COMPLETENESS AND DISPERSION RELATIONS IN THE COMPLEX ANGULAR MOMENTUM PLANE

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ABSTRACT

The set of Jost's solutions $(f(\lambda, -k, r))$ of the Schrödinger equation in the complex angular momentum plane is considered here as the set of eigenfunctions of a non-Hermitian operator. The completeness of the set $\{f(\lambda, -k, r)/r\}$ is established, and the orthogonality and completeness relations are explicitly given. It is further shown that, in contrast to the corresponding situation in the k-plane, the completeness relation does not imply here a dispersion relation for the scattering matrix.
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1 INTRODUCTION

It is known that dispersion relations for the S-matrix in the k-plane ($k^2 = E$), follow from the completeness of the set of the physical wave functions for non-relativistic potential scattering$^1$, the completeness of this set being here a consequence of the fact that the Hamiltonian is a bounded Hermitian operator. It is also known that from these dispersion relations the analytical structure of the S-matrix can be derived.

On the other hand, this analytical structure has also been shown to be a direct consequence of a suitable formulation of the causality principle in non-relativistic quantum mechanics$^2$; of course, once this structure has been obtained the dispersion relations follow at once. One thus obtains the same result using two vastly different approaches, and therefore it becomes difficult to single out the physical basis of these dispersion relations.

In this note we study the role played by the completeness relation, and for this purpose we use the completeness relation satisfied by the Jost's solutions in the complex angular momentum plane ($\lambda$-plane), a result which has recently been obtained$^3$ and of which an alternative proof is given here. In our view, the interest of this example stems from the fact that the set of functions we consider here is not a set of eigenfunctions of a quantum-mechanical operator (or even, properly, the analytical continuation of it). This remark must be understood in the following sense. For the angular momentum operator (rather, its third component) to be Hermitian, it is necessary in quantum mechanics to restrict it to the space of functions whose values are equal at both ends of the range of integration. Therefore, when we take $\lambda$ to be a
continuous index (even if we keep it real) we are already outside the framework of quantum mechanics.

In the next section we rewrite the radial Schrödinger equation in the form of an eigenvalue equation, namely,

\[ K \phi = -\lambda^2 \phi, \]  

(1)

where the operator \( K \) turns out to be non-Hermitian. The completeness of the set of eigenfunctions of this class of operators have been recently studied by FONDA et al. \(^4\), who have stressed the relevance of this problem to the theory of nuclear reactions, where effective non-Hermitian Hamiltonians occur. We here apply their procedure to the spectral decomposition of the operator \( K \) and obtain in this way a completeness relation for the Jost's solutions of Eq. (1), which is exactly the expression previously derived by BURDET et al. \(^3\) with a straightforward application of the method used by NEWTON \(^5\) (following JOST and KHON \(^6\)) to the solution of the corresponding problem in the \( k \)-plane. We further prove here the orthogonality of this set of functions.

In Section 3 we prove that no dispersion relations for the \( S \)-matrix follow from this completeness relation, which instead yields dispersion relations for a different function, a result rather unexpected in view of the corresponding situation in the \( k \)-plane.

Then, as we know, the dispersion relations in the \( \lambda \)-plane \(^7\), it is natural to ask whether or not they follow from some basic physical principle, as it can be argued in the \( k \)-plane. We do not deal with this question in the present paper, but our result is relevant to it, as it shows that in this case, if there existed such a physical principle, it would not be directly related to completeness.
2. COMPLETENESS OF THE SET OF JOST'S SOLUTIONS IN THE \( \lambda \)-PLANE

The radial Schrödinger equation is written as (Eq. (1))

\[
K \phi = -\lambda^2 \phi ,
\]

where for instance, the solution regular at the origin would be \( \phi = r \psi \), \( \psi \) being the Schrödinger wave function for physical \( \lambda \) and \( k \). In this equation \( K \) is the non-Hermitian operator

\[
K = -r^2 \frac{d^2}{dr^2} + r^2 \sqrt{v(r)} - k^2 r^2 - \frac{i}{4}
\]

and we assume the central potential \( V(r) \) to be real.

