INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

PLASMA WAVE PROPAGATION IN HOT INHOMOGENEOUS MEDIA

M. N. ROSENBLUTH
H. L. BERK
AND
R. N. SUDAN

1966
PIAZZA OBERDAN
TRIESTE
PLASMA WAVE PROPAGATION IN HOT INHOMOGENEOUS MEDIA

H. L. BERK*
M. N. ROSENBLUTH**
and
R. N. SUDAN***

TRIESTE
19 January 1966

* On leave of absence from the University of California, San Diego, California, U.S.A.
** On leave of absence from the University of California, San Diego, California, U.S.A. and from General Atomic, San Diego, California, U.S.A.
*** On leave of absence from Cornell University, Ithaca, New York, U.S.A.
PLASMA WAVE PROPAGATION IN HOT INHOMOGENEOUS MEDIA

We consider here in the WKB approximation the propagation of electrostatic plasma waves in a system described by the one-dimensional linear Vlasov equation in the presence of a slowly varying static potential, $\phi(x)$. Previous work $^{(1)}$ has also treated this problem in a similar manner but their results lack the generality of the expressions derived here, and we demonstrate the formal basis of the expansion for the Vlasov case.

The fundamental equations for the system are,

$$\frac{df}{dt}(x, v, t) = \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \nabla \phi(x) \frac{\partial f}{\partial v}$$

$$= -\frac{e}{m} v \mathcal{E}(x, t) \frac{\partial F(E)}{\partial E}$$

$$\frac{\partial \mathcal{E}}{\partial t}(x, t) + 4\pi j(x, t) = 0,$$

where $f$ is the perturbed distribution function; $F(E)$ is the equilibrium distribution function; $x, v, t$ are the space, velocity and time coordinates; $E = \frac{v^2}{2} + \phi(x)$ is the normalized particle energy; $\frac{e}{c} \phi(x)$, the static potential, is taken as zero at $x = -\infty$ and increases monotonically with $x$; $\mathcal{E}(x, t)$ is the perturbed electric field, and $e$
and \( m \) are the particles' charge and mass. The current, \( j \), is given by,

\[
j(x, t) = n_0 e \int \frac{f^+ - f^-}{\phi(x)} dE,
\]

(2a)

where \( f^+ \) and \( f^- \) are \( f(x, v, t) \) for \( v > 0 \) and \( v < 0 \) respectively. The normalization of \( F \) is chosen such that at \( x = -\infty \) the particle density is \( n_0 \) and

\[
\int_{-\infty}^{x} F \left( \frac{v^2}{2} \right) dv = 1.
\]

We imagine that the perturbation is the steady state response to a local generator at frequency \( \omega \), and we examine the solution far from the source.\(^*\)

We can integrate equation (1) by the method of characteristics. The perturbed distribution function \( f(x, v) e^{-i\omega t} \), is then found to be,

\[
f^+(x, v) = -\frac{e}{m} \frac{\partial F}{\partial E} \int_{-\infty}^{x} dx' \tilde{E}(x') \exp \left[ i \omega \int_{x'}^{x} \frac{dx''}{v(x'')} \right],
\]

(3a)

\[
f^-(x, v) = \frac{e}{m} \frac{\partial F}{\partial E} \int_{x}^{x_0} dx' \tilde{E}(x') \exp \left[ i \omega \int_{x}^{x_0} \frac{dx''}{v(x'')} \right]
\]

(3b)

\[-\frac{e}{m} \frac{\partial F}{\partial E} \exp \left[ i \omega \int_{x}^{x_0} \frac{dx'}{v(x')} \right] \int_{-\infty}^{x} dx' \tilde{E}(x') \exp \left[ i \omega \int_{x}^{x_0} \frac{dx''}{v(x'')} \right],\]

where \( v(x) = +\sqrt{2(E - \Phi(x))} \), \( \tilde{E}(x, t) = \tilde{E}(x) e^{-i\omega t} \) and \( x_0 \), is the turning point of the trajectory of a particle with energy \( E \), and is determined by the relation \( \Phi(x_0) = E \).

