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INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

THEORY OF GROUPS AND THE SYMMETRY PHYSICIST

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1965

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(Lecture delivered to the Centenary Meeting of the London Mathematical Society on 14 July 1965)

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1. INTRODUCTION

Quantum theorists have never doubted that the theory of groups was invented specifically for use in physics. An arrogant and ungrateful minority has even held that not only was the relevant theory invented for physicists but also by physicists. This claim of course is wild, irresponsible, certainly false so far as the past is concerned. It may however acquire more substance in the future. This is because one of the frontiers in mathematics today lies with theory of non-compact groups. It so happens that during the last year symmetry physics has also begun to find non-compact group theory fruitful for applications in particle physics. The physicist needs newer results. Barring an urgent interest on the part of the competent group theorists, he will doubtless make numerous conjectures and proceed to use them. I very much hope that these are not all eventually found false.

During the course of this lecture I wish to speak of some very recent successes of the group-theory method in classification schemes for the so-called elementary particles. I shall briefly sketch some particular problems in the representation theory of non-compact groups in which the physicist thinks he would like progress made. In all humility, let me say, the important point in application of a mathematical discipline is not always the insights that a physicist can specify in advance. Still more important could be those that we at present have no inkling of, and which one might successfully only bring to bear when the complete theory is available.

The first question that arises is:-

Why has group theory played such an important, such an intimate role, in the development of quantum mechanics in contrast say to classical mechanics ? The reason has been spelt out by Yang and Wigner; it lies in one basic circumstance; it lies in the basic postulate of quantum theory that the quantum states of a physical system form a linear manifold.

To illustrate, let us consider the group of space rotations, the rotation group O_3. Throughout the history of physics we have started with the assumption that laws of physics remain unchanged for space-rotations. In the final analysis it is an empirical postulate, to be tested by its consequences. We build this postulate into classical physics by demanding that any equations of motion we may write down should remain invariant with respect to rotations. In a rather subtle way the postulate leads to conservation law of angular momentum. A naive application however of this rotation invariance to classical trajectories - to planetary orbits for example - merely tells us that, given a certain orbit, we may infer by rotation the existence of other physically possible ones.

Now this is an important result, but by no means a very fruitful result. It does not lead to new insights. Contrast this with the case of quantum mechanics. The same statement can be made about quantum orbits. In quantum theory there is however the further postulate that all possible orbits form a linear manifold and that one can select from this manifold a linearly independent complete set in terms of which all the orbits can be expressed linearly;

i.e. $$| \Psi > = \sum_j a_j \, | \Psi_j > .$$

Now denote the rotation operator corresponding to a rotation g by U(g). From rotation invariance, if $| \Psi >$ is an orbit $| \Psi' > = U(g) \, | \Psi >$ is another possible one. Specialise to $| \Psi > = | \Psi_i >$. From completeness, therefore,

$$U(g) \, | \Psi_i > = \sum_j \alpha_{ij} \, | \Psi_j > .$$ Clearly the α_{ij} give us at once

a representation of the rotation group g. With the quantum postulate of the orbits forming a linear manifold, we immediately strike a level of richness with the mathematical representation theory of groups, unsuspected, unconceived of at the level of classical dynamics.

Let us pursue this further. In quantum theory we are concerned only with unitary representations. This is connected with the quantum mechanical theory of measurement. I shall not go into the measurement theory of physics in any detail but we shall accept for the rest of the lecture that we shall always deal with unitary representations. Write an infinitesimal unitary rotation operator in the form

$$U(g) \approx 1 + i\epsilon_j J_j \qquad j = 1, 2, 3$$

The standard commutation relations for the hermitian operators J_i ($J_i = J_i^+$) of 0_3 read:-

