A HIGH-ENERGY THEOREM
BY ALGEBRA OF CURRENTS

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The algebra of currents has been successfully applied to many processes. Ordinarily, they can be expressed as a kind of low-energy theorem such as the Kroll-Ruderman theorem\(^1\) or its equivalents\(^2\) in the weak interaction. Of course, there are other types of sum rules such as the ADLER-WEISBERGER relation\(^3\) which is not exactly equivalent to the low-energy theorem, although the latter could be rewritten\(^4\) as such. This note is to show that we may also obtain a high-energy theorem by the algebra of currents for the Compton scattering of the nucleon. This theorem may be used, in addition, to distinguish the validity of various models of elementary particles which have been emphasized elsewhere\(^5,6\)

For this purpose, we start from the following equation:

\[
\begin{align*}
\langle P(p') | \left[ \frac{d^2}{dx^2} \delta_\mu(x), \frac{d^2}{dy^2} \delta_\nu(y) \right] | P(p) \rangle &= \\
- \langle N(q') | \left[ \frac{d^2}{dx^2} \delta_\mu(x), \frac{d^2}{dy^2} \delta_\nu(y) \right] | N(q) \rangle &= \\
= \frac{2 G_A}{\epsilon} \frac{m}{p_0} \epsilon_{\mu \nu \lambda \sigma} \bar{u}(p) \gamma_\lambda \gamma_\sigma u(q) \delta^{(3)}(p-q') \tag{1}
\end{align*}
\]

where \(P(p)\) and \(N(p)\) refer to the proton and neutron states polarized in the direction of its momentum \(p\), respectively, and where \(\delta_\mu(x)\) represents the electric current density. Also, \(\epsilon\) is the constant which depends solely upon the model of elementary particles, i.e., \(\epsilon = +3\) for the fractionally charged quark model of Gell-Mann and Zweig\(^7\) and \(\epsilon = \pm 1\) for the integrally charged quark model of MAKI and HARA\(^8\), etc. The derivation of Eq. (1), based upon the equal-time commutation relation among electric currents, has been given elsewhere\(^5\) and we shall not repeat it except to emphasize the fact that the constant \(G_A\) appearing in the right-hand side (hereafter abbreviated as R.H.S.) of Eq. (1) is nothing but the axial-vector
\( \beta \)-decay renormalization constant in the exact SU(2) limit.

Using the standard procedure, Eq. (1) can be rewritten as

\[
\sum_{m} \frac{1}{2} \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle = - \frac{1}{2} \sum_{m} \left( \frac{2G_{A}}{\epsilon} \frac{m}{P_{0}} \right) \cdot \sum_{\lambda} \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle
\]

Furthermore, it may be transformed as

\[
\sum_{m} \frac{1}{2} \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle = - \frac{1}{2} \sum_{m} \left( \frac{2G_{A}}{\epsilon} \frac{m}{P_{0}} \right) \cdot \sum_{\lambda} \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} | N \rangle - \langle P| \hat{\sigma}^{(2)} \hat{\sigma}^{(2)} | N \rangle
\]

where \( M \) is the invariant mass of the intermediate state and \( m \) represents the nucleon mass. (Here, in this note, we neglect the mass difference between the proton and the neutron in conformity with the SU(2)-invariance.) For simplicity, let us introduce a four-vector \( \mathbf{k} \) by

\[
\mathbf{k} = \left( \frac{P_{0}^{2}}{2}, \mathbf{P}_{0} \right), \quad \mathbf{k}_{0} = \left( \frac{P_{0}^{2}}{2} + m^{2} \right) \frac{1}{P_{0}}
\]

Moreover, multiply \( G_{A} \langle \mathbf{P}_{\lambda} \rangle \mathbf{P}_{\lambda} \) to both sides of Eq. (3) to obtain the following:
where we have utilized the Dirac equation for \( u(p) \) together with the special form of the four-vector \( k \) given in Eq. (4).

Now, note that the Lorentz covariance enables us to write

\[
\frac{1}{\pi} \frac{2m}{e} \frac{\gamma_3}{\gamma_4} u(\gamma) \left( \gamma_4 \gamma_5 u(\gamma) \right)
\]

where \( u \) is given by

\[
u = -\frac{1}{m} (p,k)
\]

We emphasize for the later purpose that Eq. (6) is valid for arbitrary four-vector \( k \) which is not necessarily of the form Eq. (4). Especially, \( F_p (\nu, 0) \) can be expressed as

\[
F_p (\nu, 0) = -\frac{2m}{(2\pi)^2} \frac{1}{e^2} \frac{\nu}{\cos \theta} \left[ \sigma_p^{(\Upsilon)} (\nu, \cos \theta) - \sigma_p^{(-)} (\nu, \cos \theta) \right]
\]

where \( \sigma_p^{(\Upsilon)} (\nu, \cos \theta) \) is the total Compton scattering cross section of circularly polarized photons with the laboratory energy \( \nu \) and helicity \( \pm \) on the polarized proton at rest, and where \( \cos \theta \) is the cosine of the angle between the direction of the incident photon and the polarization vector of the proton. Note that the R.H.S. of Eq. (8) is really independent of \( \cos \theta \).
The derivation of Eq. (8) has been given elsewhere\(^5\) and we shall not go into the details.

