WKB APPROXIMATION IN THE COMPLEX PLANE
AND APPLICATIONS TO PLASMA THEORY

A. SKORUPSKI

1966
PIAZZA OBERDAN
TRIESTE
INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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A. Skorupski*

TRIESTE
May 1966

* On leave of absence from Institute for Nuclear Research, Warsaw, Poland
ABSTRACT

The following problems concerned with the WKB technique applied to the equation $d^2\psi/dz^2 + q^2(z)\psi = 0$ in the complex plane are considered: the errors and the applicability conditions along any line $L$, the properties of the Stokes and the anti-Stokes lines, and the connection formulae for zeros and simple poles of $q^2(z)$. The results obtained generalize those of N. FRÖMAN and P.O. FRÖMAN [1]. Two typical applications are considered:

(i) Ionospheric propagation of the EM waves through a region with a resonance and a cut-off. An expression for the transmission coefficient is derived and it is shown to be exact for a simple model discussed by BUDDEN [3].

(ii) Evaluation of the eigen-frequencies for the "quasi-modes" introduced in plasma stability theory [4]. For any number of zeros and poles of $q^2(z)$ in the complex plane the "quantization condition" specifying the eigen-frequencies is derived and reduced to a form convenient for approximations. It is shown that the location of poles of $q^2(z)$ defines a sector in the complex plane where the eigen-solution $\psi(z)$ is not localized.
1. INTRODUCTION

In the analysis of the collisional instabilities and the electromagnetic wave propagation in plasmas we often deal with a differential equation of the form

$$\frac{d^2 \gamma}{dz^2} + q^2(z) \gamma = 0.$$  \hfill (1.1)

This is analogous to the one-dimensional time-independent Schrödinger equation. However, unlike quantum mechanics, the function $q^2(z)$ is as a rule complex for real values of $z$. Consequently, to solve eq. (1.1) approximately by the WKB technique it is often convenient to go to the complex plane. One can find there so-called Stokes and anti-Stokes lines along which the WKB approximation takes a simple form; only for $q^2(z)$—real on the real axis do these lines coincide with the real axis.

In Section 2 we present an elementary derivation of the WKB solutions in the complex plane, analogous to that given in [1] for the real axis. We also discuss the influence of zeros and singularities of $q^2(z)$ on the so-called $\mu$-integral (cf. (2.8)), which plays an essential role in the theory. More advanced treatment is given in Section 3 where the errors of the WKB solutions are estimated. These results are then used in Section 5 to derive some useful connection formulae in the complex plane (5.9), (5.10). They connect approximately the solution $\gamma'(z)$ on adjacent Stokes and anti-Stokes lines crossing at a point $\zeta_0$. The latter can in
particular be any zero or a simple pole of $q^2(z)$ in which case the formulae are shown to have intrinsically a one-directional nature. If $\zeta_0$ is a zero of $q^2(z)$ these formulae can be used together with the known connection formulae between anti-Stokes lines emerging from $\zeta_0$; that gives a connection between any pair of a Stokes and an anti-Stokes line crossing at $\zeta_0$. In the case of a simple zero we can obtain in this way a natural generalization (cf. (5.16), (5.17)) of the known WKB connection formulae for the real axis.

As all the connection formulae in the complex plane refer to Stokes and anti-Stokes lines, some knowledge about these lines is necessary in applications. In this connection we discuss in Section 4 the properties of the Stokes and anti-Stokes lines in the vicinity of zeros and poles of $q^2(z)$ and at large distances from the origin. In these regions an approximate analysis can simply be performed. Away from these regions nothing interesting can happen.

The results of Sections 2 - 5 generalize those obtained by N. FROMAN and P.O. FROMAN [1]. We keep where possible the same notation as used in [1] and we follow the general approach of this reference.

In Section 6 we consider two typical applications:

(i) Evaluation of the transmission and the reflection coefficient for EM-waves propagating through the part of the ionosphere where one gets one resonance ($q^2 = \infty$) and one cut-off ($q^2 = 0$). For the simplest model (solved exactly by BUDDEN in [3]) the WKB result for the transmission coefficient turns out exact.

(ii) Evolution of the eigen-frequencies for the "quasi-modes" introduced in the theory of the collisional instabilities in plasmas [4]. For a fairly general form of $q^2(z)$ (cf. (6.9)) containing cases occurring in plasma stability, a generalized "quantization condition" (6.18) is derived and reduced to a form convenient for approximation. It is shown that the location of poles of $q^2(z)$ defines a sector in the complex plane in which the eigen-solution $\psi(z)$ is not localized.
2. ELEMENTARY DERIVATION OF THE WKB SOLUTIONS IN THE COMPLEX PLANE

The "traditional" derivation of the WKB solutions (cf. [1] p. 10), based on the asymptotic expansion, can be simply generalized to the complex plane. It has, however, the same disadvantage as in the case of the real axis, i.e., it does not give the error estimates. Consequently a more advanced approach is necessary to explain such important questions as a one-directional nature of the WKB approximation or the proper conditions for its applicability. That will be given in the next section.

In the following discussion we shall always assume that \( q^2(z) \) is a single-valued and analytic function which can have isolated singularities. It is convenient to introduce two other functions of \( z \):

\[
q(z) = \sqrt{q^2(z)}, \quad q^{-\nu}(z) = 1/\sqrt{q^2(z)}.
\]

We would like both of them to be single-valued and analytic, which can be achieved by introducing cuts emerging from zeros and singularities of \( q^2(z) \) and extending to infinity. We assume now that \( q^2(z) \) contains a small parameter \( \lambda \)

\[
q^2(z) = \left[ \frac{Q(z)}{\lambda} \right]^2
\]

and look for solutions of (1.1) in the form

\[
\psi(z) = \exp \int_{z_0}^{z} \left[ \lambda^{-1} q^{-\nu}(\zeta) + q^+(\zeta) + \lambda q^{-}(\zeta) + \cdots \right] d\zeta,
\]

where \( z_0 \) denotes any point in the complex plane; the path of integration \((z_0, z)\) is arbitrary but it cannot cross any cut. Inserting (2.3) into (1.1) and equating to zero coefficients of \( \lambda^{-2}, \lambda^{-4}, \ldots \) we get

\[
y_{-1} = \pm i Q, \quad y_{\nu} = \frac{d \ln Q^{-\nu}}{dz}, \quad y_{-\nu} = \pm i \frac{Q^{-\nu}}{2} \frac{d^2 Q^{-\nu}}{dz^2}, \ldots
\]
Dropping higher order terms we obtain from (2.3) and (2.4) (with a convenient normalization)

\[ \psi(z) = (Q/\lambda)^{-1/2} e^{i \int_{z_0}^{z} \frac{Q}{\lambda} d\zeta} e^{i \frac{1}{2} \int_{z_1}^{z} (Q/\lambda)^{-1/2} \frac{d^2}{d\zeta^2} (Q/\lambda)^{-1/2} d\zeta} \]

(2.5)

where \( z_1 \) denotes an arbitrary point on the path \((z_0, z)\). It is seen that \( Q \) appears in this formula everywhere as \( Q/\lambda \) so that we can go back to the original notation and replace \( Q/\lambda \) by \( q \).

Let us define

\[ \omega(z_0, z) = \int_{z_0}^{z} q(\zeta) d\zeta \]

(2.6)

\[ \epsilon(z) = q^{-1/2} \frac{d^2 q^{-1/2}}{dz^2} = \frac{1}{4q} \left[ 5 \left( \frac{d^2 q}{dz^2} \right)^2 - 4 \frac{d^2}{dz^2} q \frac{d^2}{dz^2} \right] \]

(2.7)

\[ \mu = \int_{z_1}^{z} | \epsilon(\zeta) | d\zeta \]

(2.8)

where the integration path \((z_1, z)\) of the last integral should be part of the path \((z_0, z)\).