\(^*\) The formal procedure is to consider our equations as Laplace transforms in time. The solutions are then the response to a given \( \omega \) component, where \( \omega \) approaches the real axis from above.
After equation (3) is substituted into equation (2), we obtain,

\[ -i\omega \mathcal{E}(x) = -4\pi j(x) = \omega_p^2 \int_\infty^{-\infty} dE \frac{\partial F}{\partial E} \left\{ \int_{x'}^{x} dx' \mathcal{E}(x') \exp \left[ i\omega \int_{x'}^{x} dx'' \right] \right\} 
+ \int_{x}^{x_0} dx' \mathcal{E}(x') \exp \left[ i\omega \int_{x}^{x_0} dx'' \right] 
- \exp \left[ i\omega \int_{x}^{x_0} \frac{dx'}{v(x')} \right] \int_{x}^{x_0} dx' \mathcal{E}(x') \exp \left[ i\omega \int_{x}^{x_0} dx'' \right] \right\} , \tag{4} \]

where \( \omega_p^2 = \frac{4\pi e^2 n_0}{m} \).

We now desire to use the fact that \( \Phi(x) \) is slowly varying. Hence, we postulate that the solution for \( \mathcal{E}(x) \) is of the form

\[ \mathcal{E}_0(x) \exp \left[ i \int k(x) \, dx \right] , \]

where we assume \( \frac{1}{k \mathcal{E}_0} \frac{\partial \mathcal{E}_0}{\partial x} \sim \frac{\partial k}{\partial x} \sim \frac{1}{k \Phi} \frac{\partial \Phi}{\partial x} \equiv \delta \ll 1 \). We seek an asymptotic solution in \( \delta \) of equation (4).

Integrating by parts the first \( x \) integral on the right-hand side of equation (4), we obtain,

\[ \int_{x}^{x_0} dx' \mathcal{E}(x') \exp \left[ i\omega \int_{x}^{x_0} \frac{dx'}{v(x')} \right] = \int_{x}^{x_0} dx' \mathcal{E}_0(x') \exp \left[ i \int k(x') \, dx'' \right] + i\omega \int_{x}^{x_0} \frac{dx''}{v(x'')} \]

\[ = \exp \left[ i \int dx' k(x') \right] \left\{ \frac{\mathcal{E}_0(x)}{i \left[ k(x) - \frac{\omega}{v(x)} \right]} + \frac{\mathcal{E}_0(x)}{2} \frac{d}{dx} \frac{1}{\left[ k(x) - \frac{\omega}{v(x)} \right]^2} \right\} 
+ \frac{d\mathcal{E}_0}{dx} \frac{1}{\left[ k(x) - \frac{\omega}{v(x)} \right]^2} \]

\[ -3- \]
The last integral in this expression is of $O(\delta^2)$ and may therefore be neglected with respect to the other terms which are of $O(1)$ and $O(\delta)$ respectively. The integration by parts can be repeated to generate an asymptotic power series in $\delta$ where the remainder integral is $O(\delta^n)$ after the first $n$ terms.

The second and third integrals on the right-hand side of Eq. (4) can similarly be integrated by parts to generate an asymptotic power series in $\delta$. It can be shown that the contributions at $x_0$ of the second and third integrals in Eq. (4) cancel term by term to all orders in $\delta$ (see Appendix A); and consequently these contributions are exponentially small. Thus we obtain an asymptotic expansion of the integral equation in powers of $\delta$.

The integral equation is then reduced to a set of local equations to each order in $\delta$. The solution, with the exception of an exponentially small correction, is then expressed as an asymptotic series in $\delta$.

Correct to $O(\delta)$, we have,
\[ -i \omega \xi_0(x) - \omega^2 \int_\Phi(x) dE \frac{\partial F}{\partial E} \left\{ \xi_0(x) \left[ \frac{1}{i(k(x) - \frac{\omega}{v(x)})} - \frac{1}{i(k(x) + \frac{\omega}{v(x)})} \right] \right\} \]

\[ \xi_0(x) \frac{d}{dx} \left[ \frac{1}{(k(x) - \frac{\omega}{v(x)})^2} - \frac{1}{(k(x) + \frac{\omega}{v(x)})^2} \right] = 0. \]  

We can now choose \( k(x) \) so that the first two terms (the lowest order terms) vanish. This constrains \( k(x) \) to obey the relation,

\[ \epsilon_\omega(k, k, x) = 1 - \frac{\omega^2}{\omega} \int_\Phi(x) dE \frac{\partial F}{\partial E} \left[ \frac{1}{(k) - \frac{\omega}{v(x)}} - \frac{1}{(k) + \frac{\omega}{v(x)}} \right] = 0. \]  

