is that such components in both the field theory case as well as the B-S equation do not make their appearance; we are however, unable to formulate our arguments in a convincing fashion yet.

APPENDIX

The propagator matrix

The spinor product $D^{\delta\alpha}_{\gamma\beta}$ appearing on the l. h. s. of eq. (21) can be transformed into the γ-matrix basis when it reads

$$
\begin{array}{cc}
 & \begin{array}{ccccc} S & V & T & A & P \end{array} \\
\begin{array}{c} S \\ V \\ T \\ A \\ P \end{array} &
\left(\begin{array}{c:c:c:c:c}
\frac{1}{4}p^2-q^2-m^2 & 2mq_\mu & i(q_\mu p_\nu - q_\nu p_\mu) & 0 & \\ \hdashline
2mq_\kappa & \begin{array}{l} \frac{1}{2}p_\kappa p_\mu - 2q_\kappa q_\mu \\ -(\frac{1}{4}p^2-q^2+m^2)g_{\kappa\mu} \end{array} & im(p_\mu g_{\kappa\nu} - p_\nu g_{\kappa\mu}) & i\epsilon_{\kappa\mu\rho\sigma} p_\rho q_\sigma & \\ \hdashline
-i(q_\kappa p_\lambda - q_\lambda p_\kappa) & -im(p_\kappa g_{\mu\lambda} - p_\lambda g_{\mu\kappa}) & \begin{array}{l} (\frac{1}{4}p^2-q^2-m^2)(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}) \\ -\left\{\begin{array}{l} (\frac{1}{2}p_\lambda p_\nu + p_\lambda q_\nu - 2q_\lambda q_\nu)g_{\kappa\mu} \\ +(\frac{1}{2}p_\kappa p_\mu + p_\kappa q_\mu - 2q_\kappa q_\mu)g_{\lambda\nu} \\ -(\frac{1}{2}p_\kappa p_\nu + p_\kappa q_\nu - 2q_\kappa q_\nu)g_{\lambda\mu} \\ -(\frac{1}{2}p_\lambda p_\mu + p_\lambda q_\mu - 2q_\lambda q_\mu)g_{\kappa\nu} \end{array}\right\} \end{array} & -2m\epsilon_{\kappa\lambda\mu\rho} & \begin{array}{l} i\epsilon_{\kappa\lambda\rho\sigma} \\ p_\rho q_\sigma \end{array} \\ \hdashline
0 & -i\epsilon_{\kappa\mu\rho\sigma} p_\rho q_\sigma & -2m\epsilon_{\kappa\mu\nu\rho} q_\rho & \begin{array}{l} -(\frac{1}{4}p^2-q^2-m^2)g_{\kappa\mu} \\ +\frac{1}{2}p_\kappa p_\mu - 2q_\kappa q_\mu \end{array} & imp_\kappa \\ \hdashline
0 & 0 & -i\epsilon_{\mu\nu\rho\sigma} p_\rho q_\sigma & -imp_\mu & \frac{1}{4}p^2-q^2+m^2
\end{array}\right)
\end{array}
$$

Notice that when either $p = 0$ or $q = 0$ the equation takes on a block diagonal form.

Five-dimensional tensor harmonics

We introduce four angles in the 5-dimensional Euclidean space

say
$$
\begin{aligned}
\hat{x}_4 &= \cos\chi \\
\hat{x}_5 &= \sin\chi \cos\psi \\
\hat{x}_3 &= \sin\chi \sin\psi \cos\theta \\
\hat{x}_2 &= \sin\chi \sin\psi \sin\theta \sin\varphi \\
\hat{x}_1 &= \sin\chi \sin\psi \sin\theta \cos\varphi
\end{aligned}
$$

The tensor harmonics of the unit vector $\hat{x}$ are then defined by the formula

$$Y_{Nn\ell m}(\chi,\psi,\theta,\varphi) = \frac{1}{\sqrt{2\pi}} (\sin\chi)^{(n-1)} \mathcal{C}^{n+1}_{N-n}(\cos\chi)(\sin\psi)^{\ell} \mathcal{C}^{\ell+1}_{n-\ell-1}(\cos\psi)$$
$$(\sin\theta)^{m} \mathcal{C}^{m+\frac{1}{2}}_{\ell-m}(\cos\theta)\, e^{im\varphi} , \tag{A-2}$$

with $N - 1 \geqslant n - 1 \geqslant \ell \geqslant |m|$.