If \( \left| \int_{z_1}^{z} \epsilon(\zeta) d\zeta \right| \ll 1 \) for each point \( z_2 \in (z_1, z) \)

(which is equivalent to \( \mu \ll 1 \)) the last factor in (2.5) can be replaced by unity along the path \((z_1, z)\) and we end up with the following approximate solutions of eq. (1.1) along that path:

\[ \{ f_1(z), f_2(z) \} = q^{-1/2}(z) e^{i \omega(z_0, z)} \]

(2.9)

They are usually referred to as the WKB solutions and can be simply checked to be linearly independent. Thus for the general solution \( \psi(z) \) along the path \((z_1, z)\) we get approximately

\[ \psi(z) = a_1 f_1(z) + a_2 f_2(z) \]

(2.10)

where \( a_{1,2} \) are arbitrary constants.
From the above derivation it seems that an inequality

\[
\mu \ll 1
\]  
\[(2.11)\]

is a sufficient and necessary condition for the validity of the approximate solution (2.10) along the path \((z_1, z)\). This, however, is in general not true; it is related to the fact that the series in (2.3) may not be convergent but only asymptotic. In any case, as will be shown in the next section, the \(\mu\)-integral (2.8) plays an essential role in the theory and (2.11) is always a necessary condition for the validity of the approximate solution (2.10). It is important to know the behaviour of the \(\mu\)-integral when the path \((z_1, z)\) gets close to zeros or singularities of the function \(q_2(z)\). That can be discussed simply under the assumption that in some vicinity of a zero, or pole \(\zeta_0\), \(q_2(z)\) is well approximated by the first significant term of the Taylor or Laurent expansion about \(\zeta_0\), i.e.,

\[
q_2(z) = C(z - \zeta_0)^n
\]  
\[(2.14)\]

where

\[
\begin{align*}
C &= |C| e^{i\alpha} \quad \text{a complex constant (not zero),} \\
n &= \text{an integer.}
\end{align*}
\]

Using (2.7) we get for such a case

\[
\xi(z) = \frac{n(n+4)}{4C^{\frac{1}{2}}} \left(z - \zeta_0\right)^{\frac{n+4}{2}}
\]  
\[(2.15)\]

Now we can simply calculate the \(\mu\)-integral for a small semi-circle \(L_0\) with centre at \(\zeta_0\) and radius \(R_0\):
Thus with $R_0 \to 0$ we get

\[
\mu_0 = \frac{\pi R_0 | \mathcal{E}(L_0)|}{46 |C|^2 R^{(n+2)/2}}. \tag{2.16}
\]

for $n \leq -3 \quad \mu_0 \to 0,$

for $n = -2 \quad \mu_0 = \frac{\pi}{(4 |C|^2)} = \text{const},$

for $n = +1, 2, 3, \ldots \quad \mu_0 \to \infty.$

It follows that if the path $(z_1, z)$ passes close to higher order poles ($n \leq -2$) it does not result in an appreciable increase of the $\mu$-integral. This integral does increase, however, if the path $(z_1, z)$ gets close to a first order pole or a zero of any order. Usually it is rather difficult in applications to calculate exactly the $\mu$-integral. However, we can get simply a reasonable estimate of this integral, which can then be used to verify condition (2.11). To do that we should divide the path $(z_1, z)$ into parts along which we can approximate $q^2(z)$ by (2.14). We can then estimate the $\mu$-integral along each piece by the product of some average (or maximal) value of $|\mathcal{E}(z)|$ (2.15) and the length of the corresponding piece. Let us notice that the lower limit of the $\omega$-integral in (2.9) can be chosen to coincide with a first-order pole, or any zero of $q^2(z)$ (for higher order poles the $\omega$-integral diverges at $\gamma_0$). With such a choice, however, the WKB approximation can work only at some distance from $z_0$, as the $\mu$-integral diverges at a first-order pole or any zero of $q^2(z)$.

3. **MORE ADVANCED THEORY**

Any solution $\psi(z)$ of eq. (1.1) can be written in the form (2.10) if we replace the constants $a_1, 2$ by some functions $a_1, 2(z)$:

\[
\psi(z) = a_1(z) \int_1(z) + a_2(z) \int_2(z). \tag{3.1}
\]
The fact that $\psi(z)$ is to satisfy eq. (1.1) gives us only one equation for the functions $a_{1,2}(z)$ and therefore we can impose any reasonable condition to determine these functions uniquely. It is convenient to require that

$$a_1'(z) f_1''(z) + a_2'(z) f_2''(z) = 0.$$  \hfill (3.2)

(a so-called Lagrange condition), as it leads to a simple expression for the derivative

$$\psi'(z) = a_1(z) f_1'(z) + a_2(z) f_2'(z).$$  \hfill (3.3)

Due to the linear independence of the WKB solutions $f_{1,2}(z)$, eqs. (3.1), (3.3) can be solved with respect to $a_{1,2}(z)$, which yields

$$a_1(z) = \frac{i}{2} \left[ \gamma(z) f_1'(z) - \gamma'(z) f_1(z) \right],$$
$$a_2(z) = -\frac{i}{2} \left[ \gamma(z) f_2'(z) - \gamma'(z) f_2(z) \right].$$  \hfill (3.4)

Thus it suffices to assume definite values of the functions $a_{1,2}(z)$ at some point (e.g. $z = z_1$) to specify uniquely these functions in the whole complex plane. At the same time it is obvious that the relation between $a_{1,2}(z_1)$ and $a_{1,2}(z)$ must be linear so that it can be written as

$$a(z) = \hat{F}(z_1, z_1) a(z_1),$$  \hfill (3.5)

where $a(z)$ denotes a column vector with the elements $a_{1,2}(z)$ and $\hat{F}(z_1, z_1)$ is a 2 X 2 non-singular matrix depending on $z_1$ and $z$. The properties of this matrix have been discussed in detail in [1], where it was shown, for example, that

$$\det \hat{F}(z_1, z_1) = 1.$$  \hfill (3.6)
Some of these properties follow immediately from the definition (3.5), e.g.

\[ \hat{F}(z, z_4) = \hat{F}(z, z_2) \hat{F}(z_2, z_4), \quad (3.7) \]

\[ \hat{F}(z_4, z) = \hat{F}^{-1}(z, z_4). \quad (3.8) \]

Combining (3.6) and (3.8) we can also get the relation

\[ \hat{F}(z_1, z) = \begin{pmatrix} F_{22}(z, z_4) & -F_{42}(z, z_4) \\ -F_{24}(z, z_4) & F_{44}(z, z_4) \end{pmatrix}. \quad (3.9) \]

For our purposes the following result ([1] p. 27) will be of importance. If \( L \) denotes a path emerging from a point \( z_1 \), chosen in such a way that \( |e^{i \omega(z_0, z)}| \) increases (or remains constant) as \( z \) moves from \( z_1 \) along \( L \), simple estimates exist for the \( F \)-matrix along \( L \):

\[
\begin{align*}
F_{44}(z, z_4) &= 1 + O_4(\mu), \\
F_{42}(z, z_4) &= e^{-i 2 \omega(z_0, z_4)} O_2(\mu), \\
F_{24}(z, z_4) &= e^{i 2 \omega(z_0, z_4)} O_3(\mu), \\
F_{22}(z, z_4) &= 1 + O_4(\mu) + e^{i 2 \omega(z_4, z)} O_5(\mu^2).
\end{align*}
\quad (3.10)
\]

The \( \mu \)-integral in (3.10) should be calculated along \( L \) and it is assumed that \( \mu < 1 \). \( O \)-symbols have the following meaning: for a real number \( x > 0 \) \( O(x) \) is a (complex) number whose modulus is at most of the order of \( x \). Using estimates (3.10) we can discuss simply the accuracy and the conditions for applicability of the WKB solutions along the path \( L \). These can then be generalized to an arbitrary path. Using (3.5) and (3.1) we can write

\[ \psi(z) = \hat{F}(z, z_4) \alpha(z_4) \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}. \quad (3.11) \]
For \( z \in L \) using the estimates (3.10) as well as the obvious general relations

\[
e^{\pm 2i\omega(t,z)} \int_{z_{1}}^{z} (z) = \int_{z_{1},z} (z),
\]

we can rewrite (3.11) in a form

\[
\psi(z) = a_{4}(z) \int_{1}^{z} \left[ 1 + \tilde{O}_{4}(\mu) \right] + a_{2}(z) \int_{1}^{z} \left[ 1 + O_{4}(\mu) + \tilde{O}_{2}(\mu)e^{\pm i\omega(t,z)} \right]
\]

where \( \tilde{O}_{4}(\mu) = O_{4}(\mu) + O_{3}(\mu) \), \( \tilde{O}_{2}(\mu) = O_{2}(\mu) + O_{5}(\mu) \).

