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# RELATIVISTIC EXTENSION OF NON-COMPACT SYMMETRY GROUPS*

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## ABSTRACT

The problem of relativistically boosting the unitary representations of a non-compact spin-containing rest-symmetry group is solved by starting with non-unitary infinite-dimensional representations of a relativistic extension of this group, by adjoining to this extension four space-time translations and by then applying Bargmann-Wigner equations to guarantee a unitary norm. The boosting problem considered here is the first step towards the solution of the problem of coupling of such infinite-dimensional representations which is briefly investigated.

RELATIVISTIC EXTENSION OF NON-COMPACT SYMMETRY GROUPS

## 1. INTRODUCTION

It appears reasonably well established that known baryons and mesons at rest can be classified as multiplets of a compact spin-containing symmetry group SU(6) (SAKITA, GÜRSEY and RADICATI). In analogy with the well-known energy-level structure in atomic* and nuclear physics, it has further been conjectured (BARUT, BUDINI and FRONSDAL, DOTHAN, GELL-MANN and NE'EMAN) that there may indeed exist a very large number of such multiplets and that these may correspond in an idealized limit to the infinite-dimensional unitary representations of a non-compact rest-symmetry group containing SU(6) as a subgroup.

A basic problem which arises with all spin-containing rest-symmetry groups (compact or non-compact) is that of their relativistic extension. For SU(6) this problem was solved by FULTON and WESS (1965), RUHL (1965) and BACRY and NUYTS (1965) - the relativistic extension found being $T_4 \otimes SL(6, C)$ ** - and for $U(6) \otimes U(6)$ by SALAM, DELBOURGO and STRATHDEE (1965). Indeed for $U(6) \otimes U(6)$ it was the relativistic extension $T_4 \otimes \tilde{U}(12)$ which was postulated first. Only later was it recognized (SALAM, DELBOURGO, RASHID and STRATHDEE, II addendum and III; KATO, DASHEN and GELL-MANN) that the $\tilde{U}(12)$ multiplets had the content*** of $U(6) \otimes U(6)$. The present

*For example with no spin-orbit coupling the hydrogen atom exhibits SO(4) symmetry at rest. When this compact group is embedded in the non-compact group SO(4, 1) the degeneracy structure is correctly reproduced for a special class of unitary irreducible representations (BARCY, BARUT, BUDINI and FRONSDAL) though nothing can be inferred about the actual energy values.

** $T_4$ stands for the manifold of four space-time translations.

*** Historically MARSHAK and OKUBO (1964) were among the first to emphasize the importance of the non-chiral compact group U(6) x U(6).

paper is devoted to the corresponding problem of relativistic extension for possible non-compact rest symmetries. Once solved, it becomes possible in principle to couple such infinite dimensional multiplets in motion.

In Section 2 we outline the general procedure for embedding a rest-symmetry into a relativistic manifold. In Section 3 we consider as an example the non-compact rest-symmetry U(6, 6); and show that the required extension is $T_4 \otimes$ (U(6, 6) $\otimes$ U(6, 6). The rest-symmetry U(6, 6) was chosen simply because it happens to contain the successful compact rest-symmetry U(6) $\otimes$ U(6) as a subgroup. The point of departure of our method in constructing infinite-dimensional representations of U(6, 6) $\otimes$ U(6, 6) is to reduce the group with respect to non-unitary representations of a spin-containing subgroup U(6, 6). The space-time translations $T_4$ are then introduced in a fundamental manner to induce a unitary norm through the standard application of Bargmann-Wigner equations. The final symmetry after the application of the equations for one-particle states is of course the rest-symmetry U(6, 6). In Section 4 we show that the maximal residual symmetries for S-matrix elements involving two, three or four independent momenta are GL(6, C), U (3, 3) and GL(3, C). The Clebsch-Gordan problem for three-particle coupling has in general not been solved. There is however one simple soluble example which we give in Section 4. We wish to stress that the interest of this paper is mainly methodological and that it may not provide a realistic model.

## 2. THE GENERAL PROCEDURE

In this section we review some of the relevant concepts in non-compact group theory and also summarize the essential content of papers I and III pertaining to the relativistic extension of any rest-symmetry group.

Let us denote the compact and non-compact generators of a symmetry group G by $G_c$ and $G_{nc}$ respectively. Thus, symbolically,

$$
\begin{aligned}
[G_c\ ,\ G_c] &= i\,G_c \\
[G_c\ ,\ G_{nc}] &= i\,G_{nc} \\
[G_{nc}\ ,\ G_{nc}] &= -i\,G_c
\end{aligned}
\tag{2.1}
$$

The standard procedure in non-compact theory is to decompose unitary irreducible representations of G relative to those of the maximal compact subgroup, viz. $G_c$ itself. Since the former are infinite-dimensional in any one representation there will necessarily occur an infinite number of (finite-dimensional, irreducible and unitary) representations of $G_c$. Thus G will be represented by infinite-dimensional hermitian matrices: $G_c$ are block-diagonal with each block relating to one $G_c$ representation, while $G_{nc}$ are block off-diagonal thereby acting as shift or transition operators between nearby $G_c$ representations.

To define precisely the concepts of a rest-symmetry group, whether compact or not, one must introduce translations $P$. These, in general, satisfy the symbolic commutation relations

$$
[P\,,P] = 0 \quad , \quad [G\,,P] = i\,P\,. \tag{2.2}
$$

The rest-symmetry subgroup $G(\hat{p})$ of G will, by definition, commute with the energy $P_0$,

$$
[G(\hat{p})\,,\,P_0] = 0 \tag{2.3}
$$

$G(\hat{p})$ may thus be termed "the little group". In paper I G was U(6, 6) and $G(\hat{p})$ was the compact U(6) $\otimes$ U(6).

