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INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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ON THE COUPLING OF NON-COMPACT GROUP REPRESENTATIONS

1. In analogy with the case of atomic and nuclear physics, a hope has been expressed (Barut 1965, Fronsdal 1965, Dothan, Gell-Mann and Ne'eman 1965) that "towers" of elementary particle multiplets may possibly correspond to the (infinite-dimensional) unitary representations of certain dynamical non-compact rest-symmetry groups. A second hope could be that if "elementary particles" resemble a liquid drop the dynamics responsible for their binding and structure is also the dynamics responsible for their scattering, so that the same symmetry group or its subgroups manifest themselves in the S-matrix elements.

In a recent paper (Delbourgo, Salam and Strathdee 1965, referred to as I) it was shown that one can always find a relativistic extension of a given rest-symmetry and that from this boosted symmetry (intrinsically broken by kinetic energy terms) there follows a chain of possible symmetry subgroups for a certain class of S-matrix elements. For the rest-symmetry U(6, 6), the relativistic extension was found to be U(6, 6) x U(6, 6) with GL(6, C) as the subgroup for collinear processes, U(3, 3) for co-planar processes and GL(3, C) for processes with four independent momenta.

Now for the case of the compact rest-symmetry U(6) x U(6) with the relativistic extension $\tilde{U}(12)$, it appears experimentally, for reasons one does not yet clearly understand, that the collinear subsymmetry $SU(6)_W$ works reasonably for three-point functions. This is notwithstanding the symmetry-breaking produced by non-collinear intermediate states as well as the symmetry-breaking for such states arising from mass inequalities in the multiplets. One may conceivably hope that the same thing may happen for the case of non-compact rest-symmetries and the maximally allowed collinear symmetry (e. g. GL(6, C)) may exhibit itself for the trilinear couplings of three infinite towers (of e. g. U(6, 6)). In the earlier paper (I) a beginning was made to consider such trilinear couplings for SL(2, C) towers. In the present paper we consider the more realistic case of GL(6, C) and show that a new complication arises; in general it appears that an infinity of independent amplitudes are involved in the three-point function and further approximations must be made to extract physically meaningful predictions from the formalism. We propose to return to this crucial aspect of the approximation problem in another paper; the interest of the present paper is methodological. We wish particularly to emphasise those unresolved problems in the theory of non-compact groups which are at present relatively dark and where more mathematical work is called for.

2. Some Representations of U(ν, ν) and GL(ν, c)

In the hierarchy of subgroups mentioned in Section 1 two types of group occur, U(ν, ν) and GL(ν, c). In order to be able to discuss the decomposition and coupling problems it is necessary to have some detailed information about representations. The purpose of this section is to provide the essential formalism whereby these manipulations can be undertaken at least for a rather simple class of unitary representation.

The only representations of U(ν, ν) which we shall consider are those of the discrete type discussed by Dothan et al. This type of representation is most easily described in terms of a basis consisting of an infinite sequence of symmetrized tensors,

$$\left\{ \Psi(0),\ \Psi(1),\ \ldots,\ \Psi(\ell),\ \ldots \right\}$$

where

$$\Psi(\ell) = \Psi^{\hat{\beta}_1 \cdots \hat{\beta}_\ell}_{\alpha_1 \cdots \alpha_{\gamma+\ell}}, \qquad \alpha, \beta \ldots = 1, 2, \ldots, \nu \; . \tag{2.1}$$

each tensor belonging to an irreducible representation of the maximal compact subgroup $U(\nu) \times U(\nu)$.

