DISCRETE DEGENERATE REPRESENTATIONS OF NON-COMPACT ROTATION GROUPS

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Formulae (3,15), (3,16), (4,4), (4,5), (5,4) and the last line but one of page 23 should read

\[ L = \frac{p+q-4}{2}, \frac{p+q-4}{2} + 1, \ldots \]

with corresponding \( q \).

Page 13: Multiply the right-hand side of the expression for \( N \) in (4,8) by \( \frac{4}{\ell} [1 + \ell_1 + \ell_2] \).

Page 19: Line 6 should read: \( B \rho, \rho, \rho \).

Line 8 should read: \( m_2 = \ldots = m_{[\frac{\ell_1}{2}]} = \bar{m}_2 = \ldots = \bar{m}_{[\frac{\ell_2}{2}]} = 0 \) and \( |m_1|, |\bar{m}_1|, \ldots \).

Page 22: Line 7 should read: \( m_2 = \ldots = m_{[\frac{\ell_1}{2}]} = 0 \) and \( |m_1|, |\bar{m}_1|, \ldots \).

Page 31: (A, 3) should read: \( r = 2_\rho, \Delta(X') = \frac{2^\ell}{(2_\rho)^{\frac{\ell}{2}}} \).

Page 32: (A, 6), (A, 9) should read: with \( _\rho \psi_m \). \( M_{\ell}, \ell_\rho - 1 \).

Page 34: Line 6: Multiply the right-hand side of the equation for \( \phi \) by \( \frac{c^{\mid \rho \rangle}}{\sqrt{2\pi}} \).
DISCRETE DEGENERATE REPRESENTATIONS
OF NON-COMPACT ROTATION GROUPS+

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ABSTRACT

The discrete most degenerate principal series of irreducible unitary one-valued representations of an arbitrary non-compact rotation group \( SO(p, q) \) are derived. The properties of these representations are discussed and the explicit form of the corresponding harmonic functions is given.
1. **INTRODUCTION**

Properties of representations of a semi-simple Lie group are closely related with properties of its Cartan subgroup. That is, the number of different principal series of irreducible unitary representations is equal to the number of non-isomorphic Cartan subgroups of a considered semi-simple Lie group [1]. Moreover, if a given Cartan subgroup is isomorphic to a direct product of a k-dimensional linear space and an r-dimensional torus, then there exists a corresponding series of irreducible unitary representations determined by k real numbers and r integers [1]. Hence, if a semi-simple Lie group has a compact Cartan subgroup, there exists a discrete principal series of irreducible unitary representations characterized only by integers.

The number $N$ of non-isomorphic Cartan subgroups contained in an arbitrary non-compact rotation group $SO(p,q)$ is enumerated in Table I.

<table>
<thead>
<tr>
<th>Class</th>
<th>p</th>
<th>q</th>
<th>$N$</th>
<th>Compact Cartan subgroup</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>even</td>
<td>even</td>
<td>$\min(p,q) + 1$</td>
<td>yes</td>
</tr>
<tr>
<td>(ii)</td>
<td>even</td>
<td>odd</td>
<td>$\min(p,q) + 1$</td>
<td>yes</td>
</tr>
<tr>
<td>(iii)</td>
<td>odd</td>
<td>even</td>
<td>$\min(p,q) + 1$</td>
<td>yes</td>
</tr>
<tr>
<td>(iv)</td>
<td>odd</td>
<td>odd</td>
<td>$\frac{\min(p,q) + 1}{2}$</td>
<td>no</td>
</tr>
</tbody>
</table>

Table I
We see that there is a discrete principal series of irreducible unitary representations only in the class (i), (ii) and (iii) [2].

The discrete principal series of irreducible unitary representations of the $SO(p,q)$ group have only been constructed in three special cases: for the $SO(2,1)$ group by Bargmann [4], for the $SO(3,1)$ group by Gel'fand and Graev [5] and for the $SO(4,1)$ group by Dixmier [6]. In the present work we consider the properties of the discrete most degenerate principal series of irreducible unitary one-valued representations for an arbitrary $SO(p,q)$ group. We restrict ourselves to the discrete most degenerate principal series of representations since these representations seem to be of great importance in quantum mechanics and in elementary particle physics [7].

The main idea behind our construction method of the most degenerate representations of $SO(p,q)$ groups is explained in Section 2. In Section 3 the properties of the discrete most degenerate representations of $SO(p,q)$ groups for $p>q>2$ are discussed and the explicit form of harmonic functions is given. The properties of the discrete degenerate representations of the $SO(p,2)$ groups, $p>2$, are considered in Section 4. It is shown that there are three principal series of discrete most degenerate representations in this case. Section 5 contains the construction of the discrete representations of the so-called Lorentz-type groups, i.e. $SO(p,1)$ groups. In Section 6 the proof of irreducibility and unitarity of our representations is given. In Section 7, we discuss properties of the derived discrete representations. Finally, in the Appendix the most degenerate representations of an arbitrary compact rotation group $SO(p)$ is derived.

2. DISCRETE MOST DEGENERATE REPRESENTATIONS OF GROUPS

Different principal series of irreducible unitary representations
of a semi-simple Lie group may be created in the Hilbert space $\mathcal{H}(X)$ of functions where the domain $X$ is a homogeneous space of the type

$$X = \frac{G}{G_0},$$

where $G_0$ is a closed subgroup of $G$.

The number of non-vanishing invariant operators characterizing different irreducible representations by their eigenvalues is equal to the rank of the space $X$ \[8\], \[9\] (Chap. X), \[10\]. Consequently, the most degenerate representations, which are determined by one invariant operator, are representations on the Hilbert space of functions, the domain of which is a homogeneous space of rank one.

In a homogeneous space of rank one the invariant operator is just the Laplace-Beltrami operator of the form \[11\]

$$\Delta(X) = \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x} g^{\alpha\beta}(x) \sqrt{g(x)} \frac{\partial}{\partial x},$$

where $g_{\alpha\beta}(x)$ is the left-invariant metric tensor on $X$ and $g(x) = \det[g_{\alpha\beta}(x)]$ \[12\].

If the metric tensor $g_{\alpha\beta}(x)$ on $X$ is induced by the Cartan metric tensor $g_{ik}$ in the Lie algebra $\mathfrak{g}$ of the group $G$, then the Laplace-Beltrami operator $\Delta(X)$ is equal to the second order Casimir operator $Q_2 = g_{ik} \chi^i \chi^k$ \[9\] (Chap. X, § 7).

Thus the construction of most degenerate irreducible unitary representations may be reduced to the following \[13\]:

i) Construction of a convenient coordinate system on $X$, in which the metric tensor $g_{\alpha\beta}(x)$ is diagonal.

ii) Solution of the eigenvalue problem for the Laplace-Beltrami operator.
\( \Delta (\chi) \cdot \Psi_\lambda = \lambda \cdot \Psi_\lambda \).

iii) Proof of the irreducibility and unitarity of the representations related with the set of harmonic functions \( \Psi_\lambda \).

As homogeneous spaces \( X \) we may take the quotient spaces \( G_\circ \) with a compact or non-compact stability group \( G_\circ \). The homogeneous spaces of rank \( k \) with the compact stability group related with \( S O(p,q) \) groups are ([9] Chap. IX), [14]:

\[
X^k = \frac{S_0(p,q)}{S_0(p) \times S_0(q)}.
\]

Since the rank \( k \) of these Cartan symmetric spaces is equal to \( \min(p,q) \), we may construct in these spaces the most degenerate representation only of the Lorentz type groups \( S O(p,q) \). For an arbitrary \( S O(p,q) \) group we have to consider more general spaces of rank one. We may take these spaces as homogeneous spaces of rank one of the following form:

\[
X^{p+q-1} = \frac{S_0(p,q)}{S_0(p-1,q)} \quad \text{and} \quad X^{p+q-1} = \frac{S_0(p,q)}{S_0(p,q)}, \tag{2.2}
\]

where the superscript \( p+q-1 \) denotes the dimension of the space \( X^{p+q-1} \).

3. DISCRETE MOST DEGENERATE REPRESENTATIONS OF \( S O(p,q) \) GROUPS \( (p \geq q \geq 2) \).

To choose a suitable coordinate system we have to introduce some convenient model of the space \( X^{p+q-1} \). This means we have to introduce a manifold, which has the same dimension and the same stability group as \( X^{p+q-1} \) itself and on which the group \( S O(p,q) \) acts transitively.