Given a spin-containing rest-symmetry $G(\hat{p})$ our problem is to find G as its relativistic generalization. The procedure applied in I to compact rest-symmetries was essentially to assign well-defined

Lorentz tensor transformation properties to the generators of $G(\hat{p})$ and thereby to generate the full G by acting on the $G(\hat{p})$ with pure Lorentz transformations; for example $G(\hat{p}) = U(6)$ is isomorphic to the 36 matrices. $T^i$, $\sigma_{rs}T^i$, where $\sigma_{rs}T^0$ corresponds to the pure spin transformation. The action of Lorentz transformations $\sigma_{0r}T^0$ on these little-group generators is thereby specified and completes $G(\hat{p})$ to $G = GL(6, C)$ with the 72 matrices

$$T^i\ ,\ \gamma_5 T^i\ ,\ \sigma_{\mu\nu}T^i\ .$$

For non-compact rest-symmetries our procedure will be identical. We perform pure Lorentz transformations (lying outside $G(\hat{p})$ but forming part of G) on the $G(\hat{p})$ generators and thereby close on the algebra of G. The case $G(\hat{p}) = U(6, 6)$ is treated in the next section.*

Having found G we shall next supplement it with translations; this is an integral part of the relativistic boosting procedure. We have used in I and shall again in this paper use the translations in an absolutely fundamental way to define a unitary norm for multiplets of the little group $G(p)$, <u>when in motion</u>. Since this point has not been fully appreciated we wish to go over it carefully once again.

All finite-dimensional representations of a non-compact group like $\mathcal{L}_4$ (the homogeneous Lorentz group) are non-unitary. Only for infinite-dimensional representations can one define a unitary norm. Thus, if $\mathcal{L}_4$ were a rest-symmetry at least certain unitary representations would correspond to particles with spins ranging over all integers or half-integers $j_0, j_0+1, \ldots\ldots \infty$. These representations may or may not be useful in physics but in the past the more urgent problem was to describe a relativistically moving particle of one definite spin $j$, and the relativistic aspects of the problem in any case called for the use of the non-unitary finite-dimensional representations of $\mathcal{L}_4$. It was Dirac's and Wigner's

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* We must stress that the various U(6, 6) subgroups of G have completely different connotations. There is the little-group $G(\hat{p}) = U(6, 6)$; then there is the $\mathcal{L}_4$-containing subgroup of (Dirac) matrices U(6, 6). These must be carefully distinguished, though they possess the common subgroup $U(6) \otimes U(6)$.

great contribution to recognize that a physically satisfactory unitary norm could indeed be defined in such cases because one adjoined the space-time translations $T_4$ to $\mathcal{L}_4$ and considered the full inhomogeneous Lorentz group $I\mathcal{L}_4 = T_4 \otimes \mathcal{L}_4$ . This group equally is non-compact but its unitary (infinite-dimensional in the momentum variable) representations are constructed (in the BARGMANN-WIGNER (1948) method) from certain momentum-dependent finite-dimensional representations of $\mathcal{L}_4$ , a positive definite norm for which is guaranteed by imposing on the representation functions certain equations of motion. These Bargmann-Wigner equations have one purpose: to project out just the positive definite class of these functions. To take a concrete example, if the spinor $\psi_\alpha$ corresponds to a non-unitary representation of $\mathcal{L}_4$ , or more properly SL(2, C), the Dirac spinor $\psi_\alpha(p)$ which satisfies $\not{p}\,\psi(p) = m\psi(p)$ corresponds to a perfectly unitary representation of the Poincaré group $T_4 \otimes \mathcal{L}_4$ with the norm given by

$$(\psi,\psi) = \sum_\alpha \int_{p_0>0} \frac{d_3 p}{p_0} \bar{\psi}^\alpha(p)\,\psi_\alpha(p) \tag{2.4}$$

This norm pertains to moving states of spin $\frac{1}{2}$.

Now in I we started with the non-unitary representations of the non-compact group U(6, 6) (= $\tilde{U}$(12)). By adjoining the four translation* operators $P_\mu$ of $T_4$ and after imposing Bargmann-Wigner equations, we once again recovered a unitary probability-conserving norm similar to (2.4) for moving particles with $\alpha = 1, \ldots\ldots 12$. The important point - and this is shown in detail in III - is that one-particle states possess not just SU(3) $\otimes$ $I\mathcal{L}_4$ but the larger symmetry $(U(6) \otimes U(6))_p$ . This may indeed be states as a theorem:

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* It must be recognized however that whereas the algebra of $T_4 \times \mathcal{L}_4$ closes on itself, that of a structure like $T_4 \otimes U(6,6)$ does not. As shown in III the full closure requires 143 momenta, and since these momenta possess no physical significance $T_4$ x U(6, 6) is an intrinsically broken symmetry. As stated before, one-particle (rest) states show U(6) $\otimes$ U(6) symmetry nevertheless, which is further reduced for two and higher particle states. The problem of residual symmetries is considered in Section 4.

Given a structure $\mathcal{T}_4 \times G$, with commutation relations (2.1) and (2.2), the appropriate unitary norm for moving one-particle states is of the type (2.4) which have $G(\hat{p})$ as the little-group symmetry in the rest frame $\underset{\sim}{p} = 0$ (cf. equation (2.3)).

## 3. THE HYPER-SYMMETRY METHOD

Since the generators of the spin-internal symmetry group can only be incomplete Lorentz tensors it is necessary to adjoin new generators so as to make the tensors whole. This leads in a unique way to an enlarged algebra.

The operators which must be brought in to make the relativistic completion will not provide any additional conserved quantities. They do not commute with $P_0$ : all such commuting members being, by definition, in the algebra of the little group. In order to discover what they are one can apply the following criteria:

1) The generators $J$ of the little group $G(\hat{p})$ commute with the energy

$$[J, P_0] = 0 \qquad (3.1)$$

2) The generators $J$ include the spin rotations $J_{ij}$ which in any realistic relativistic theory are coupled to the momenta, hence

$$[J_{ij}, P_k] = i(\delta_{ik} P_j - \delta_{jk} P_i) \qquad (3.2)$$

3) The pure Lorentz transformations $K_{i0}$ which, by definition, affect the energy, cannot be found among the $J$.

4) The relativistically complete algebra G must include the Lorentz group in the form $(J_{ij}, K_{i0})$.