Different values of γ label distinct representations of $U(\nu, \nu)$. $\gamma = 0$, for example, corresponds to the well-known meson tower conventionally written as $[(1, \bar{1}); (6, \bar{6}); (21, \overline{21}); \ldots ;]$, while $\gamma = 3$ gives the baryon-tower $[(56, 1); (126, \bar{6}); \ldots]$ if we are dealing with $U(6,6)$ rest symmetry $(\nu = 6)$. The generators of infinitesimal trensformations may be taken in the form

$$M^{\beta}_{\alpha} \; , \; M^{\hat{\beta}}_{\hat{\alpha}} \quad ; \quad M^{\beta}_{\hat{\alpha}} \; , \; M^{\hat{\beta}}_{\alpha} \tag{2.2}$$

cr if one combines α and $\hat{\alpha}$ into one index A taking the values 1, ..., 2ν, simply as

$$M^{B}_{A} = \begin{pmatrix} M^{\beta}_{\alpha} & M^{\hat{\beta}}_{\alpha} \\ M^{\beta}_{\hat{\alpha}} & M^{\hat{\beta}}_{\hat{\alpha}} \end{pmatrix} \tag{2.3}$$

$$(M^{\beta}_{\alpha})^{+} = M^{\alpha}_{\beta} \, , \, (M^{\beta}_{\hat{\alpha}})^{+} = - M^{\hat{\alpha}}_{\beta}$$

These generators satisfy the commutation relations

$$[M^{B}_{A} , M^{D}_{C}] = \delta^{D}_{A} M^{B}_{C} - \delta^{B}_{C} M^{D}_{A} \; . \tag{2.4}$$

The subset M^{β}_{α}, $M^{\hat{\beta}}_{\hat{\alpha}}$ generate the compact part $U(\nu) \times U(\nu)$ while the remainder $M^{\beta}_{\hat{\alpha}}$ and $M^{\hat{\beta}}_{\alpha}$ connect neighboring $U(6) \times U(6)$ multiplets in the sequence (2.1). In detail, these operators are defined by the following relations:

$$
\left.
\begin{aligned}
M_{\alpha}^{\beta}\ \Psi^{\hat{\beta}_1\cdots\hat{\beta}_\ell}_{\alpha_1\cdots\alpha_{\ell+r}} &= -\sum_i \delta^{\beta}_{\alpha_i}\ \Psi^{\hat{\beta}_1\cdots\hat{\beta}_\ell}_{\alpha\alpha_1\cdots(i)\cdots\alpha_{\ell+r}} - \delta^{\alpha}_{\beta}\,\Psi^{\hat{\beta}_1\cdots\hat{\beta}_\ell}_{\alpha_1\cdots\alpha_{\ell+r}} \\
M_{\hat{\alpha}}^{\hat{\beta}}\ \Psi^{\hat{\beta}_1\cdots\hat{\beta}_\ell}_{\alpha_1\cdots\alpha_{\ell+r}} &= \sum_j \delta^{\hat{\beta}_j}_{\hat{\alpha}}\ \Psi^{\hat{\beta}\hat{\beta}_1\cdots(j)\cdots\hat{\beta}_\ell}_{\alpha_1\cdots\alpha_{\ell+r}}
\end{aligned}
\right\}
\qquad (2.5)
$$

$$
\left.
\begin{aligned}
M_{\alpha}^{\hat{\beta}}\ \Psi^{\hat{\beta}_1\cdots\hat{\beta}_\ell}_{\alpha_1\cdots\alpha_{\ell+r}} &= \Psi^{\hat{\beta}\hat{\beta}_1\cdots\hat{\beta}_\ell}_{\alpha\alpha_1\cdots\alpha_{\ell+r}} \\
M_{\hat{\alpha}}^{\beta}\ \Psi^{\hat{\beta}_1\cdots\hat{\beta}_\ell}_{\alpha_1\cdots\alpha_{\ell+r}} &= -\sum_{i,j} \delta^{\beta}_{\alpha_i}\,\delta^{\hat{\beta}_j}_{\hat{\alpha}}\ \Psi^{\hat{\beta}_1\cdots(j)\cdots\hat{\beta}_\ell}_{\alpha_1\cdots(i)\cdots\alpha_{\ell+r}}
\end{aligned}
\right\}
\qquad (2.6)
$$

where $\ell = 0, 1$ and the notation $\alpha_1\cdots(i)\cdots\alpha_{\ell+r}$ indicates that α_i is removed from the sequence $\alpha_1\cdots\alpha_{\ell+r}$. With the formulae (2.5), which completely fix the representation, it is possible to evaluate the Casimir operators.