We use the notation of BOTTINO et al. \(^3\) and their results for a Yukawa-like potential. In the coordinate representation, the resolvent \( G(z) \) defined by

\[
G(z) = \frac{1}{z - K} \quad , \quad z = -\lambda^2
\]

yields the Green's function

\[
G(z, k, r, r') = \frac{-\varphi(\sqrt{-z}, k, r) \varphi(\sqrt{-z}, -k, r')}{2 r r' \varphi(\sqrt{-z}, -k)}
\]

which has single poles at the zeros of the Jost's function appearing in the denominator and a branch cut starting at \( z = 0 \) and running along the positive real axis. Here \( \varphi(\lambda, k, r) \) and \( f(\lambda, k, r) \) are of course the well-known solutions of the Schrödinger equation defined by

\[
\varphi(\lambda, k, r) \propto r^{\lambda + \frac{1}{2}} \quad , \quad r \to 0
\]
and

\[ f(\lambda, k, r) \sim e^{-ikr} \quad r \to \infty \]  \hspace{1cm} (6)

Further, recalling the asymptotic expressions

\[ \Psi(\lambda, k, r) \sim \Psi_0(\lambda, k, r) \sim r^{\lambda + \frac{1}{2}} \quad |\lambda| \to \infty \]  \hspace{1cm} (7)

where \( \Psi_0(\lambda, k, r) \) is the "free" (no potential) regular solution,

\[ f(\lambda-k) \sim f_0(\lambda-k) \sim \frac{2(\frac{\lambda}{ek})^\lambda}{\sqrt{\lambda k}} e^{i \frac{\pi}{4} \lambda} e^{-i \frac{\pi}{4}} \quad |\lambda| \to \infty \]  \hspace{1cm} (8)

and

\[ f(\lambda-k, r) \sim \sqrt{\frac{kr}{\lambda}} e^{i \frac{\pi}{4} \lambda} \left[ \frac{2(\frac{\lambda}{ek})^\lambda}{\sqrt{\lambda k}} \right] e^{-i \frac{\pi}{4}} \lambda - \left( \frac{ek}{\lambda} \right)^\lambda e^{i \frac{\pi}{4} \lambda} \]  \hspace{1cm} (9)

it is immediately verified that \( G(z, k, r, r') \) satisfies the conditions under which FONDA et al. have shown that the following relation holds

\[ \sum_{\phi} \langle \psi | P_{\phi} | \psi \rangle + \int d\zeta \langle \psi | P(\zeta) | \psi \rangle = \langle \psi | \psi \rangle \]  \hspace{1cm} (10)

Here we take the kets \( |\psi\rangle \) and \( |\psi\rangle \) to be arbitrary vectors of the physical Hilbert space.

In Eq. (10) \( P_{\phi} \) is given by

\[ P_{\phi} = \int (z + \lambda^2) G(z, k, r, r') \]  \hspace{1cm} (11)
and, by definition, \( P(z) \) satisfies

\[
-2\pi \lambda \mathcal{P}(z) = \lim_{\varepsilon \to -\lambda^2 + i0} G(z) - \lim_{\varepsilon \to -\lambda^2 - i0} G(z) \quad \text{(12)}
\]

Now, from Eqs. (4) and (11) we obtain

\[
\mathcal{P}_f = \frac{-\Psi(\lambda_p, k, \xi) f(\lambda_p, -k, \rho)}{\partial f(\lambda_p, -k) \bigg|_{\lambda = \lambda_p}} = \frac{\frac{f(\lambda_p, -k, \rho)}{\partial (-\lambda^2)}}{\lambda_p} \overline{M}^2(\lambda_p, k) \quad \text{(13)}
\]

where

\[
\overline{M}^2(\lambda_p, k) = \int_0^\infty \frac{f^2(\lambda_p, -k, r)}{\lambda^2} \, dr \quad \text{(14)}
\]

the \( \lambda_p \) being the zeros of the Jost's function \( f(\lambda, -k) \) in the half-plane \( \text{Re} \lambda > 0 \).