If the terms containing the \( O \)-symbols in (3.12) can be neglected as compared to unity we recognize the WKB result (2.10) (with \( a_{1} = a_{1,2}(z_{1}) \)); in general the terms with the \( O \)-symbols represent the relative errors of the WKB solutions \( f_{1,2}(z) \). Having in mind the definition of the path \( L \), it can be noticed that the exponential factor \( |e^{\pm i\omega(t,z)}| \) in (3.12) never decreases as \( z \) moves from \( z \) and its initial value is unity. This factor is identically unity only if \( \text{Im} \omega(z_{1}, z) = 0 \) for \( z \in L \). Noticing that \( \omega(z_{1}, z) = \omega(z_{0}, z) - \omega(z_{0}, z_{1}) \), the last condition becomes

\[
\text{Im} \omega(z_{0}, z) = \text{const}, \quad \text{for } z \in L.
\]

This condition defines a family of lines in the complex plane, all of them having the following important properties:

(i) The exponential factors \( e^{\pm i\omega} \) are pure oscillatory (i.e. \( |e^{\pm i\omega}| = \text{const} \) along these lines,

(ii) The relative errors of both WKB terms are there at most of the order of the \( \mu \)-integral.

In the literature usually only those of the lines (3.13) which emerge from zeros (or simple poles) are taken into consideration and they are then called anti-Stokes lines. However in more complicated applications it happens frequently that an anti-Stokes line emerging from a zero or a pole passes close by another zero or pole, say \( \gamma_{4} \). To understand the behaviour of this line in the vicinity of \( \gamma_{4} \), it is not sufficient to restrict ourselves only to those lines (3.13) in this region which emerge from \( \gamma_{4} \).
For that reason and also because all the lines (3.13) have the properties (i), (ii) we find it convenient to call all of them the anti-Stokes lines. Similarly we shall call the Stokes lines all the lines in the complex plane along which

$$\Re e \omega(z_0, z) = \text{const.}$$  \hspace{1cm} (3.14)

It will be convenient to write simply: S-lines, aS-lines, or S,aS-lines, when referring to the lines (3.14) (3.13), or both of them, respectively.

From the adopted definitions it follows that apart from zeros and singularities of \( q^2(z) \), the S-lines and the aS-lines are orthogonal to each other.

For the validity of the estimates (3.10) we had to assume that \( |e^{i\omega(z_0, z)}| \) increases (or is constant) along the curve \( L \). However, we can also get a result similar to (3.12) if \( |e^{i\omega(z_0, z)}| \) decreases (or remains constant) along \( L \). (In that case we can apply the estimates of the type (3.10) to the matrix \( \hat{P}(z_0, z) \) and then determine \( \hat{P}(z, z_0) \) from (3.9).) Both these results can be written in a unified form as

$$\psi(z) = a_\gamma(z_0) \left\{ \frac{1}{2} \left[ 1 + O_\gamma(\mu) \right] \right\} a_d(z) \left\{ \left[ 1 + O_d(\mu) + O_3(\mu) \right] e^{2i\text{Im} \omega(z_0, z)} \right\}$$  \hspace{1cm} (3.15)

where \( g = 1 \) or \( 2 \) - labels a growing WKB term and \( d = 2 \) or \( 1 \), respectively, labels a decreasing WKB term. This formula gives a generalization of the expression for \( \psi(z) \) in a "classically forbidden region" (cf. (8.6) in [1]). An essential condition for its applicability is that \( |e^{i\omega(z_0, z)}| \) changes monotonically along \( L \), or that the same property holds for \( \text{Im} \omega(z_0, z) \). The last requirement has a simple geometrical interpretation, i.e., it means that the curve \( L \) cannot cross more than once any aS-line, though either \( L \) or a part of it can coincide with such a line.

It is obvious that any curve \( L \) in the complex plane can always be divided into parts satisfying this requirement. Using (3.15) along these parts we can thereby trace the solution \( \psi(z) \) along the whole curve \( L \).

We would like now to draw some conclusions from formula (3.15) in the case where \( \text{Im} \omega(z_0, z) \) is a strictly
monotonic function along $L$ (i.e., $L$ does not coincide with an aS-line).

(A) It is seen that the decreasing WKB term gives a reasonable approximation only along some part of the line $L$ up to some point $z_2$ where

$$
\mu e^{12 \Im \omega(z_1, z_2)} \sim 1, \quad \text{or} \quad |\Im \omega(z_1, z_2)| \sim -\frac{\ln \mu}{2\ell}. \quad (3.16)
$$

The length of the segment $(z_1, z_2)$ depends on the angles at which the line $L$ crosses the aS-lines and is the smallest if $L$ coincides with a S-line. It should also be mentioned that if $q^2(z)$ contains a small parameter $\lambda$ (cf. (2.2)) in which case $\mu \sim \lambda$ and $\omega \sim 1/\lambda$ the length of the segment $(z_1, z_2)$ becomes infinitesimally small for $\lambda \to 0$.

(B) If $a_g(z_2) \neq 0$, the growing WKB term in (3.15) will always dominate the decreasing term for $z \in L$ satisfying

$$
e^{12 \Im \omega(z_1, z)} \gg \frac{a_0(z_1)}{a_g(z_1)} \frac{f_4(z_1)}{f_4(z_1)} \quad (3.17)
$$

For such values of $z$ we can neglect in (3.15) the first two terms related to the decreasing term, and formula (3.15) can be rewritten as

$$
\gamma(z) = a_g(z_1) f_4(z) \left[ 1 + O_{\ell}(\mu) + O_{\ell}(\mu) \frac{a_0(z_1)}{a_g(z_1)} \frac{f_4(z_1)}{f_4(z_1)} \right]. \quad (3.18)
$$

Thus the relative error of the WKB approximation in the region where the growing WKB term dominates, depends on the initial ratio of the two WKB terms. In particular this error is at most of the order of $\mu$ if the decreasing WKB term is initially at most of the order of the growing one.

The main conclusion of this section is that there is a necessity to distinguish between the aS-lines (where (2.11) is a sufficient and necessary condition for the validity of the WKB solution) and any other lines in the complex plane, especially the S-lines. In the latter case we have usually very restricted possibilities to make use of the decreasing WKB term; the growing term, however, provides us with a good approximation if the $\mu$-integral is sufficiently small.

- 12 -
4. STOKES- AND ANTI-STOKES LINES IN THE VICINITY OF ZEROS AND POLES OF $q^2(z)$

Let $\zeta_o$ denote a zero, or a pole of the function $q^2(z)$. Assuming that $q^2(z)$ is given approximately by (2.14) in some vicinity of $\zeta_o$ and that both $z_0$ and $z$ in (2.6) belong to this vicinity, we obtain

$$\omega(z_0, z) = \int_{z_0}^z C_{\frac{1}{2}} (\zeta - \zeta_o)^{n/2} d\zeta = \begin{cases} C_{\frac{1}{2}} \ln \frac{z - \zeta_o}{z_0 - \zeta_o}, & \text{for } n = -2 \\ C_{\frac{1}{2}} \frac{2}{n+2} \left[ (z - \zeta_o)^{n+2} - (z_0 - \zeta_o)^{n+2} \right], & \text{for } n \neq -2. \end{cases}$$

(4.1)

Introducing

$$Re^{i\psi} = z - \zeta_o, \quad Ro e^{i\psi} = z_0 - \zeta_o,$$

(4.2)

we can rewrite (4.1) for $n \neq -2$ as

$$\omega(z_0, z) = |C|^{1/2} \frac{2}{n+2} \left[ R_{\frac{1}{2}} e^{\frac{(n+2)\psi + \alpha}{2}} - R_{0 \frac{1}{2}} e^{\frac{(n+2)\psi + \alpha}{2}} \right].$$

(4.3)