One observes that \( \epsilon_\omega(k, k, x) \) is the dielectric function of a homogeneous medium with density \( n(x) \), the local density of our system. We now use the relations,

\[ \frac{\partial \epsilon_\omega(k, k, x)}{\partial k} = \frac{\omega^2}{\omega} \int_\Phi(x) dE \frac{\partial F}{\partial E} \left[ \frac{1}{(k - \frac{\omega}{v})^2} - \frac{1}{(k + \frac{\omega}{v})^2} \right], \]  

\[ \frac{d}{dx} \frac{\partial \epsilon_\omega(k, k, x)}{\partial k} = \frac{\omega^2}{\omega} \int_\Phi(x) dE \frac{\partial F}{\partial E} \frac{d}{dx} \left[ \frac{1}{(k - \frac{\omega}{v})^2} - \frac{1}{(k + \frac{\omega}{v})^2} \right], \]

where we have used the fact that the integrand vanishes when \( E = \Phi(x) \). Equation (5) then takes the form,

\[ \frac{\xi_0(x)}{2} \frac{d}{dx} \frac{\partial \epsilon_\omega(k(x), k(x))}{\partial k} + \frac{\xi_0(x)}{2} \frac{d}{dx} \frac{\partial \epsilon_\omega(k(x), k(x))}{\partial k} = 0. \]
The solution to this equation is

\[ \mathcal{E}_0(x) = \frac{C}{\partial (\varepsilon(\omega, k(x), x))} \]

where \( C \) is an arbitrary constant of integration.

Thus, the electric field \( \mathcal{E}(x) \) is given by the relation,

\[ \mathcal{E}(x) = \frac{C e^{i \int k(x) dx}}{\partial (\varepsilon(\omega, k(x), x))} \]

where \( k(x) \) satisfies the relation \( \varepsilon(\omega, k(x), x) = 0^* \).

It is well known that for a nearly dissipationless system, the energy flux \( T \) is given by

\[ T = \frac{\omega \mathcal{E}^2(x)}{16\pi} \frac{\partial \text{Re} (\varepsilon(\omega, k))}{\partial k} \mathcal{E}(x). \]

Thus we see that if \( \varepsilon(\omega, k) \) is nearly real, the energy flux is,

\[ T = -\frac{\omega}{16\pi} |C|^2 \exp \left[ -2 \int \text{Im} k(x) dx \right], \]

and

\[ \frac{dT}{dx} = -2 \text{Im} k(x) T. \]

We see that in the limit \( \text{Im} k(x) \to 0 \), the wave energy which consists of both electric field energy and coherent kinetic energy, is conserved by our solution.

* Alternatively, we could have chosen to keep \( \mathcal{E}_0 \) constant in space, and generate an asymptotic series for \( k \). In this case, the equations for the \( k \) corrections are algebraic rather than differential and the spatial variation of the electric field is then given through \( k \).
It is interesting to recover directly from Eq. (4) the effect of resonant particles in slightly inhomogeneous media, i.e. Landau damping. For this purpose, we assume \( \omega \) and \( k \) to be real and compute the contribution of the resonant particles to the current \( j(x) \). From Eq. (4) \( j(x) \) may be rewritten as

\[
-4\pi j(x) = \omega_p^2 \exp \left[ i \int k(x') \, dx' \right] \int dE \frac{\partial F}{\partial E} \left\{ \int dx' \xi_0(x') \right\}
\]

\[
\left[ \exp \left[ i \int \limits_{x_0(E)}^{x''} \left( k(x'') - \frac{\omega}{v(x'')} \right) \, dx'' \right] - \exp \left[ i \int \limits_{x_0(E)}^{x'} \left( k(x'') + \frac{\omega}{v(x'')} \right) \, dx'' \right] \right] 
\]

\[
\int \limits_{x_0(E)}^{x'} \left( k(x') + \frac{\omega}{v(x')} \right) \, dx' \xi_0(x') \left[ \exp \left[ i \int \limits_{x_0(E)}^{x'} \left( k(x'') - \frac{\omega}{v(x'')} \right) \, dx'' \right] - \exp \left[ i \int \limits_{x_0(E)}^{x'} \left( k(x'') + \frac{\omega}{v(x'')} \right) \, dx'' \right] \right] 
\]