For the space \( X^{p+q-1} \) such a model can be realized by the hyperboloid \( H^p \) determined by the equation
As an appropriate model for the space $X^{p+q-1}$ we take the hyperboloid $H^2_p$ defined by the equation

$$(x^1)^2 + \ldots + (x^p)^2 - (x^{p+1})^2 - \ldots - (x^{p+q})^2 = 1. \quad (3.1)$$

If we introduce internal coordinates $\{\theta^1, \ldots, \theta^{p+q-1}\}$ on the space $H^2_p$ (or $H^2_p$), which is imbedded in the flat Minkowski space $\mathcal{M}^{p+q}$, then the metric tensor $g_{ab}(H^2_p)$ on the hyperboloid $H^2_p$ is induced by the metric tensor $g_{ab}(\mathcal{M}^{p+q})$ on the Minkowski space $\mathcal{M}^{p+q}$ and is defined as

$$g_{ab}(H^2_p) = g_{ab}(\mathcal{M}^{p+q}), \quad \partial_\alpha x^a(\Omega), \partial_\beta x^b(\Omega), \quad (3.3)$$

where $a, b = 1, 2, \ldots, p+q$ and $\alpha, \beta = 1, 2, \ldots, p+q-1$.

Generally, we may choose a large number of different coordinate systems on the hyperboloid $H^2_p$ (or $H^2_p$), in which the Laplace-Beltrami operator can be separated. However, as follows from our previous work [15], the most convenient coordinate system is the biharmonic one because in this system the maximal Abelian compact subalgebra of the considered $SO(p,q)$ group is automatically contained in the maximal set of commuting operators.

The biharmonic coordinate system on the hyperboloid $H^2_p$ is constructed as follows:

$$x^k = x^{'k} \cosh \theta, \quad k = 1, 2, \ldots, p; \quad x^{p+q} = x^{'q} \sinh \theta, \quad \ell = 1, 2, \ldots, q; \quad \theta \in [0, \infty). \quad (3.4)$$
where the form of the $x^i$ and $x^k$ depends on whether $p$ and $q$ are even or odd. We must distinguish four cases:

1. \( p = 2r \); \( q = 2s \)
2. \( p = 2r \); \( q = 2s + 1 \)
3. \( p = 2r + 1 \); \( q = 2s \) \( r, s = 1, 2, \ldots \) (3, 5)
4. \( p = 2r + 1 \); \( q = 2s + 1 \)

Then, if \( p \) is even \((p = 2r)\), the corresponding $x^k \ (k = 1, 2, \ldots, 2r)$ are given by recursion formulae

\[
\begin{align*}
\text{for } r & = 1 \\
x^1 & = \cos \varphi^1, \\
x^2 & = \sin \varphi^1, \\
\varphi^1 & \in [0, 2\pi), \quad (3, 6)
\end{align*}
\]

\[
\begin{align*}
\text{for } r > 1 \\
x^{i+1} & = x^i \sin \varphi^{r+1}, \\
x^{2r+1} & = \cos \varphi^{r+1}, \\
\varphi^{r+1} & \in [0, 2\pi), \quad i = 1, 2, \ldots, 2r - 1,
\end{align*}
\]

and if \( p \) is odd \((p = 2r+1)\) we first construct the $x^i \ (i = 1, 2, \ldots, 2r)$ by using the above-mentioned method for $p = 2r$; we then obtain the corresponding $x^k \ (k = 1, 2, \ldots, 2r+1)$ as

\[
\begin{align*}
\text{for } r & > 1 \\
x^i & = x^i \sin \varphi^{r+1}, \\
x^{2r+1} & = \cos \varphi^{r+1}, \\
\varphi^{r+1} & \in [0, \pi], \quad i = 1, 2, \ldots, 2r \quad (3, 7)
\end{align*}
\]

The recursion formulae for $x^k$ even or odd, are the same as those for $x^i$, $p$ even or odd respectively, except angles $\varphi^i, \varphi^{r+1}$ in $x^k$ are replaced by $\tilde{\varphi}^i, \tilde{\varphi}^{r+1}$.

Choosing the parametrization $\Omega = \{\omega, \tilde{\omega}, \theta\}$ on the hyperboloid $H^p_q$ as [16]
\[ \omega = \{ \varphi', \ldots, \varphi_{[\xi]}, \theta', \ldots, \theta_{[\xi]} \} , \quad \bar{\omega} = \{ \bar{\varphi}', \ldots, \bar{\varphi}_{[\xi]}, \bar{\theta}', \ldots, \bar{\theta}_{[\xi]} \} \] (3.8)

and denoting
\[ \{ \partial_x \} \equiv \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots, \frac{\partial}{\partial P}, \frac{\partial}{\partial P_x}, \frac{\partial}{\partial P_y}, \ldots, \frac{\partial}{\partial P_{[\xi]}}, \frac{\partial}{\partial P_{[\xi]}} \right\} \quad \text{for} \quad x = 1, 2, \ldots, p + q - 1, \]

we can calculate the metric tensor \( g_{\theta \theta} (H^p \mid H^q) \) as well as the Laplace-Beltrami operator \( \Delta (H^p \mid H^q) \) on the Hilbert space \( \mathcal{H} (H^p \mid H^q) \).

Since in all four cases \((3, 5)\) the variables in the Laplace-Beltrami operator \((2, 1)\) are separated in the same way due to the properties of the metric tensor \((3, 3)\), we can write the operator \( \Delta (H^p \mid H^q) \) in the form
\[ \Delta (H^p \mid H^q) = -\frac{1}{\partial^2 / \partial \theta^2} \cdot \partial / \partial \theta - \partial^2 / \partial \theta^2 + \Delta (X^{p-1}) \Delta (X^{q-1}) \] (3.10)

where \( \Delta (X^{p-1}) \Delta (X^{q-1}) \) is the Laplace-Beltrami operator of the rotation group \( SO(p) \mid SO(q) \). If we represent the eigenfunctions of \( \Delta (H^p \mid H^q) \) as a product of the eigenfunctions of \( \Delta (X^{p-1}) \Delta (X^{q-1}) \) and a function \( \psi^\lambda (\theta) \), we obtain the following equation:

\[ \left[ -\frac{1}{\partial^2 / \partial \theta^2} \cdot \partial / \partial \theta - \partial^2 / \partial \theta^2 - \frac{\ell (\ell + p - 2)}{\theta^2} + \frac{\ell (\ell + q - 2)}{\theta^2} - \lambda \right] \psi^\lambda (\theta) = 0, \] (3.11)

where \( \ell (\ell + p - 2) \) and \( \ell (\ell + q - 2) \) are eigenvalues of the operator \( \Delta (X^{p-1}) \Delta (X^{q-1}) \) with \( \ell \mid \ell \) being certain non-negative integers for \( p > 2 \mid q > 2 \).

* For more details, see Appendix.
A discrete series of representations exist if there exist solutions of (3, 11), which are square integrable functions \( \psi^a(\theta) \), \( \theta \in (0, \infty) \), with respect to the measure [17]

\[
d\mu(\theta) = \chi^{\alpha,\beta} \theta, \theta \, d\theta, \tag{3, 12}
\]

which is induced by the measure \( d\mu(\Omega) \) on the hyperboloid \( H^p_q \) [17]:

\[
d\mu(\Omega) = \sqrt{g(\mu)} d\Omega = d\mu(\omega) \, d\mu(\tilde{\omega}) \, d\theta. \tag{3, 13}
\]

As the differential equation (3, 11) has meromorphic coefficients which are regular in the interval \((0, \infty)\), any two linearly independent solutions are also regular analytic in this interval [18]. Since at the origin and at infinity the coefficients are singular, the solutions are not generally regular there and we easily find two essentially distinct behaviours of the solutions at the origin: \( \psi_1^0 \sim \theta^{\frac{q}{2}} \), \( \psi_2^0 \sim \theta^{-\frac{q}{2}} \)

and at infinity: \( \psi_1^\infty \sim \exp\{-\left(\frac{q - 2}{2}\right) + \sqrt{\left(\frac{p q - 2}{2}\right)^2 - \lambda}\} \theta \).

The only satisfactory solution, i.e. the solution which is square integrable with respect to our measure \( d\mu(\theta) \) (3, 12), is that which behaves like \( \psi_1^0(\theta) \) at the origin and like \( \psi_2^\infty(\theta) \) at infinity. It turns out that the solution of (3, 11) with these properties is

\[
\psi^\lambda(\theta) = t^\frac{\beta - q}{2} \chi^\lambda \theta \, \left(\frac{q - 2}{2} + \frac{p q - 2 n}{2} - \lambda\right) \, 2F_1\left(-n + \lambda, n; \frac{q - 2}{2}, \theta^2, -\theta^2\right),
\]

where a non-negative integer \( n \) is connected with our \( \lambda \) by the condition that \( 2F_1 \) be a polynomial, i.e.

\[
\lambda \equiv \frac{q - 2}{2} + \sqrt{\left(\frac{p q - 2}{2}\right)^2 - \lambda} - p + 2, n = 0, 1, 2, \ldots \tag{3, 14}
\]

From this restrictive condition we can find that the discrete spectrum of the operator \( \Delta(H^p_q) \) is of the form

-8-
\[ \lambda = -L \{ L + p + q - 2 \} \quad L = 1, 2, \ldots \quad (3.15) \]

where

\[ L = \ell \{ m \} - \ell \{ \ell \} - q - 2n \]

Of course it does not mean that \( n \) is limited as \( \ell \{ \ell \} \) is an arbitrary integer larger than 2.