It is seen that $\omega(z_0, z)$ in that case is a periodic function of the angle $\psi$, which implies a periodic structure of the $S, S$ lines in the vicinity of $\zeta_o$. Half a period of $\omega$ as a function of $\psi$ is equal to

$$\Delta \psi = \frac{2\pi}{|n+2|}.$$  \hspace{0.9cm} (4.4)

The $S$-line emerging from the point $z_0$ (i.e., the lower limit of the $\omega$-integral) is characterized by a condition $\Im \omega(z_0, z) = 0$ (cf. (3.13)) which, using (4.3), can be written as

$$R_{\frac{1}{2}}^{n+2} \sin \frac{(n+2)\psi + \alpha}{2} - R_{0 \frac{1}{2}}^{n+2} \sin \frac{(n+2)\psi + \alpha}{2} = 0.$$  \hspace{0.9cm} (4.5)
Similarly, for the S-line emerging from $z_0$, we obtain an equation analogous to (4.5) but with $\cos$ instead of $\sin$. Such an equation can also be obtained formally from (4.5) if we replace $\psi$ by

$$\psi + \frac{\pi}{n+2} \quad \text{(and the same for $\varphi$)}.$$  

Thus the S-lines in the vicinity of $\zeta_0$ (for $n \neq -2$) can be obtained from the corresponding aS-lines by a rotation around $\zeta_0$ at an angle $\frac{\pi}{n+2}$. If $\psi_0$ in (4.5) is chosen in such a way that $\sin \left(\frac{(n+2)\varphi_0 + \alpha}{2}\right) = 0$ (and $R \neq 0$) we can satisfy eq. (4.5) only for $\psi = \varphi_0$, i.e., along the straight lines emerging from $\zeta_0$ at the angles

$$\psi_{ak} = -\frac{\alpha}{n+2} + k \frac{2\pi}{n+2}. \quad (4.6)$$

The relative angles between these aS-lines are equal to $\Delta \varphi$ given by (4.4). For $\psi_0 \neq \psi_{ak}$ (and $R \neq 0$) we can satisfy eq. (4.5) for any $\psi \neq \psi_{ak}$; with $\psi \to \psi_{ak}$ we get $R \to \infty$

i.e., $R \to \infty$ for $n > -1$ and $R \to 0$ for $n \leq -3$. Thus in the vicinity of zeros, or simple poles, the aS-lines become parallel to radii (4.6) at large distances from $\zeta_0$, whereas for higher order poles that is the case at the small distances.

Some examples of the S, aS-lines obtained from eq. (4.5) are shown in Figs. 1, 2. These figures correspond to a coordinate system rotated around $\zeta_0$ at an angle $\psi_{ak}$. To discuss the case $n = -2$ (a second order pole) let us denote $u + iv = \omega(z_0, z)/|z_0|^2$. The aS-line emerging from $z_0$ is characterized by $v = 0$. Using (4.1) and (4.2) we can write an equation of this line as

$$\frac{R}{R_0} e^{i(\psi - \varphi_0)} = e^{u \cos \frac{\alpha}{2}} e^{-i u \sin \frac{\alpha}{2}}. \quad (4.7)$$

Similarly, for the S-line emerging from $z_0$ we get

$$\frac{R}{R_0} e^{i(\psi - \varphi_0)} = e^{v \sin \frac{\alpha}{2}} e^{i v \cos \frac{\alpha}{2}}. \quad (4.8)$$
Both eqs. (4.7) and (4.8) represent logarithmic spirals except when \( \alpha = 0 \) and \( \alpha = \pi \); in the latter situations one equation describes the radii \( \varphi = \text{const} \) and the other the circles \( R = \text{const} \). In any case the lines described by these eqs. have a rotational symmetry (cf. Fig. 3).

Let us notice that the results of this section can often be used to examine the asymptotic properties of the \( S, aS \)-lines at large distances from the origin. That is the case if \( q^2(z) \) can be approximated by a formula of the form (2.14) for \( |z| \) sufficiently large. For example if

\[
q^2(z) = \frac{P_l(z)}{P_m(z)},
\]

where \( P_l(z), P_m(z) \) are polynomials of order \( l \) and \( m \), respectively, we have for \( |z| \) large enough

\[
q^2(z) \approx C \, z^{l-m}.
\]

Thus in that region the \( S, aS \)-lines are approximately described by eqs. (4.5) - (4.8) (with \( n = l - m \) and \( \gamma_0 = 0 \)).

5. THE WKB CONNECTION FORMULAE IN THE COMPLEX PLANE

As we have explained in Section 2, the WKB approximation breaks down in the vicinity of any zero or a pole of the function \( q^2(z) \). Also it turns out that in general to represent the same solution \( \psi(z) \) along different curves \( L \) emerging from such a critical point we must use different constants \( a_1, a_2 \) in formula (2.10). The relations between these constants, usually referred to as the connections formulae, play an important role in applications (such as boundary value, or eigen-value problems, or evaluation of the transmission coefficient). The connection formulae derived in this section should provide one with all possible connections in the complex plane.
Let \( \zeta_0 \) denote any point at which the \( \omega \)-integral (2.6) is convergent, i.e., \( \zeta_0 \) can be a regular point of \( q^2(z) \) (e.g., any zero), or a simple pole. Let \( s \) and \( a \) denote a \( S \)-line and an \( aS \)-line, respectively, which cross at \( \zeta_0 \); the lines should be chosen in such a way that it is possible to get from \( a \) to \( s \) without crossing any other \( S \), \( aS \)-line emerging from \( \zeta_0 \). We would like first to derive the connection formulae for the lines \( a \) and \( s \).

Let us consider the mapping \( z \rightarrow \omega \) defined by the integral (2.6) \( \omega(\zeta_0, z) \) and denote by \( L \) a curve in the \( z \)-plane along which \( |\omega(\zeta_0, z)| \) = const. The region \( \Delta \) in the \( z \)-plane bounded by the curves \( a \), \( s \) and \( L \) can be mapped into the \( \omega \)-plane in two different ways (Fig. 4) depending on the sign of \( \omega_a = \omega(\zeta_0, z) \) for \( z \in a \). (In what follows we assign subscripts \( a \) or \( s \) to any function of \( z \) taken along the lines \( a \) or \( s \), respectively; it follows from (3.13) and (3.14) that \( \omega_a \) is real and \( \omega_s \) is pure imaginary.) For both locations of the region \( (\Delta) \) we have \( \omega(z_1) = i \omega(z_2) \), which is related to the fact that we move around \( \zeta_0 \) in a positive sense when going from \( z_1 \) to \( z_2 \). If the rotation in a negative sense was necessary to get from \( z_1 \) to \( z_2 \) we would have \( \omega(z_2) = -i \omega(z_1) \). Thus in general

\[
i \omega(\zeta_0, z) = \begin{cases} 
i \omega_a(z_1), & \text{for } z = z_1, \\ i \omega_s(z_2) = \mp \omega_a(z_1), & \text{for } z = z_2. \end{cases} \quad (5.1)
\]

Multiplying both sides of this relation by \( \mp \text{ sign } \omega_a(z_1) \) and noticing that \( |\text{ sign } \omega_a(z_1)| \omega_a(z_1) = |\omega_a(z_1)| = |\omega_s(z_2)| \) we obtain the following correspondence of the exponential factors \( e^{i\omega} \) at the points \( z_1 \) and \( z_2 \):

\[
e^{i|\omega_a(z_1)|} \quad \longleftrightarrow \quad e^{i|\omega_s(z_2)|}, \quad (5.2)
\]

where the upper sign corresponds to a positive rotation from \( a \) to \( s \). Now we can simply make use of formula (3.15) along the curve \( L \) (it is obvious from Fig. 4 that \( \text{Im}(\zeta_0, z) \) changes monotonically along \( L \)). Assuming

\[
\varphi(z_2) = q^{-1/2} \left( A e^{-|\omega_s|} + B e^{i|\omega_s|} \right), \quad (5.3)
\]

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we obtain in this way

\[ \psi(z_i) = q_i^{\frac{1}{2}} \left\{ A e^{i|\omega_0|} \left[ 1 + O_i(\mu) \right] + B e^{-i|\omega_0|} \left[ 1 + O_2(\mu) + O_3(\mu) e^{2|\omega_0|} \right] \right\} (5.4) \]