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

with $\underline{J}^2 = J_1^2 + J_2^2 + J_3^2$ as the Casimir invariant which commutes with all three J_i. Now it is well known, that the compact group 0_3 possesses finite-dimensional two-valued unitary representations, labelled by two numbers j and j_3; symbolically

$$J^2 \,|\, j, j_3 > = j(j+1) \,|\, j, j_3 >$$

$$J_3 \,|\, j, j_3 > = j_3 \,|\, j, j_3 >$$

where j can take integral or half-integral values; $0, \frac{1}{2}, 1, 3/2, \ldots$ and j_3 ranges between +j and -j. The invariance postulate for space rotations, combined with the completeness of the basic states $|\Psi_i>$ allows us therefore to state that the physical (Hilbert) space of any quantum mechanical system can be realized in terms of a discrete basis; as a direct sum of the irreducible unitary representations of the rotation group 0_3 :

$$|H> = \ldots . \oplus \,|\, j, j_3 > \oplus \ldots . .$$

The group description if you like has become an essential part of the "kinematic" structure of quantum theory. The quantum number

j is called the total angular momentum - the spin - of a pure state $|j, j_3>$ (in units of Planck's constant $\hbar$), j_3 is the component of the spin along the z-axis. Both j_3 and j are quantised; both are integers or half-integers. We all know that quantisation, the discreteness of physical quantities is the essence of quantum mechanics. The contact with representation theory of groups automatically guarantees that j and j_3 are quantised.

To state the vocabulary I shall henceforth use :- the totality of the $(2j + 1)$ states (i.e. the $(2j + 1)$ vectors)

$$|j, j>, \; |j, j-1>, \; \ldots\ldots \; |j, -j>$$

labelled with the two quantum numbers j and $-j \leq j3 \leq j$ will be designated as a $(2j+1)$-fold multiplet consisting of $(2j + 1)$ components <u>corresponding</u> to a representation of 0_3 of dimensionality $(2j + 1)$. The multiplet of particles is directly associated with the basic set of independent vectors which correspond to a given irreducible representation of the group. In the sequel we shall be faced with the converse problem, given the dimensionality of a multiplet and the labelling quantum numbers for the states, find the underlying invariance group. In particle physics the task of the experimental physicist in recent past has been to classify more and more particles, more and more resonances, in multiplets, each distinct multiplet being characterised by the value of some (one suspects) Casimir invariant. The task of the symmetry physicist has consisted in identifying the possible invariance group for the underlying dynamics. Once the group is known, the conservation laws - like the conservation of angular momentum - which are implicit in the invariance postulate, then provide a whole host of new and testable relations between experimental magnitudes.

2. "KINEMATIC" CLASSIFICATION OF PARTICLES - THE INTERNAL SYMMETRIES

I have discussed the quantisation of spin of a system at rest in terms of the compact group 0_3. The automatic manner in which representation theory of groups takes care of the discreteness of physical quantities, and gives rise to a multiplet structure set the pattern for all future classification schemes in particle physics. To illustrate I shall have to go briefly over the experimental situation in the subject as it has developed from the year 1926 onwards.

Around 1926, two so-called elementary particles were known to physics; the proton and the electron. These are tiny chunks of matter; the proton with a mass of around 10^{-24} grams, the electron some 2000 times lighter. Both are electrically charged, the proton positively and the electron negatively. They were elementary in the sense that all matter - all the 92 atoms - were then (erroneously) believed to be made from just these two objects.

Protons and electrons are not just simple chunks of matter. Both these particles carry intrinsic spin; group theoretically the particles corresponded to the spinor representation of 0_3, with $j = 1/2$, $j_3 = 1/2, -1/2$. In the vocabulary I have used earlier the proton (or the electron) formed a two-fold ($2j + 1 = 2$) multiplet. The two possible states of a proton represented by $|j = 1/2, j_3 = +1/2>$ and $|j = 1/2, j_3 = -1/2>$ were also called the states of left and the states of right polarisation by the physicist.

The astonishing thing about these two particles was - and still is - the numerical equality of the electric charge they carried. The masses of the electron and the proton are so different. The electric charge, like angular-momentum, seems however to be quantised - quantised in integral multiples of just one fundamental unit. This is a most mysterious fact but one which we must incorporate in our description of nature.

