GENERAL EIGENFUNCTION EXPANSIONS
AND GROUP REPRESENTATIONS

(Lecture Notes)

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In this paper I shall give some results concerning a general theory of
eigenfunction expansions for families of commuting self-adjoint oper-
ators in a separable Hilbert space $H$ ($§2$). As was remarked (in a special
case) by Gelfand and Kostyuenko in the case of an arbitrary spectrum
the (generalized) eigenvectors are elements of $\Phi'$ - the dual of $\Phi$,
where $\Phi$ is a dense linear subset of $H$ equipped with such nuclear topo-
logy that the identical imbedding

$$i: \Phi \to H$$

is continuous.

Thus we have always to do with a triplet of

$$\Phi \subseteq H \subseteq \Phi'$$  (0.1)

(locally convex) vector spaces, each dense in the following one, but only
$H$ is a Hilbert space. The most classical example of the triplet (0.1) is

$$\mathcal{D}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$$   (0.2)

where $\mathcal{D}(\mathbb{R}^n)$ is the L. Schwartz space of infinitely differentiable
functions with compact supports and $\mathcal{D}'(\mathbb{R}^n)$ the space of all distribu-
tions on $\mathbb{R}^n$.

In § 1 we recall the fundamental notion of the Hilbert-Schmidt (H-S)
mapping, and give the definition of a nuclear space by means of H-S
mappings, which, as was shown by A. Pietsch,is equivalent to the
original (much more involved) one by Grothendieck. A proof of the nuclear-
ity of $\mathcal{D}(\mathbb{R}^n)$ - or more generally $\mathcal{D}(\mathcal{M}_n)$ - follows.
In § 3 it is shown how the fundamental theorem of § 2 can be applied to obtain a decomposition of a unitary representation of a locally compact (l.c.) group into irreducible (or factor) representations. The first step in this direction has been taken by Mautner and V. Neumann. In § 4 it is shown that the irreducible spaces $H(\lambda)$ are common generalized eigenspaces of some self-adjoint operators constructed in a natural way.

§ 1. HILBERT-SCHMIDT (H-S) MAPS. NUCLEARITY

All Hilbert spaces considered here are separable, i.e., they have a countable orthonormal basis.

Let $H, K$ be Hilbert spaces with a scalar product $(\cdot | \cdot)$ and let $(e_j)_{j=1}^{\dim H}$ be an orthonormal basis of $H$. For any linear continuous mapping $A : H \to K$, the number $|A|^2 = \sum \|Ae_j\|^2$ is independent of the basis $(e_j)$.

Definition 1. If $|A| < \infty$ then the map $A : L(H_1, H_2)$ is called an H-S map.

Lemma 1. The identical operator $I : L(H, H)$ is H-S if and only if the dimension of $H$: $\dim H$ is finite.

Proof: $|I|^2 = \sum \|e_j\|^2 \cdot \dim H$.

Let $\Omega$ be an open subset of $\mathbb{R}^n$ or (more generally) a differentiable (separable) manifold of dimension $n$. $C_0^\infty(\Omega)$ is the set of all infinite differentiable functions on $\Omega$ with compact supports. Let $\Omega_j$ be a precompact open subset of $\Omega$. $H^p = H^p(\Omega_j)$ is the Hilbert space of measurable (complex) functions on $\Omega$ with the scalar product

$$(f, h)_p = \sum_{\Omega_j} D^\alpha f D^\alpha h,$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The derivations are understood in the sense of the theory of distributions.
Theorem 1. (cf. MAURIN [6])

1° If \( \mathcal{S}_j \) is a pre-compact domain of an \( n \)-dimensional differentiable manifold, and if \( m > n/2 \), \( \kappa > 0 \) then the canonical imbeddings

\[
H^m_0(\mathcal{S}_j) \to H^k_0(\mathcal{S}_j)
\]

are H-S.

2° If \( \mathcal{S}_j \) possesses a regular boundary then the imbeddings

\[
H^m_0(\mathcal{S}_j) \to H^k(\mathcal{S}_j)
\]

are H-S.