Similarly, assuming

\[ \psi(z_s) = q_s^{\frac{1}{2}} \left( A e^{i|\omega_0|} + B e^{-i|\omega_0|} \right), \quad (5.5) \]

we get

\[ \psi(z_s) = q_s^{\frac{1}{2}} \left\{ B e^{i\omega_s} \left[ 1 + O_s(\mu) \right] + A e^{-i\omega_s} \left[ 1 + O_2(\mu) + O_3(\mu) e^{2|\omega_s|} \right] \right\}. (5.6) \]

The results (5.4) and (5.6) can then be traced along a and s, respectively, using formula (3.15). It can be noticed that though the correspondence between the approximate WKB solutions on a and s is very simple, as just given by (5.2), the errors associated with each term depend on the direction in which the connection is made; as a rule only in one direction can a useful connection be obtained. To see it better let us choose the curve L (Fig. 4) in such a way that

\[ \mu \ll 1, \quad (5.7) \]
\[ e^{2|\omega_{s,a}|} \gg 1, \quad (5.8) \]

and consider two special choices for the coefficients A and B

(i) \[ |B| e^{\omega_s} \lesssim |A| e^{-\omega_s}, \]

(ii) \[ |A| \lesssim |B|. \]

(The sign \( \lesssim \) means that the left-hand side is at most of the order of the right-hand side.) Assuming the first choice for A and B in formulae (5.3), (5.4) and the second one in (5.5), (5.6) we get in both cases the decreasing WKB term initially at most of the order of the growing term. Thus (cf. (3.17), (3.18), (5.8)) in both cases the growing WKB term will dominate at the final point and the
relative error of this term will be at most of the order of \( \mu \).
That leads to the following connection formulae:

\[
q_{\nu_k - \frac{1}{2}} (A e^{-i|\omega|} + B e^{i|\omega|}) \longrightarrow q_{\nu_k - \frac{1}{2}} A e^{i|\omega|} \quad \text{for the case (i)} (5.9)
\]

\[
q_{\nu_k - \frac{1}{2}} (A e^{i|\omega|} + B e^{-i|\omega|}) \longrightarrow q_{\nu_k - \frac{1}{2}} B e^{-i|\omega|} \quad \text{for the case (ii)} (5.10)
\]

They hold for any admissible choice of the intersection point (cf. p. 16). On the other hand, taking the values (i) in (5.5), (5.6) and (ii) in (5.3), (5.4), we end up with the results where the second WKB term (which is at least of the order of the first one) has the relative error of the order of \( \mu \exp(2|\omega|) \gg \mu \) (cf. (5.8)).
This error is usually even greater than unity so that the corresponding connection formulae are then useless. For example, if \( \zeta \) coincides with a zero, or a simple pole of \( q(z) \) and \( q(z) \) can be approximated by (2.14), one can simply show that

\[
2|\omega_{\nu, \alpha}| = \frac{\pi (n+4)|n|}{4(n+2)^2} \mu > \frac{0.4}{\mu}, \quad \text{for } n = +1, 2, 3, \ldots (5.11)
\]

Thus in that case \( \mu \exp(2|\omega_{\nu, \alpha}|) > \mu \exp(0.4/\mu) > 1 \) for any value of \( \mu \). (Notice that by virtue of (5.11) condition (5.8) is satisfied automatically if \( \mu \ll 1 \).) It follows that in the case of a zero, or a simple pole, the connection formulae (5.9), (5.10) have a one-directional nature, i.e., can be used only from the left to the right, as indicated by the arrows.

We would like now to discuss the connection formulae of another type, i.e., those connecting the WKB solutions on different \( \alpha \)-lines emerging from zeros of \( q^2(z) \). The derivation of these formulae for an arbitrary zero of \( q^2(z) \) can be found in [1]. For the applications we find it convenient, however, to change slightly the notation used in this reference. Assuming \( q^2(z) \) to be given exactly by (2.14) (where \( n > 0 \)) and taking the lower limit in the \( w \)-integral to coincide with \( \gamma_0 \) we can write in analogy to (5.5)
\( \psi(z_k) = \left| \frac{1}{\sqrt{\mu}} \left[ b_1(z_k) e^{i\omega_1} + b_2(z_k) e^{-i\omega_1} \right] \right|, \) \hspace{1cm} (5.12)

where \( z_k \) denotes some point on an aS-line emerging from \( \gamma_0 \), i.e.,

\[ z_k = R_0 e^{i\psi_k} \quad \text{(cf. (4.6))}. \]

With this notation we can then write by analogy to (3.5)

\[ b(z_k) = \hat{G}(z_k, z_k) b(z_k). \] \hspace{1cm} (5.13)

Using this \( \hat{G} \)-matrix (which differs from the corresponding \( F \)-matrix only in the order of elements) we can write both formulae (7.11 a) and (7.11 b) of the reference [1] in a unified form, i.e.,

\[ \hat{G}(z_{k+1}, z_k) = \begin{pmatrix} 0 & 1 \\ 1 & 2i \cos \frac{n}{n+2} \end{pmatrix}. \] \hspace{1cm} (5.14)

This formula is only approximate but as follows from the analysis performed in [1], the absolute errors of the \( \hat{G} \)-matrix elements are at most of the order of the \( \mu \)-integral along a part of the circle \( R_0 e^{i\psi} \) in between \( z_k \) and \( z_{k+1} \). It is obvious that when \( q^2(z) \) is given only approximately by (2.14), relation (5.14) should also be approximately true. The error, however, for this more general case and for a zero of an arbitrary order has not yet been estimated. For a simple zero this error has been shown in [1] to be at most of the order of the \( \mu \)-integral along the whole circle \( R_0 e^{i\psi} \). From (5.13) and (5.14) it follows that

\[ \hat{G}(z_k, z_{k+1}) = \hat{G}(z_{k+1}, z_k) = \begin{pmatrix} -2i \cos \frac{n}{n+2} & 1 \\ 1 & 0 \end{pmatrix}, \] \hspace{1cm} (5.15)

Formulae (5.14) and (5.15) define the connections between any aS-lines emerging from a zero \( \gamma_0 \). Combining these results with formulae (5.9) and (5.10) enables one to trace approximately the solution \( \psi(z) \) from any S-line to any aS-line (both emerging from \( \gamma_0 \)) and vice versa. In particular, for a S-line and an aS-line emerging from a simple zero in opposite directions we get the connection formulae which can be written in the following...
The relative error of these formulae is at most of the order of the $\mu$-integral along the whole circle $R_0 e^{i\nu}$ and it is assumed that $\mu \ll 1$. The distance $R_0$ at which this $\mu$-integral is already small can be evaluated approximately from a simple formula following from (2.15), i.e.,

$$
\mu = \frac{5 \pi}{8 |C|^{\frac{1}{2}} R_0^{3/2}}. \quad (5.18)
$$

If $q^2(z)$ is real on the real axis the parts of this axis where $q^2$ has a constant sign are the $\alpha$-lines (if $q^2 > 0$), or the $\beta$-lines (if $q^2 < 0$). In the region where $q^2(x) > 0$ we can write

$$
q^{-\nu_2}(x) = |q(x)|^{-\nu_2} e^{i k \pi/2}, \quad (5.19)
$$

where $k = 0, 1, 2, 3$, depending on the choice of the phase of $q^2(x)$. Crossing a simple zero of $q^2(z)$ this function gains a factor $e^{-i \pi}$ where the sign depends on the location of the cut emerging from the zero (upper sign should be chosen if the cut enables one to reach the $\beta$-line by a positive rotation). Thus on the opposite side of the considered zero we get

$$
q^{-\nu_2}(x) = |q(x)|^{-\nu_2} e^{i k \pi/2} e^{-i \pi/4}. \quad (5.20)
$$

Inserting (5.19) and (5.20) into the connection formulae (5.16), (5.17) and leaving out the common factor $e^{i k \pi/2}$ we reduce the formulae to the standard form for the real axis (cf., e.g., formulae (8.19) and (8.21) in [1]). That form is independent of the location of the cut emerging from $\gamma_p$. 