Returning to our original discussion, Eqs. (5) and (6) give us the following:

\[
\frac{1}{\rho_0} \int_{\mathcal{M}_-}^{\mathcal{M}_+} \frac{dM}{(p^2 + M^2)^{\nu/2}} \left\{ \frac{F_p(\nu, k^2) - F_N(\nu, k^2)}{E} \right\} = \frac{1}{(2\pi)^4} \frac{2\pi A}{\varepsilon} \tag{9}
\]

where \( \nu \) and \( k^2 \) are now functions of \( M \) by Eqs. (4) and (7). Let \( p \to \infty \) in this expression. In this limit, it is obvious that

\[
\nu \to 0, \quad k^2 \to 0, \quad \nu \to \frac{1}{2m} (M^2 - m^2) \tag{10}
\]

Hence, if the order of the limit \( p \to \infty \) and of the integration in Eq. (9) can be interchanged, then one finds that the left-hand side (hereafter abbreviated as L.H.S.) of Eq. (9) tends to zero while the R.H.S. remains a constant. Therefore, this leads to a contradiction unless \( G_A \neq 0 \), which is ridiculous experimentally. This means that we cannot justify the interchange of the integral and the limit unless we abandon Eq. (1) and hence the simplest type of quark models which led to that. In this connection, one may remark that if one restricts oneself to only a few (finite) low-lying isobaric intermediate states, as is commonly done, then this interchange of the order is certainly justifiable and leads to a contradiction. This fact has been already noted elsewhere\(^6\) and has been used to support an argument that in such an approximation we should choose the reference system in which \( p = 0 \) rather than \( p = \infty \) in order to saturate the commutation relation.

At any rate, in order to obtain a consistent answer, we have to modify our procedure as follows. First, let us integrate in part the L.H.S. of Eq. (9); then one finds

\[
\frac{1}{\rho_0} \lim_{M \to \infty} M \cdot \left[ F_p(\nu, k^2) - F_N(\nu, k^2) \right] - \rho_0 \cdot \left[ F_p(\nu, 0) - F_N(\nu, 0) \right] -
\]
Note that for $M \to \infty$, one gets $\nu \to \infty$, $k^2 \to \infty$, and hence $F_\nu^3(\nu, k^2)$, for this limit is proportional to the difference of the Compton-scattering cross section, as in Eq. (8), of the infinitely heavy photon in the infinitely high-energy limit by the proton. If such a limit exists and if it is independent of the momentum $p$ (which may be plausible from the dimension argument), then this term will not contribute at all when we let $p \to \infty$ further. Probably this is reminiscent of the fact that the disconnected vacuum loop diagrams should not contribute at all, as we see from

$$\langle \int \frac{d^4 \mathbf{x}}{2^3} \langle \mathcal{J}_\nu(\mathbf{x}) \mathcal{J}_\nu(\mathbf{y}) \rangle \rangle = \text{const.} \, \frac{e q \nu}{A^2} \int \frac{d^3 \mathbf{x}}{2^3} < \nu, \mu, \nu > = 0$$

This is because these disconnected loop diagrams will appear as a contribution from the infinite mass $M \to \infty$ in the limit $p \to \infty$ as has already been noted by DASHEN and GELL-MANN in another connection.

At any rate, demanding also that the limit $p \to \infty$ can be interchanged with both integration and differentiation operations in the L.H.S. of Eq. (11), then one obtains in this way:

$$- \int \frac{d^4 \mathbf{x}}{2^3} \left[ F_\nu(\nu, 0) - F_\nu^0(\nu, 0) \right] - \int \frac{d^4 \mathbf{x}}{2^3} \left[ F_\nu(\nu, 0) - F_\nu^0(\nu, 0) \right] = - \frac{1}{2 \pi^2} \frac{2 \mathcal{G}_A}{\epsilon} (12)$$

where $\nu$ is now given by

$$\nu = \frac{1}{2m} \left( M^2 - m^2 \right) \quad (12')$$

Integrating the L.H.S. of Eq. (12) in part once more, one finally finds

$$\lim_{\nu \to \infty} \left[ F_\nu(\nu, 0) - F_\nu^0(\nu, 0) \right] = \frac{1}{2 \pi^2} \frac{2 \mathcal{G}_A}{\epsilon} \quad (13)$$

At this point, we may remark that we could probably justify our procedure
of interchanging the order of the limit and of the differentiation if we have a perturbation-theoretical representation of the form:

\[ F_P(\nu, k^2) = \int dx \int d\nu \int d\nu \frac{\sigma(\nu, \nu, t)}{[\nu, \nu + k^2 \nu + i]} \]  