These requirements are sufficient to make the procedure unique. If, for example, the little group is taken as U(6) $\otimes$ U(6) with generators

$$\left( \tfrac{1}{2}(1+\gamma_0)\,\underline{\sigma}\,T^j \quad , \quad \tfrac{1}{2}(1-\gamma_0)\,\underline{\sigma}\,T^j \right) \tag{3.3}$$

then the Lorentz transformations must be $\sigma_{i0}$ and the translations $\gamma_\mu$ . The full group is then easily shown to incorporate the 144 ( $\gamma_R T^j$ ) (R = 1,...., 16; $j$ = 0,...., 8), this being just the non-compact U(6, 6). This case was dealt with in I. There it was shown that the natural way to employ the finite-dimensional (and therefore non-unitary) representations of U(6, 6) was to make use of equations of motion (Bargmann-Wigner equations) which served to restrict the physical states to positive definite sectors of the Hilbert space. We shall adopt the same attitude here for the infinite-dimensional representations.

Suppose now that the little group is taken to be U(6, 6) itself. This group has been chosen to provide illustrative example only and may not be the group of final physical interest. Exploiting the isomorphism of the group with Dirac-like matrices, represent the little group generators $J_R^j$ by

$$J_c^j = \gamma_c T^j \times 1 \qquad \text{and} \qquad J_{nc}^j = \gamma_{nc} T^j \times \tau_1 \tag{3.4}$$

where the subscripts $c$ and $nc$ label the compact and non-compact parts of the algebra respectively. For the translations we can take

$$\gamma_\mu \times \tau_3 \tag{3.5}$$

since the time-like component of this matrix evidently commutes with the little group matrices. The pure Lorentz transformations must then be

$$K_{i0}^0 = \sigma_{i0} \times 1 \tag{3.6}$$

Filling out the algebra by taking repeated commutators we find it necessary to include the operators

$$K_c^j = \gamma_c T^j \times 1 \qquad \text{and} \qquad K_{nc}^j = \gamma_{nc} T^j \times 1 \tag{3.7}$$

which go to make up the algebra of U(6, 6) ⊗ U(6, 6):

$$\tfrac{1}{2}\left( J^{j}_{k} + K^{j}_{k} \right) \otimes \tfrac{1}{2}\left( J^{j}_{k} - K^{j}_{k} \right) \tag{3.8}$$

The form (3.5) adopted for the translations means that the momenta transform like components of the (12, $\overline{12}$) and ($\overline{12}$, 12) representations of U(6, 6) ⊗ U(6, 6).

The next question to be settled concerns the nature of the representations of U(6, 6) ⊗ U(6, 6) to be employed. Since we are following the viewpoint of I in which the physical states involve a number of redundant components determined by the Bargmann-Wigner equations, we can afford to accept non-unitary representations of the full algebra, requiring only that the representations of the little group $U(6, 6)_p$ contained therein shall be unitary. The little group $U(6, 6)_p$ appropriate to a particle with 4-momentum $p_\mu$ is obtained from the $J^j_k$ by applying a suitable Lorentz boost . It can otherwise be defined as the set of matrices which commute with $\not p \times \tau_3$.

There is in addition to the subgroups $U(6, 6)_p$, another U(6, 6) which contains the Lorentz transformations. This has been called the Dirac subgroup, $U(6, 6)_D$ in Section 2. It is generated by

$$J^{j}_{k} \quad \text{and} \quad K^{j}_{k}$$

As explained earlier, the Bargmann-Wigner equations will guarantee the existence of a unitary norm even though one starts with non-unitary representations of $U(6, 6)_D$ . Thus it will be quite sufficient to take finite-dimensional representations for this subgroup and to build the infinite matrices corresponding to the full group with block-diagonal representations of $U(6, 6)_D$ . The subgroup $U(6, 6)_D$ in our work plays the role of the maximal "compact subgroup in conventional theory of non-compact groups.

The construction of representations of U(6, 6) ⊗ U(6, 6) is considered next. We shall employ the method of Feynman et al. (DOTHAN, GELL-MANN and NE'EMAN (1965)) which has the advantage of directness. More important, it produces only the "degenerate" representations, i.e. those associated with discrete eigenvalues of the Casimir operators. These are more amenable to physical interpretation than the continuum representations.

A representation of the algebra (3.4) and (3.7) may be taken in the form

$$J^j_c = \bar{\psi}\gamma_c T^j \psi \quad , \quad J^j_{nc} = \bar{\psi}\gamma_{nc} T^j \tau_i \psi$$
$$K^j_c = \bar{\psi}\gamma_c T^j \tau_i \psi \quad , \quad K^j_{nc} = \bar{\psi}\gamma_{nc} T^j \psi \qquad (3.9)$$

where symbols $\psi$ and $\bar{\psi}$ denote respectively 24-component column and row vectors and we require Bose-like commutation relations for $\psi$ and $\bar{\psi}$ ;

$$\left.\begin{array}{l} [\psi , \psi] = 0 \quad , \quad [\bar{\psi} , \bar{\psi}] = 0 \\ \text{and} \\ [\psi , \bar{\psi}] = 1 \end{array}\right\} \qquad (3.10)$$

It is simplest to think of $\psi$ and $\bar{\psi}$ as Bose-like quarks. Since the little group, generated by the $J^j_R$ , must be represented by unitary matrices we require that

$$(J^j_R)^\dagger = J^j_R \qquad (3.11)$$

Supposing now that $\bar{\psi}$ is related to the adjoint of $\psi$ by an equation of the form

$$\bar{\psi} = \psi^\dagger M \qquad (3.12)$$

where $M = M^T$ defines the metric, then (3.11) requires that

$$[M , \gamma_c] = 0$$
$$\{M , \gamma_{nc}\tau_i\} = 0 \qquad (3.13)$$

This fixes $M$ (apart from trivial modifications) to be of the form

$$M = \gamma_0 \tag{3.14}$$

and so we find in addition

$$\left(K^j_R\right)^\dagger = K^j_R \tag{3.15}$$

Unfortunately the condition (3.11) is not enough to assure unitarity for the little group since the metric (3.14) is indefinite. In order to fix this we first split $\psi$ into two parts $\psi_\pm$ according to the sign taken by $\gamma_0\tau_3$:

$$\psi = \psi_+ + \psi_- \tag{3.16}$$

where

$$\gamma_0\tau_3\,\psi_\pm = \pm\,\psi_\pm \tag{3.17}$$

The commutation relations between these parts are then

$$\begin{aligned} \left[\psi_+ , \psi_-^\dagger\right] &= \tfrac{1}{2}\left(1 + \gamma_0\tau_3\right)\gamma_0 \\ \left[\psi_+ , \psi_-^\dagger\right] &= 0 \\ \left[\psi_- , \psi_-^\dagger\right] &= \tfrac{1}{2}\left(1 - \gamma_0\tau_3\right)\gamma_0 \end{aligned} \tag{3.18}$$