Corollary. Let \( \mathcal{S}_j \to \mathcal{S} \), i.e., \( \Omega_1 \subset \Omega_2 \subset \ldots \), be a sequence of pre-compact domains exhausting \( \mathcal{S} \) and let \( H^m_j := H^m(\mathcal{S}_j) \) (resp. \( H^m_0(\mathcal{S}_j) \)). Then to each \( m, j \in \mathbb{N} \) there exist such \( m', j' \in \mathbb{N} \) such that the imbedding

\[
H_j^{m'} \to H_j^m
\]

is of H-S type.

The spaces \( \mathcal{D}(\mathbb{R}) \), \( \mathcal{E}(\mathbb{R}) \), \( \mathcal{D}'(\mathbb{R}) \), \( \mathcal{E}'(\mathbb{R}) \), are defined in the following way:

Definition 2. \( \mathcal{E}(\mathbb{R}) \) is the set \( \mathcal{C}^\infty(\Omega) \) with (locally convex) topology defined by the norms \( \| \cdot \|_{m,j} \), \( m, j = 1, 2, \ldots \), where

\[
\| f \|_{m,j}^2 = \int_{\Omega_j} \sum_{|\alpha| \leq m} \left| D^\alpha f \right|^2,
\]

\( \Omega_j \to \mathcal{R} \).

We introduce now the notion of nuclearity.

Definition 3. A locally-convex vector space \( \mathcal{F} \) is nuclear if there exists an equivalent system of seminorms \( \| \cdot \|_\beta \), \( \beta \in \mathcal{B} \), such that:

1° For every \( \beta \in \mathcal{B} \) is the (normed) quotient space

\[
\mathcal{R}_\beta := \mathcal{F} / \mathcal{N}_\beta,
\]

where \( \mathcal{N}_\beta = \{ \phi \in \mathcal{F} : \| \phi \|_\beta = 0 \} \)

a prehilbert space.

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For each \( \beta \in \mathcal{B} \) there exists such \( \alpha \in \mathcal{B} \) that the imbedding 
\[
\mathcal{H}_\beta \rightarrow \mathcal{H}_\alpha
\] is H-S.

**Proposition 1.** A Hilbert space \( H \) is nuclear if and only if \( \dim H < \infty \).

**Proof:** The topology of \( H \) is given by one norm \( \| \cdot \| \) only. Hence \( N = \{0\} \) and \( \mathcal{H} = H \). In virtue of \( 2^0 \) the identical imbedding \( H \rightarrow H \) is H-S. Thus Proposition 1 follows from Lemma 1.

**Proposition 2.** As was proved by Grothendieck, nuclearity is preserved by the most important operations on locally-convex vector spaces:

a) taking the quotient by a closed subspace \( \mathcal{N} \subset \mathcal{F} \) (i.e., if \( \mathcal{F} \) is nuclear) then \( \mathcal{F}/\mathcal{N} \) is nuclear too;

b) the completion \( \mathcal{F} \) of a nuclear \( \mathcal{F} \) is nuclear;

c) the strict inductive limit \( \mathcal{F} = \lim_{\rightarrow} \mathcal{F}_i \) of nuclear spaces \( \mathcal{F}_i \) is nuclear, i.e., \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \), \( \mathcal{F} = \bigcup \mathcal{F}_i \), and if \( \mathcal{F}_m \subset \mathcal{F}_n \) then the topology induced on \( \mathcal{F}_m \) by the topology of \( \mathcal{F}_n \) is identical to that of \( \mathcal{F}_m \). The topology of \( \mathcal{F} \) is - by definition - the finest topology on \( \mathcal{F} \) such that for each \( i \) the imbedding \( \mathcal{F}_i \rightarrow \mathcal{F} \) is continuous;

d) the (topological) direct sum \( \bigoplus \mathcal{F}_i \) of nuclear spaces is nuclear.

**Definition 4.** \( \mathcal{D}(\mathcal{L}) \) is the open space \( C^\infty(\mathcal{L}) \) provided with the following topology: \( \mathcal{F}_j := \{ \varphi \in C^\infty(\mathcal{L}) : \text{the support of } \varphi \subset \mathcal{L}_j \} \), \( \mathcal{L}_j \not\supset \mathcal{L} \). \( \mathcal{F}_j \) has the (nuclear) topology of the space \( C(\mathcal{L}_j) \), \( \mathcal{L}_j \)-precompact. \( \mathcal{D}(\mathcal{L}) = \lim_{\rightarrow} \mathcal{F}_i \).