With the success of the group theory ideas in understanding quantisation of spin, the pattern seemed clear for a group-theoretic description of charge-quantisation. The group-representations of a rotation-group in two-dimensions are labelled as is well known by positive or negative integers. Assume that there exists a two-dimensional "internal" space - call it the "charge space", and assume that all equations of motion are invariant for rotations in this space. This rotation-invariance will imply, through the quantum mechanical procedure sketched earlier both charge-quantisation as well as charge-conservation. The pattern is the same as that for the three-dimensional rotation group; the logical argument is the same. The difference however is that contrasted to the three-dimensional physical space the new twin-dimensions of "charge space" apparently possess no direct physical significance. The space has conveniently been called the "internal space", associated perhaps with some internal structure of protons or electrons. Even so no-one has had the courage actually to write equations of motion displaying an explicit dependence on the co-ordinates of this space. Put it another way; in the case of space-rotations, the rotation group itself is physically significant and not only the infinitesimal generators J_1, J_2, J_3. For the internal space, the rotation itself is hardly significant at all; it's only the Algebra of the infinitesimal generators and their eigenvalues which make physical sense. The internal-symmetry physicist has no use for the Group; he has ample use for its Algebra.

Till 1930, the only known "internal" characteristic of an elementary particle was just this one quantised entity - the electric charge. In 1930, with Chadwick's discovery of the neutron, there came a break. The neutron was the third "elementary particle"; it was almost as massive as the proton but carried no electric charge. The proton and the electron attracted when close together through the intermediacy of the classical electrostatic force. For a proton-neutron system, the electrical force was naturally irrelevant since the neutron carried no electrical charge. Two neutrons, or a proton and neutron, however, did exhibit a strong attraction when close to each other. This clearly was a new force of nature. At comparable distances one found empirically that it was at least one hundred times stronger than the electrical force. Thus to an excellent approximation - to the approximation that one could neglect electrical forces relative to the nuclear - protons and neutrons were two states of just one single entity - the so-called nucleon.

Now one had encountered the situation of a particle existing in one of two possible (polarisation) states before. One had seen that a spin $\frac{1}{2}$ proton or an electron possesses two polarisation states $|j = \frac{1}{2}, j_3 = +\frac{1}{2}>$ and $|j = \frac{1}{2}, j_3 = -\frac{1}{2}>$. Could one once again invent a new "internal" space - three dimensional this time - and invent three infinitesimal rotation generators I_1, I_2, I_3, with commutation relations

$$[I_i, I_j] = i \epsilon_{ijk} I_k$$

The spinor representations of this new group could then be identified with the nucleon. This suggestion originated with Kemmer, Heisenberg, Breit and several others around 1934-38. The new "internal-space" was named the "isotopic space"; the nuclei, which are composites of nucleons, formed multiplets, corresponding to the irreducible representations of the isotopic-rotation group. All nuclei carried isotopic-spin, in addition of course to the ordinary spin which henceforth I shall call Poincaré spin.

The next development in particle physics came in 1935 with some speculations of Yukawa. Yukawa recalled that all accelerating electric charges emit electro-magnetic radiation in accordance with Maxwell's laws. The quantum aspects of the electro-magnetic force are the photons. Yukawa raised the question; what is the analogue of a photon for the nuclear force? What type of radiation do nucleons emit when they are accelerated? He conjectured that there exist in nature photon-like objects, the so-called mesons, particles with masses intermediate between electrons and nucleons, which are emitted by accelerating nucleons. From the group-theoretic point of view, these particles,if they did exist,would once again correspond to irreducible representations of the isotopic group. Further,if like photons these are emitted - shed by nucleons - singly at a time, conservation of isotopic spin would demand that their I-spin be integral and not half-integral.