The generators (3.9) when expressed in terms of $\psi_\pm$ take the forms

$$\begin{aligned} J_c &= \bar{\psi}_+\gamma_c\psi_+ + \bar{\psi}_-\gamma_c\psi_- \;, \quad & J_{nc} &= \bar{\psi}_+\gamma_{nc}\tau_1\psi_+ + \bar{\psi}_-\gamma_{nc}\tau_1\psi_- \\ K_c &= \bar{\psi}_+\gamma_c\tau_1\psi_- + \bar{\psi}_-\gamma_c\tau_1\psi \;, \quad & K_{nc} &= \bar{\psi}_+\gamma_{nc}\psi_- + \bar{\psi}_-\gamma_{nc}\psi_+ \end{aligned} \tag{3.19}$$

so that the $J_R$ are even (++ or --) and the $K_R$ odd (+- or -+). This means that if we can arrange that $\psi_+$ operate within the negative and positive sectors of the Hilbert space while $\psi_-$ makes transitions between them, then the $J_R$ - being even - will have non-vanishing matrix elements only between states in the same sector whereas the will have them only between states in opposite sectors. This, together with (3.11) will assure that the $J_R$ generate unitary representations of the little group.

The crucial step then is to fix the metrical character of $\psi_+$ and $\psi_-$. Considered as (fictitious) field operators the components of $\psi_+ (\psi_-)$ must create or annihilate "particle states" of positive (negative) norm. In the two-component $\tau$-space if we write

$$\psi = \begin{pmatrix} b_\alpha \\ \bar{a}_\alpha \end{pmatrix} \quad , \quad \bar{\psi} = (\bar{b}^\alpha , a^\alpha) = (b^\dagger_\beta (\gamma_0)^\alpha_\beta , \ \bar{a}^\dagger_\beta (\gamma_0)^\alpha_\beta) \tag{3.20}$$

then (3.14) reads

$$[b_\alpha , b^\dagger_\beta] = (\gamma_0)^\beta_\alpha \quad , \quad [\bar{a}_\alpha , \bar{a}^\dagger_\beta] = (\gamma_0)^\beta_\alpha \tag{3.21}$$

and hence, using (3.17),

$$\left. \begin{array}{ll} [b_+ , b^\dagger_+] = 1 & [b_- , b^\dagger_-] = -1 \\ [\bar{a}_+ , \bar{a}^\dagger_+] = -1 & [\bar{a}_- , \bar{a}^\dagger_-] = 1 \end{array} \right\} \tag{3.22}$$

This means that <u>the $\bar{a}_\alpha$ are to be regarded as creation operators and the $b_\alpha$ as annihilation operators</u>. The adjoints $a^\alpha$ and $\bar{b}^\alpha$ are then, respectively, annihilation and creation operators.

The operators $J^i_c$ and $K^i_{kc}$ of the Dirac group are seen to be made of products of one creation and one annihilation operator while the others, $J^i_{kc}$ and $K^i_c$ , are made of products of two creation or two annihilation operators. This means that the operators of the

Dirac group will be represented in block-diagonal form while the others will consist of elements connecting adjacent blocks. The representation appears as a tower of finite-dimensional (non-unitary) representations of $U(6, 6)_D$.

For the construction of representations it is convenient to regroup the generators into two sets

$$\left.\begin{aligned} M^B_A &= \bar{b}^B b_A + a^B \bar{a}_A \\ N^B_A &= \bar{b}^B \bar{a}_A + a^B b_A \end{aligned}\right\} \qquad (3.23)$$

where $A$ , $B$ take the values 1, 2, ..., 12. In terms of these generators the old set can be expressed by

$$\begin{aligned} J^j_c &= M^B_A (\gamma_c T^j)^A_B \quad , \quad & J^j_{nc} &= N^B_A (\gamma_{nc} T^j)^A_B \\ K^j_c &= N^B_A (\gamma_c T^j)^A_B \quad , \quad & K^j_{nc} &= M^B_A (\gamma_{nc} T^j)^A_B \end{aligned} \qquad (3.24)$$

The $M^B_A$ are seen to be the generators of $U(6, 6)_D$ . The representations may be constructed by fixing on a "lowest level" and applying the generators $M^B_A$ and $N^B_A$ to it repeatedly. The $N^B_A$ will lead to new levels while the $M^B_A$ fill out each one. This method of generating representations with $M$ and $N$ is discussed in the appendix. Typically, a representation starting off with the $U(6, 6)_D$ singlet $\underline{\Psi}(1)$ will be found to contain the sequence

$$\underline{\Psi}(1) \oplus \underline{\Psi}^B_A (143) \oplus \underline{\Psi}^{\{B_1 B_2\}}_{\{A_1 A_2\}} (5940) \oplus \dots \qquad (3.25)$$

i.e. traceless tensors symmetric in upper and lower indices. Implicit in this is the restriction to quarks of type 1, i.e. we are keeping quarks and anti-quarks of type (6, 1) and (1, $\bar{6}$) and not using (1, 6) and ($\bar{6}$, 1).

We shall deal with the problem of picking out physical states from these representations in the same fashion as was done previously (see I) for the case of the compact little group $(U(6) \times U(6))_p$. That is, we shall reduce the full multiplet with respect to the little group $U(6, 6)_p$ and project out one of the resulting (unitary) multiplets. Consider firstly the system at rest, for which the little group is generated by the $J^j_R$. Since the distinction between $\psi_+$ and $\psi_-$ defined by (3.16) and (3.17) is invariant so far as the $J^j_R$ are concerned, we can impose the condition

$$\psi_- = 0 \tag{3.26}$$

and produce a representation of the little group at rest $U(6, 6)_{\hat{p}}$ by operating with the surviving generators, namely

$$J^j_c = \bar{\psi}_+ \gamma_c T^j \psi_+ \quad \text{and} \quad J^j_{nc} = \bar{\psi}_+ \gamma_{nc} T^j \tau_2 \psi_+ \tag{3.27}$$