Summarizing, we have proved that there exist discrete most degenerate series of representations of an arbitrary \( SO(p, q) \) group \((p \geq q > 2)\) on the Hilbert space \( \mathcal{H}_p^q \), i.e. on the space of square integrable functions \( \psi_{m, n, \ell \{ \ell \}}(x, \theta) \) with respect to the measure \((3.13)\) and with \( \lambda \) given in \((3.15)\). We shall denote such representations by \( D^\ell(\mathcal{H}_p^q) \).

The basis of the Hilbert space \( \mathcal{H}_p^q \) is formed by the orthonormal functions:

\[ Y_{m, n, \ell \{ \ell \}}(\theta, \phi) = Y_{\ell \{ \ell \}}(\theta) Y_{m}(\phi) \quad (3.17) \]

where

\[ Y_{\ell \{ \ell \}}(\theta) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{N_r}} \int_{k=2}^{r} \sin^{2-k}(\phi) \frac{d_2^k(2\phi)}{M_{k}, M_{\ell}} \int_{k=1}^{r} \frac{d_2^k(2\phi)}{M_{k}} & \text{if } p=2r \\
\frac{1}{\sqrt{N_r}} \int_{k=0}^{r} \frac{d_2^k(2\phi)}{M_{k}} & \text{if } p=2r+1 
\end{array} \right. \quad (3.18) \]
are eigenfunctions of $\Delta (x^{\alpha - \epsilon})$ derived in Appx; $\varphi^{(\alpha)}_{\ell}^{(\epsilon)}$ are eigenfunctions of $\Delta (x^{\alpha - \epsilon})$ expressed as the product of the usual d-functions of angular momenta and exponential functions exactly as (3,18), but with variables $\varphi^{(\alpha)}, S^{\epsilon}$ and $I_{\ell}, m_{\ell}$ instead of $\varphi^{(\alpha)}, S^{\epsilon}$ and $I_{\ell}, m_{\ell}$ respectively, and $V^{(\alpha)}_{L}(\theta)$ is the solution of (3,11) given by

$$\sqrt{L} \left( \frac{\theta}{\ell+1} \right) = \frac{1}{\sqrt{N}} \cdot t_{\ell}^{(\alpha)}(\theta) \cdot c_{\ell}^{(\alpha)}(\theta) \cdot 2^{\frac{L+2L_{\ell}+L_{\ell}}{2}} \cdot 2^{\frac{L+2L_{\ell}+L_{\ell}}{2}} \cdot \frac{\xi_{\ell}^{(\alpha)} + 2 \epsilon_{\ell}^{(\alpha)}}{2},$$

(3,19)

where for a definite representation $L$ is fixed and $\xi_{\ell}^{(\alpha)}, \xi_{\ell}^{(\alpha)}$ are restricted by the condition that $t_{\ell}^{(\alpha)}$ be a polynomial, i.e.

$$\xi_{\ell}^{(\alpha)} - \xi_{\ell}^{(\alpha)} = L + q + 2n, \quad n = 0, 1, 2, \ldots \quad (3,20)$$

$N_r, N_{r+1}, N$ are normalization factors given by

$$N_r = 2 \pi \frac{r}{\ell+1-k-1}, \quad N_{r+1} = 4 \pi \frac{r}{2(\ell+1-k-1)}, \quad r = 0, 1, 2, \ldots \quad (3,21)$$

$$N = \frac{\Gamma\left(\frac{L}{2} + \xi_{\ell}^{(\alpha)} + \xi_{\ell}^{(\alpha)} - L + 2\frac{k+1}{2}\right) \Gamma\left(\frac{L}{2} + \xi_{\ell}^{(\alpha)} + \xi_{\ell}^{(\alpha)} - L + 2\frac{k+1}{2}\right) \Gamma\left(\frac{L}{2} + \xi_{\ell}^{(\alpha)} + \xi_{\ell}^{(\alpha)} - L + 2\frac{k+1}{2}\right)}{\Gamma\left(\frac{L}{2} + \xi_{\ell}^{(\alpha)} + \xi_{\ell}^{(\alpha)} - L + 2\frac{k+1}{2}\right) \Gamma\left(\frac{L}{2} + \xi_{\ell}^{(\alpha)} + \xi_{\ell}^{(\alpha)} - L + 2\frac{k+1}{2}\right) \Gamma\left(\frac{L}{2} + \xi_{\ell}^{(\alpha)} + \xi_{\ell}^{(\alpha)} - L + 2\frac{k+1}{2}\right)}$$

(3,22)

and the indices $J_{\ell}, M_{\ell}, M'_{\ell}$ are defined in the Appx. as:

$$J_{\ell} = \frac{1}{2} (\ell + k - 2), \quad M_{\ell} = \frac{1}{2} (m_{\ell} + \ell_{\ell} + k - 2), \quad M'_{\ell} = \frac{1}{2} (m_{\ell} + \ell_{\ell} + k + 2) \quad \text{for } k = 2, 3, \ldots, r \quad (3,23)$$

$$J_{r+1} = \ell_{r+1} + r - 1, \quad M_{r+1} = \ell_{r+1} + r - 1$$

$l_{\epsilon}, k = 2, \ldots, r+1,$ are non-negative integers, $m_{\ell}, k = 1, \ldots, r$, are integers restricted as follows (See Appx.).
There exists also a discrete series of representations $D^t(H_p^q)$ on the Hilbert space $\mathcal{H}(H_p^q)$ with $H_p^q$ given in (3, 2). This series is obtained from the previous one $D^t(H_q^p)$ formally by exchanging $p, \ell_p^t$ for $q, \ell_q^t$ respectively and vice versa. The most degenerate representations $D^t(H_p^q)$ created on $\mathcal{H}(H_p^q)$ are not equivalent except in the case $\rho = q$, when both Hilbert spaces coincide.

Finally, we would like to mention that the representations $D^t(H_p^q)$ and $D^t(H_q^p)$ are irreducible and unitary as it will be proved in Section 6.

4. DISCRETE MOST DEGENERATE REPRESENTATIONS OF $SO(p,2)$ GROUPS ($p \geq 2$).

For the "de Sitter type groups" $SO(p,2)$ the homogeneous spaces are [14]

\[ X^+ = \frac{SO(p,2)}{SO(p-1,2)} \quad \text{and} \quad X^- = \frac{SO(p,2)}{SO(p,1)} \quad (4,1) \]

and can be represented by hyperboloids $H^p_2$ and $H^q_p$ respectively.

The biharmonic coordinate system is introduced again in the same way as in Section 3. Hence, for the Laplace-Beltrami operator on the Hilbert space $\mathcal{H}(H_p^q)$ we obtain

\[ \Delta (H_p^q) = -\frac{1}{\ell \ell_o \ell_o^{p+1}} \frac{\partial}{\partial \ell} \ell \ell_o \ell_o^{p+1} \frac{\partial}{\partial \ell} + \frac{1}{\ell \ell_o (\ell_o^{p+1})^2} - \frac{\Delta (X^+)}{\ell \ell_o^{p+1}} \quad (4,2) \]
The main difference with respect to the equation for $\Delta(H^2_P)$ is rooted in the fact that instead of the operator $\Delta(k^{2}\cdot \gamma / \gamma + 2 \cdot \gamma)$ appearing there, in equation (4,2) the operator $\Delta(k') = \partial^2 / \partial \eta^2$ appears, which has eigenvalues $(-\tilde{n}_\gamma)^2$ with $\tilde{n}_\gamma$ an arbitrary integer. By using the same procedure as in Section 3 we obtain finally the following equation for the function of $\theta$

$$
\left[ \frac{-d}{d\theta} \frac{d}{d\theta} - \frac{(\tilde{m}_\gamma)^2}{\hbar^2 \theta} + \frac{\ell(\ell + 1 + p - 2)}{\hbar^2 \theta} - \lambda \right] \psi(\theta) = 0. 
$$

(4,3)

The discrete series of representations exist again due to the fact that there exist solutions of (4,3) which are square integrable in $\theta \in (0, \infty)$ with respect to the measure $d\mu(\theta) = \hbar^2 \theta \cdot d\theta$. The discrete spectrum of the operator $\Delta(H^2_P)$ looks like

$$
\lambda = -L(L + p), \quad L = 1, 2, \ldots \quad (4,4)
$$

where

$$
L = |\tilde{m}_\gamma| - |\ell_{\frac{1}{2}}| - 2n - p, \quad |\tilde{m}_\gamma| = p+1, p+2, \ldots, \quad |\ell_{\frac{1}{2}}| = 0, 1, \ldots, |\tilde{m}_\gamma| - p - 1 \quad (4,5)
$$

$n = 0, 1, \ldots, \left\lfloor \frac{|\tilde{m}_\gamma| - |\ell_{\frac{1}{2}}| - p}{2} \right\rfloor - 1$.