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We would like to mention that the existence of the connection formulae (5.14) is related to the fact that any solution \( \psi(z) \) of eq. (1.1) is a single-valued function in the vicinity of a zero of \( q^2(z) \). This fact is reflected in the following property of the matrix (5.14):

\[
\begin{pmatrix}
0 & 1 \\
-2i \cos \frac{n \pi}{n+2} & 1
\end{pmatrix}^{(n+1)} = e^{i \frac{2 \pi n}{n+2}} I,
\]

where \( I \) denotes a unit matrix. Thus after a whole rotation around the zero in a positive sense the coefficients \( b_{1,2} \) in (5.12) gain the phase factor \( e^{i \frac{2 \pi n}{n+2}} \). The latter cancels with the factor \( e^{-i \frac{2 \pi n}{n+2}} \) gained by \( q^{-2}(z) \) (cf. (2.14)) so that \( \psi(z) \) goes back to its initial value.

In applications of the connection formulae in the complex plane the following relation is usually useful:

\[
|\omega_{s \alpha}(z_1, z)| + |\omega_{s \alpha}(z_2, z)| = |\omega_{s \alpha}(z_1, z_2)|.
\]

It holds along any \( s \), \( a \)-line provided that the point \( z \) lies between \( z_1 \) and \( z_2 \) and there are no zeros or poles in between \( z_1 \) and \( z_2 \) (we admit, however, the possibility that \( z_1 \), or \( z_2 \), or both coincide with any zero or a simple pole of \( q^2(z) \)). The proof of (5.21) follows from an obvious identity \( w(z_1, z) + w(z, z_2) = w(z_1, z_2) \) (cf. (2.6)) and from the fact that under the assumptions made all the terms in this identity are either pure imaginary or real and are all of the same sign.

6. EXAMPLES

As a first example let us take

\[
q^2(z) = \kappa^2 \frac{z}{z-p},
\]

where \( \kappa \) and \( p \) are some real and positive constants. It is the simplest case of a function which has only one simple zero (at \( z = 0 \)) and one simple pole (at \( z = p \)).
Physically, eq. (1.1) with such a function can be interpreted as that describing the electromagnetic wave propagation in a medium where, due to a certain spatial dependence of the refractive index, one gets one cut-off ($q^2 = 0$) and one resonance ($q^2 = \infty$). Such a physical situation can occur in ionosphere (cf. [2] p. 245) and therefore it is of interest to find the corresponding reflection and transmission coefficients. That can be done exactly, as for $q^2(z)$ given by (6.1) eq. (1.1) can be reduced to the standard form for the confluent hypergeometric function [3]. Nevertheless we find it instructive to perform the same calculation with the WKB technique and to compare both results. A typical procedure in that case consists first in finding the $S$, $aS$-lines and then in using appropriate connection formulae. Using the results of Section 4, we can construct the $S$, $aS$-lines in the vicinity of $z = 0$ (where $q^2(z) \propto (-K/p) z$) and $z = p$ (where $q^2(z) \propto (K^2 p) (z-p)^{-1}$). Also it follows from formula (4.6) that the anti-Stokes lines at large distances from the origin (where $q^2(z) \propto \omega^2$) must go parallel to the real axis; consequently the Stokes lines must be there parallel to the imaginary axis and the final configuration of the $S$, $aS$-lines is as shown in Fig. 5. Consider first a left-moving wave which passes first the resonance at $x = p$ and then the cut-off at $z = 0$. On the anti-Stokes line $a$ (Fig. 5) there will be only the left-moving wave. Thus if the time dependence of the wave is taken in the form $e^{-i\omega t}$ we should put $B = 0$ in the connection formula (5.17) applied at $z = 0$. Choosing the upper sign in the exponent (i.e., assuming that the cut emerging from $z = 0$ is located in the upper half of the $z$-plane) and leaving out the common factor $e^{i\pi/4}$ we get

$$
\psi_a = q_{\psi a} A e^{i \omega_a (0, x)}, \quad x < 0 \tag{6.2}
$$

$$
\psi_s = q_{\psi s} A e^{i \omega_s (0, x)} = q_{\psi s} \left[ A e^{i \omega_s (0, p)} \right] e^{i \omega_s (p, x)}, \quad 0 < x < p \tag{6.3}
$$

where in the last equality use was made of (5.22). Using now (5.9) at $x = p$ we should choose the lower sign in the exponent, as otherwise we should not get any left-moving wave. That corresponds
again to the cut in the upper half of the \( z \)-plane, or to the values of \( \gamma \) from the lower lip of the cut chosen along the real axis, as in Fig. 5.) The result is

\[
\gamma_{\alpha} = q_{\alpha}^2 \left[ A e^{i \omega_{a}(q,p)} \right] e^{i \omega_{a}(p,x)}, \quad p < x. \tag{6.4}
\]

Noticing that \( q_{\alpha}^{-\frac{1}{2}}(-\infty) = q_{\alpha}^{-\frac{1}{2}}(\infty) \) we obtain from (6.2) and (6.4) the following results for the reflection and the transmission coefficients

\[
R = 0, \tag{6.5}
\]

\[
T = \frac{A}{(A e^{i \omega_{a}(0,p)})} = e^{-\int_{0}^{x} |q(x)| \, dx}. \tag{6.6}
\]

The integral in the exponent is elementary and the final result is

\[
T = e^{-\pi K p/2}. \tag{6.7}
\]

It turns out that this result as well as (6.5) are both exact (cf. [2] p.245).

The similar kind of WKB analysis for the right-moving wave gives the same result for \( T \) (6.7) and \( R = 1 \). Again the expression for the transmission coefficient is exact, whereas the actual value of \( R \) is less than unity and equal to \( 1 - \exp(-\pi K p) \).

(In both cases \( |R|^2 + |T|^2 < 1 \), which means that the wave energy is partly absorbed in the region of the resonance.)

Concluding, we can expect that the WKB technique should give a reasonable approximation (at least for the transmission coefficient) also in more complicated cases, e.g.,

\[
q_{\alpha}(z) = K \frac{z}{z-p} f(z), \tag{6.8}
\]

where \( f(x) > 0 \), and \( \lim_{x \to \pm \infty} f(x) = 1 \). For such a function formula (6.6) should be approximately true provided that the zeros and the singularities of \( f(z) \) are not too close to the real axis.
As the next example let us take

\[ q^2(z) = \frac{P_n(z^2)}{P_d(z^2)} = \kappa^2 \frac{(z^2 - z_1^2) \cdots (z^2 - z_n^2)}{(z^2 - p_1^2) \cdots (z^2 - p_d^2)}, \quad \kappa^2 = |\kappa|^2 e^{i2\delta}, \quad (6.9) \]

where \( P_n, P_d \) are polynomials of order \( n \) and \( d \), respectively, \( z_1, z_2, \ldots \) are zeros and \( p_1, p_2, \ldots \) poles of \( q^2(z) \) (all of them of the first order).

Special cases of such a function occur in plasma stability theory, e.g., those with \( n - 1, d = 0 \) (in which case (1.1) is the Weber equation and can be treated without approximations), \( n = 2, d = 0 \), or \( n - d = 2 \) [4]. In all these cases we deal with a "non-linear eigenvalue problem" which means that the coefficients of the polynomials \( P_n, P_d \) are some functions of the unknown eigenvalue \( s = \gamma + i\omega \); the eigenvalue should be determined in such a way as to get \( \psi(x) \to 0 \) for \( x \to \pm \infty \).

We shall derive the "quantization condition" from which to determine the eigenvalues, and discuss the question of the localization of the eigen-solutions along the real axis. If \( |z| \) is large enough, (6.9) is approximately given by

\[ q^2(z) = \kappa^2 z^{2(n-d)}. \]

Assuming \( n \geq d \) it follows (cf. (4.6)) that the \( aS \)-lines become asymptotically parallel to the straight lines defined by the angles

\[ \psi_{\alpha \infty} = -\frac{\delta}{n-d+1} + k \frac{\pi}{n-d+1}. \quad (6.10) \]

Let us consider first the case \( n = d \), where \( \psi_{\alpha \infty} = -\frac{\delta}{d} \) and the asymptotic direction of the \( S \)-lines is given by \( \psi_{s \infty} = \pi/2 - \delta \).

