SOME REMARKS
ON THE CONSTRUCTION OF INVARIANTS
OF SEMI-SIMPLE LOCAL LIE GROUPS

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OF INVARIANTS OF SEMI-SIMPLE LOCAL LIE GROUPS*

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ABSTRACT

A general form of the \( \ell \) invariants of compact semi-simple local Lie groups of rank \( \ell \), as the traces of the powers of the "velocity potential" operator is suggested. The connection of this form of the invariants with those of Ref. 3 is described. The possible generalization beyond those of adjoint group and its connection with the one of BIEDENHARN is discussed.
Several attempts have been made in recent years to obtain the invariants of semi-simple local Lie groups. They all consist in generalizing Racah-type invariants and to show that there are only \( \ell \) independent invariants for a group of rank \( \ell \). Such invariants have been constructed in the earlier literature for the special case of adjoint groups. The inadequacy of these invariants especially to suit the covariant and contravariant representations has been pointed out in Ref. 1. The object of this paper consists in realizing that these nth-order invariants of the semi-simple local Lie group of rank \( \ell \) can be expressed as the spur of the nth power of the velocity potential \( U \) operator of the group of the infinitesimal generators. Also it is shown that since this velocity potential operator has always an expansion in terms of the self-representation of the infinitesimal generators, one can always choose the self-representation for the infinitesimal generators without loss of generality. The connection of the present work with that of Ref. 3 is given. The tensor behaviour of \( U \) is pointed out. We essentially follow the treatment of Ref. 4 for notation and subject.

Since later we are going to deal with the "velocity potential" of the adjoint group, let us introduce its properties here. It is defined to be

\[
U^i_\alpha (x) = \left[ \frac{\partial \phi^i(x,y)}{\partial y^\alpha} \right]_{y=0}
\]

(1)

where the \( \phi \)'s are the transformation functions of the Lie group.
In fact, as is well known, the whole analysis and the classification of continuous groups are accomplished by the study of $\mathcal{U}$. The infinitesimal generators of the group $X$ are defined by

$$X^\alpha = \sum_i \mathcal{U}_i^\alpha (x) \frac{\partial}{\partial x_i}.$$  \hfill (2)

The functions $\phi$ are analytic and have an expansion

$$\phi_\alpha (x, y) = x_\alpha + y_\alpha + \alpha_\rho^\alpha x_\rho y_\chi + \cdots.$$  \hfill (3)

The structure constants of the group $C_{\rho}^\chi$ are related to $\alpha$'s through the relation

$$C_{\rho}^\chi = \alpha_{\rho}^\alpha - \alpha_{\chi}^\alpha.$$  \hfill (4)

These things are indeed well known and are introduced just for continuity and notation. The $\mathcal{U}$ operator for the group $O(3)$, for example, looks like

$$\mathcal{U}_i^\alpha (x) = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \quad x = (x, y, z).$$  \hfill (5)

The adjoint group $\mathcal{P}$ of a group $G$ is defined through the homomorphism of $G$ on the group of matrices. So, to every element $x \in G$, there corresponds a matrix $\mathcal{P} \in \mathcal{P}$. The adjoint group of the infinitesimal group is called infinitesimal adjoint group.

Let us start with the Casimir operator

$$\mathcal{I}_2 = \alpha^\alpha \beta^\beta.$$  \hfill (6)

Now, if one wants to express this as a trace which could later be
generalized, one naturally introduces the concept of a matrix. This association is sometimes achieved by taking the adjoint representation in which

$$ C^\sigma_{\alpha \sigma_2} = \left( X^\alpha \right)^\sigma_{\sigma_2} $$

and so $I_2$ can be expressed as

$$ I_2 = \text{Tr} \left( X^\alpha X^\beta \right) X^\alpha X^\beta. \quad (8) $$

It is then looked for replacing the adjoint representation by some general representation, so that when $I_2$ is generalized, it is really independent of the choice of the representation. What we want to emphasize is that we need essentially an association (whether with adjoint representation or otherwise) of $(C X)$ with a matrix in order to express $I_2$ as a trace. In other words, if we define

$$ \gamma^\sigma_{\alpha \sigma_2} = C^\sigma_{\alpha \sigma_2} X^\alpha \quad (9) $$

then

$$ I_2 = \text{Tr} \left( \eta \right)^{\omega} \quad (10) $$

so that the generalized nth-order invariant may just be written as

$$ I_n = \text{Tr} \left( \eta \right)^{\gamma}. \quad (11) $$

These are, of course, known to be the invariants of the adjoint group (see Ref. 4 for extensive information) defined as the coefficients in the expansion of the characteristic equation

$$ \Delta \left( x, \rho \right) = \det \left( \eta^\alpha \left( x \right) - \rho \delta^\alpha \right) = 0 \quad (12) $$

as powers of $\rho^3$. This parameter is supposed to define the invariant directions. The theorem of Killing is that the coefficients $\psi$ in the characteristic equation