In the same way, Eqs. (4) and (12) lead to

\[
-2\pi \lambda \mathcal{P}(z) = \frac{-\Psi(-\sqrt{z}, k, \xi) f(-\sqrt{z}, -k, \rho)}{\partial f(-\sqrt{z}, -k)} - \frac{-\Psi(\sqrt{z}, k, \xi) f(\sqrt{z}, -k, \rho)}{\partial f(\sqrt{z}, -k)} \quad \text{(15)}
\]

whence the result

\[
\mathcal{P}(-\lambda^2) = \frac{-\lambda \frac{f(\lambda, -k, \rho)}{\partial f(\lambda, -k)} f(\lambda, -k, \rho)}{\prod \frac{f(\lambda, -k) f(-\lambda, -k)}} \quad \text{(16)}
\]

follows.

Now it is clear that from Eq. (10) the spectral decomposition of the operator \( K \), namely,
follows, provided the interchanging of the order of integration is allowed. To prove that this is the case in our problem we insert Eq. (16) into the integral in the left-hand side of Eq. (10) and work in the coordinate representation to show that this is given by

\[
\sum_{\rho} P_{\rho} + \int_{0}^{\infty} d\tilde{z} \left( \frac{d\tilde{z}}{\pi} \right)^2 \int_{0}^{\infty} dr dr' \frac{\psi^*(r)}{r} \frac{\psi(r')}{r'} \phi\left(\sqrt{\tilde{z}}, k, r\right) \phi\left(\sqrt{\tilde{z}}, -k, r\right) \right)
\]

Moreover, from the physical meaning of the wave function \(\phi(r)\) it follows that the ratio \(\phi(r)/r\) is finite at the origin and tends to zero at infinity. The same of course is true for the wave function \(\psi(r)\). Then, the integral over \(r\) in Eq. (18) exists, as the Jost's solutions \(\phi(\lambda, -k, r)\) are given by Eq. (6) when \(r \to -\infty\) and at the origin we have

\[
\phi(\lambda, -k, r) \propto \frac{1}{2\lambda} \left[ f(\lambda, -k) r^{-\lambda + \frac{1}{2}} - f(-\lambda, -k) r^{-\lambda + \frac{1}{2}} \right]
\]

which (\(\lambda\) being pure imaginary) vanishes when \(r\) goes to zero; Eq. (17) therefore holds.

Inserting now Eqs. (16) and (13) into Eq. (17) we finally obtain
As already remarked, this is exactly the completeness relation for the Jost's solutions formally derived in reference 3) using a different procedure.

Let us now prove that the set of these solutions is also orthogonal. For the discrete spectrum, from Eqs. (1) and (2) we first obtain in the standard way (k being either real or pure imaginary) the relation

\[
\frac{2\pi}{(\lambda^2 - \lambda^2_p)} \int_0^\infty \frac{\lambda^2 d\lambda}{\bar{f}(\lambda - k, r)} \frac{\bar{f}(\lambda - k, r')}{r'} = \delta(r - r')
\]

(20)

As in the integral of the right-hand side the upper primes indicate a first derivative and the shorthand notation used is obvious. At the lower limit \(r = 0\) this integral vanishes, on account of the relation

\[
\int_0^\infty \frac{\bar{f}(\lambda - k, r)\bar{f}(\lambda - k, r')}{r} dr = \int_0^\infty \left\{ \bar{f}(\lambda - k)\varphi'(-\lambda - k, r) - \bar{f}(-\lambda - k)\varphi(\lambda, k, r) \right\}
\]

(21)

where in the integral of the right-hand side the upper primes indicate a first derivative and the shorthand notation used is obvious. At the lower limit \(r = 0\) this integral vanishes, on account of the relation

\[
\bar{f}(\lambda - k, r) = \frac{1}{2\lambda} \left[ \bar{f}(\lambda - k)\varphi(-\lambda - k, r) - \bar{f}(-\lambda - k)\varphi(\lambda, k, r) \right]
\]

(22)

which at a pole of the S-matrix reduces to

\[
\bar{f}(\lambda - k, r) = -\frac{1}{2\lambda} \left[ \bar{f}(-\lambda - k)\varphi(\lambda, k, r) \right]
\]