In that case it is convenient to introduce a new o.s. rotated around the origin at the angle \( \psi_{s \infty} \), so that the \( S \)-lines and the \( aS \)-lines be asymptotically parallel to the new real and imaginary axes, respectively. In the following discussion of the case \( n = d \) we shall always be referring to the new o.s. when talking of a real or imaginary axis without further specification. The transformation to the new o.s.

\[ \bar{z} = z e^{-i(\pi/2 - \delta)}, \quad (6.11) \]
leaves the form of eq. (1.1) unchanged and the new function $\bar{q}^2(\bar{z})$ is given by

$$
\bar{q}^2(\bar{z}) = -|K|^2 \frac{(\bar{z}^2 - \bar{z}_1^2) \cdots (\bar{z}^2 - \bar{z}_n^2)}{(\bar{z}^2 - \bar{p}_1^2) \cdots (\bar{z}^2 - \bar{p}_d^2)}.
$$

(6.12)

There exist some configurations of zeros and poles of $\bar{q}^2(\bar{z})$, which play a special role in the discussed eigenvalue problem. One can see it by considering the WKB solutions of eq. (1.1) at large distances from the origin, where

$$
\tilde{f}_{ij}(\bar{z}) \sim e^{\frac{1}{\bar{q}^2}(\bar{z} + i\bar{q})}
$$

(6.13)

The lower sign solution is "well behaving" for $\bar{x} > 0$, the upper sign for $\bar{x} < 0$, but they cannot be matched continuously along the imaginary axis. Thus the cuts are necessary in order to obtain a single-valued eigen-solution. It is well known that these cuts should emerge from poles of $\bar{q}^2(\bar{z})$ and therefore the most natural location of the poles is on the imaginary axis. The natural location of zeros is on the real or on the imaginary axis (in an arbitrary order with respect to the poles); for such a configuration $S$, as-lines emerging from the most distant zeros or poles have immediately the required directions at infinity (cf. Fig 6(a)). From the physical point of view the most interesting situation arises if there is only one pair of zeros (e.g., $\pm \bar{z}_1$) on the real axis. That will be our "unperturbed geometry" (Fig. 6(a)). The "perturbation" will consist of a rotation of the zeros $\bar{z}_i$ at some angles $\delta \psi_{\bar{z}_i}$ and the poles $\bar{p}_i$ at the angles $\delta \psi_{\bar{p}_i}$ (none of the angles being necessarily small).

The detailed structure of the $S$, as-lines in the perturbed geometry is usually rather involved and it depends not only on the perturbation angles but also on the positions of zeros and poles in the unperturbed geometry. Let us notice, however, that the slope of the as-line at the origin is defined by the perturbation angles only:

$$
\psi_{a.o} = \sum_{i=1}^{n} (\delta \psi_{\bar{p}_i} - \delta \psi_{\bar{z}_i}),
$$

(6.14)
which can be shown using Taylor expansion of $q' (z)$ about zero, and formula (4.6).

Let us consider first a configuration where mostly the zeros $\pm \bar{z}_4$ are perturbed, so that $\psi_{a_0} \simeq - \delta \psi_{z_1}$ (Fig. 6(b)). One can simply see that it cannot correspond to an eigen-solution. First of all one should realize that if the solution $\psi(z)$ increases from zero as $z$ moves from infinity along the $S$-line $s_1$, the same type of behaviour will take place along any $S$-line in the region $D_1$.

To show it we assume that there is only a decreasing WKB term at some point on $s_1$ to the right from $\gamma_1'$, use formula (5.7) at $\gamma_2$ and then (5.10) at $\gamma_3'$ (after taking (5.22) into account). As a result we obtain only a growing WKB term on $s_3$ to the left of $\gamma_3'$; the coefficient of this term differs only in a phase factor from that on $s_1$. Thus when $z$ moves to the left from $\gamma_3'$ along $s_3$, $\psi(z)$ increases exponentially in the same way as it does along $s_1$. Repeating now the same procedure but for the points $\gamma_1'$ and $\gamma_2'$ we can show that the solution $\psi(z)$ which is "well behaving" in $D_1$ will grow up to infinity along the $S$-line $s_2$, i.e., along any $S$-line in the region $D_2$. Similarly, the solution "well behaving" in $D_1$ will grow up in $D_2$.

A similar proof can be applied for any perturbed geometry where the $aS$-line $a$ passing through the origin extends to infinity. Also, a situation where the line $a$ terminates at some poles can be excluded in a similar way. Thus there is only one possibility left as far as the line $a$ is concerned, namely that it terminates at some zeros, e.g., $\pm \bar{z}_4$. In that case there should be approximately $\delta \psi_{z_1} = \psi_{a_0}$, or using (6.14)

$$\delta \psi_{s_1} \approx 2 \delta \psi_{z_1} + \sum_{i=2}^{n} (\delta \psi_{z_1} - \delta \psi_{s_1}) , \quad (6.15)$$

(of Fig. 7 corresponding to $n = d = 1$, where $\delta \psi_{s_1} \approx 2 \delta \psi_{z_1}$).

The unperturbed geometry is the simplest example of such a situation, in which case all the terms in (6.15) are zero. Relation (6.15) is only approximate. An exact condition expressing the fact that both zeros $\pm \bar{z}_4$ lie on the same $aS$-line is

$$\text{Im} \omega (-z_4, z_4) = 0 . \quad (6.16)$$
A more precise condition for the existence of an eigen-solution can be obtained using the connection formula (5.16) at \( \pm z_1 \). Assuming in both cases \( B = 0 \) and requiring both results obtained for \( \mathcal{V}_a(z) \) to be equal, we get an identity

\[
A_1 \cos \left( m \omega_a(z_1, z) - \pi/4 \right) = A_2 \cos \left( m \omega_a(-z_1, z) - \pi/4 \right),
\]

which obviously requires \( |A_1| = |A_2| \). Noticing that

\[
|\omega_a(-z_1, z)| = |\omega_a(z_1, z)| \quad \text{(cf. (5.22))}
\]

one can simply show that (6.17) is equivalent to a condition: \( |\omega_a(-z_1, z)| = \pi (m + 1/2), \quad m = 0, 1, 2, \ldots \), which in a more explicit form (cf. (2.6)) becomes

\[
\int_{-z_1}^{z_1} q(z) \, dz = \pm \pi (m + 1/2), \quad m = 0, 1, 2, \ldots. \quad (6.18)
\]

This is the "quantization condition" specifying approximately the eigenvalues \( s \). It differs from an analogous condition of quantum mechanics only in the fact that the limits of the integral \( \pm z_1 \) and the integrand are in general complex. Also the eigenvalues defined by (6.18) will usually be complex.

Let us now come back to the initial requirement for the eigen-solution, i.e., that \( \mathcal{V}(z) \to 0 \) for \( z \) going to infinity along the old real axis. Remembering that the \( \alpha \)-lines become asymptotically parallel to the imaginary axis and that \( \mathcal{V}(z) \to 0 \) as \( z \) goes to infinity along any \( S \)-line in the regions \( D_{1,2} \) (Fig. 7) it seems that the same should be true along the old real axis -- whatever the direction of this axis. That, however, is not true if the old real axis belongs to the region \( D \) (Fig. 7). In that case we must choose the cut emerging from \( \mathcal{P}_1 \) in such a way that it does not cross the old real axis, i.e., to the left of this axis. That means, however, that the values of \( \mathcal{V}(z) \) on the old real axis in \( D \) are a continuation of those from the region \( D_1 \); consequently \( |\mathcal{V}(z)| \) increases with \( z \to \infty \) along the considered part of the old real axis.

Thus for the perturbed geometry there always exists a definite sector in the complex plane in which the eigen-solution is not localized. This sector is defined by the asymptotic direction of the
aS-lines $\psi_{aS} = \mathcal{L} = \arg(1/k)$ and the straight lines connecting the origin and the poles of $q^2(z)$. In the case of the unperturbed geometry this sector shrinks to a straight line passing through the origin and a point $1/k$.