The $\mathcal{C}^{\mu+\nu}_{\lambda-\mu}(z)$ are normalized Gegenbauer polynomials:

$$\mathcal{C}^{\mu+\nu}_{\lambda-\mu}(z) = N(\lambda,\mu,\nu)\; C^{\mu+\nu}_{\lambda-\mu}(z)$$

$$(1-2zy+y^2)^{-\alpha} = \sum_{r=0}^{\infty} C^{\alpha}_{r}(z)\, y^{r} \tag{A-3}$$

$$\left[N(\lambda,\mu,\nu)\right]^{-2} = \frac{\pi\, 2^{-2\mu-2\nu}\,\Gamma(\lambda+\mu+2\nu+1)}{(\lambda+\nu+\frac{1}{2})\;(\lambda-\mu)!\left[\Gamma(\mu+\nu+\frac{1}{2})\right]^{2}}$$

The so-defined harmonics are orthogonal over the 5-dimensional unit sphere

$$\int Y_{Nn\ell m}(\hat{x})\, Y^{*}_{N'n'\ell'm'}(\hat{x})\, d^4\Omega(\hat{x}) = \delta_{NN'}\,\delta_{nn'}\,\delta_{\ell\ell'}\,\delta_{mm'}$$
$$d^4\Omega(\vec{\eta}) = \sin^3\chi\, d\chi\, \sin^2\psi\, d\psi\, \sin\theta\, d\theta\, d\varphi . \tag{A-4}$$

Finally we note the differential properties

$$\partial_K \partial_K Y_{Nn\ell m}(\hat{x}) = -\frac{(N-1)(N+2)}{x^2}\, Y_{Nn\ell m}(\hat{x})$$
$$x_K \partial_K Y_{Nn\ell m}(\hat{x}) = 0 \tag{A-5}$$
$$x_K = |x|\,\hat{x}_K , \qquad \partial_K = \frac{\partial}{\partial x_K}$$

$$\left[(1-z^2)\frac{d^2}{dz^2} - (2\alpha+1)\, z\frac{d}{dz} + r\,(r+2\alpha)\right] C^{\alpha}_{r}(z) = 0 \tag{A-6}$$

and the general theorem[14]

$$\int d^{p+1}\Omega(\vec{\eta}')\, Y_N(\vec{\eta}')\, F(\vec{\eta}\cdot\vec{\eta}') =$$
$$= \frac{(4\pi)^{\frac{1}{2}p}\, N!\,\Gamma(\frac{1}{2}p)\, Y_N(\vec{\eta})}{(N+p-1)!} \int_{-1}^{1} (1-z^2)^{\frac{1}{2}(p-1)}\, C^{\frac{1}{2}p}_{N}(z)\, F(z)\, dz \tag{A-7}$$

Tensor harmonics are derivable from the above by differentiation or multiplication with the unit vector $\hat{X}$. We shall only require the simplest ones:

$$\left.\begin{aligned} Y^{(1)}_{K,Nn\ell m}(\hat{X}) &= \hat{X}_K\, Y_{Nn\ell m}(\hat{X}) \\ Y^{(2)}_{K,Nn\ell m}(\hat{X}) &= \frac{|x|\,\partial_K}{\sqrt{(N-1)\,(N+2)}}\, Y_{Nn\ell m}(\hat{X}) \end{aligned}\right\} \text{vector}$$

$$Y_{KL;Nn\ell m}(\hat{X}) = \frac{(x_K\partial_L - x_L\partial_K)}{\sqrt{(N-1)(N+2)}}\, Y_{Nn\ell m} \qquad \text{tensor.} \tag{A-8}$$

These are also orthonormalized in the sense that

$$\begin{aligned} \int Y^{(\tau)}_{K,N}(\hat{X})\, Y^{(\tau')}_{K,N}(\hat{X})\, d\Omega(\hat{X}) &= \delta_{\tau\tau'}\,\delta_{NN'} \\ \tfrac{1}{2}\int Y_{KL,N}(\hat{X})\, Y_{KL,N}(\hat{X})\, d\Omega(\hat{X}) &= \delta_{NN'} \end{aligned} \tag{A-9}$$

In the course of calculation given in §5, the following formulae were used:

$$\begin{aligned} \frac{1}{8\pi^2}\int \frac{Y^{(1)}_{K,N}(\hat{X}')\, d\Omega(\hat{X}')}{1-\hat{X}\cdot\hat{X}'} &= \left[-\delta_{N1} + \frac{1}{N(N+1)}\right] Y^{(1)}_{K,N}(\hat{X}) \\ &+ \frac{1}{\sqrt{(N-1)(N+2)}}\left[-3\delta_{N1} + \frac{2}{N(N+1)}\right] Y^{(2)}_{K,N}(\hat{X}) \end{aligned} \tag{A-10}$$

$$\begin{aligned} \frac{1}{8\pi^2}\int \frac{Y^{(2)}_{K,N}(\hat{X}')\, d\Omega(\hat{X}')}{1-\hat{X}\cdot\hat{X}'} &= \frac{1}{\sqrt{(N-1)\,(N+2)}}\left[-3\delta_{N1} + \frac{2}{N(N+1)}\right] Y^{(1)}_{K,N}(\hat{X}) \\ &+ \left[-9\delta_{N1} + \frac{4}{N(N+1)}\right] Y^{(2)}_{K,N}(\hat{X}) \end{aligned} \tag{A-11}$$