In a definite representation the value $L$ is fixed and equation (4,5) imposes the following restriction on $|\tilde{m}_\gamma|$

$$
|\tilde{m}_\gamma| \geq L + p \quad (4,6)
$$

Since generators of an $SO(p, q)$ group can change the quantum number $\tilde{m}_\gamma$ only by one (see Section 6), we create on $\mathcal{H}^L(H^2_P)$ two discrete non-equivalent series of representations. One of them, corresponding to $\tilde{m}_\gamma \geq L + p$, we denote by $\mathcal{D}_+^L(H^2_P)$ and the other
with \( \tilde{m}_v \equiv -(L+p) \) by \( \mathcal{D}^L(H^2_p) \).

The representations \( \mathcal{D}^L(H^2_p) \) are representations on different invariant subspaces of the Hilbert space \( \mathcal{H}^L(H^2_p) \), with basis formed by the following orthonormal functions

\[
\mathcal{Y}_{(\omega, \vec{\phi}, \vec{\theta})} = \frac{1}{\sqrt{N}} \mathcal{F}_L^{(\omega, \vec{\phi}, \vec{\theta})} \mathcal{H}_L(\omega) \mathcal{G}_L(\vec{\phi}, \vec{\theta}) \mathcal{J}_L, \quad \text{where} \quad \mathcal{Y}_{(\omega, \vec{\phi}, \vec{\theta})} \quad \text{is given in (3, 18)}
\]

and

\[
\mathcal{Y}_{(\omega, \vec{\phi}, \vec{\theta})} = \frac{1}{\sqrt{N}} \mathcal{F}_L^{(\omega, \vec{\phi}, \vec{\theta})} \mathcal{H}_L(\omega) \mathcal{G}_L(\vec{\phi}, \vec{\theta}) \mathcal{J}_L, \quad \text{where} \quad \mathcal{Y}_{(\omega, \vec{\phi}, \vec{\theta})} \quad \text{is given in (3, 18)}
\]

where \( \omega, \vec{\phi}, \vec{\theta} \) are restricted by condition that \( \mathcal{J}_L \) be a polynomial, i.e.

\[
|\tilde{m}_v| = |q_{(\omega, \vec{\phi}, \vec{\theta})}| = L + p + 2n, \quad n = 0, 1, \ldots
\]  

The discrete series of representations on the Hilbert space \( \mathcal{H}^L(H^2_p) \) are constructed by the same method, but (except \( p=2 \)) we obtain only one series because now \( q_{(\omega, \vec{\phi}, \vec{\theta})} \) plays the role of \( \tilde{m}_v \) and for \( p>2 \) \( q_{(\omega, \vec{\phi}, \vec{\theta})} \) is a non-negative integer. For \( p=2 \) \( q_{(\omega, \vec{\phi}, \vec{\theta})} \) we find again two discrete non-equivalent series as both Hilbert spaces \( \mathcal{H}^L(H^2_p) \) and \( \mathcal{H}^L(H^2_p) \) coincide.

5. DISCRETE MOST DEGENERATE REPRESENTATIONS OF SO(p,1) GROUPS.

The homogeneous spaces of rank one for the Lorentz type groups are [14].
where the $X^p_+$ space is the Cartan symmetric one. As their models we take the hyperboloids $H^p_1$ and $H^p_4$ respectively.

The biharmonic coordinates on $H^p_1$ and $H^p_4$ are introduced again by the method explained in Section 3, but on the hyperboloid $H^p_1$ the range of $\theta$ is $(-\infty, \infty)$. On the hyperboloid $H^p_4$ the range of $\theta$ is from zero to infinity since we restrict ourselves to the upper sheet of the hyperboloid $H^1_4$. Of course, the upper sheet of $H^1_4$ is a transitive manifold only under the proper $SO(p,1)$ group, i.e. under the group of transformations $g=(g_{\mu\nu})$, for which $g_{tt}$ is positive.

The Laplace-Beltrami operator on the Hilbert space $L^2(H^p_4)$ has the form

$$\Delta (H^p_4) = \frac{-1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \sin^2 \theta \frac{\partial}{\partial \theta} + \frac{\Delta (H^p_1)}{\partial^2 \theta} \ , \ \theta \in (-\infty, \infty) , \ (5.2)$$

where $\Delta (H^p_1)$ is the Laplace-Beltrami operator for the $SO(p)$ group given in the Appx.

The eigenvalue problem of $\Delta (H^p_4)$ is reduced to

$$\left[ \frac{-1}{\sin^2 \theta} \frac{d}{d\theta} \sin^2 \theta \frac{d}{d\theta} - \frac{\xi_4 (\xi_4 + p - 2)}{\sin^2 \theta} - \lambda \right] \psi_{\xi_4}^2 (\theta) = 0 \ . \ (5.3)$$

Analogously to the previous cases, we find the discrete spectrum of $\Delta (H^p_4)$ to be of the form

$$\lambda = -L \left( L + p - 1 \right), \quad L = 0, 1, 2, \ldots \ , \ (5.4)$$

-14-
where

\[ L = |\ell_{\xi}^-| - 1 - n \]

\( n = 0, 1, \ldots, \frac{L_{\xi}^- - 1}{2} \) .

\( \ell_{\xi}^- \) a positive integer for \( p > 2 \), and for \( p = 2 \) an arbitrary non-zero integer, \( m \). Hence, there is an exceptional case for \( p = 2 \) and we obtain again two types of discrete non-equivalent series of representations \( D^L_p(H^3_l) \) and \( D^L_p(H^3_r) \) on different invariant subspaces of the Hilbert space \( \mathcal{H}^L_p(H^3) \). For \( SO(2,1) \) these results were obtained by Bargmann [4], for \( SO(3,1) \) by Gel'fand and Graev [5] and for \( SO(4) \) by Dixmier [6].

The basis of the Hilbert space \( \mathcal{H}^L_p(H^3) \) is formed by the orthonormal functions

\[
Y_{(\omega, \theta)}^{L, \ell_{\xi}^+, \ell_{\xi}^-} = \begin{cases} 
Y_{(\omega, \theta)}^{L, \ell_{\xi}^+, \ell_{\xi}^-} = \frac{1}{N} \sqrt{(L\ell_{\xi}^+ + 1; L - \ell_{\xi}^- + 1; \frac{1}{2}; \theta^2)} 
& \text{if } L - \ell_{\xi}^- - \ell_{\xi}^+ - 1, \\
Y_{(\omega, \theta)}^{L, \ell_{\xi}^+, \ell_{\xi}^-} = \frac{1}{N} \sqrt{(L\ell_{\xi}^+ + 1; L - \ell_{\xi}^- + 1; \frac{1}{2}; \theta^2)} 
& \text{if } L - \ell_{\xi}^- - \ell_{\xi}^+ - 1,
\end{cases}
\]

(5.6)

where \( Y_{(\omega, \theta)}^{L, \ell_{\xi}^+, \ell_{\xi}^-} \) is explicitly given in equation (3,18) and

\[
N_{\xi}^{L,\ell_{\xi}^+,\ell_{\xi}^-} = \frac{2}{\sqrt{N}} \begin{cases} 
2^{-(L+p-1)} (L + \ell_{\xi}^+ + 1; L - \ell_{\xi}^- + 1; \frac{1}{2}; \theta^2) 
& \text{if } L - \ell_{\xi}^- - \ell_{\xi}^+ - 1, \\
2^{-(L+p-1)} (L + \ell_{\xi}^+ + 1; L - \ell_{\xi}^- + 1; \frac{1}{2}; \theta^2) 
& \text{if } L - \ell_{\xi}^- - \ell_{\xi}^+ - 1,
\end{cases}
\]

(5.7)

Here, normalization factors \( N, 2^N \) are of the form

\[
N = \frac{2\pi^{(\ell_{\xi}^- - 1)/2} \Gamma(\ell_{\xi}^+ + 1)}{(L + p - 1)!}, \quad 2^N = \frac{4(2L + p - 1) \cdot 2\pi^{(\ell_{\xi}^- - 1)/2} \Gamma(\ell_{\xi}^+ + 1)}{(L + p - 1)!}
\]

and for definite representation \( L \) is fixed and \( \ell_{\xi}^- \) must satisfy the
restrictive condition that \( z \in \mathbb{F} \) be a polynomial, i.e.

\[ | \ell_{\{2\}}^L | = L + 1 + n , \quad n = 0, 1, 2, \ldots \]  

(5, 9)

The discrete series of representations on the Hilbert space \( \mathcal{H}_{\mathcal{P}} \) does not exist, because the Laplace-Beltrami operator

\[ \Delta (\mathcal{H}_{\mathcal{P}}) = \frac{-1}{\sin^2 \theta} \frac{d}{d\theta} \sin^2 \theta \frac{d}{d\theta} - \frac{\ell_{\{2\}}^L (\ell_{\{2\}}^L + p - 2)}{\sin^2 \theta} , \]

\( \theta \in [0, \infty) \) has no discrete spectrum.