The case $n > d$ will not be discussed here in such detail. We would like only to mention that also in that case the quantization condition (6.18) can be used to determine approximately the eigenvalues $s$; for $s$ satisfying this condition the zeros $\pm z_i$ lie both on the aS-line passing through the origin; the corresponding eigenfunction is now localized in two sectors bounded by the straight lines (6.10), those containing the zeros $\pm z_i$.

Whatever the values of $n$ and $d$ we can reduce the condition (6.18) to a form convenient for approximations or a numerical analysis. Introducing a new integration variable

$$t = \sqrt{1 - z^2/z^2_i}, \quad (6.19)$$

the left-hand side of (6.18) can be rewritten as

$$2 \omega(0, z_i) = 2K i z_i^2 \int_0^1 \frac{t^2 \, dt}{1 - t^2} \sqrt{\frac{[z_i^2(1-t^2)] \cdots [z_i^2(1-t^2) - z_i^2]}{[z_i^2(1-t^2) - p_i^2] \cdots [z_i^2(1-t^2) - p_i^2]}}. \quad (6.20)$$

Let us define $(d + n - 1)$ complex variables $k_j$:

$$k_j^2 = \frac{z^2_i}{z^2_i - p^2_j}, \quad \text{for } j = 1, 2, \ldots, d, \quad (6.21)$$

$$k_{d+1}^2 = \frac{z^2_i}{z^2_i - z_j^2}, \quad \text{for } j = 2, 3, \ldots, n,$$

and a function of these variables

$$\left(\frac{4}{\pi^2}\right) \int_0^1 \frac{t^2 \, dt}{1 - t^2} \sqrt{\frac{(1 - k_{d+1}^2 t^2) \cdots (1 - k_{d+n-1}^2 t^2)}{(1 - k_d^2 t^2) \cdots (1 - k_d^2 t^2)}}. \quad (6.22)$$
With these definitions and taking (6.20) into account the quantization condition (6.18) can be written as:

$$A \sum_j \left( k_j^2 \right) + \left( 2m + 1 \right)^2 = 0, \quad m = 0, 1, 2, \ldots$$

where

$$A = \sum_j p_j^2 \frac{(z_j^4 - z_j^2) \cdots (z_j^4 - z_j^2)}{(z_j^4 - p_j^4) \cdots (z_j^4 - p_j^4)} = \sum_j \frac{p'_j(z_j^2)}{p_j(z_j^2)} = z_j^{2(n-d+1)} \frac{k_2^2 \cdots k_d^2}{k_{d+2} \cdots k_{d+n-1}},$$

$$\left( p'_n(z_j^2) = \frac{d_p(z^2)/d(z^2)}{z_j^2} \right). \quad (6.23)$$

The only problem now is to evaluate the integral (6.22). If there is only one \( k_j \neq 0 \) this integral can be expressed in terms of the complete elliptic integrals \( E(k_j) \) and \( K(k_j) \) (cf. [5] formulae 110.06, 110.07, 219.11):

$$\int (o, \ldots, i, \ldots, 0) = \frac{4}{\pi} \left\{ \begin{array}{ll}
- \frac{K(k_j^2) - E(k_j^2)}{k_j^2}, & 1 \leq j \leq d, \\
(2k_j^2 - 1)E(k_j^2) + (1 - k_j^2)K(k_j^2), & d+1 \leq j \leq d+n-1.
\end{array} \right. \quad (6.24)$$

Such a situation arises if \( n = 2, d = 0 \) or \( n = d = 1 \), also it is approximately so if there is only one pair of zeros (apart from \( \pm z_1 \)) or one pair of poles at a distance from the origin comparable with \( |z_1| \), whereas all the other zeros and singularities are very far away (cf. (6.21)). In all the other cases the exact value of (6.22) can be obtained only numerically. Let us notice, however, that a simple approximate expression for this integral can be obtained if \( |k_j^2| \) are all much less than unity. To show it we expand the second square root in (6.22) in the power series in \( t^2 \) making use of the expansions:

$$(1 - z)^{-1/2} = \sum_0^\infty a_k z^k, \quad (1 - z)^{1/2} = \sum_0^\infty \frac{a_k z^k}{2k-1},$$

where \( |z| < 1 \), \quad (6.25)$$

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and
\[ a_k = \frac{(2k)!}{2^{2k}(k!)^2}, \quad k = 0, 1, 2, \ldots \]  
(6.26)

Integrating term by term and taking into account the known result
\[ \int_0^1 \frac{t^{2k}}{t^2 - 1} dt = \frac{\pi}{2} \alpha_k, \quad (\alpha_k \text{ given by (6.26)}) \]
we end up with the following expansion
\[ \frac{\rho}{(k_j^2)^d} = 1 + \frac{3}{8} \left( k_1^2 + \cdots + k_d^2 - k_{d+1}^2 - \cdots - k_{d+n-1}^2 \right) + \]
\[ + \frac{5}{64} \left[ (k_1^2 + \cdots + k_d^2 - k_{d+1}^2 - \cdots - k_{d+n-1}^2)^2 \right] \]
valid if \( |k_j^2| < 1, \quad j = 1, 2, \ldots, d + n - 1. \)
(6.27)

The higher order terms can be neglected if \( |k_j^2| \) are sufficiently small.

The general form of the expansion (6.27) is rather complicated; for example for \( d, n \ll 2 \) one gets
\[ \frac{\rho}{(k_1^2, k_2^2, k_3^2)} = -2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_k}{2k \cdot j} \cdot \frac{k_3^2}{k_1^2 + k_2^2} \]
where
\[ b_j = \sum_{i=0}^{\infty} a_i a_i \left( k_i^2 k_j^2 \right) \cdot \frac{E(j/2)}{4 \cdot \delta_{i,j} - 2} \]

\( E(x) = \text{Entier } (x) \) - the largest integer not greater than \( x \),
\( \delta_{i,k} - \text{Kronecker delta.} \)
(6.28)

It is obvious that \( |k_j^2| \) will all be less than unity if \( |z_1|, \ldots, |z_n|, |p_1|, \ldots, |p_d| \) are all much greater than \( |z_1| \). Let us notice, however, that even if some of them are of the order of \( |z_1| \) the corresponding \( |k_j^2| \) should be less than unity if the perturbation
of the geometry is not too large (for the unperturbed geometry:
\[ p_j = i \alpha_j z_j, \quad j = 4, 2, \ldots, d \]
and \[ z_j = e^{i \alpha_{d+j-1}} z_{d+j-1}, \quad j = 2, 3, \ldots, n, \]
where \( \alpha_j \) are some real numbers; thus in that case \( k_j^2 = 1/(1+\alpha_j^2) \)
are all real and less than unity). Thus also in that case use can
be made of the expansion (6.27) if an appropriate number of the
higher order terms is taken into account.

ACKNOWLEDGMENT

The author is grateful to Professor Abdus Salam and the IAEA
for hospitality at the International Centre for Theoretical Physics,
Trieste. He would like to thank Dr. Coppi for suggesting the problem
and for his encouragement. He is also grateful to Drs. Cavaliere and
Kennel for useful discussions during the course of this work.
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Fig. 1 Stokes and anti-Stokes lines in the vicinity of zeros of $q^2(z)$
(a) - simple zero, (b) - double zero
Solid curves denote the anti-Stokes lines
Fig. 2 Stokes and anti-Stokes lines in the vicinity of poles of $q^2(x)$
(a) – simple pole, (b) – third order pole
Fig. 3 Stokes and anti-Stokes lines in the vicinity of a second order pole of \( q^2(z) \).

(a) \( \alpha = \pi/2 \) (cf. (2.14)),
(b) \( \alpha = \pi \),
(c) \( \alpha = 0 \)
Fig. 4 Mapping defined by the w-integral (2.6)

Fig. 5 Stokes and anti-Stokes lines for $q^2(z)$ given by (6.1)
Fig. 6 (a) An example of the unperturbed geometry. $n = d = 2$.

(b) A general form of the perturbed geometry if mostly zeros $\pm z_4$ are perturbed.

Fig. 7 A typical configuration corresponding to an eigen-solution. $n = d = 1$.
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