We can give now a simple proof of the following fundamental result of Grothendieck.

**Theorem 2.** (Grothendieck)

1° \( C(\mathcal{L}) \) is nuclear;

2° \( \mathcal{D}(\mathcal{L}) \) is nuclear.
Proof: 1° follows immediately from Definition 1 and the Corollary, 2° follows from 1° and Proposition 2c.

If $\mathcal{F}$ is a locally-convex vector space (l.c.v.s.) then $\mathcal{F}'$ denotes the dual space of $\mathcal{F}$, i.e., $\mathcal{F}' = \mathcal{L}(\mathcal{F}, \mathcal{C})$, i.e., the elements of $\mathcal{F}'$ are linear continuous functionals on $\mathcal{F}$.

**Definition 5.** Elements of $\mathcal{D}'(\mathcal{R})$ and $\mathcal{C}'(\mathcal{R})$ are called distributions on $\Omega$ and distributions with compact supports in $\Omega$ respectively. It can be proved that $\mathcal{D}'$ and $\mathcal{C}'$ are nuclear.

$\S 2. \text{GENERAL EIGENFUNCTION EXPANSIONS}$

Let us recall rapidly the notion of a direct integral of Hilbert spaces. Let $(\Lambda, \mathcal{C})$ be a locally-compact (l.c.) separable measure space. For each $\lambda \in \Lambda$ there exists a Hilbert space $(\hat{H}(\lambda), (\cdot, \cdot)_\lambda)$. Let $k(\cdot)$ be a vector field on $\Lambda$, i.e., $k(\cdot)$ is a map: $\Lambda \ni \lambda \rightarrow k(\lambda) \in \hat{H}(\lambda)$.

A countable family of vector fields is called a fundamental family if:

1° all functions $\Lambda \ni \lambda \rightarrow (k_i(\lambda), k_j(\lambda))_\lambda \in \mathcal{C}$ are $\sigma$-measurable for all $i, j = 1, 2, \ldots$

2° for each $\lambda \in \Lambda$ the set $\{k_j(\lambda) : j = 1, 2, \ldots\}$ spans the space $\hat{H}(\lambda)$. A vector field $k(\cdot)$ is $\sigma$-measurable if each function $\lambda \rightarrow (k(\lambda), k_j(\lambda))_\lambda, j = 1, 2, \ldots$ is $\sigma$-measurable.

**Definition 6.** The direct integral $\hat{H}$ of Hilbert spaces $\hat{H}(\lambda), \lambda \in (\Lambda, \sigma)$ is the Hilbert space of equivalence classes of $\sigma$-measurable vector fields $k(\cdot)$ for which

$$\int \|k(\cdot)\|^2 d\sigma < \infty.$$  

The scalar product on $\hat{H}$ is defined by

$$\langle u(\cdot), v(\cdot) \rangle := \int \langle u(\lambda), v(\lambda) \rangle_\lambda d\sigma.$$
\[ H \] is denoted by \( \int \hat{H}(\lambda) d\sigma \).

### Diagonal and Decomposable Operators

An operator field

\[ \lambda \rightarrow A(\lambda) \in L(\hat{H}(\lambda)) \]

is measurable if all functions

\[ \lambda \rightarrow (A(\lambda)h_j(\lambda) / h_i^* (\lambda)) \]

are \( \sigma \)-measurable where \( (h_j(\cdot)) \) is the fundamental family. A measurable operator field \( (A(\cdot)) \) is called a decomposable operator in the Hilbert space \( \hat{H} = \int \hat{H}(\lambda) d\sigma \).