6. IRREDUCIBILITY AND UNITARITY

The Lie algebra \( \mathfrak{g} \) of the group \( SO(p, q) \) can be expressed in the form of two types of operators with commutation relations:

\[
\begin{align*}
[l_{ij}, l_{rs}] &= -\delta_{ir} l_{js} + \delta_{is} l_{jr} + \delta_{jr} l_{is} - \delta_{js} l_{ir} , \\
[l_{ij}, b_{rs}] &= -\delta_{ir} b_{js} - \delta_{is} b_{jr} + \delta_{jr} b_{is} + \delta_{js} b_{ir} , \\
[b_{ij}, b_{rs}] &= \delta_{ir} l_{js} + \delta_{js} l_{ir} + \delta_{jr} l_{is} + \delta_{is} l_{jr} ,
\end{align*}
\]

(6, 1)

where \( l_{ij} \) and \( b_{ij} \) are the generators of the compact and non-compact one-parameter subgroups respectively. We represent the Lie algebra \( \mathfrak{g} \) on the linear manifold \( \mathcal{L} \) composed of the set of harmonic functions \( \gamma (\omega, \omega', \theta) \)

\[ m_1, \ldots, m_q, \tilde{m}_1, \ldots, \tilde{m}_q \]

the Lie algebra of operators:
where \( i, j = 1, 2, \ldots, p \); and \( x^1, \ldots, x^p \) are defined by (3, 4). For the remaining operators \( L_{ij} \), \( i, j = p+1, p+2, \ldots, p+q \), we obtain analogous expressions. The generators of the non-compact type we represent by:

\[
B_{ij} = \frac{x^i x^j}{\partial \theta \partial \theta} \frac{\partial}{\partial \theta} + \ldots + \frac{x^i (\partial x^j)}{\partial x^j} \frac{\partial}{\partial \theta} + \ldots + \frac{x^i (\partial x^j)}{\partial x^j} \frac{\partial}{\partial \theta} = (6, 3)
\]

where \( i = 1, 2, \ldots, p \) and \( j = p+1, p+2, \ldots, p+q \). It turns out that the differential operators \( L_{ij} \) do not contain the derivatives with respect to \( \theta \) and their coefficients depend only on the parameters \( \omega \) or \( \tilde{\omega} \) of the corresponding subgroup respectively. The differential operators \( B_{ij} \) contain the derivative with respect to \( \theta \) and their coefficients generally depend on all the parameters \( \omega, \tilde{\omega} \) and \( \theta \).

1. Irreducibility. To prove the irreducibility of our representations \( \mathcal{D}^i(H_q^p) \) on the Hilbert space \( \mathcal{H}^i(H_q^p) \) we show that there is no invariant subspace of the space \( \mathcal{H}^i(H_q^p) \) with respect to the representation (6, 2), (6, 3) of the Lie algebra \( R \).

a) The case \( p \neq q > 2 \). The Hilbert space \( \mathcal{H}^i(H_q^p) \) has
the structure:

\[
\mathcal{H}^L (H^p_q) = \sum_{i} \oplus \mathcal{H}^L (H^p_q)_{\ell_{[i]}, \tilde{\ell}_{[i]}}
\]

(6, 4)

where the sum is taken over all such non-negative integers \( \ell_{[i]}, \tilde{\ell}_{[i]} \), which satisfy

\[
\ell_{[i]} - \tilde{\ell}_{[i]} - 2n = L + q, \quad n = 0, 1, 2, \ldots
\]

Here the subspaces \( \mathcal{H}^L (H^p_q)_{\ell_{[i]}, \tilde{\ell}_{[i]}} \) are the finite-dimensional spaces spanned by all the harmonic functions (3, 17) with fixed values of pairs of integers \( \ell_{[i]}, \tilde{\ell}_{[i]} \). The representation (6, 2) of the algebra of the maximal compact subgroup \( SO(p) \times SO(q) \) is irreducible on the space \( \mathcal{H}^L (H^p_q)_{\ell_{[i]}, \tilde{\ell}_{[i]}} \), as is proved in the Appx.

The structure of the Hilbert space \( \mathcal{H}^L (H^p_q) \) can be graphically illustrated by use of nets. A characteristic detail of the net is drawn in Fig. 1. Every knot of the net represents a subspace \( \mathcal{H}^L (H^p_q)_{\ell_{[i]}, \tilde{\ell}_{[i]}} \) and every unit step in the net connects the two nearest neighbouring subspaces.

---

Fig. 1.
Thus to prove that there is no invariant subspace of the space $\mathcal{H}^L(H^2_\xi)$ with respect to the operators $(6,2)$ and $(6,3)$, it is sufficient to find one operator $B_{ij}$ and one element $Y_{\xi,\xi}^L$, $\xi \in \mathcal{L}^{L+1}(H^2_\xi)$ such that $B Y_{\xi,\xi}^L$ has non-vanishing components in four neighbouring subspaces (see Fig. 1).

Let us show that $B Y_{\xi,\xi}^L$ has the desired properties if $\sum_{\xi}^L a_{ij} Y_{\xi,\xi}^L = Y_{\xi,\xi}^L$, where $m = \cdots = m_{\xi} = \cdots = \tilde{m}_{\xi} = 0$ and $\xi, \cdots, \xi_{\xi-1}$ have the minimal possible values. We omit the indices $\{\xi\}, \{\xi\}$ whenever such omission does not lead to misunderstanding and we obtain the following expression:

$$
(B Y_{\xi,\xi}^L)(\Omega) =
$$

$$
- \frac{(\tilde{\xi} + \xi + p + q - 2)(\tilde{\xi} + \xi - 1)}{2 \tilde{\xi} + q} A_\xi(\xi) A_{\xi}(\xi) \frac{N(\xi + 1, \xi + 1)}{N(\xi, \xi)} Z_{\xi,\xi}^L (\Omega) +
$$

$$
+ (2 \tilde{\xi} + q - 2) A_\xi(\xi) A_{\xi}(\xi) \sqrt{\frac{N(\xi + 1, \xi + 1)}{N(\xi, \xi)}} Z_{\xi,\xi}^L (\Omega) -
$$

$$
- \frac{(\tilde{\xi} - \xi + p + q)}{2 \tilde{\xi} + q} A_\xi(\xi) A_{\xi}(\xi) \sqrt{\frac{N(\xi, \xi + 1)}{N(\xi, \xi)}} Z_{\xi,\xi}^L (\Omega) +
$$

$$
+ (2 \tilde{\xi} + q - 2) A_\xi(\xi) A_{\xi}(\xi) \sqrt{\frac{N(\xi, \xi + 1)}{N(\xi, \xi)}} Z_{\xi,\xi}^L (\Omega)
$$

(6, 5)
where

\[
Z_{L, L', \xi_1}^{\xi_2, \xi_2'} (\Omega) = \begin{cases} 
Y_{\xi_1, \xi_2}^{L, \xi_2'}(\Omega) & \text{if } p = 2r, \quad q = 2s+1, \\
\frac{1}{2i} Y_{\xi_1, \xi_2}^{L, \xi_2'}(\Omega) - \frac{1}{2i} Y_{-\xi_1, -\xi_2}^{L, -\xi_2'}(\Omega) & \text{if } p = 2s, \quad q = 2s+1,
\end{cases}
\]

(6.6)

Here \( N (\xi_1, \xi_2) = N_1^{\xi_1} \cdot N_2^{\xi_2} \cdot N \) where \( N_1^{\xi_1}, N_2^{\xi_2}, N \) are defined by the expressions (3.21) and (3.22).

Then if \( p \) is even:

\[
A_+ \left( \xi_1, \xi_2 \right) = C \left( J, \frac{1}{2}, J, \frac{1}{2} \right), C \left( J, \frac{1}{2}, J, \frac{1}{2} \right); C \left( J, \frac{1}{2}, J, \frac{1}{2} \right),
\]

(6.7)

and if \( p \) is odd then

\[
A_+ \left( \xi_1, \xi_2 \right) = C (J, J, M, M, 0), C (J, J, M, M, 0),
\]

(6.8)

where the Clebsch-Gordan coefficients \( C (J, J, J, J, M) \) are taken from ref [19], and \( J^{\xi_1, \xi_2}, M^{\xi_1, \xi_2} \) are defined by (3.23).

Substituting the corresponding values of the constants appearing in the expression (6.5), we check that no term vanishes if the corresponding values of integers \( \xi_1, \xi_2 \) satisfy \( \xi_1 - \xi_2 - 2n = \) \( L + n \).
except in the case \( \ell \neq 1, \ell \{\xi\} = L + q \pm 2n \), \( q \) even. In that case \( B_0^{\ell, L + q + 2n} \) has no components in \( \mathcal{H}^L_{\ell, q + 2n} (H_{\rho}^0) \) and \( \mathcal{H}^L_{L - q + 2n} (H_{\rho}^0) \) but this does not mean that the representation is reducible on \( \mathcal{H}^L (H_{\rho}^0) \) as \( B \) is skew-symmetric on \( \mathcal{L} \) and \( B_0^{\ell, L + q + 2n} \) and \( B_0^{\ell, L - q + 2n} \) have non-vanishing components in \( \mathcal{H}^L_{\ell, q + 2n} (H_{\rho}^0) \).