(23)
which vanishes at the origin (Eq. (5)). We then evaluate the upper
limit using Eq. (6) to show trivially that it vanishes for all \( \lambda \); this
proves the orthogonality for \( \lambda_p \neq \lambda_{p'} \). For \( \lambda_p = \lambda_{p'} \), the integral in the
left-hand side of Eq. (21) can be directly evaluated using Eq. (23). We
find
\[
\int_0^\infty \frac{f(\lambda_p, k, r)}{r} \, dr = \frac{f(-\lambda_{p'}, k)}{4 \lambda_p^2} \int_0^\infty \frac{\varphi^2(\lambda_{p'}, k, r)}{r^2} \, dr \quad \quad \quad \quad (24)
\]
which exists for \( \text{Re} \lambda > 0 \) (Eq. (5)). Note that this normalization
integral is just the function \( M(\lambda_p, k) \) appearing in the completeness
relation and defined by Eq. (14). Summarizing, the orthogonality re-
lation for the discrete spectrum reads then
\[
\int_0^\infty \frac{f(\lambda_p, k, r) f(\lambda_{p'}, r)}{M^2(\lambda_p, k) r^2} \, dr = \delta_{p, p'} \quad \quad \quad \quad (25)
\]

It is interesting to remark that in Eq. (25) the orthogonality is
between \( f_{\lambda_p}/r \) and \( f_{\lambda_{p'}}/r \), and not \( f_{\lambda_p}/r \) and \( f_{\lambda_{p'}}^*/r \), as in the usual
case. This situation, however, is not new in complex eigenvalue problems
appearing in physics. We recall, for instance, the KAPUR-PEIERLS
theory of nuclear reactions \(^9,10\) (which leads to complex energy eigen-
values) where a similar situation arises.

Let us finally prove the orthogonality of the continuous spectrum.
We consider again Eq. (21), which, as discussed above, can be written
as
\[
(\lambda^2 - \lambda'^2) \int_0^\infty \frac{f(\lambda, k, r) f(\lambda', k, r)}{r^2} \, dr = -\left( f'' \frac{f}{\lambda} - f' \frac{f}{\lambda'} \right)_{\gamma=0} \quad \quad \quad \quad (26)
\]

Here \( \lambda = i |\lambda| \). Using Eqs. (22) and (6), a straightforward cal-
culation yields
To summarize, we have shown in this section that any function \( \phi_k(r) \) of the space considered can be expanded in terms of the Jost's solutions, namely,

\[
\phi_k(r) = \sum \frac{A_k(\lambda_p)}{r} \frac{f(\lambda_p, k, r)}{\sqrt{\lambda_p(k)}} + \int B_k(\lambda) \frac{f(\lambda, k, r)}{r} d\lambda,
\]

where the expansion coefficients are given by

\[
A_k(\lambda_p) = \int_0^\infty \frac{\phi_k(r) f(\lambda_p, k, r)}{M^2(\lambda_p, k)} d\lambda,
\]

and

\[
B_k(\lambda) = \frac{2i}{\pi} \int_0^\infty \frac{\phi_k(r) f(\lambda, k, r)}{f(\lambda, k) f(-\lambda, k)} d\lambda
\]

respectively.

3. DISPERSION RELATIONS

Let us call \( I \) the integral in the left-hand side of Eq. (20). Then, using relation (22) we can write

\[
I = \frac{i}{2\pi r r'} \left[ \int_{-\infty}^{\infty} d\lambda \left\{ \frac{f(-\lambda, k)}{f(\lambda, k)} \psi(\lambda, k, r') \psi(\lambda, k, r) - \psi(-\lambda, k, r') \psi(\lambda, k, r) \Theta(r-r') \right. \\
- \left. \psi(\lambda, k, r') \psi(-\lambda, k, r) \Theta(r'-r) \right\} \right] \]
This integral we evaluate applying Cauchy's residue theorem, using a contour consisting of the imaginary axis and a semicircle \((\Gamma)\) of infinite radius in the half-plane \(\Re \lambda > 0\). The integral over \((\Gamma)\) is easily calculated recalling Eqs. (7) and (8). The first term of the integrand in Eq. (31) then vanishes and we are left with the result

\[
\frac{i}{2\pi r r'} \int_{-\infty}^{\infty} d\lambda \left[ Y(\lambda, k, r) Y(\lambda, k, r') \Theta(r-r') + Y(\lambda, k, r') Y(-\lambda, k, r) \Theta(r-r') \right] = \delta(r-r')
\]  