Having decided that the physical states at rest are to be generated by the operators (3.27) we can re-arrange these operators into the form

$$\left.\begin{aligned} M^B_A(\hat{p}) &= \bar{b}^B(\hat{p})\, b_A(\hat{p}) + a^B(\hat{p})\,\bar{a}_A(\hat{p}) \\ N^B_A(\hat{p}) &= \bar{b}^B(\hat{p})\,\bar{a}_A(\hat{p}) + a^B(\hat{p})\, b_A(\hat{p}) \end{aligned}\right\} \tag{3.28}$$

where we have adopted the suggestive notation

$$\psi_+ = \begin{pmatrix} b_A(\hat{p}) \\ \bar{a}_A(\hat{p}) \end{pmatrix}, \quad \bar{\psi}_+ = \left(\bar{b}^A(\hat{p}) \quad a^A(\hat{p})\right) \tag{3.29}$$

and correspondingly

$$\psi_- = \begin{pmatrix} b_A(-\hat{p}) \\ \bar{a}_A(-\hat{p}) \end{pmatrix} \quad , \bar{\psi}_- = \left( \bar{b}^A(-\hat{p}), \; a^A(-\hat{p}) \right) \tag{3.30}$$

though, for the present discussion we shall need only the operators (3.29) with $\hat{p}_0 > 0$. The creation operators contained in (3.29) satisfy the conditions

$$\left. \begin{aligned} (\gamma_0 - i)_A^{\ B} \; \bar{a}_B(\hat{p}) &= 0 \\ (\gamma_0 + i)_B^{\ A} \; \bar{b}^B(\hat{p}) &= 0 \end{aligned} \right\} \tag{3.31}$$

From this it follows that the states created by applications of $M(\hat{p})$ and $N(\hat{p})$, say $\bar{\Psi}_{A_1A_2\ldots}^{B_1B_2\ldots}(\hat{p})$, will satisfy the conditions

$$(\gamma_0 - 1)_{A_1}^{A_1'} \; \bar{\Psi}_{A_1'A_2\ldots}^{B_1\ldots} = 0 \qquad \text{for lower indices}$$

and (3.32)

$$(\gamma_0 + i)_{B_1'}^{B_1} \; \bar{\Psi}_{A_1A_2\ldots}^{B_1'\ldots} = 0 \qquad \text{for upper indices}$$

These states can now be set in motion by applying a Lorentz boost $L_p$. Since the Lorentz transformations come within $U(6, 6)_D$ they will be represented by block-diagonal matrices, for example

$$\bar{\Psi}_{A_1A_2\ldots}^{B_1\ldots} \rightarrow L_{A_1}^{A_1'} \, L_{A_2}^{A_2'} \cdots \bar{\Psi}_{A_1'A_2'\ldots}^{B_1'\ldots} \; (L^{-1})_{B_1'}^{B_1} \tag{3.33}$$

If we adopt a family of boosts $L_p$ with the property

$$L_p \, m\gamma_0 \, L_p^{-1} = \not{p} \tag{3.34}$$

then the little group at rest $U(6, 6)_{\hat{p}}$ is carried into $U(6, 6)_p$ = $= L_p \; U(6, 6)_{\hat{p}} \; L_p^{-1}$. The representation (3.29) of $U(6, 6)_{\hat{p}}$ is correspondingly carried into a representation of $U(6, 6)_p$, namely

$$\bar{\Psi}^{B_1\cdots}_{A_1A_2\cdots}(p) = (L_p)^{A_1'}_{A_1}(L_p)^{A_2'}_{A_2}\cdots\ \bar{\Psi}^{B_1'\cdots}_{A_1'A_2'\cdots}(\hat{p})(L_p^{-1})^{B_1}_{B_1'}\cdots \qquad (3.35)$$

which satisfies the Bargmann-Wigner equations:

$$(\not p - m)^{A_1'}_{A_1}\ \bar{\Psi}^{B_1\cdots}_{A_1'A_2\cdots}(p) = 0 \qquad \text{for lower indices}$$

$$(\not p + m)^{B_1}_{B_1'}\ \bar{\Psi}^{B_1'\cdots}_{A_1A_2\cdots}(p) = 0 \qquad \text{for upper indices} \qquad (3.36)$$

The states $\bar{\Psi}(p)$ may be generated directly with the operators

$$M^{B}_{A}(p) = \bar{b}^{B}(p)\, b_A(p) + a^{B}(p)\, \bar{a}_A(p)$$

$$N^{B}_{A}(p) = \bar{b}^{B}(p)\, \bar{a}_A(p) + b_B(p)\, a^{A}(p) \qquad (3.37)$$

where

$$(\not p + m)^{A}_{B}\ \bar{b}^{B}(p) = 0 \quad \text{and} \quad (\not p - m)^{B}_{A}\ \bar{b}_B(p) = 0 \qquad (3.38)$$

these operators being the boosted forms of $\bar{b}^A$, $\bar{a}_B$, i.e.

$$\left.\begin{aligned} \bar{b}^{A}(p) &= \bar{b}^{B}(\hat{p})(L_p^{-1})^{A}_{B} \\ \bar{a}_A(p) &= (L_p)^{B}_{A}\ \bar{a}_B(\hat{p}) \end{aligned}\right\} \qquad (3.39)$$

To summarize this discussion we stress that the essential content of the above is to say that $\bar{\Psi}\gamma_c\Psi$ and $\bar{\Psi}\gamma_{nc}\tau_i\Psi$ are hermitian forms and their hermiticity is guaranteed in an arbitrary Lorentz frame by the Bargmann-Wigner equations.

At this stage we are requiring U(6, 6) symmetry in the rest frame so that there can be no mass breaking. The levels in the tower are completely degenerate. The lifting of the degeneracy could be effected by supposing that the mass operator has components proportional to Casimir operators of various subgroups. The U(6) ⊗ U(6) subgroup is the most obvious candidate. For the representations we are considering which are made out of fully symmetrized tensors, the U(6) ⊗ U(6) Casimir operators are determined by the numbers of upper and lower indices at each level in the tower. Since only the sum of these numbers varies from one level to the next, it (and its powers) are the only numbers on which the mass operator could depend. In a linear approximation this could give a mass formula of the form

$$m_n = a + nb$$

where $n$ denotes the number of quarks (indices) comprising the particle. Some evidence has been given by FREUND (1965) that such a relation of physical masses with $a = 0$ indeed exists.