The proof of the irreducibility of the representation \( D^\ell (H_{\rho}^0) \) on the Hilbert space \( \mathcal{H}^L (H_{\rho}^0) \) is analogous.

b) The case \( \rho \neq q = 2 \). The proof of irreducibility of the representation \( D^\ell (H_{\rho}^0) \) on the Hilbert space \( \mathcal{H}^L (H_{\rho}^0) \) is the same as in the previous case.

The representation of the group \( SO(p, q) \) on the Hilbert space \( \mathcal{H}^L (H_{\rho}^0) \) reduces to two irreducible parts \( D^\ell (H_{\rho}^0) \) on the subspaces \( \mathcal{H}^L_{\pm} (H_{\rho}^0) = \sum \oplus \mathcal{H}^L_{\ell, q + 2n} (H_{\rho}^0) \), where the sum is taken over all non-negative integers \( \ell, q \) such that \( \ell - 2n = L + p \), \( n = 0, 1, 2, \ldots \). The proof of the irreducibility of both representations \( D^\ell (H_{\rho}^0) \) on \( \mathcal{H}^L (H_{\rho}^0) \) is the same as in the previous case. (The reducibility of the representation on the space \( \mathcal{H}^L (H_{\rho}^0) = \mathcal{H}^L_{\pm} (H_{\rho}^0) \oplus \mathcal{H}^L (H_{\rho}^0) \) can be easily understood from (4, 6) and the fact that every operator \( B_{ij} \) (6, 3) maps the subspace \( \mathcal{H}^L_{\ell, q + 2n} (H_{\rho}^0) \) only into those four subspaces \( \mathcal{H}^L_{\ell, q + 2n} (H_{\rho}^0) \) for which \( \ell - 2n = \pm 1 \), \( q = \pm 1 \).

This follows from (6, 1) and (6, 5).
c) The case $\mathcal{g} = 4$. If $p > 2$ the structure of the Hilbert space $\mathcal{H}^l(H^p_0)$ is

$$\mathcal{H}^l(H^p_0) = \sum \mathcal{H}^l_{l_1 l_2 \ldots l_p}(H^p_0),$$

where the sum is taken over all non-negative integers $l_1, l_2, \ldots, l_p = l + 1, l + 2, \ldots$.

The irreducibility is proved as in the previous case using the operator $B_{l_1 l_2 \ldots l_p}$ and the element $(5, 6)$

$$\gamma_{l_1 l_2 \ldots l_p}(\Omega) \equiv \gamma_{m_1 \ldots m_p}(\Omega),$$

where $m_1 = \ldots = m_p = 0$ and $\epsilon_1, \ldots, \epsilon_p - 1$ have minimal possible values. The element $B\gamma_{l_1 l_2 \ldots l_p}$ has non-vanishing components in $\mathcal{H}^l_{l_1 l_2 \ldots l_p}$ as is expressible in the form:

$$\mathcal{Y}^l_{l_1 l_2 \ldots l_p}(\Omega) = \frac{1}{2} (l + \ell + p - 1)(l - \ell) A_{l/2}(\ell) \sqrt{\frac{2N(l+\ell)}{N(\ell)}} \cdot Z^{l,\ell+1}(\Omega) -$$

$$- \frac{1}{2} (l - \ell + 1)(l + \ell + p - 2) A_{l/2}(\ell) \sqrt{\frac{2N(l-\ell)}{N(\ell)}} \cdot Z^{l,\ell-1}(\Omega),$$

$$\mathcal{Y}^l_{l_1 l_2 \ldots l_p}(\Omega) = -2 A_{l/2}(\ell) \sqrt{\frac{N(\ell+\ell)}{N(\ell)}} \cdot Z^{l,\ell+1}(\Omega) -$$

$$-2 A_{l/2}(\ell) \sqrt{\frac{N(\ell-\ell)}{N(\ell)}} \cdot Z^{l,\ell-1}(\Omega),$$

where

$$Z^{l,\ell}(\Omega) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2i}} \gamma^l_{l_1 l_2 \ldots l_p}(\Omega) & \text{for } p \text{ odd}, \\
\frac{1}{\sqrt{2i}} \gamma^l_{l_1 l_2 \ldots l_p}(\Omega) - \frac{1}{\sqrt{2i}} \gamma^{l_1 l_2 \ldots l_p}(\Omega) & \text{for } p \text{ even}. \end{array} \right.$$
Here \( N(l) = N_{l;j}^i \), where the constants \( N_{l;j}^i \) and \( N_l \) are defined by the expressions (3, 21) and (5, 5). The non-vanishing of the coefficients can be checked as before. (Again there exists an exceptional case: \( L = 0 \), \( \ell = 1 \) and \( p \) even, which can be treated as in the previous case).

If \( p = 2 \), there exist two irreducible representations \( D_2^L(H_2^p) \) on the spaces \( H_2^p(H_2^p) = \sum_{m \in L} \otimes H_2^p(H_2^p) \), which can be proved as before.

\section{II. Unitarity.} The representation \( T_g \) of a group element \( g \in SO(p,2) \) on the Hilbert space \( H_2^p(H_2^p) \) is determined by the left-translation:

\[
T_g \gamma_{l,j}^{m_{l,j},m_{l,j},z_{l,j}}^{m_{l,j},m_{l,j},z_{l,j}} = \gamma_{l,j}^{m_{l,j},m_{l,j},z_{l,j}}^{m_{l,j},m_{l,j},z_{l,j}} \quad (6,13)
\]

as follows from the representation of the Lie algebra given by (6, 2) and (6, 3). Here the symbol \( g \Omega \) represents the set of parameters \( \phi, \ldots, \theta, \theta' \) of the point \( \Omega = g \Omega \) on \( H_2^p \) and \( \gamma_{l,j}^{m_{l,j},m_{l,j},z_{l,j}}^{m_{l,j},m_{l,j},z_{l,j}} \) is a harmonic function defined in the expressions (3, 17), (4, 7) or (5, 6). Therefore, the unitarity follows from the left-invariance of the measure \( dq \) on the corresponding hyperboloid \( H_2^p \).

\section{7. CONCLUSIONS}

We devote this section to a brief review and discussion of the derived representations \( D_2^L(H_2^p) \) of the group \( SO(p,2) \).

\section{A) The case \( p > 2 \) (Section 3).} There exist two series of representations: \( D_2^L(H_2^p) \) and \( D_2^L(H_2^p) \) \( L = 1, 2, \ldots \) related to hyperboloids \( H_2^p \) (3, 1) and \( H_2^p \) (3, 2), respectively. The non-
negative integers \( \ell_{\frac{1}{2}}^i, \ell_{\frac{1}{2}}^j \), which determine the irreducible representations of the subgroup \( SO(p) \) and \( SO(q) \) respectively, are not independent as in the case of continuous most degenerate representations [15], but are restricted by

\[
\ell_{\frac{1}{2}}^i - \ell_{\frac{1}{2}}^j - 2n = L + q \quad \text{for } H^p_\ell, \quad (7,1)
\]

\[
\ell_{\frac{1}{2}}^i - \ell_{\frac{1}{2}}^j - 2n = L + p \quad \text{for } H^q_\ell, \quad (7,2)
\]

where \( \ell_{\frac{1}{2}}^i, \ell_{\frac{1}{2}}^j \) and \( n \) range through every such triplet of non-negative integers which satisfy (7,1) and (7,2) respectively. These two conditions are illustrated graphically in Fig. 2 and Fig. 3 respectively. Every knot of the net in the figures represents a subspace \( \mathcal{H}_\ell \) of an irreducible representation of the maximal compact subgroup \( SO(p) \times SO(q) \) determined by a pair of integers \( \ell_{\frac{1}{2}}^i, \ell_{\frac{1}{2}}^j \). Generators of the compact type act inside the subspace \( \mathcal{H}_\ell \). On the other hand, the generator of the non-compact type maps the subspace \( \mathcal{H}_\ell \) into four neighbouring subspaces \( \mathcal{H}_{\ell+1}^l, \mathcal{H}_{\ell+1}^l, \mathcal{H}_{\ell+1}^l, \mathcal{H}_{\ell+1}^l \) graphically represented in Fig. 2.
All the representations $D^c(H^p_2)$ and $D^c(H^p_0)$ are inequivalent except for $p=2$ when we have only one series of representations $D^c(H^p_0)$.

B) The case $p \geq 2$ (Section 4). Generally, there exist three series of representations. Before proceeding with their description we wish to stress that the irreducible representations of the subgroup $SO(2)$ are characterized by an integer $m$ which also takes on negative values. Instead of the conditions (7,1) and (7,2) we have now

$$|\ell_{\pm \frac{1}{2}}| - |m| - 2n = L + 2 \quad \text{for } H^p_2, \quad (7,3)$$

$$|\tilde{m}| - |\ell_{\pm \frac{1}{2}}| - 2n = L + p \quad \text{for } H^p_0, \quad (7,4)$$

where $|\ell_{\pm \frac{1}{2}}|, |\tilde{m}|$ and $n$ range through all such non-negative integers that (7,3) and (7,4) are satisfied respectively. It follows from these conditions and conclusions of Section 4, that there exists
only one series $\mathcal{D}_L(H^0_\lambda)$ of representations related with the hyperboloid $H^0_\lambda$, while there exist two series of representations $\mathcal{D}_R(H^0_\lambda)$ and $\mathcal{D}_L(H^0_\lambda)$ related with the hyperboloid $H^0_\lambda$. Their graphical representations are given in Fig. 4, Fig. 5 and Fig. 6 respectively.