For the $GL(\nu, C)$ we shall consider only those representations whose bases can be taken in the form of a sequence of symmetrized and traceless tensors,

$$
\{\Psi(\ell)\} = \{\phi_{\alpha_1\cdots\alpha_r},\ \phi^{\beta_1}_{\alpha_1\cdots\alpha_{r+1}},\ \phi^{\beta_1\beta_2}_{\alpha_1\cdots\alpha_{r+2}},\ \cdots,\ \phi^{\beta_1\cdots\beta_\ell}_{\alpha_1\cdots\alpha_{r+\ell}},\ \cdots\}
\qquad (2.7)
$$

each tensor belonging to an irreducible representation of the maximal compact subgroup, $U(\nu)$, $\alpha, \beta, \ldots = 1, 2, \ldots, \nu$. The generators of the infinitesimal transformations of this group may be taken in the form

$$
M_{\alpha}^{\beta}\ ,\ N_{\alpha}^{\beta}\ . \qquad (2.8)
$$

They satisfy the commutation relations,

$$
\left.\begin{aligned}
[M_\alpha^\beta, M_\gamma^\delta] &= \delta_\alpha^\delta M_\gamma^\beta - \delta_\gamma^\beta M_\alpha^\delta \\
[M_\alpha^\beta, N_\gamma^\delta] &= \delta_\alpha^\delta N_\gamma^\beta - \delta_\gamma^\beta N_\alpha^\delta \\
[N_\alpha^\beta, N_\gamma^\delta] &= -\delta_\alpha^\delta M_\gamma^\beta + \delta_\gamma^\beta M_\alpha^\delta
\end{aligned}\right\} \tag{2.9}
$$

The subset M_α^β generate the compact part $U(\nu)$ while the N_α^β connect neighboring $U(\nu)$ multiplets. These operators are defined by the relations

$$
M_\alpha^\beta \phi^{\beta_1 \dots \beta_\ell}_{\alpha_1 \dots \alpha_{r+\ell}} = \sum \delta_\alpha^{\beta_j} \phi^{\beta\beta_1 \dots (j) \dots \beta_\ell}_{\alpha_1 \dots \alpha_{r+\ell}} - \sum \delta^\beta_{\alpha_i} \phi^{\beta_1 \dots \beta_\ell}_{\alpha\alpha_1 \dots (i) \dots \alpha_{r+\ell}}
$$