Yukawa's ideas were put forward in 1935. These were persuasive ideas; the search for the particles was interrupted by the war, but just after, in 1947, Professor F. C. Powell announced the discovery of the Yukawa partices - the so-called "pions". There were three pions, corresponding to an isotopic spin I = 1,

$$\pi^+ \rightarrow | I = 1, I_3 = 1 >$$

$$\pi^0 \rightarrow | I = 1, I_3 = 0 >$$

$$\pi^- \rightarrow | I = 1, I_3 = -1 >$$

The Poincaré spin J of the particles turned out to equal zero. These particles,in contrast to nucleons,were spinless. Summarising, the classification scheme of particles concerned with the nuclear force (and let us recall that the electron is not one of such particles) proceeds in terms of three characertistics:-

(1) An "external" characteristic, Poincaré spin J, Group-space 0_3

(2) Two "internal" characteristics,

(a) Isotopic spin, Group-space O'_3

(b) Electric charge Q, Group-space O''_2.

The internal symmetry associated with isotopic spin was not an exact symmetry; it was exact only if electromagnetic forces were neglected in comparison with the nuclear forces.

Note the linear relation :-

$$Q = I_3 + \tfrac{1}{2} \quad \text{for nucleons}$$

$$= I_3 \quad \text{for mesons}$$

We will encounter the general relation of which the above are special cases later; it is clear,however, that a relation of this type would imply a connection between the charge-space O''_2 and the isotopic space O'_3.

After 1947 came further experimental discoveries. A whole host of new objects were discovered; particles with different masses, charges, Poincaré and isotopic spins. By no stretch of imagination could one call these "elementary particles", any more. But whether these were or were not composites of any simpler entities one had to find a quantum description for these. In January 1964 the situation could be summarised as follows:-

As a result of patient and painstaking experimentation, both with cosmic rays and the giant accelerators at CERN, Brookhaven, Dubna, Berkeley and elsewhere, one could classify the newly-discovered particles into the following multiplets:-

(1) 8 particles of Poincaré spin $\frac{1}{2}$ (the nucleon family)
(2) 9 particles of Poincaré spin 3/2 (excited nucleons)
(3) 8 mesons of Poincaré spin zero
(4) 9 excited mesons of Poincaré spin one.

It is irrelevant for my purposes to go into details concerning these multiplets; in particular, the identifying nomenclature etc. within a multiplet is completely irrelevant. However there is one common point I need to illustrate about these multiplets, and for this I shall refer to the 8-fold nucleon multiplet which consisted of the following "components" :-

	I	I_3	Q	$Y = 2(Q-I_3)$
p	$\frac{1}{2}$	$\frac{1}{2}$	1	1
n		$\frac{1}{2}$	0	1
Λ^o	0	0	0	0
Σ^+	1	1	1	0
Σ^o		0	0	0
Σ^-		-1	-1	0
Ξ^o	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
Ξ^-		$-\frac{1}{2}$	-1	-1

TABLE 1

This 8-fold of particles - all of nearly equal mass, all of same Poincaré spin - consists of 4 distinct isotopic multiplets (p, n), (Λ), ($\Sigma^+\Sigma^o\Sigma^-$) and (Ξ^o, Ξ^-). For each multiplet empirically the quantum number Y ($=2(Q - I_3)$) - the so-called hypercharge - possesses the same (integral eigenvalue). Fixing on I_3 and Y, clearly there must exist a higher symmetry group, a higher invariance, perhaps a group of rank 2 since we are dealing with at least two simultaneously diagonalisable operators I_3 and Y.

That there was some higher symmetry at work in the physics of the nuclear interaction was clear quite early, around 1956-1957. That the way to progress lay along a systematic search for a compact group of rank 2 was only very imperfectly understood till 1961. The major uncertainty in the approach was of course always experimental. I have blandly stated that the nucleon multiplet was found to consist of 8 members, all of Poincaré spin $\frac{1}{2}$, and I have specified the I-spins and hypercharge Y for each particle as if every experimentally discovered object carried a chain around it with a label on which one could read off its characteristics. In real life things are never like this; till this day for example one does not experimentally know with more than 95% statistical confidence that the Poincaré spin of the Ξ^0 and Ξ^- particles is indeed $\frac{1}{2}\hbar$. My colleague Professor P. T. Matthews has given an illuminating illustration of the difficulties which beset the work of an experimental physicist. There is only one experiment a particle physicist can perform; he can scatter one set of particles off another and by counting the numbers which fly off in a collision in a given direction try to find the spins and isotopic-spins,etc.,of the different end-products. It's like playing a hose of water on a statue in a dark room and being allowed to collect the water that splashes off from the statue's face. One can appreciate the hardships of the experimental physicist if one were posed the problem of delineating the statue's features by measuring only the quantities of water that splashed off per square inch of its surface.