Fig. 4

Fig. 5

Fig. 6
The representations are inequivalent except for the case $p = 2$. In the latter case two subgroups $SO(2)$ of the group $SO(2,2)$ are indistinguishable and we have only two inequivalent representations drawn in solid lines in Fig. 7. The representations which appear after changing $m_1$ and $\tilde{m}_1$ are equivalent to a pair of previous representations. We represent them by dotted lines in Fig. 7.

![Diagram](Fig. 7)

C) The case $q = 1$ (Section 5). Generally, there exists only one series of discrete most degenerate representations $D^1(H^p_1)$. However, in the case of $SO(2,4)$ we have obtained two series of irreducible representations, i.e., $D^1_+ (H^p_2)$ and $D^1_- (H^p_2)$. The condition on the number $\xi$, which determines the irreducible representations of the maximal compact subgroup $SO(p)$, has the form

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\( |(\ell, \ell)\rangle = L + 1 + 2n = 0, 1, 2, \ldots \) \hspace{1cm} (7.5)

It is interesting that we have found the discrete most degenerate representations even for the groups \( SO(p,q) \) with \( p \) and \( q \) odd, which have no discrete principal series of representations (see Table I). Let us explain this unexpected fact, for example, for the Lorentz group \( SO(3,1) \). The action of two Casimir operators \( \Delta_1 = \tilde{M}^2 - \tilde{N}^2 \) and \( \Delta_2 = \tilde{M} \cdot \tilde{N} \) on the basis \( f_r^k \) of the Hilbert space, which realizes the irreducible representation, can be written in the form \([20]\):

\[
\Delta_1 f_r^k = -2(k_0^2 + c^2 - 1) f_r^k, \quad k_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \hspace{1cm} (7.6)
\]

\[
\Delta_2 f_r^k = -4izk_0 c f_r^k, \quad c = ip, \quad p \in (0, \infty). \hspace{1cm} (7.7)
\]

If we take the hyperboloid \( H^3_t \) as the domain of the functions \( f_r^k \) the second Casimir operator \( \Delta_2 \) vanishes identically and the first one admits the discrete spectrum for \( c = 0 \):

\[
\Delta_1 f_r^k = -2(k_0^2 - 1) f_r^k
\]

The result agrees with our result derived in Section 5, if we put

\[ L = k_0 - 1. \]

For applications to physical problems with the \( SO(p,q) \) symmetry the derived discrete most degenerate representations \( D^l(H^p_q) \) or \( D^l(H^q_p) \) are especially convenient due to the following facts:

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i) The maximal set of the commuting operators is maximally reduced in these representations of $SO(p,q)$ groups. That is, for the discrete most degenerate representations of the $SO(p,q)$ group the maximal set of commuting operators consists of

$$
\Delta \left[ SO(p,q) \right],
$$

$$
C_p = \begin{cases} 
\Delta \left[ SO(p) \right], \Delta \left[ SO(p-2) \right], ..., \Delta \left[ SO(4) \right] & \text{for } p \text{ even} \\
\Delta \left[ SO(p) \right], \Delta \left[ SO(p-1) \right], \Delta \left[ SO(p-3) \right], ..., \Delta \left[ SO(4) \right] & \text{for } p \text{ odd} 
\end{cases}
$$

$$
\tilde{C}_q = \begin{cases} 
\Delta \left[ SO(q) \right], \Delta \left[ SO(q-2) \right], ..., \Delta \left[ SO(4) \right] & \text{for } q \text{ even} \\
\Delta \left[ SO(q) \right], \Delta \left[ SO(q-1) \right], \Delta \left[ SO(q-3) \right], ..., \Delta \left[ SO(4) \right] & \text{for } q \text{ odd} 
\end{cases}
$$

$$
H = \left\{ L_{2k,2k-1} = -\frac{\partial}{\partial \phi}, L_{2k,2k-1} = -\frac{\partial}{\partial \psi}, k = 1, 2, ..., \left[ \frac{q}{2} \right] \right\}
$$

where $\Delta \left[ SO(p,q) \right]$ represents the Casimir operator of $SO(p,q)$ and $C_p$ and $\tilde{C}_q$ the sequence of corresponding Casimir operators of the maximal compact subgroup $SO(p) \times SO(q)$.

The set $H$ contains operators of the Cartan subalgebra except in the case $p$ and $q$ odd where $H$ represents the maximal abelian compact subalgebra of $SO(p,q)$ (see Table I).

The number of operators contained in the maximal set of commuting operators for the discrete most degenerate representations of $SO(p,q)$ is equal to

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\[ N = p + q - 1 \]

while the corresponding number for principal non-degenerate representations is

\[ N' = \frac{r + \ell}{2} = \frac{1}{4} \left[ N(N + 1) + 2\ell \right] \]

where \( r \) and \( \ell \) are the dimension and rank of \( \text{SO}(p, q) \) respectively.

ii) The additive quantum numbers may be related to the eigenvalues of the set \( H \). It turns out that the set \( H \) is largest in the biharmonic coordinate system, which we have used.

iii) The eigenfunctions of the maximal commuting set of operators are given in explicit form by formulae (3, 17), (4, 7), (5, 6) and the range of the numbers \( L, \xi_1, \ldots, \xi_{L/2}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{L/2}, m_1, \ldots, m_{L/2}, \tilde{m}_1, \ldots, \tilde{m}_{L/2} \) which may play the role of quantum numbers, is determined by (3, 20), (3, 24), (4, 9) and (5, 9).

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APPENDIX - THE MOST DEGENERATE REPRESENTATIONS OF THE COMPACT ROTATION GROUP \( SO(p) \).

The most degenerate representations of the compact rotation group \( SO(p) \) were derived in [15]. In this Appendix we shall briefly review the main results and moreover we shall prove the irreducibility and unitarity of these representations.

For an arbitrary compact rotation group \( SO(p) \) there exist the most degenerate representations on the finite-dimensional Hilbert space \( \mathcal{H}(X) \) of functions the domain \( X \) of which is the following Cartan symmetric space of rank one [9]

\[
X^{p-1} = \frac{SO(p)}{SO(p-1)}.
\]  

(A, 1)

The model of this space is a \( (p-1) \)-dimensional sphere \( S^{p-1} \)

\[(r^1)^2 + \ldots + (r^p)^2 = 1,
\]

(A, 2)

which is imbedded in \( p \)-dimensional Euclidean space \( E^p \).

Introducing the biharmonic coordinate system on the sphere \( S^{p-1} \) by formula (3.6) or (3.7) we can calculate the metric tensor

\[ g_{\alpha\beta}(S^{p-1}) \]

on the sphere \( S^{p-1} \) and for the Laplace-Beltrami operator \( \Delta(S^{p-1}) \) defined by (2.1) we then obtain

\[
\Delta(x^{2r}) = \frac{4}{\sin^2 \frac{\pi r}{2}} \frac{\partial^2}{\partial x^2} + \frac{4}{\sin \frac{\pi r}{2}} \frac{1}{\sin \frac{2\pi r}{2}} \frac{\partial}{\partial x} \frac{\partial}{\partial \theta} + \frac{\Delta(x^{2r-2})}{\sin^2 \frac{\pi r}{2}} \quad \text{for} \quad p = 2r, r = 1, 2, \ldots
\]

(A, 3)

and

\[
\Delta(x^{2r+1}) = \frac{1}{\sin^{2r+1}} \frac{\partial^2}{\partial x^2} + \frac{1}{\sin \frac{2\pi r}{2}} \frac{\partial}{\partial x} + \Delta(x^{2r-1}) \quad \text{for} \quad p = 2r+1, r = 1, 2, \ldots
\]

(A, 4)

where \( \Delta(x^{2r-2}) \) and \( \Delta(x^{2r-1}) \) are again invariant operators of \( SO(2r-2) \) and \( SO(2r) \) respectively. Using induction, both operators \( \Delta(x^{2r-2}) \) and \( \Delta(x^{2r-1}) \) can be decomposed in the same way as \( \Delta(S^{p-1}) \) in (A, 3),