The simplest decomposable operators are diagonal operators: if \( f \in L^\infty(\lambda, \sigma) \), i.e., if \( f \) is an essentially bounded \( \sigma \)-measurable function then

\[ \lambda \rightarrow T_f(\lambda) := f(\lambda) I(\lambda) \]

where \( I(\lambda) \) is the identity in \( \hat{H}(\lambda) \) and is called diagonal operator. Of course, one could introduce diagonal operators

\[ \lambda \rightarrow g(\lambda) I(\lambda) \]

with unbounded measurable \( g \), but since such operators are not bounded their domain is not the whole space.

\[ \int \hat{H}(\lambda) d\sigma \]

If \( \mathcal{A} \) is a weakly closed algebra in \( L(\hat{H}) \) then \( \mathcal{A}' \) denotes the commutant of \( \mathcal{A} \). It is the set of all bounded operators in \( \hat{H} \) commuting with every operator in \( \mathcal{A} \). A weakly closed \( * \)-algebra in \( L(\hat{H}) \) is called a von Neumann \( (v, N) \) algebra.

The following fundamental result due to von Neumann gives a nice characterization of algebras of diagonal and decomposable operators.

**Lemma 2. (v. Neumann)**

1° The algebra \( \mathcal{D} \) of diagonal operators is a commutative \( v, N \) algebra.

2° The map \( L^\infty(\sigma') \ni f \rightarrow T_f \in \mathcal{D} \) is a topological isomorphism of the \( * \)-algebra \( L^\infty(\sigma) \) onto \( \mathcal{D} \) (both algebras are provided with weak topologies):

\[ T_f g^* = T_f T_g^* \]

etc.

3° The commutant \( \mathcal{D}' \) of the algebra \( \mathcal{D} \) of diagonal operators is the \( v, N \) algebra \( \mathcal{R} \) of (all) decomposable operators: \( \mathcal{D}' = \mathcal{R} \) and the commutant \( \mathcal{R}' \) of \( \mathcal{R} \) is the algebra \( \mathcal{D} \):

\[ \mathcal{R}' = \mathcal{D}, \quad \mathcal{D}' = \mathcal{R} \]
Thus an operator is decomposable if and only if it is commuting with all diagonal operators.

In the spectral theory of unbounded self-adjoint (s.a.) operators the notion of strong commutativity is of paramount importance.

Definition 7. Two s.a. operators $A_1$, $A_2$ are strongly commuting if their spectral families $\{ E_k(\lambda) \}$, $\kappa = 1, 2$ are commuting.

This is equivalent to the commutativity of Cayley transforms of $A_1$ and $A_2$: $(A_k + iI)(A_k - iI)^{-1}$, $\kappa = 1, 2$.

We can now formulate the principal result of this section.

Theorem 3. (Nuclear Spectral Theorem cf MAURIN [2, 6])

Let $A_\beta$, $\lambda \in B$ be a denumerable family of strongly commuting s.a. operators having a common invariant domain $D$ dense in $H$:

1. there exists such a nuclear space that $\Phi \subset H \subset \Phi'$ is a Gelfand triplet;

2. each $A_\beta$ maps $\Phi$ continuously into $\Phi'$: $A_\beta : \Phi \rightarrow \Phi'$;

3. there exists a direct integral $H = \int_\lambda H(\lambda) d\nu$ and such a Hilbert isomorphism $\mathcal{F} : H \rightarrow H$ that for $\sigma$-almost all $\lambda$, $H(\lambda) \subset \Phi'$ and $H(\lambda)$ are common generalized eigenspaces of all $A_\beta$: if $\phi (\lambda) \in H(\lambda)$, then

$$<A_\beta \phi, e(\lambda)> = \hat{A}_\beta (\lambda) <\phi, e(\lambda)> \quad (2.1)$$

identically for $\phi \in \Phi$, where $\{ \hat{A}_\beta (\lambda) : \lambda \in \Lambda \}$ is the spectrum of $A_\beta$.

(2.1) can be written shorter $A'_\beta e(\lambda) = \hat{A}_\beta (\lambda) e(\lambda)$, where $A'_\beta$ is the natural extension of $A_\beta$ given by the identity

$$<A_\beta \phi, \psi'> = <\phi, A'_\beta \psi'> \quad (2.1')$$
or plainly $A'_1 \geq A'_2 = A'_3$.