Returning to the classification problem, there are but four semi-simple Lie groups of rank two, A_2, B_2, C_2 and G_2. One of the most misleading aspects of the situation was the tradition that seemed established from the days of classical physics in the subject of using rotation groups like O''_2 and O_3. It may sound extremely trite to you but one of the important break-through's was to realise that since one was using two-valued representations of the groups concerned, the isomorphism of O_3 with $A_1 \approx U_2$ could be exploited to pose the problem of internal symmetries in the following manner:-

Associating I-spin with the group structure U_2
and associating hypercharge Y with the group structure U_1,

find a group of rank two which possesses both U_2 and U_1 as sub-groups, and has an 8-dimensional representation with the independent vectors with characteristics indicated in the Table 1.

Once again let me stress, no working physicist can ever state a physical problem in this form except after the event. There are so many reservations, so many hesitations, so much one must take on trust and, most difficult of all, so much one must discount. But this, don't forget, is after all the joy of discovery in physics.

Stated in the form I have used above, the problem had one solution; the higher symmetry group must be $A_2 \approx U_3$; this solution was first proposed by the Japanese physicists - Oknuki, Ogawa and Sawada in 1959. The U_3 symmetry was named "unitary symmetry". Unfortunately the Japanese authors made a wrong physical identification of particles; the representations of the group they happened to choose did not decompose in the manner of Table 1. A later version given simultaneously and independently in 1961 by M. Gell-Mann in California and by Y. Ne'eman working at Imperial College fitted facts better.

This was encouraging, but not encouraging enough. In addition to the 8-fold multiplet of nucleons, there was also the other multiplet consisting of 9 excited nucleons with the following assignments:-

	I	Y
N^{*++}, N^{*+}, N^{*o}, N^{*-}	3/2	1
Y^{*+}, Y^{*o}, Y^{*-}	1	0
Ξ^{*o}, Ξ^{*-}	$\frac{1}{2}$	-1

TABLE 2

Now there is no 9-component irreducible representation of the group U(3); the nearest whose ($U_2 \times U_1$) decomposition would yield the eigenvalues shown in Table 2 for the generators I_3 and Y must contain 10 objects. One crucial particle was missing; a particle which from the group-theoretic structure of the representation concerned must possess $I = 0$, $Y = -2$. The empirical formula $Q = I_3 + Y/2$ would give its electric charge as -1. The particle was given the name Ω^- in absentia. The fate of the U(3) symmetry hung on its discovery.

This as I said earlier was the situation in January 1964. In February 1964, among a scan of millions of photographic bubble-chamber pictures, two Ω^-'s were discovered at Brookhaven. The production and decay of these particles is spectacular; the fact that there are just two known specimens of the particle made them highly precious. The higher symmetry scheme U_3 was fully vindicated; the "isotopic-spin" group U_2 had been generalised to the "unitary-spin" group U_3 as the still higher, the still more embracing, symmetry of the nuclear force.

The next lot of developments came pretty soon after, in September 1964. And these were still more spectacular. At this stage, as I said earlier, one had a total of 4 complete multiplets:-

N	(nucleon)	$J = \frac{1}{2}$,	number of particles 8
N^*	(excited nucleon)	$J = 3/2$,	number of particles 10
M	(meson)	$J = 0$.	number of particles 8
M^*	(excited mesons)	$J = 1$.	number of particles 9

TABLE 3

One had two varieties of quantum numbers:-

(1) External, Poincaré-spin; group structure $0_3 \approx U_2$;

(2) Internal, I-spin and hypercharge Y; group structure U_3

Now when counting the numbers of particles in any internal U(3) multiplet, no account had been taken of spin polarizations, of spin multiplicity. Supposing we do treat each Poincaré spin polarization as distinct; the count would then be as follows:-