$$
\begin{aligned}
N_\alpha^\beta \phi^{\beta_1 \dots \beta_\ell}_{\alpha_1 \dots \alpha_{r+\ell}} = {} & A_\ell \phi^{\beta\beta_1 \dots \beta_\ell}_{\alpha\alpha_1 \dots \alpha_{r+\ell}} \\
& + B_\ell \Bigg\{ \sum_j \delta_\alpha^{\beta_j} \phi^{\beta\beta_1 \dots (j) \dots \beta_\ell}_{\alpha_1 \dots \alpha_{r+\ell}} + \sum_i \delta^\beta_{\alpha_i} \phi^{\beta_1 \dots \beta_\ell}_{\alpha\alpha_1 \dots (i) \dots \alpha_{r+\ell}} \\
& \qquad - \frac{2}{x_\ell} \sum_{ij} \delta^{\beta_j}_{\alpha_i} \phi^{\beta\beta_1 \dots (j) \dots \beta_\ell}_{\alpha\alpha_1 \dots (i) \dots \alpha_{r+\ell}} - \frac{2\ell + r}{\nu} \delta_\alpha^\beta \phi^{\beta_1 \dots \beta_\ell}_{\alpha_1 \dots \alpha_{r+\ell}} \Bigg\} \\
& + A_{\ell-1} \Bigg\{ \sum_{ij} \left(\delta_\alpha^{\beta_j} \delta^\beta_{\alpha_i} - \frac{1}{x_\ell} \delta_\alpha^\beta \delta^{\beta_j}_{\alpha_i} \right) \phi^{\beta_1 \dots (j) \dots \beta_\ell}_{\alpha_1 \dots (i) \dots \alpha_{r+\ell}} \\
& \qquad - \frac{1}{x_\ell} \sum_{\substack{i \neq i' \\ j}} \delta^{\beta_j}_{\alpha_i} \delta^\beta_{\alpha_{i'}} \phi^{\beta_1 \dots (j) \dots \beta_\ell}_{\alpha\alpha_1 \dots (i) \dots (i') \dots \alpha_{r+\ell}} \\
& \qquad - \frac{1}{x_\ell} \sum_{\substack{j \neq j' \\ i}} \delta^{\beta_j}_{\alpha_i} \delta_\alpha^{\beta_{j'}} \phi^{\beta\beta_1 \dots (j) \dots (j') \dots \beta_\ell}_{\alpha_1 \dots (i) \dots \alpha_{r+\ell}} \\
& \qquad + \frac{2}{x_\ell (x_\ell - 1)} \sum_{\substack{i > i' \\ j' > j}} \left(\delta^{\beta_j}_{\alpha_i} \delta^{\beta_{j'}}_{\alpha_{i'}} + \delta^{\beta_{j'}}_{\alpha_i} \delta^{\beta_j}_{\alpha_{i'}} \right) \\
& \qquad\qquad \phi^{\beta\beta_1 \dots (j) \dots (j') \dots \beta_\ell}_{\alpha\alpha_1 \dots (i) \dots (i') \dots \alpha_{r+\ell}} \Bigg\}
\end{aligned} \tag{2.10}
$$

where $x_\ell = 2\ell + r + \nu - 2$.

The coefficients A_ℓ and B_ℓ are given by

$$
\begin{aligned}
A_\ell &= \sqrt{\frac{(2\ell + r + \nu)^2 + \rho^2}{(2\ell + r + \nu)(2\ell + r + \nu + 1)}} \\
B_\ell &= \frac{\rho}{2\ell + r + \nu}
\end{aligned}
\qquad (2.11)
$$

The constant ρ is an arbitrary real number which fixes the representation. The second order Casimir operators for example are expressed in terms of ρ (and r) by

$$
\begin{aligned}
M^\beta_\alpha M^\alpha_\beta - N^\beta_\alpha N^\alpha_\beta &= r(r+\nu-1) - (\nu-1)\,\frac{(r+\nu)^2 + \rho^2}{r\nu} \\
M^\beta_\alpha N^\alpha_\beta &= -\rho\,\frac{r(r+\nu-1)}{r+\nu}
\end{aligned}
\qquad (2.12)
$$

Finally it may be remarked that the unitarity of the representations of $U(\nu, \nu)$ and $GL(\nu, c)$ given above is verified by noting the invariance of the positive bilinear forms

$$
\sum_\ell \sum_{\alpha\beta} \frac{1}{\ell!(\ell+r)!} \left| \Psi^{\beta_1 \dots \hat{\beta}_\ell}_{\alpha_1 \dots \alpha_{r+\ell}} \right|^2 \qquad (2.13)
$$

for $U(\nu, r)$, and

$$
\sum_\ell \sum_{\alpha\beta} \frac{1}{\ell!(\ell+r)!} \left| \phi^{\beta_1 \dots \beta_\ell}_{\alpha_1 \dots \alpha_{r+\ell}} \right|^2 \qquad (2.14)
$$

for $GL(\nu, c)$. That these sums are indeed invariant may be verified by applying the generators as defined in (2.5) and (2.10) respectively.

3. The Decomposition of $U(\nu, \nu)$ Representations

In this section the problem of decomposing the particular $U(6, 6)$ representations discussed in Section 2 is formulated. The complete solution to this problem has yet to be worked out.