This is analogous to the treatment of the hydrogen atom level system as a representation of the non-compact O(4, 1) with the energy given as a function of the Casimir operators of the compact subgroup O(4).

Another possibility for breaking the mass is to use instead of U(6) ⊗ U(6) one of the non-compact subgroups, SL(6, C), for example.

## 4. RESIDUAL SYMMETRIES AND THE COUPLING PROBLEM

(i) Assuming that the rest states of a system furnish a representation of the little group $G(\hat{p}) = U(6, 6)_{\hat{p}}$, we wish to enquire into the maximum residual symmetry that may be expected for many-particle states $|p_1 \cdots p_n\rangle$ and for the corresponding S-matrix elements.

(ii) For two-particle states we select the frame for which

$$p_{1x} = p_{2x} = p_{1y} = p_{2y} = 0$$

and look for those transformations $J^i_R$ of $U(6, 6)_{\hat{p}}$ that commute with $K^0_{03}$, the Lorentz boost along the $z$-axis. These are simply:

$$J^i, \; J^i_{12}, \; J^i_{51}, \; J^i_{52} \; ; \; J^i_5, \; J^i_{03}, \; J^i_1, \; J^i_2$$

and constitute a $GL(6, c)_W$ group*. The compact subgroup $\dot{J}^j$, $\dot{J}^j_{12}$, $\dot{J}^j_{51}$, $\dot{J}^j_{52}$, is the W-spin collinear group .

(iii) For three-particle states we may set $p_{1y} = p_{2y} = p_{3y} = 0$. The generators of $GL(6, c)$ which commute further with $K^i_{2v}$ are $J^j$, $J^j_{15}$, $J^j_5$, $J^j_1$ giving $U(3, 3)$. Thus, the compact subgroup is here the coplanar group $U(3) \otimes U(3)$.

(iv) Finally for four-particle states and higher we just have $GL(3, C)$ as the non-compact extension of $U(3)$.

Thus we encounter the following chain of little group symmetries in general

$$G \to G(p) \to G(p_1 p_2) \to G(p_1 p_2 p_3) \to G(p_1 p_2 p_3 p_4) \to \cdots$$

which for our case read

$$U(6,6) \otimes U(6,6) \to U(6,6)_p \to GL(6,c)_{p_1 p_2} \to$$
$$\to U(3,3)_{p_1 p_2 p_3} \to GL(3,c) \qquad (4.1)$$

When translated to the compact sugroups $G_c(p_1 \ldots p_n)$ with respect to which the infinite-dimensional representations of $G(p_1 \ldots p_n)$ are decomposed, we have the familiar chain

$$[U(6) \otimes U(6)]_p \to U(6)_{p_1 p_2} \to [U(3) \otimes U(3)]_{p_1 p_2 p_3} \to U(3)$$

If one wishes to set up a phenomenological S-matrix theory which exhibits these maximal residual symmetries one is faced with finding the $G(p_1 \ldots p_n)$ content of the discrete hypermultiplet representations of $G(\hat{p}) = U(6, 6)_{\hat{p}}$ (and restricting oneself to (boson) quarks of first kind) one has in mind the following hypermultiplet sequences:

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* This can be obtained using the simple procedure outlined by HARARI and LIPKIN (1965) by looking for those $\gamma_R$ which commute with $\gamma_3$ .

Meson Hypermultiplet

$G(\hat{\mathfrak{p}})$ Series : $(1, \bar{1}) \oplus (6\ \bar{6}) \oplus (21, \overline{21}) \oplus (56, \overline{56}) \oplus \ldots$

Boosted Series: $1 \oplus 143 \oplus 5940 \oplus 126412 \oplus \ldots$

Baryon Hypermultiplet

$G(\mathfrak{p})$ Series : $(56, 1) \oplus (126, \bar{6}) \oplus (252, \overline{21}) + \ldots$

Boosted Series: $364 \oplus 16016 \oplus 411684 + \ldots$

(4.2)

One can construct a fully $[U(6, 6)]^{L}$ invariant type of coupling; in that case one would use the fully boosted series, with the powerful result then that there is just <u>one</u> fundamental (zeroth order) coupling constant for all members of the hypermultiplet. On the other hand, if one wishes to take into account higher order corrections to this fundamental couplings so that the maximal residual coupling is GL $(6, C)_W$ one must first solve the problem of finding the $GL(6, C)_W$ content of $G(\hat{\mathfrak{p}}) = U(6, 6)$. The general reduction problem from a non-compact group relative to one of its non-compact subgroups is to our knowledge not solved in literature. However, we believe that among the various $GL_W(6, C)$ hypermultiplet series which occur are the following:

$$\text{Mesons} \quad : \quad 1 \oplus 35 \oplus 405 \oplus \ldots$$

$$\text{Baryons} \quad : \quad 56 \oplus 700 \oplus 4536 \oplus \ldots \qquad (4.3)$$

The next piece of information required is the Clebsch-Gordan expansion for coupling two or more infinite-dimensional hypermultiplets of $GL(6, C)_W$. There is certainly no indication of this problem being tackled in the literature. However the case of Clebsch-Gordan coefficients for $SL(2, C)_W$ has been worked out (BISIACCHI and FRONSDAL (1965), DOLGINOV and TOPTYGIN (1960)) and for illustration we give the appropriate formalism.