(14)

since one can then easily verify

\[ \lim_{t \to \infty} \frac{2}{\nu M} \frac{1}{[\nu, \nu + k^2 \nu + i]} = \frac{2}{\nu M} \lim_{t \to \infty} \frac{1}{[\nu, \nu + k^2 \nu + i]} \]

where \( \nu \) and \( k^2 \) are functions of \( M \) by Eqs. (4) and (7) again. Now, returning to our problem, if such assumptions are justifiable, then Eq. (13) can be finally rewritten as a high-energy theorem:

\[ \lim_{\nu \to \infty} \frac{\nu}{\cos \theta} \left[ \sigma_P^{(\nu)}(\nu, \nu, \cos \theta) - \sigma_C^{(\nu)}(\nu, \nu, \cos \theta) - \sigma_N^{(\nu)}(\nu, \nu, \cos \theta) + \sigma_N^{(\nu)}(\nu, \nu, \cos \theta) \right] = \frac{G_\mu}{\nu} \cdot \epsilon^2 \]

(15)

for any fixed \( \cos \theta \). Hence, comparing both sides of Eq. (15), one can experimentally determine the value of \( \epsilon \) so as to enable us to distinguish the validity of various models of elementary particles.

Note that if Eq. (15) is correct, then the unsubtracted dispersion-relation technique given in the previous paper 5) will not be justifiable. Indeed, in that paper, we derived the following sum rule in that way:

\[ \int_{\mu(1+\epsilon \mu)}^{\infty} d\nu \frac{\nu}{\cos \theta} \left[ \sigma_P^{(\nu)}(\nu, \nu, \cos \theta) - \sigma_C^{(\nu)}(\nu, \nu, \cos \theta) - \sigma_N^{(\nu)}(\nu, \nu, \cos \theta) + \sigma_N^{(\nu)}(\nu, \nu, \cos \theta) \right] = \frac{G_\mu}{\nu} \cdot \epsilon^2 \]

(16)

whose right-hand side integral will be meaningless if Eq. (15) is valid. Of course, in the derivation of Eq. (15), some rather questionable procedures, such as interchanging the order of the limit and the integral, have been
freely used, so that Eq. (15) need not be correct after all. The final choice between Eqs. (15) and (16) must be ultimately decided by experimental verification of these equations.

Finally, we may remark that a similar high-energy theorem may be derived in the neutrino reaction on the nucleon. Assuming the quark model, and setting

$$\frac{\delta^2}{\delta \mu^2}(x) = \int d^2 x \, \overline{\phi}_\mu(x) \, \gamma_\mu(1 + \gamma_5) \, \gamma_5(0)$$

we obtain

$$\mathcal{L} \left( \int d^2 x \, \overline{\phi}_\mu(x), \int d^2 y \, \overline{\phi}_\nu(2) \right) + (\mu \leftrightarrow \nu) = - 4 \frac{e}{\mu} \int d^2 x \, \left[ \frac{1}{14(x)} \overline{\phi}_\mu(x) \gamma_5 \gamma_5(0) \right] \left( \mu, \nu \neq 4 \right)$$

(17)

Taking a matrix element of both sides with respect to one-proton state with momentum $p$ and using a similar trick as before, we get, for instance,

$$\lim_{\nu \to \infty} F_{11}(\nu, 0) = \frac{e}{2 \pi \mu} \frac{1}{m}$$

$$\lim_{\nu \to \infty} G_{11}(\nu, 0) = - \frac{e}{2 \pi \mu} \frac{1}{m} \, G_{11}$$

(18)

where $F_{1}(\nu, k^2)$ and $G_{1}(\nu, k^2)$ are defined by

$$\frac{\sum}{n} \left\{ <\phi_0| \overline{\phi}_{12}(0)| \nu > \right\} - \sum \left\{ <\phi_0| \overline{\phi}_{21}(0)| \nu > \right\} \left\{ <\phi_0| \overline{\phi}_{12}(0)| \nu > \right\} +$$

$$+ (\mu \leftrightarrow \nu) \, \gamma \, S^{(4)}(p + P_0 - P_n) =$$

$$= \frac{m}{P_0} \cdot \overline{\mu}(\nu) \cdot \frac{d}{d \omega} \left[ F_{1}(\nu, k^2) + (\lambda \lambda P_3) \gamma_5 \, G_{1}(\nu, k^2) \right]$$

$$+ (\lambda \lambda P_2 + \lambda \lambda P_0) \left[ F_{2}(\nu, k^2) + (\lambda \lambda k) \gamma_5 \, G_{2}(\nu, k^2) \right]$$

$$+ (\lambda \lambda P_2 + \lambda \lambda P_0) \left[ F_{3}(\nu, k^2) + (\lambda \lambda k) \gamma_5 \, G_{3}(\nu, k^2) \right]$$

$$+ \lambda \lambda P_0 \left[ F_{4}(\nu, k^2) + (\lambda \lambda k) \gamma_5 \, G_{4}(\nu, k^2) \right] \mu(\nu)$$

(19)
and \( \nu = -\frac{p_2}{m_N} \) as in Eq. (7). These quantities \( F_1(\nu, k^2) \) and \( G_2(\nu, k^2) \) can in principle be determined experimentally from the neutrino-reaction cross section on the proton.

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