Taking in each $H(\lambda)$ the orthonormal basis $e_k(\lambda), \ k = 1, \ldots, \dim H(\lambda)$, we obtain the generalized Fourier-Plancherel equation

$$
(\phi, \psi) = \int \sum_{k=1}^{\dim H(\lambda)} <\phi, e_k(\lambda)> <\psi, e_k(\lambda)> d\sigma(\lambda), \quad \phi, \psi \in \mathcal{F}.
$$

4° the isomorphism $\mathcal{F}: H \rightarrow \hat{H}$ is given by

$$
H \ni \phi \mapsto (\phi(\lambda)) = (\hat{\phi}(\lambda)) \in \hat{H},
$$

where

$$
\hat{\phi}(\lambda) = \sum_{k=1}^{\dim H(\lambda)} <\phi, e_k(\lambda)> e_k(\lambda).
$$

5° the spectral synthesis of $\phi \in \mathcal{F}$ is given by

$$
\phi = \int \hat{\phi}(\lambda) d\sigma, \quad (2.4)
$$

where $\hat{\phi}$ is given by (2.3)

In view of the applications to the theory of unitary group representations the following corollary is of great importance (cf § 3.). We can consider $\hat{H}$ as a subspace of $\hat{\mathcal{F}}$ so we have the following:

Corollary. Let $\mathcal{U}$ be such a unitary decomposable operator in $\hat{H} \subset \hat{\mathcal{F}}$ that $\mathcal{U}, \mathcal{U}^*: \hat{\mathcal{F}} \rightarrow \hat{\mathcal{F}}$ are continuous (cf Remark 2° below); and let

$$
\tilde{\mathcal{U}} := (\mathcal{U}^*)' \quad \text{be the natural extension of } \mathcal{U} \quad \text{to } \hat{\mathcal{F}} \text{ (precisely, } \tilde{\mathcal{U}} := (\mathcal{U}^* / \mathcal{F})' \text{) and let } \mathcal{U}(\lambda) := \tilde{\mathcal{U}} | H(\lambda),
$$

then

$$
\mathcal{U} = \int \mathcal{U}(\lambda) d\sigma.
$$

(This integral is not a symbolic way of writing a decomposable operator but a genuine strong integral).
Remarks.

1°: The isomorphism \( \mathcal{F} \) diagonalizes simultaneously all operators \( A_\lambda \), i.e.

\[
(\mathcal{F} A_\lambda \mathcal{F}^{-1})(\lambda) = \hat{A}_\lambda(\lambda) \hat{I}(\lambda), \quad \lambda \in \Lambda
\]

and is called the \((\hat{A}_\lambda)\)-Fourier transform. We obtain the classical Fourier transformation taking \( H = L^2(\mathbb{R}^n), A_k = -i \frac{\partial}{\partial x_k}, \quad \phi = \mathcal{D}(\mathbb{R}^n) \) (on \( \mathcal{S}(\mathbb{R}^n) \)).

Then \( \Lambda = \mathbb{R}^n \cong \mathbb{R}^n \), and \( e(\lambda) = e(\lambda_j) \)

where \( e(\lambda, x) = \exp(-i(\lambda_1 x_1 + \ldots + \lambda_n x_n)) \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \), \( x = (x_1, \ldots, x_n) \).

2°: von Neumann proved "only" that to every abelian \( * \rightarrow v. N.-\)algebra there exists an isomorphism \( \mathcal{F} : H \rightarrow \hat{H} \) diagonalizing simultaneously all operators \( A_\lambda \in \mathcal{A} \), i.e., for which \((\ast)\) is satisfied for \( \sigma \)-almost all \( \lambda \in \Lambda \). In this way we can identify \( \mathcal{A} \) with the algebra \( \mathcal{H} \) of diagonal operators and hence the commutant \( \mathcal{H}' \) of \( \mathcal{A} \) with the algebra of decomposable operators.