These may be verified by operating on each side of the relations with X_K and ∂_K. These formulae do not appear to be in standard references.[14]

Four-dimensional tensor harmonics

The scalar harmonics are contained in the $Y_{Nn\ell m}$ and need be discussed no further except to note the properties

$$\partial_\mu \partial_\mu Y_{n\ell m}(\hat{x}) = -\frac{n^2-1}{x^2} Y_{n\ell m}(\hat{x})$$

$$\int Y_{n\ell m}(\hat{x}) Y_{n'\ell' m'}(\hat{x}) \, d\Omega(\hat{x}) = \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'} \quad . \qquad \text{(A-12)}$$

Thence one derives generalized harmonics

$$\left.\begin{aligned} Y^{(1)}_{\mu, n\ell m}(\hat{x}) &= \hat{x}_\mu Y_{n\ell m}(\hat{x}) \\ Y^{(2)}_{\mu, n\ell m}(\hat{x}) &= \frac{|x| \partial_\mu}{\sqrt{n^2-1}} Y_{n\ell m}(\hat{x}) \end{aligned}\right\} \text{vector}$$

$$\text{(A-13)}$$

$$\left.\begin{aligned} Y^{(1)}_{\mu\nu, n\ell m}(\hat{x}) &= \frac{(x_\mu \partial_\nu - x_\nu \partial_\mu)}{\sqrt{n^2-1}} Y_{n\ell m}(\hat{x}) \\ Y^{(2)}_{\mu\nu, n\ell m}(\hat{x}) &= \epsilon_{\mu\nu\kappa\lambda} Y^{(1)}_{\kappa\lambda, n\ell m}(\hat{x}) \end{aligned}\right\} \text{tensor}$$

which are orthonormal over the unit sphere

$$\int Y^{(r)}_{\mu, n}(\hat{x}) \, Y^{(r')}_{\mu, n'}(\hat{x}) \, d\Omega(\hat{x}) = \delta_{rs} \delta_{nn'}$$

$$\text{(A-14)}$$

$$\frac{1}{2}\int Y^{(r)}_{\mu\nu, r}(\hat{x}) \, Y^{(r')}_{\mu\nu, n'}(\hat{x}) \, d\Omega(\hat{x}) = \delta_{rs} \delta_{nn'}$$

The computation given in §4 of the text has required the following differential properties:

$$\partial^2 \left[F(x^2) Y^{(1)}_\mu(\hat{x})\right] = Y^{(1)}_\mu(\hat{x}) \left[\Box_n - \frac{3}{x^2}\right] F(x^2) + \frac{2\sqrt{n^2-1}}{x^2} F(x^2) \, Y^{(2)}_\mu(\hat{x})$$

$$\text{(A-15)}$$

$$\partial^2 \left[F(x^2) Y^{(2)}_\mu(x^2)\right] = \frac{2\sqrt{n^2-1}}{x^2} F(x^2) Y^{(1)}_\mu(x) + \left[\Box_n + \frac{1}{x^2}\right] Y^{(2)}_\mu(\hat{x})$$

$$\partial^2 \left[F(x^2) Y^{(r)}_{\mu\nu}(\hat{x})\right] = Y^{(r)}_{\mu\nu}(\hat{x}) \; \Box_n F(x^2)$$

$$\Box_n \equiv \frac{d^2}{dx^2} + \frac{3}{x}\frac{d}{dx} - \frac{n^2-1}{x^2}$$

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5 This particular differential equation was first derived in reference 4. Our method is different and in further sections is generalized to the spinor case.

6 If one were dealing with different masses for the quarks the appropriate definitions would be

$$m_1 = m(1+\Delta) \quad , \qquad m_2 = (1+\Delta)$$
$$p_1 = \tfrac{1}{2}(1+\Delta)\,p + q' \quad , \qquad p^2 = \tfrac{1}{2}(1-\Delta)\,p - q \; .$$

The final equation would then assume the same form as given in the text except for the reinterpretations

$$q_5^2 \rightarrow \pm\left(\tfrac{1}{4}p^2 - m^2\right)\,\left(1-\Delta^2\right)$$

$$\lambda \rightarrow g^2\left(1-\Delta^2\right)/8\pi^2$$

$$p\cdot\eta \rightarrow \vec{P}\cdot\vec{\eta} \qquad \text{with} \qquad \vec{P} = \left[\left(1-\Delta^2\right)p_\mu ,\, 2q_5\,\Delta\right]$$

for $\mu = 0$, the little group, consisting of those η rotations which leave $\vec{P}\cdot\vec{\eta}$ invariant, is still O(4).

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