The infinite-dimensional $U(6, 6)$ tower whose components describe particles at rest must be rearranged into subsets which are irreducible under the hybrid group $GL(6, c)$. This subgroup, we may suppose, is generated by the following combinations of $U(\nu, \nu)$ generators,

$$\left.\begin{aligned} m_\alpha^\beta &= M_\alpha^\beta + M_{\hat\alpha}^{\hat\beta} \\ n_\alpha^\beta &= M_{\hat\alpha}^\beta + M_\alpha^{\hat\beta} \end{aligned}\right\} \tag{3.1}$$

so that m_α^β, which generates $U(\nu)$, will operate within the finite-dimensional $U(\nu) \times U(\nu)$ multiplets and n_α^β will connect neighboring ones. The decomposition under $U(\nu)$ of each level $\bar{\Psi}^{\hat\beta_1 \cdots \hat\beta_\ell}_{\alpha_1 \cdots \alpha_{\ell+r}}$ in the $U(\nu, \nu)$ tower is effected by extracting traces since the distinction between hatted and unhatted indices disappears for the subgroup defined by (3.1). For example, for the meson tower ($r = 0$), $\bar{\Psi}$, $\bar{\Psi}^{\hat\beta}_\alpha$, $\bar{\Psi}^{\hat\beta_1 \hat\beta_2}_{\alpha_1 \alpha_2}$, ... or, in the case $\nu = 6$, for example:

$$(1, \bar{1}),\ (6, \bar{6}),\ (21, \overline{21}),\ \ldots \tag{3.2}$$

we write

$$\begin{aligned} (1, \bar{1}) &= 1 \\ (6, \bar{6}) &= 1' + 35' \\ (21, \overline{21}) &= 1'' + 35'' + 405'', \ \ldots \text{ etc.} \end{aligned} \tag{3.3}$$

the decomposition on the right being unique and invariant for U(6). The primes are used in (3.3) to emphasize that each U(6) representation occurring there corresponds to a distinct set of physical particles (at rest). There are for example an infinite number of singlets $1, 1', 1'', \ldots$ which must ultimately be distributed, in appropriate combinations, among the different GL(6, c) towers that appear in the decomposition. These towers will be of the general type

$$1\ ,\ 35\ ,\ 405\ ,\ \ldots$$

repeated indefinitely though with different values of the Casimir*, ρ. The problem of deciding what values of ρ occur in the decomposition is considered next.

From the commutation rules (2.9) it follows that m_α^α and n_α^α are invariants of GL(6, c) and so must take unique values in each irreducible representation. For m_α^α this gives no information since $m_\alpha^\alpha = r$ on every state in the U(6, 6) tower. For n_α^α, however, the information contained is non-trivial. The method is best demonstrated for the case of the meson tower, $r = 0$, the generalization to $r \neq 0$ being straightforward. It is simplest to begin by trying to construct the U(6) singlet member of a GL(6, c) tower by requiring

$$n_\alpha^\alpha \ \Phi = \lambda \Phi \tag{3.4}$$

where λ is a number. The rest of the tower can then be generated by repeated operations with m_α^β and n_α^β . Since Φ is a singlet

* We are here assuming that the only GL(ν, c) representations appearing in the reduction are of this special type. This assumption must of course be justified by constructing the complete solution of the reduction problem.

it must be a linear combination of 1 , $1'$, $1''$, ... etc.

$$\Phi = \sum_{\ell} a(\lambda,\ell) \; \Psi^{\hat{\alpha}_1 \dots \hat{\alpha}_\ell}_{\alpha_1 \dots \alpha_\ell} \tag{3.5}$$