§ 3. UNITARY REPRESENTATIONS OF LOCALLY-COMPACT GROUPS

Let \( G \) be a separable l.c. unimodular group, i.e., a group with a bi-invariant Haar measure. On every l.c. group one can introduce nuclear function spaces \( \mathcal{E}(G), \mathcal{D}(G) \) respectively which in the case of a Lie group \( G \) are simply the (infinitely) differentiable functions and regular functions with compact supports respectively (See Appendix). The dual spaces \( \mathcal{D}'(G) \) resp. \( \mathcal{E}'(G) \) are called spaces of distributions resp. distributions with compact supports in \( G \). Clearly we have inclusions

\[
\mathcal{D}(G) \subset L^2(G) \subset \mathcal{D}(G), \quad \mathcal{D}(G) \subset \mathcal{E}(G) \subset \mathcal{D}'(G).
\]

One can define in the usual way the convolution of distributions

\[
T \ast S, \quad T \in \mathcal{D}', \quad S \in \mathcal{E}.
\]

Since in the case of a Lie group the invariant differential operators
can be given by distributions with support in \( \mathfrak{e} \):

\[ T \varphi = T \ast \varphi, \quad T \in \mathfrak{E}'. \]

In that way, the invariant enveloping algebra of any l.c. group \( \mathcal{G} \) can be defined as an algebra \( \mathfrak{E}'_e \) of distributions with carriers in \( \{ e \} \), the multiplication in \( \mathfrak{E}'_e \) being the convolution. We have at our disposal a much bigger algebra: the whole space \( \mathfrak{E}'(\mathcal{G}) \). The unit in \( \mathfrak{E}' \) is the Dirac measure \( \delta_e \). This algebra is an involution algebra.

**Definition.** Let \( \varphi(g) := \varphi(g^r) \), then for each \( T \in \mathcal{D}'(\mathcal{G}) \), \( T^+ \in \mathcal{D}(\mathcal{G}) \) is defined by

\[ \langle \varphi, T^+ \rangle = \langle \overline{\varphi^+}, T \rangle \quad (3.1) \]

Let \( \mathcal{U} = (\mathcal{U}, \mathcal{H}) \) be a unitary representation of a l.c. group \( \mathcal{G} : \mathcal{G} \ni g \rightarrow \mathcal{U}_g \in \mathcal{L}(\mathcal{H}) \). Our aim is to construct to each unitary representation \((\mathcal{U}, \mathcal{H})\) a Gel'fand triplet \( \mathcal{F} \subset \mathcal{H} \subset \mathcal{F}' \), and a set \((\mathcal{U}(\mathcal{T}))\) of operators in \( \mathcal{H} \) such that

1° \( \mathcal{U}_g \) acts continuously in \( \mathcal{F} : \mathcal{U}_g : \mathcal{F} \rightarrow \mathcal{F} \);

2° the operators \( \mathcal{U}(\mathcal{T}) \) are mapping continuously \( \mathcal{F} \) into itself

\[ \mathcal{U}(\mathcal{T}) : \mathcal{F} \rightarrow \mathcal{F} ; \]

3° the operators \( \mathcal{U}(\mathcal{T}) \) are symmetric: \( \mathcal{U}(\mathcal{T}) \subset \mathcal{U}(\mathcal{T})^\ast \) and their closures are self-adjoint;

4° \( \mathcal{U}(\mathcal{T}_k) \) are strongly commuting, \( k = 1, 2 \);

5° \( \mathcal{U}(\mathcal{T}) \) are commuting with the operators \( \mathcal{U}_g, g \in \mathcal{G} \).

Applying our fundamental Theorem 3 we obtain thus a decomposition of the representation \((\mathcal{U}, \mathcal{H})\) into a direct integral of irreducible representations \( \int (\mathcal{U}(\lambda), \mathcal{H}(\lambda)) d\lambda \) and the irreducible spaces \( \mathcal{H}(\lambda) \) are common.
(generalized) eigenspaces of the operators $U(T)$.

The Gårding space $H_G$ of the representation $(U, H)$ is defined as the set of linear combinations of vectors $U(\varphi_i)\mathbf{h}_k = \int_G \varphi_i(g) U_g h_k dg$, $\varphi_i \in \mathcal{D}(G)$, $\mathbf{h}_k \in H$. Plainly $H_G$ is dense in $H$.