In this example we neglect unitary indices, corresponding to a rest-symmetry U(2, 2), and consider the three-meson interaction only. The meson hypermultiplet series is

$$(1, 1) \oplus (2, 2) \oplus (3, 3) + \cdots\cdots$$

or (4.4)

$$1 \oplus 15 \oplus 84 + \cdots\cdots$$

after applying the Lorentz boost. In performing the SL(2, C) reduction we encounter the series

$$1 \oplus 3 \oplus 5 + \cdots\cdots$$

an infinite number of times. Since we have no physical criteria for selecting one of these series rather than another, we choose from this SL(2, C) set one arbitrary series for which one Casimir operator takes the value $n$ , (the other Casimir operator is zero since the series starts with $j_c = 0$). It is this particular case which has been studied extensively and for which Clebsch-Gordan coefficients have been provided. Following BISIACCHI and FRONSDAL (1965) we couple three such irreducible representations of SL(2, C) in the form

$$\sum_{j_1 j_2 j_3} \begin{bmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle \quad (4.5)$$

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the well-known Wigner $3j$ - symbol and the first few $\begin{bmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$ functions have been tabulated. For example the coupling of the $SU(2)_W$ scalar $\rho_0$ to two $SU(2)_W$ vector particles $\pi$ having $W_z = 0$ involves the Clebsch-Gordan coefficient

$$\begin{bmatrix} 1 & 1 & 0 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\delta_{n_1}^2(1) + \delta_{n_2}^2(1) - \delta_{n_3}^2(0)}{2\,\delta_{n_1}(1)\,\delta_{n_2}(1)}$$

where $\delta_n(j) = [j^2 - n^2]^{\frac{1}{2}}$

This represents the ratio of the $\rho_v \pi\pi$ coupling to the $3\rho_v$ coupling coefficient $\begin{bmatrix} 0 & 0 & 0 \\ n_1 & n_2 & n_3 \end{bmatrix} = 1$.

Other coupling relations are straightforwardly deduced from (4.5) and the detailed formulae given by DOLGINOV and TOPTYGIN and BISIACCHI and FRONSDAL.

## SUMMARY AND OUTLOOK

The major contribution of this paper is the procedure developed in Section 3 for relativistic boosting of infinite-dimensional hypermultiplets of a non-compact rest-symmetry group $G(\hat{p})$. The procedure involves construction of infinite-dimensional non-unitary representations of the full relativistically-extended symmetry group G relative to its (spin-containing) non-compact Dirac sugroup $G_D$, the unitarity of the norm for the hypermultiplet in motion being guaranteed by the introduction of momenta and an application of Bargmann-Wigner equations. Restricting oneself to the subspace of quarks of first kind, the boosted hypermultiplets consist of a series or a tower each component of which corresponds to a $\tilde{U}(12)$ multiplet whose physical content (after application of Bargmann-Wigner equations) is precisely that of a compact $U(6) \otimes U(6)$. Before the application of the equations the symmetry is $U(6, 6) \otimes U(6, 6)$. The final symmetry, however, is intrinsically broken. For the multiplet at rest it is precisely the symmetry of the little group $U(6, 6)$ and it reduces progressively to its smaller subgroups for S-matrix elements involving two, three or four independent momenta. The theory developed in this paper is completely analogous to the $T_4 \otimes \tilde{U}(12)$ theory of Paper I, where $G(\hat{p})$ equalled the compact $U(6) \otimes U(6)$. Although the hypermultiplets possess a perfectly unitary norm, the unitarity of the exact S-matrix is incompatible with any symmetry higher than SU(3) and even that in the limit when mass differences are neglected. In this respect the present theory shares all the failing or virtues of the $\tilde{U}(12)$ theory.

Before considering the physical outlook for a hypermultiplet scheme of the type described, let us list the unsolved mathematical problems which must be tackled before further progress can be made. These are:

1) The reduction of a hypermultiplet of $G(\hat{p})$ relative to a (non-compact) subgroup of $G(\hat{p})$.

2) The Clebsch-Gordan problem.

From the physical point of view, the major value of a theory like that of this paper lies in its prediction of higher multiplets and their respective coupling coefficients. Even though the intrinsic symmetry breaking could automatically lead to mass differences among the various components of the hypermultiplet, it would be a miracle if the resulting mass spectrum had any relevance to physics; for example it is too much to hope that the much-to-be-desired result of increasing mass values for increasing spin could follow simply from the intrinsic symmetry breaking. The situation here is rather analogous to the hydrogen atom, where the use of the non-compact group $O_{4+1}$ gives correctly the level sequence, though its use provides no information in respect of the level values. The point is that the origin of all symmetries we are discussing must lie in the idealizations and accidents of the underlying dynamics: the dynamics both of the forces which give rise to the hypermultiplet in the first place, and also of the forces which makes two or more hypermultiplets interact with each other. The hope that both these types of forces give rise to the same symmetry pattern is a hope which only further experiment can justify.

## APPENDIX

On Discrete Representations of $U(\nu, \nu) \otimes U(\nu, \nu)$

DOTHAN, GELL-MANN, NE'EMAN and FEYNMAN have given a method for constructing some discrete representations of U(6, 6), the rest symmetry, through the application of Fock space techniques. In their case the reduction is carried out relative to the compact subgroup $U(6) \otimes U(6)$ by introducing creation and annihilation operators and defining U(6, 6) group generators in terms of these. We shall adopt the same method to obtain some of the discrete series for our overall group $U(6, 6) \otimes U(6, 6)$ and carry out the reduction relative to non-unitary representations of the subgroup $U(6, 6)_D$. In the second stage of applying Bargmann-Wigner equations the structure represents a boosted Feynman tower.

Since the case $U(\nu, \nu) \otimes U(\nu, \nu)$ is no harder to treat we begin with two types of creating and annihilation operators which obey the commutation rules ($A, B = 1, \ldots \nu$)

$$[\bar{a}_A, a^B] = \delta_A^B \quad , \quad [b_A, \bar{b}^B] = \delta_A^B$$

$$[b_A, a^B] = [b_A, \bar{a}_B] = \ldots = 0 . \tag{A.1}$$

These relations are equivalent to those for the $\psi$ stated in Section 3. Next we construct the operators (cf. Eq. (3.23)).

$$M_A^B = \bar{b}^B b_A + a^B \bar{a}_A \quad , \quad N_A^B = \bar{b}^B \bar{a}_A + a^B b_A \tag{A.2}$$