Using the formulae (2.10)one then finds

$$\begin{aligned}
(n^{\alpha}_{\alpha} - \lambda)\, \Phi &= \sum_{\ell} a(\lambda,\ell) \left(M^{\hat{\alpha}}_{\alpha} + M^{\alpha}_{\hat{\alpha}} - \lambda \right) \Psi^{\hat{\alpha}_1 \dots \hat{\alpha}_\ell}_{\alpha_1 \dots \alpha_\ell} \\
& \sum_{\ell} a(\lambda,\ell) \Big(\Psi^{\hat{\alpha}\hat{\alpha}_1 \dots \hat{\alpha}_\ell}_{\alpha\alpha_1 \dots \alpha_\ell} - \lambda \Psi^{\hat{\alpha}_1 \dots \hat{\alpha}_\ell}_{\alpha_1 \dots \alpha_\ell} \\
& \qquad + \ell(\ell+\nu-1) \Psi^{\hat{\alpha}_2 \dots \hat{\alpha}_\ell}_{\alpha_2 \dots \alpha_\ell} \\
&= \sum_{\ell} \Big(a(\lambda,\ell-1) - \lambda a(\lambda,\ell) \\
& \qquad + (\ell+1)(\ell+\nu)\, a(\lambda,\ell+1) \Big) \Psi^{\hat{\alpha}_1 \dots \hat{\alpha}_\ell}_{\alpha_1 \dots \alpha_\ell} \\
&= 0
\end{aligned}$$

That is

$$a(\lambda,\ell+1) - \lambda a(\lambda,\ell) + (\ell+1)(\ell+\nu)\, a(\lambda,\ell+1) = 0 \quad . \tag{3.6}$$

This recursion relation may be looked upon as an eigenvalue problem whose solution gives the possible values of λ together with the associated coefficients $a(\lambda,\ell)$. In other words,one is looking for a new basis in the representation space, one which diagonalizes the operator n^{α}_{α} . Each vector of the new basis will then be the lowest state of an irreducible representation of the subgroup GL(6, c).

The recursion relations (3.6) have yet to be solved. A possibly fruitful approach may be to replace them with an equivalent differential equation. Thus one could define the function

$$\psi(\lambda,x) = \sum_{\ell} a(\lambda,\ell)\, x^{\ell} \tag{3.7}$$

and derive from (3.6) the equation

$$x\,\psi'' + \nu\,\psi' + (\lambda+x)\,\psi = 0 \tag{3.8}$$

which is closely related to the confluent hypergeometric equation. In order to formulate this as a Sturm-Lionville problem it is necessary to fix the boundary conditions. This must be done in such a way that the solutions satisfy the orthogonality conditions

$$(\lambda-\lambda') \sum_{\ell} \ell!\,(\ell+\nu+1)!\;\, a^{*}(\lambda,\ell)\; a(\lambda',\ell) = 0 \,. \tag{3.9}$$

The authors have not yet succeeded in doing this.

Assuming that the problem can be solved, much depends on whether the spectrum of λ is discrete or not. If it is discrete then the sequence of numbers $a(\lambda,\ell)$, $a(\lambda',\ell)$, ... will decrease rapidly for "large" λ and it may be a feasible approximation for practical calculations to terminate it at some fixed λ . If there is a continuum in λ as well as the discrete spectrum it may yet be reasonable to retain only the discrete part. Finally, however, if there is no discrete part then the problem is hopeless. Any process would require an infinite number of amplitudes for its description.

4. The Coupling Problem

An invariant constructed out of the product of three irreducible towers

$$\{\psi(0),\ \psi(1),\ \dots,\ \psi(\ell),\ \dots\}$$

may in general be written in the form

$$I = \sum_{\ell_1 \ell_2 \ell_3} \sum_{\xi} [\ell_1, \ell_2, \ell_3]_{\xi} \; (\psi(\ell_1)\, \psi(\ell_2)\, \psi(\ell_3))_{\xi} \tag{4.1}$$

where $(\psi(\ell_1)\psi(\ell_2)\psi(\ell_3))_{\xi}$ denotes an invariant of the maximal compact subgroup, $U(\nu) \times U(\nu)$ or $U(\nu)$: ξ indicates the distinct invariants which may be obtainable with the same three tensors. The problem is to determine the coefficients, $[\ell_1, \ell_2, \ell_3]_{\xi}$, so as to make the sum an invariant of the full group, $U(\nu, \nu)$ or $GL(\nu, c)$.