We can now transplant the representation $U$ to a *-representation of the algebra $\mathcal{B}'(G)$, whence to the enveloping Lie algebra $\mathcal{G}'$ of the group $G$ in the following way.

**Theorem 4.** (cf MAURIN [4])

For each $T \in \mathcal{B}'(G)$ and $\alpha \in H_G$

$$U(T)\alpha := \int U_\alpha \alpha^T(dg)$$

(3.2)

or, more precisely, for every $\ell \in H_G^\prime$, $\langle U(T)\alpha, \ell \rangle := \int \langle U_\alpha \alpha^T, \ell \rangle dg$.

Then 1° $\mathcal{B}' \to U(T)$ is a *-representation of the algebra $\mathcal{B}'(G)$ by operators in $H_G$: $U(T^S)a = U(T)U(S)a$, $a \in H_G$; $U(T^*) = U(T)^*$.

Thus for symmetric operators $T = T^\prime$ the operators are symmetric.

2° The representation $(U(T))$ is an extension of the representation $(U, H)$ since $U(\mathcal{G}) = U_G$.

Let $Z(\mathcal{G})$ denote the centre of an algebra $\mathcal{A}$

**Theorem 5.** The construction of a Gel'fand triplet adapted to $(U, H)$.

Let $H_0 \subseteq H$ be any Gel'fand pair we can provide the set of linear combinations $\Phi := \sum a_i U_\alpha \mathbf{h}_i$, $\varphi_i \in \mathcal{D}(G)$, $\mathbf{h}_i \in H$

with such nuclear topology that $\Phi \subseteq H \subseteq \bar{\Phi}'$ is such a Gel'fand triplet with all desired properties 1° - 5°.

**Proof:** Let us prove, for example, that...
\( U_g, \mathcal{U}(T) : \Phi \rightarrow \Phi : \)

\( \mathcal{U}(T) \Sigma \alpha_i^j \mathcal{U}(\varphi^i) \mathcal{H}_i = \sum \alpha_i^j \mathcal{U}(T) \mathcal{U}(\varphi^i) \mathcal{H}_i = \sum \alpha_i \mathcal{U}(T \times \varphi^i) \mathcal{H}_i ; \)

taking \( T = \delta_g \) we obtain by virtue of Theorem 4.

\( U_g \sum \alpha_i \mathcal{U}(\varphi^i) \mathcal{H}_i = \mathcal{U}(\delta_g) \sum \alpha_i \mathcal{U}(\varphi^i) \mathcal{H}_i \in \Phi . \)

From the continuity of the maps \( \mathcal{D} \ni \varphi \rightarrow T \times \varphi \in \mathcal{D} \) follows the continuity of the operators \( \mathcal{U}(T) : \Phi \rightarrow \Phi . \)

In order to obtain a decomposition of \( (\mathcal{U}, \mathcal{H}) \) into a direct integral of irreducible representations \( \int_G (\mathcal{U}(\lambda), \mathcal{H}(\lambda)) d\sigma \) we take any maximal abelian algebra in \( \mathcal{U} \) - the commutant of the v. N. algebra generated by the operators \( U_g, g \in G \), which contains all spectral families of \( \mathcal{U}(T), T = T^t \in \mathcal{Z}(\mathcal{E}_x) \) or \( T \in \mathcal{Z}(\mathcal{E}(G)) \).

Then the irreducible spaces \( \mathcal{H}(\lambda) \in \Phi ' \) can be considered as common generalized eigenspaces of the operators \( \mathcal{U}(T) \).

The most important models of unitary representations are provided by shift-operators on a homogeneous space \( X = G/\mathcal{H} \) \( (U_g f)(x) : = f(gx) \) (where \( f \) are functions on the space \( X \)). If we take an invariant measure \( \mu \) on \( X \), we take \( \mathcal{H} = L^2(X, \mu) \), our construction leads to distributions on the space \( X : \Phi ' \ni \mathcal{D}'(X) \). The operators \( \mathcal{U}(T), T \in \mathcal{E}_x \) are then invariant "differential" operators (they are differential operators in case of differentiable homogeneous space \( X \) and the irreducible spaces \( \mathcal{H}(\lambda) \) are eigenspaces of the "differential operators" \( \mathcal{U}(T) \). We want to stress once more that our whole theory does not suppose either the differentiability or the compactness of the group. If \( G \) is compact and we take for \( T \in \mathcal{Z}(\mathcal{E}(G)) \) a function, we obtain integral convolution operators from the theory of Weyl and Peter.