It is then easy to verify from (A.1) that

$$[M_A^B, M_C^D] = \delta_A^D M_C^B - \delta_C^B M_A^D$$

$$[M_A^B, N_C^D] = \delta_A^D N_C^B - \delta_C^B N_A^D$$

$$[N_A^B, N_C^D] = \delta_A^D M_C^B - \delta_C^B M_A^D \tag{A.3}$$

With the further specifications (cf. Eq. (3.20))

$$\bar{b}^B = b_C^{\dagger} (\gamma_0)_C^B \quad , \quad \bar{a}_B = a^{C\dagger} (\gamma_0)_C^B \tag{A.4}$$
$$\gamma_0 = (\underbrace{1 \ldots 1}_{\nu}, \underbrace{-1 \ldots -1}_{\nu})_{diag}$$

the reality conditions

$$M^{\dagger} = \gamma_0 M \gamma_0 \quad , \quad N^{\dagger} = \gamma_0 N \gamma_0$$

show that $M$, $N$ together generate the group $U(\nu,\nu) \otimes U(\nu,\nu)$. $M$ on its own generates the $U(\nu,\nu)_D$ subgroup and it is with respect to this group that we intend to decompose some discrete $U(\nu,\nu) \otimes U(\nu,\nu)$ representations. The latter we obtain by the action of creation operators contained in $N$ on some lowest state. If we define the vacuum state $|0\rangle$ by

$$a^A |0\rangle = b_A |0\rangle = 0 \tag{A.5}$$

we find that the Casimir operator $N_A^A$ takes us out of the vacuum state into $(a_A \bar{b}^A) |0\rangle$. Thus it is necessary to define our lowest state by

$$|f\rangle = \sum_n f_n (\bar{a}_A \bar{b}^A)^n |0\rangle \tag{A.6}$$

and impose the conditions

$$N_A^A |f\rangle = \lambda |f\rangle \quad , \quad (M_A^B + \nu \delta_A^B M_C^C) |f\rangle = 0 \tag{A.7}$$

to guarantee that we are dealing with an irreducible meson representation*.

Let us concentrate on the meson series which is built up from $|f\rangle$ by repeated action of the operators $N$. We assert that subject to (A.7) we do in fact arrive at an irreducible meson representation of $U(\nu,\nu) \otimes U(\nu,\nu)$ in the form

$$\Psi(1) \oplus \Psi_A^B(\text{"}143\text{"}) \oplus \Psi_{AB}^{CD}(\text{"}5940\text{"}) \oplus \cdots \tag{A.9}$$

where

$$\begin{aligned}\Psi_{AB\cdots}^{CD\cdots} &= \left(N_A^C N_B^D \cdots + N_B^C N_A^D \cdots + N_A^D N_B^C \cdots + N_B^D N_A^C \cdots - \text{Trace}\right)|f\rangle \\ &\equiv \{N_A^C N_B^D \cdots\}|f\rangle\end{aligned} \tag{A.10}$$

To prove this assertion we must show that $N_A^B N_B^C$ when acting on a state in the representation is proportional to $N_A^C$ or $\delta_A^C$ and that $(N_A^B N_C^D - N_C^B N_A^D)$ creates no new antisymmetric tensor.

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* Similar conditions should of course be imposed for all other Casimir operators. We do not know if these follow from (A.7) in general. Also analogous definitions of the lowest states have to be supplemented for the baryon hypermultiplet series, etc. For instance the lowest quark state will read

$$|q_A\rangle = \sum_n q_n \bar{a}_A (\bar{a}_B \bar{b}^B)^n |0\rangle \tag{A.8}$$

To show this we offer the following inductive argument:

(i) Condition (A.7) is in the nature of an eigenvalue equation for $\lambda$. It leads to the following recurrence relations between the expansion coefficients $f_n$ of (A.6),

$$f_{n-1} - \lambda f_n - (n+1)(n+\nu) f_{n+1} = 0 \quad ; n = 0, 1, \ldots \qquad (A.11)$$

with $f_{-1} = 0$. The spectrum of $\lambda$ can in principle be determined by converting (A.11) into a differential equation. However, the solution with $\lambda = 0$ is easy to obtain and for this case we have the rapidly converging series

$$f_{2k} = \left(\tfrac{1}{2}\right)^{2k} \frac{\left[\frac{1}{2}(\nu-1)\right]!\ f_0}{k!\ \left[k + \frac{1}{2}(\nu-1)\right]!} \qquad (A.12)$$

Therefore the eigenvalue $\lambda = 0$ belongs to the discrete spectrum and for simplicity we adhere to this value for the rest of the discussion.

(ii) Consider next the expression $N_A^B N_B^C$. On the lowest state this gives after some traightforward manipulation,

$$\begin{aligned} N_A^B N_B^C \Psi(1) &= \sum_n f_n \left(\bar{b}^B \bar{a}_A + a^B b_A\right)\left(\bar{b}^C \bar{a}_B + a^C b_B\right)(\bar{a}\bar{b})^n |0\rangle \\ &= (1-\nu)\, \delta_A^C\, \Psi(1) \end{aligned} \qquad (A.13)$$

That is to say we do indeed produce no new 143 representation.

(iii) Finally consider the expression $\left(N_A^B N_C^D - N_C^B N_A^D\right)$. On the lowest state it produces

$$\begin{aligned} \Psi^{BD}_{[AC]} &= \left(N_A^B N_C^D - N_C^B N_A^D\right) |f\rangle \\ &= \left(\delta_A^D \delta_C^B - \delta_C^D \delta_A^B\right) \Psi(1) \end{aligned} \qquad (A.14)$$

Hence, on $|f\rangle$, only the symmetrized part produces a new tensor.

(iv) We now wish to extend conclusions (ii) and (iii) to apply to every one of the hypermultiplet tensor components $\bar{\Psi}^{CD\ldots}_{AB\ldots}$. We will not attempt to give a complete proof but will offer instead the following plausibility argument: Since the commutator of two $N$ 's is an $M$ we can always re-arrange factors in a product of several $N$ 's so as to bring a selected pair into adjacent positions at the right-hand end. In the course of this, a number of states with fewer $N$ 's multiplied by $M$ 's appear. The $M$ multiplication is a trivial reshuffling and can be ignored. By an inductive argument we may assume that products with fewer $N$'s have been already dealt with. Thus no generality is lost by bringing the selected pair to the right-hand end, where by arguments (ii) and (iii) only the symmetrized traceless part of the pair is relevant. Since this procedure can be carried out with all indices we deduce thatfrom the products of $N$ 's acting on the lowest state only the symmetrized and traceless parts need be retained. This justifies the series (A.9).

The Feynman construction of U(6,6) representations depended on the assumption of Bose statistics for the creation and annihilation operators. One could construct other representations with different symmetry types by generalizing to para-Bose statistics. This will be shown in detail elsewhere.

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