For \( \omega \) and \( k \) positive real it can be seen that only the first and third terms can have points of stationary phase in the integration. The additional phase factor \( \exp \left[ i \int \limits_{x_0(E)}^{x'} \left( k(x') - \frac{\omega}{v(x')} \right) \, dx' \right] \) in the third term oscillates rapidly for all resonant particles and hence the \( E \) integration makes this an exponentially small term, so that we need only consider the first term. The main contribution in the \( x' \) integration of the first term comes from the vicinity of the point of stationary phase \( x_s \) given by \( k(x_s) - \frac{\omega}{v(x_s)} = 0 \) which is the point where a particle of energy \( E \) is in local resonance with the wave. We expand (see Appendix B) the integrand of the \( x' \) integration about \( x_s \) and transform the energy integration to one over \( x_s \) by means of the transformation

\[
E = \Phi(x_s) + \frac{1}{2} \frac{\omega^2}{k^2(x_s)} 
\]
The resulting expression for the "resonant" component of the current
\( j_r(x) \) is

\[
-4\pi j_r(x) = \omega_p^2 \exp \left[ i\int k(x') \, dx' \right] \int_\infty^\infty dx_s \frac{dE}{dx_s} \frac{\partial F}{\partial E}.
\]

\[
\mathcal{E}_0(x_s) \exp \left[ i\int k(x') \left( \frac{\omega}{\sqrt{E(x_s), x'}} \right) \, dx' \right]
\]

\[
\int_\infty^x dx' \exp \left[ i\beta(x_s) (x' - x_s)^2/2 \right]
\]

where

\[
\beta(x_s) = \frac{d}{dx} \left( k - \frac{\omega}{\sqrt{E(x_s)}} \right) \bigg|_{x=x_s} = -\frac{k^3}{\omega^2} \frac{dE}{dx_s}
\]

The choice of limits on the \( x_s \) integration depends upon the sign of \( \beta \) and for definitiveness we have taken it to be positive. The point of stationary phase in the \( x_s \) integration now occurs at \( x_s = x \) and on expanding the integrand around this point we have

\[
4\pi j_r(x) = \omega_p^2 \mathcal{E}(x) \exp \left[ i\int k dx' \right] \left( \frac{dE}{dx_s} \frac{\partial F}{\partial E} \right)_{x_s=x}.
\]

\[
\int_\infty^\infty dx_s \exp \left[ -i\beta(x) (x_s - x)^2/2 \right] \int_\infty^x dx' \exp \left[ i\beta(x) \frac{(x'-x_s)^2}{2} \right]
\]

\[
= -\pi \omega_p^2 \frac{\omega^2}{k^3(x)} \frac{dF}{dE} \bigg| \mathcal{E}(x) \exp \left[ i\int k dx' \right]
\]

\[
E = \Phi(x) + i\frac{\omega^2}{k^3(x)}
\]

* which is the usual Landau result and is contained in Eqs. (5) and (6) if

* The double integral has been evaluated as follows:

\[
\int_\infty^\infty dx_s \int_\infty^x dx' \exp \left[ i\beta \left( (x_3 - x')^2 - (x - x_3)^2 \right) \right]
\]

\[
= \int_\infty^x dx' \exp \left[ i\beta \left( (x' - x)^2/2 + i\Phi(x) \right) \right] \int_\infty^\infty dx_s \exp \left[ -i\beta x_s (x' - x) \right]
\]

\[
= \frac{2\pi}{\beta} \int_\infty^x dx' \delta(x' - x) \exp \left[ i\beta \left( x'^2 - x^2 \right) /2 \right] = \frac{\pi}{\beta}.
\]
the Landau prescription for the integrals is followed.

If we consider waves that propagate at an angle to the direction of the inhomogeneity i.e. of the form \( \xi(x) \exp \{ i k_y y \} \), then a parallel analysis yields,

\[
\xi(x) = \frac{Ck}{\frac{\partial \epsilon}{\partial k_x}} \exp \left[ i \int x(x') \, dx' \right].
\]  \hspace{1cm} (17)

A more detailed discussion of this subject will appear shortly.

ACKNOWLEDGMENTS

The authors would like to thank Prof. C. Oberman for valuable discussion.