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APPENDIX

Distributions on l.c. groups

1. The spaces $\mathcal{D}(G)$, $\mathcal{F}(G)$, $\mathcal{E}(G)$, $\mathcal{F}^0(G)$. Let $G$ be a separable l.c. group. We define here only the distribution spaces for the most important limit cases: a) $G$-connected, b) $G$-discrete.

a) Let $G$ be a connected separable group. A famous theorem of Yamabe asserts that to each neighbourhood $\mathcal{O}_k$ of the neutral element $e$ of $G$ there exists such a (compact) normal subgroup $n_k \subset G$ that $G_k := G/n_k$ is a Lie group. We know (cf §1) that the spaces $\mathcal{D}(G_k)$ are nuclear. Taking $n_k \supset n_k$ we can identify $\mathcal{D}(G_k)$ with a subspace $\mathcal{D}(G_k^0)$. Identifying $\mathcal{D}(G_k)$ with $\mathcal{D}(G_k^0)$ - a space of continuous functions on $G$ which are constant on such left coset $G / n_k$) we obtain an increasing sequence of nuclear spaces $\mathcal{D}(G) = \lim_{\rightarrow} \mathcal{D}(G_k)$. Let $\mathcal{D}(G) = \mathcal{D}(G) / n_k$. Then $\mathcal{D}(G)$ (as a strict inductive limit of nuclear spaces) is nuclear (cf §1).

b) $G$ is discrete. Let $K_j \uparrow G$ be an increasing sequence of compact subsets of $G$ exhausting $G$. Since the sets $K_j$ are finite the spaces $\mathcal{D}(K_j)$ are of finite dimension and hence they are nuclear. $\mathcal{D}$ is now defined as a strict inductive limit of $\mathcal{D}(K_j)$: $\mathcal{D}(G) = \lim_{\rightarrow} \mathcal{D}(K_j)$.

2. Nuclear topology on $\mathcal{D}$ can be introduced in the following way.
Let $H_o \subset H$ be a Gel'fand pair, whence the (topological) product $\mathcal{D}(G) \times H_o$ is nuclear. Consider the mapping $T : \mathcal{D}(G) \times H_o \rightarrow \mathcal{E}(G) \times H_o$ defined by

$$T(\varphi, h_o) = U(\varphi) h_o \in \mathcal{E}(G) \times H.$$ 

Let $N$ be the kernel of $T : N = \mathcal{T}^{-1}(0)$. Then $T$ induces a linear isomorphism $[T]$ of $(\mathcal{D}(G) \times H_o) / N$ onto $\mathcal{D}$. Since $N$ is closed, the space $(\mathcal{D}(G) \times H_o) / N$ is nuclear (Proposition 2a) carrying by $[T]$ that nuclear topology on $\mathcal{D}$ - the image of $[T]$ - we obtain the desired nuclear space $\mathcal{D}$.

Let $r_k : G \rightarrow G / n_k = G_k$ where $r_k(\varphi) = \varphi n_k$, then $r_k^* : C(G_k) \rightarrow C(G)$ is given by $r_k^* : f \rightarrow f o r_k \in C(G)$. Let $\mathcal{D}(G_k) = \mathcal{D}(G) / n_k$. Then $\mathcal{D}(G)$ is a nuclear space (cf §1).
BIBLIOGRAPHY AND REFERENCES

A. Books

All proofs of the theorems mentioned can be found in the author's monograph: "General Eigenfunction Expansions and Unitary Representations of Topological Groups", Warsaw (Monografie Matematyczne), in the Press.

Von Neumann and Mautner Theorems are presented in monographs of J. Dixmier (Gauthier-Villars, Paris, 1957 and 1964) and Naimark "Normed Rings" (Noordhof, Groeningen, 1964).

B. Papers