(32)

Therefore,

\[
\frac{i}{2\pi r r'} \int_{-\infty}^{\infty} d\lambda \left[ Y(\lambda, k, r) Y(\lambda, k, r') \Theta(r-r') + Y(\lambda, k, r') Y(-\lambda, k, r) \Theta(r-r') \right] = -\delta(r-r') + 2\pi i \sum_{\nu} R(\lambda_{\nu}, k, r, r'),
\]

(33)

where \(R(\lambda_{\nu}, k, r, r')\) are the residues of the function appearing in the integrand of Eq. (32), the index \(\nu = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots\) labelling the poles of the function \(\varphi(-\lambda, k, r)\). Introducing now Eqs. (31) and (33) into Eq. (20) we finally obtain

\[
\frac{i}{2\pi} \int_{-\infty}^{\infty} d\lambda \frac{f(\lambda, k)}{f(\lambda, -k)} Y(\lambda, k, r) Y(\lambda, k, r') = 2\pi i \sum_{\nu} \frac{f(\lambda_{\nu}, -k, r)}{M^2(\lambda_{\nu}, k)} - 4\pi^2 r r' \sum_{\nu} R(\lambda_{\nu}, k, r, r')
\]

(34)

which can be further transformed into

\* See, for example, reference 11.

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where in the last step we have used the relation

\[ e^{-i\pi \lambda} S(\lambda, k) - e^{i\pi \lambda} S(-\lambda, k) = \frac{4\lambda k}{f(\lambda, -k)f(-\lambda, -k)} \]

which is valid on the imaginary axis. Eq. (35) shows that a dispersion relation for the scattering matrix does not follow from the completeness relation (Eq. (20)), but that instead a dispersion relation for a different function (Eq. (34)) is obtained.*

The poles \( \lambda_p \) appearing in the first term of the right-hand side of Eq. (34), being zeros of the Jost's function \( f(\lambda, -k) \), are physical poles (bound states or resonances); the poles \( \lambda_v \), instead, are not poles of the S-matrix and do not appear to have any physical significance. To illustrate this point we can apply Eq. (34) to the case of zero potential. Then the physical poles disappear, and using the explicit form of the "free" regular solution

\[ \Phi_0(\lambda, k, r) = 2^{\lambda} \frac{\Gamma(\lambda+1)}{\kappa} \sqrt{\kappa} \int_{\lambda}^\infty (k r) \]

* Analogously, no inequalities physically meaningful follow from the orthogonality relation (Eq. (27)), as it happens in the k-plane ("causal" inequalities). See Reference 1.)
the right-hand side of Eq. (34) is easily shown to reduce to

\[-2\pi \sqrt{r} \sum_{m=1}^{\infty} m \int_{-\infty}^{\infty} (kr) \int_{-\infty}^{\infty} (kr')\]

as only the poles located at the negative integers appear in this case.

Finally, let us recall that the dispersion relations for the S-matrix in the \(\lambda\)-plane read \(^7\)

\[
S(\lambda, k) = 1 + \frac{1}{\iota \pi} \int_{-\infty}^{\infty} \frac{S(\lambda, k) - 1}{\lambda^2 - \lambda'^2} d\lambda' + 2 \sum_{m} \frac{\lambda_m S_m(k)}{\lambda^2 - \lambda_m^2} \tag{38}
\]

which is to be compared with Eq. (35). There is no relationship between these two expressions, but instead one notices that Eq. (38) strongly resembles the corresponding relation in the \(k\)-plane, a remark which leads us straight back to the question we asked in the introduction.

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