Let us for the sake of definiteness restrict attention to $GL(\nu, c)$ with infinitesimal generators M_{α}^{β} and N_{α}^{β}. Since each term in the sum (4.1) is an invariant of the compact group $U(\nu)$, it is evident that

$$M_{\alpha}^{\beta} \, I = 0. \tag{4.2}$$

The remaining conditions,

$$N_{\alpha}^{\beta} \, I = 0, \tag{4.3}$$

which assure that I is an invariant of $GL(\nu, c)$, imply relations between the various coefficients $[\ell_1, \ell_2, \ell_3]_{\xi}$. They are, in fact, recursion relations. Writing

$$N_{\alpha}^{\beta} \left(\psi(\ell_1)\; \psi(\ell_2)\, \psi(\ell_3) \right) =$$

$$= (N_{\alpha}^{\beta}(1)\, \psi(\ell_1)\psi(\ell_2)\psi(\ell_3) + \psi(\ell_1)\, (N_{\alpha}^{\beta}(2)\, \psi(\ell_2)\, \psi(\ell_3)$$

$$+ \psi(\ell_1)\, \psi(\ell_2)\; (N_{\alpha}^{\beta}(3)\, \psi\, (\ell_3))$$

where

$$N_\alpha^\beta \psi(\ell) = A_\ell \psi(\ell+1) + B_\ell \psi(\ell) + A_{\ell-1} \psi(\ell-1)$$

and equating to zero the various terms in the sum leads to an elaborate set of relations. In general they are highly overdeterminate. The basic problem of proving that there exists a solution in a given case is not considered here. Going by analogy with the SL(2, c) case which has been dealt with in the literature we shall assume that for any three representations of GL(ν, c) - of the type discussed in Section 2 - specified by parameters ρ_1, ρ_2 and ρ_3 there exists exactly one invariant coupling. It is then a simple matter to pick out from the abundance of recursion relations sufficient to enable us to calculate any particular coefficient. In practice it is only the first few that can be of interest.

By way of illustration we compute a coefficient for the coupling of two baryon-like towers to a meson-like tower.

Examples

(1) Baryon-Meson

Let us obtain the first few coefficients in the coupling of the towers

$$\left\{ \phi, \quad \phi_{\alpha_1}^{\beta_1}, \quad \phi_{\alpha_1\alpha_2}^{\beta_1\beta_2}, \quad \ldots \right\}$$

$$\left\{ \psi_{\alpha_1\alpha_2\alpha_3}, \quad \psi_{\alpha_1\ldots\alpha_4}^{\beta_1}, \quad \ldots \right\}$$

$$\left\{ \bar{\psi}^{\beta_1\beta_2\beta_3}, \quad \bar{\psi}_{\alpha_1}^{\beta_1\ldots\beta_4}, \quad \ldots \right\}$$

$$I = \Big\{ [000]\ \bar{\Psi}^{\gamma_1\gamma_2\gamma_3}\ \psi_{\gamma_1\gamma_2\gamma_3} + [110]\ \bar{\Psi}_{\delta_1}^{\gamma_1 \cdots \gamma_4}\ \psi^{\delta_1}_{\gamma_1 \cdots \gamma_4} + \\ + \cdots \Big\}\ \phi$$

$$+ \Big\{ [001]\ \bar{\psi}^{\gamma_1\gamma_2\beta_1}\psi_{\gamma_1\gamma_2\alpha_1} + [101]\ \bar{\psi}_{\alpha_1}^{\gamma_1\gamma_2\gamma_3\beta_1}\ \psi_{\gamma_1\gamma_2\gamma_3} \\ + [011]\ \bar{\psi}^{\gamma_1\gamma_2\gamma_3}\ \psi^{\beta_1}_{\gamma_1\gamma_2\gamma_3\alpha_1} + \cdots \Big\}\ \phi^{\alpha_1}_{\beta_1}$$

$$+ \Big\{ [002]\ \bar{\psi}^{\gamma\beta_1\beta_2}\ \psi_{\gamma\alpha_1\alpha_2} \quad + \quad \cdots \quad \Big\}\ \phi^{\alpha_1\alpha_2}_{\beta_1\beta_2}$$

$$+ \quad \cdots$$

Terms proportional to $\bar{\psi}^{\gamma_1\gamma_2\beta}\ \psi_{\gamma_1\gamma_2\alpha}\ \phi$ are:-

$$3\big(B_0(1) + B_0(2)\big)\ [0\,0\,0] \quad + \quad C_1(3)\ [0\,0\,1] \quad = \quad 0$$