$$ \Delta \left( x, \tilde{s} \right) = \rho^r - \psi_1(x) \rho^{r-1} + \cdots + (-1)^{r-1} \psi_{r-1}(x) \rho^1 + \cdots + (-1)^{r-1} \psi_{r-1}(x) \rho^1, \quad (13) $$
of the group are in fact the invariants of the adjoint group. Also it has been shown that there are only \( \ell \) independent \( \psi \)'s where \( \ell \) is the rank of the group. The operator \( \eta(x) \) is nothing but the operator \( (X \times X) \) defined in Ref. 3 and is in fact independent of the choice of \( \hat{X} \). They are the velocity potential operators for the group of the infinitesimal generators \( X \) of the group. For the case of \( O(3) \), the operator \( \eta \) is obtained by replacing in the velocity potential \( U(x) \) of the group, the elements of the group by the infinitesimal generators. So, for \( O(3) \) we get

\[
\eta = \begin{pmatrix}
0 & x_3 & -x_2 \\
-x_3 & 0 & x_1 \\
x_2 & -x_1 & 0
\end{pmatrix} = U(x)
\]

which is in fact the operator \( (X \times X) \) of Ref. 3. The invariants of the group are then

\[
\Gamma_n = \Gamma_n \begin{pmatrix}
0 & x_3 & -x_2 \\
-x_3 & 0 & x_1 \\
x_2 & -x_1 & 0
\end{pmatrix}^n
\]

It is easy to show that for \( O(3) \), \( I_3 = f(I_2) \) so that there is only one Casimir operator for \( O(3) \).

So, the method consists in putting in the \( U \) matrix of the group defined through Eq. (1) the infinitesimal generators (defined in Eq. (2)) of the group in the place of the \( X \)'s.
Then take the traces of the powers of this new matrix. It is clear that the number of $X$'s is indeed equal to the order of the group. It is also easy to see that trace $U$ is the same even if one permutes the $X$'s in $U$. Of course, the choice of $U(X)$ strongly reflects that the corresponding group function is

$$\phi^\alpha_{\beta} (x, y) = C_{\beta, \gamma} X_\beta Y_\gamma,$$

so that

$$U^\alpha_{\beta, i} (x) = \left[ \frac{\partial \phi^\alpha_{\beta}}{\partial y_i} \right]_{y=0} = C_{\beta, i} X_\beta,$$

and hence

$$I_2 = \text{tr} (U)^2 = U^\alpha_{\beta, i} U^\alpha_{\beta, i} = C_{\beta, i} X_\beta X_\beta.$$

The form of $U^\alpha_{\beta, i} (x)$ and hence that of $\phi^\alpha_{\beta} (x, y)$ immediately tells us that in fact we are dealing with the invariants of the adjoint group and that one does not get anything general by replacing $X$ by a general representation $\hat{X}$ since we are dealing with the product $\sum_{\alpha} \hat{X}_d \times X^\alpha$. This is independent of choice of the representation $\hat{X}_d$ and thus the particular choice of the $\hat{X}_d$ as the self-representation, (natural basis), made in Ref. 3, is fully justified.

Since $U(X)$ is a transformation function, it can be shown to
be a tensor operator. We will not discuss here about the completeness of these invariants and their explicit construction for special cases, as these problems have been carried out in Ref. 3.

One final remark may be worth pointing out. If one wants to generalize these invariants beyond the adjoint group, one can still retain the form

\[ I_n = T_n (U)^n. \]

But now, the \( U \)'s are defined through the relation

\[ U^\alpha_i = A^\alpha_{\beta i} X_\beta, \]

where the \( A \)'s are not the structure constants. They are the second-order coefficients occurring in the expansion of \( \phi'_\alpha \) in the general case. We know, however, that the structure constants are related to the \( A \)'s by the relation

\[ C^\alpha_{\beta \gamma} = A^\alpha_{\beta \gamma} - A^\alpha_{\gamma \beta}, \]

which is the antisymmetric part.

Normally, in what is called the normal parameter system, one makes the symmetric part of \( A \) vanish so that one can replace \( A \)'s occurring in the expansion of \( \phi' \) by the structure constants which we have done earlier. The generalization of the general invariants beyond those of adjoint group consists in retaining both the symmetric and antisymmetric parts of \( A \); in other words, having the general expansion for \( \phi' \). In this case,

\[ A^\alpha_{\beta \gamma} = \frac{1}{2} (a^\alpha_{\beta \gamma} + a^\alpha_{\gamma \beta}) + \frac{1}{2} (a^\alpha_{\beta \gamma} - a^\alpha_{\gamma \beta}), \]

where \( d^\alpha_{\beta \gamma} \) are symmetric structure constants occurring in the
anticommutator of the \( X \)'s

\[
\{ X_\alpha , X_\beta \} = d^\gamma_{\alpha \beta} X_\gamma
\]

and \( C \)'s are the usual structure constants (antisymmetric) occurring in the commutator of the \( X \)'s

\[
[ X_\alpha , X_\beta ] = C^\gamma_{\alpha \beta} X_\gamma .
\]

So, to conclude, the generalization of \( I_n \) beyond the adjoint group is achieved as

\[
I_n = T_r (U)^n ,
\]

\[
U^i_\alpha = a^i_{\alpha \gamma} X_\gamma ; \quad a^i_{\alpha \gamma} = \frac{1}{2} (d^i_{\alpha \gamma} + c^i_{\alpha \gamma})
\]

Incidentally, the \( a \)'s have been used earlier by BIEDENHARN\(^1\) to construct the general invariants \( I_n \).

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