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The eigenvalues $\lambda_{\ell^2}$ of the Laplace-Beltrami operator on the sphere $S^{r-1}$ are of the very well-known form

$$
\lambda_{\ell^2} = -\ell_{\ell^2} \left( \ell_{\ell^2} + p - 2 \right)
$$

Due to the inductive construction of the Laplace-Beltrami operators we can separate variables in the eigenvalue problem for the operator $\Delta(S^{r-1})$ and finally we obtain the differential equations:

$$
\frac{d}{d \theta} \sin \theta \frac{d}{d \theta} \sin \theta \frac{d}{d \theta} \sin \theta - m_r^2 \sin^2 \theta + \ell_{\ell^2}(\ell_{\ell^2} + 2r - 4) \psi_{m_r, \ell_{\ell^2}}(\theta) = 0 \quad \text{if} \quad p = 2r
$$

and

$$
\frac{d}{d \theta} \sin \theta \frac{d}{d \theta} \sin \theta \frac{d}{d \theta} \sin \theta - \ell_{\ell^2}(\ell_{\ell^2} + 2r - 2) \psi_{m_r+1, \ell_{\ell^2}+1}(\theta) = 0 \quad \text{if} \quad p = 2r+1.
$$

Solutions of the equations (A, 6) or (A, 7) belonging to the Hilbert space of square integrable functions $\psi(\omega)$ with respect to the measure

$$
d\mu(\omega) = \frac{\Gamma^r}{\Gamma^r - 1} \frac{\sin \theta}{\sin \theta} d\theta d\phi d\varphi
$$

are given as follows:

For $p = 2r, r = 1, 2, \ldots$

$$
\psi_{m_r, \ell_{\ell^2}}(\omega) = \frac{\Gamma^r}{\Gamma^r - 1} \frac{\sin \theta}{\sin \theta} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!}
$$

where $\ell_r, \ell_{r-1}, m_r$ are restricted by the condition that $\ell_r$ be a polynomial, i.e.

$$
-\ell_r + (\ell_{r-1} + m_r) = -2n, \quad n = 0, 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor
$$

and for $p = 2r+1, r = 1, 2, \ldots$

$$
\psi_{m_r, \ell_{\ell^2}}(\omega) = \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!} \frac{\ell_{\ell^2}!}{\ell_{\ell^2}!}
$$

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with the restriction

\[ \ell_r - \ell_{r+1} = -n \quad n = 0, 1, \ldots, \left\lfloor \frac{\ell_{r+1}}{2} \right\rfloor. \]  

(A, 12)

Both solutions (A, 9) and (A, 11) can be expressed in terms of d-functions and exponential functions (using e.g. [19]). The basis of the Hilbert space \( \mathcal{H}_r^{\ell}(S^{r-1}) \) is then given by expressions (3, 18).

Irreducibility a) \( \rho = 2\pi, r = 1, 2, \ldots \). The proof is based on induction.

The representation \( D^m(S^r) \) of the group \( SO(2) \) is irreducible on the one-dimensional space \( \mathcal{H}_r^m(S^r) \) determined by the vector

\[ Y_{m_r}^{\ell}(\varphi) = \frac{1}{\mathcal{D}^{\ell}(S^{r-1})} \exp(i m_r \varphi) \].

Let us suppose that the representation \( D^m(S^{r-1}) \) is irreducible on \( \mathcal{H}_r^{\ell}(S^{r-1}) \) and then let us show that the representation \( D^r(S^{r-1}) \) must be irreducible on \( \mathcal{H}_r^r(S^{r-1}) \).

Denoting \( \mathcal{H}_r^{\ell}, m_r = \mathcal{H}_r^{\ell}(S^{r-1}) \otimes \mathcal{H}_r^m(S^1) \) we can represent the space \( \mathcal{H}_r^r(S^{r-1}) \) in the form

\[ \mathcal{H}_r^r(S^{r-1}) = \sum \mathcal{H}_{\ell_r, m_r}, \]  

(A, 13)

where the sum is taken over all such integers \( \ell_r, m_r \), which satisfy the condition

\[ |\ell_r| + |m_r| + 2n = \ell_r, \quad n = 0, 1, 2, \ldots, \left\lfloor \frac{\ell_r}{2} \right\rfloor. \]

(In the following we use the convention \( \ell_{r-1} = m_r \) for \( r = 2 \)). The decomposition (A, 13) is represented by the net in Fig. 8,
where the knots correspond to the subspaces $\mathcal{H}_{\ell - 1, m + 1}$. Hence to prove the irreducibility of the representation $D_{(\ell - 1, m + 1)}$ it is sufficient to show that the vector 

$$(L_{r - 1, r - 2} \phi_{m, r + 1, o})(w)$$

has non-vanishing components in all four possible neighbouring subspaces

$\mathcal{H}_{\ell, \ell + 1}$. Here $L_{r - 1, r - 2}$ is defined by (6, 2) and

$$\phi_{m, r + 1, o}(w) = \psi_{m, r + 1, o}(w).$$


(A, 14)

where

$$Z_{m, r - 1}(w) = \frac{1}{2i} \phi_{m, r + 1, o} - \frac{1}{2i} \phi_{m, r - 1, o}.$$

The coefficients $A_{\pm}(\ell - 1)$ are defined by the expression (6, 7),

$$A_{\pm}(\ell - 1) = \frac{1}{2i} \quad \text{and} \quad N(\ell - 1) = N_{r - 1} \quad \text{is defined by (3, 21).}$$
The coefficients in the expression \((A, 14)\) do not vanish for any two non-negative integers \(\ell_r, m_r\) which satisfy the condition
\[
|\ell_r| + |m_r| + 2n > \ell_r, \quad n = 0, 1, \ldots, \left[\frac{\ell_r}{2}\right],
\]
except \(A_-(\ell_r^2)\) for \(\ell_r = 1\). However the mapping \(\mathcal{H}_{\ell_r, m_r}^{\ell_r, 2} \rightarrow \mathcal{H}_{0, m_r+1}^{\ell_r, 2}\) is possible as the operator \(L_{\ell_r, m_r}^{\ell_r, 2}\) is skew-symmetric on \(\mathcal{H}_{\ell_r, m_r}^{\ell_r, 2}\) and \(\mathcal{H}_{0, m_r+1}^{\ell_r, 2}\), where

\[
\phi_{\ell_r, m_r}^{\ell_r, 0}, Z_{m_r}^{\ell_r, 1} 
\]

and

\[
\phi_{\ell_r, m_r}^{\ell', 0}, Z_{m_r}^{\ell', 1} 
\]

are non-zero.

b) \(\rho = 2\ell_1, \ell_1, \ell_2, \ldots\). Using the operator \(L_{\ell_1, \ell_2}^{\ell_1, 2}\) and the element
\[
\phi_{\ell_1, \ell_2}^{\ell_1, \ell_2, \ldots} = \psi_{\ell_1, \ell_2}^{\ell_1, \ell_2, \ldots} (\nu_1, \nu_2, \ldots) \psi_{\ell_1, \ell_2, \ldots}^{\ell_1, \ell_2, \ldots, \ell_{\rho-n}} \]

where \(\ell_1, \ldots, \ell_{\rho-n}\) take the minimal possible values, we prove the irreducibility of the representation \(D^{\ell_1}_{\ell_2} (S^{\rho-2})\) on the space \(\mathcal{H}_{\ell_1, \ell_2}^{\ell_1, 2} (S^{\rho-2})\) from the irreducibility of the representations \(D^{\ell_1}_{\ell_2} (S^{\rho-2})\) on \(\mathcal{H}_{\ell_1, \ell_2}^{\ell_1, 2} (S^{\rho-2})\) similarly as before. The irreducibility of the representations
\[
D^{\ell_1}_{\ell_2} (S^{\rho-2}) \quad \text{on} \quad \mathcal{H}_{\ell_1, \ell_2}^{\ell_1, 2} (S^{\rho-2})
\]

have been proved in the previous case.

**Unitarity.** Due to the left-invariance of the measure \(d\mu(\omega) (A, 8)\) on the sphere \(S^{\rho-1}\) the representations \(D^{\ell_1}_{\ell_2} (S^{\rho-1})\) and \(D^{\ell_r, 2}_{\ell_1, \ell_2} (S^{\rho-1})\) are unitary. (See the analogous proof in Section 6, II).
REFERENCES


[2] Of course, besides the principal series, there also exists a supplementary series, which we do not consider. Their existence is closely related to the existence of a double point measure [3].


In fact this theorem has been proved for the space $X$ being Euclidean or a global symmetric one of rank 1 ([9], chap. X, §2). However, we may extend it on homogeneous spaces of rank 1 by using results of Gel'fand and Graev [10].

If no other indication is given we employ the Einstein summation convention.


SO$_0(a,b)$ denotes a component of a unity of the group SO($a,b$).

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Here and elsewhere, we use brackets for indices defined as follows:

\[
\begin{align*}
\left[ \frac{a}{2} \right] & = \begin{cases} 
\frac{a}{2} & \text{if } a = 2r, \\
\frac{a-r}{2} & \text{if } a = 2r+1,
\end{cases} \\
\left\{ \frac{a}{2} \right\} & = \begin{cases} 
\frac{a}{2} & \text{if } a = 2r, \\
\frac{a+r}{2} & \text{if } a = 2r+1,
\end{cases}
\end{align*}
\]

The measure $d\mu(\Omega) = \sqrt{H_2^a} d\Omega$ is the Riemannian measure, which is left invariant under the action of SO($a,b$) on $H_2^a$ [9].

E. L. Ince, Ordinary Differential Equations (Dover publications, 1956).