The authors are grateful to the IAEA for the hospitality extended to them at the International Centre for Theoretical Physics, Trieste, to the U.S. Atomic Energy Commission and Cornell University for financial support.
APPENDIX A

In order to demonstrate that the contribution of the integrals at $x_0$ in Eq. (4) vanishes to all orders in $\delta$ we combine the terms involving $x_0$ in the following manner. We denote this contribution at the upper limit $x_0$ by $G(x_0)$,

$$G(x_0) \equiv - \exp \left[ i \int_{x}^{x_0} \frac{dx'}{x} \int dx' \, \mathcal{E}_o(x') \left( \exp \left[ i \int^{x'}_{x} \left( k(x') - \frac{\omega}{v(x')} \right) dx'' \right] - \exp \left[ i \int^{x'}_{x} \left( k(x') + \frac{\omega}{v(x')} \right) dx'' \right] \right) \right]$$

$$\equiv - \exp \left[ i \int_{x}^{x_0} \left( k(x') + \frac{\omega}{v(x')} \right) dx' \right] \left[ g(\omega) - g(-\omega) \right].$$

By repeated parts integration we can expand $g(\omega)$ in an asymptotic series,

$$g(\omega) = \lim_{x \to x_0} \left[ D^{(u)} - D^{(b)} + D^{(n)} \ldots (-1)^n D^{(n)} \right] \frac{\mathcal{E}_o(x)}{i(k(x) - \frac{\omega}{v(x)})}$$

$$+ (-1)^{n+1} \int dx' \, \exp \left[ i \int_{x}^{x_0} \left( k(x') - \frac{\omega}{v(x')} \right) dx'' \right] \frac{d}{dx'} \left[ D^{(n)} \frac{\mathcal{E}_o(x')}{i(k(x') - \frac{\omega}{v(x')})} \right]$$

where the operator $D^{(n)}$ is the operator $D$ repeated $n$ times and $D$ is defined by

$$D \equiv \frac{1}{i(k(x) - \frac{\omega}{v(x)})} \frac{d}{dx}. $$

-10-
On evaluating the series explicitly and keeping in mind that \( v(x_0) \) vanishes, the \((r + 1)\)th term is

\[
\left(\frac{-1}{\kappa}\right)^{r+1}\frac{(2r-1)(2r-3)\cdots 3\cdot 1}{\kappa}\left(\frac{\Phi'(x)}{\omega^2}\right)^r \epsilon_0(x_0).
\]

We observe that the terms in the \( g(o) \) series are even in \( \omega \), and so all the terms in \( g(o) \) cancel those in \( g(-\omega) \) individually, leaving only an exponentially small residue.
APPENDIX B

We would like to establish that the resonant contribution to the integral

\[ I \equiv \int_{-\infty}^{\infty} dx' \hat{E}(x') e^{i \int_{x}^{x'} dx'' \left( k(x'') - \frac{\omega}{V(x'')} \right)}, \] (B.1)

can be expanded in an asymptotic series by a local expansion about the stationary point \( x_s \), where \( k(x_s) = \frac{\omega}{V(x_s)} \). Unlike the usual stationary phase analysis where the parameter in the ensuing exponential is large, here the parameter will be small and approaches zero as the spatially homogeneous limit is approached.

If we expand the integrand in Eq. (B.1) about the point \( x_s \), we obtain

\[ I = \int_{-\infty}^{\infty} dx' \hat{E}(x_s) \left\{ 1 + (x' - x_s) \frac{\hat{E}'(x_s)}{\hat{E}(x_s)} + \cdots \right\} \exp \left[ i \beta \left( x' - x_s \right)^2 + i \frac{\beta'}{6} (x' - x_s)^3 + \cdots \right] \] (B.2)

where, as in the text,

\[ \beta \equiv \left. \frac{d}{dx} \left( k(x) - \frac{\omega}{V(x)} \right) \right|_{x = x_s}. \]

Now, the dominant contribution to the integrand comes from the region where \( \beta (x' - x_s)^2 \ll 1 \). Hence the region in the integration that contributes to the integral is \( (x' - x_s) \sim \frac{1}{\beta^{\frac{3}{2}}} \sim \frac{1}{k \delta^3} \). In this region, the order of the first correction terms are \( \delta^{\frac{3}{2}} \) since \( \hat{E}'(x_s) \sim \hat{E}(x_s) \) and \( \beta^{\frac{3}{2}} \sim \delta^{\frac{2}{2}} k^3 \).
Hence we see that these terms are small where the dominant con-
tribution to the integrand arises. Similarly, the $n^{th}$ order terms are
small by a factor $\delta^{n\frac{1}{2}}$, so that an asymptotic series for this integral
can be generated in powers of $\delta^{\frac{1}{2}}$.

We also notice that there are stationary point contributions to
this integral, even if $x_s$ falls outside the interval between $x_1$ and $x$,
as long as $|x_s - x| \leq \frac{1}{k\delta^{\frac{1}{2}}}$. Hence, formally, the limits of $x_s$ are
taken from $-\infty$ to $\infty$ as $\delta \to 0$. 

-13-
REFERENCES