And

$$\bar{\psi}^{\gamma_1\beta_1\beta}\ \ \psi_{\gamma_1\alpha_1\alpha}\ \ \phi^{\alpha_1}_{\beta_1}\ :$$

$$C_1(1)\left(-\frac{6}{x_1}\right)[101] \quad + \quad C_1(2)\left(-\frac{6}{x_1}\right)[0\ 1\ 1]$$

$$+ 2\left(B_0(1)+B_0(2)\right)\ [0\,0\,1] + 4\,C_2(3)\ [0\,0\,2] = 0$$

And $\quad \bar{\psi}^{\gamma_1\gamma_2\gamma_3} \quad \psi_{\gamma_1\gamma_2\gamma_3} \quad \phi^{\beta}_{\alpha}$:

$$C_1(1)\ [1\,0\,1] + C_1(2)\ [0\,1\,1]$$

$$+ A_0(3)\ [0\,0\,0] + B_1(3)\left(-\frac{2}{x_1}\right)[0\,0\,1] + C_2(3)\ \frac{4}{x_2(x_{2-1})}[0\,0\,2] = 0$$

Putting in numbers - these three equations become

$$\frac{3(\rho_1+\rho_2)}{\nu+1}\ [0\,0\,0] + \sqrt{\frac{\nu^2+\rho_0^2}{\nu(\nu+1)}}\ [0\,0\,1] = 0 \tag{i}$$

$$-\frac{6}{\nu+1}\left\{\sqrt{\frac{(\nu+1)^2+\rho_1^2}{(\nu+1)(\nu+2)}}\ [1\,0\,1] + \sqrt{\frac{(\nu+1)^2+\rho_2^2}{(\nu+1)(\nu+2)}}\ [0\,1\,1]\right\}$$

$$+ \frac{2(\rho_1+\rho_2)}{\nu+1}\ [0\,0\,1] + 4\sqrt{\frac{(\nu+2)^2+\rho_3^2}{(\nu+2)(\nu+3)}}\ [0\,0\,2] = 0 \tag{ii}$$

$$\left\{\sqrt{\frac{(\nu+1)^2+\rho_1^2}{(\nu+1)(\nu+2)}}\;[1\,0\,1] + \sqrt{\frac{(\nu+1)^2+\rho_2^2}{(\nu+1)(\nu+2)}}\;[0\,1\,1]\right\}$$

$$+ \sqrt{\frac{\nu^2+\rho_3^2}{\nu(\nu+1)}}\;[0\,0\,0] - \frac{2\rho_3}{\nu(\nu+2)}\;[0\,0\,1] +$$

$$+ \frac{4}{(\nu+1)(\nu+2)} \sqrt{\frac{(\nu+2)^2+\rho_3^2}{(\nu+2)(\nu+3)}}\;[0\,0\,2] = 0 \qquad \text{(iii)}$$

For the case $\rho_1 = \rho_2 = \rho, \rho_3 = \rho'$, these give:-

$$[0\,0\,0] : [0\,0\,1] : [0\,0\,2] =$$

$$= -\frac{\nu+1}{6\rho}\sqrt{\frac{\nu^2+\rho'^2}{\nu(\nu+1)}} : 1 : \frac{1}{4}\sqrt{\frac{(\nu+2)(\nu+3)}{(\nu+2)^2+\rho'^2}}\left\{-\frac{\nu^2+\rho'^2}{\nu\rho} + \frac{4\rho}{\nu+1} - \frac{12\rho'}{2(\nu+2)}\right\}$$

as the ratios of the relevant coupling coefficients.

The same sort of method may be applied to the coupling of infinite unitary representations to finite-dimensional non-unitary ones. In this case the finite-dimensional representation would correspond to the momentum "spurion" or kineton.

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