N	(nucleons) $J = \frac{1}{2}$	$8 \times (2J+1)$	=	16 distinct particles	
N^*	(excited nucleons) $J = \frac{3}{2}$,	$10 \times (2J+1)$	=	40	" "
M	(mesons) $J = 0$,	$8 \times (2J+1)$	=	8	"
M^*	(excited mesons) $J=1$,	$9 \times (2J+1)$	=	27	"

In September 1964, the question was raised; is it conceivable that one did have in nature a symmetry higher than U(3), a symmetry comprising both "external" Poincaré spin and the "internal" unitary spin for which all nucleons - both those which we have called excited and those that are unexcited - appear as just one multiplet; and the same thing happening possibly for mesons. Could we possibly obliterate the distinction of external and internal symmetries? Could we think up a symmetry group with both U(3) and U(2) as subgroups? The obvious candidate was U(6). This was tried in September 1964 by F. Gürsey, L. Radicati, and B. Sakita. Among its irreducible representations U(6) does possess two representations, with dimensionalities 35 = 8 + 27 and 56 = 16 + 40. These precisely are the numbers of distinct known particles. On the face of it U(6) could be the still higher symmetry of nuclear interactions we were looking for.

The importance of U(6) was this: other particles may now be discovered experimentally; we would expect them to correspond to higher representations of the U(6) group. The key-symmetry seems to have been discovered. The compact group U(6) seems to combine internal as well as external spin attributes.

3. DYNAMICAL CONSIDERATIONS AND NON-COMPACT GROUPS

So far so good. One was making progress but where was the dynamics? How was one to understand this merging of the "external" Poincaré and the "internal" unitary spin? But even before one could look at the problem in depth, one should give the technical problem connected with spin a clearer perspective.

I presented earlier the theory of Poincaré spin, starting with the symmetry group of rotations in three-dimensional space. This may be adequate for a system at rest but for a moving system one must use relativistic kinematics. One must consider invariance for space-time rotations; one must consider the Lorentz group.

Now the Lorentz group is a non-compact group. All its unitary representations are infinite-dimensional. For identification with the finite multiplets of spinning particles we would like to use the finite-dimensional representations.

The Dirac-Wigner resolution of this perplexing dilemma is well known. Within the group-theoretic context, Wigner was the first to realize clearly that for spin-representations, it was not the homogeneous Lorentz group which was relevant; it was the invariance for the inhomogeneous group - the group which includes four translations in addition to the 6 rotations, the so-called Poincaré group - which was of importance. Wigner was the first to study in a classic paper of 1939 in Annals of Mathematics, the unitary representations of the Poincaré group. As is well known, this was the beginning of the theory of induced representations. Wigner was able to show that one could indeed define a unitary norm for the finite-dimensional representations of the homogeneous Lorentz group provided the four translations were properly taken into account. For particle-like representation of the inhomogeneous Lorentz group - i.e, for representations for which the Casimir invariant $P_\mu^2 = P_o^2 - P_1^2 - P_2^2 - P_3^2$ possesses positive eigenvalues, the second Casimir invariant

$$(J_{\alpha\beta})^2 \quad - \quad \frac{(P_\alpha J_{\alpha\beta})^2}{(P_\alpha)^2}$$

reduces to the 0_3 invariant $J_1^2 + J_2^2 + J_3^2$ we have encountered before, in the rest frames of the particles concerned. Dirac of course, with his great physical insight, had generated precisely such representations of the Poincaré group in 1928 using his electron equation.

The moral for the systematics of particle physics was clear. If one wished to work with a relativistically invariant symmetry scheme which should (1) incorporate U(3), (2) provide a unitary norm for finite-dimensional representations, one must consider as the minimal acceptable extension, the non-compact inhomogeneous group SL(6, C). This would be the minimal group with both SL(2, C) (the homogeneous Lorentz group) as well as U(3) as subgroup. And its maximal compact subgroup is indeed U(6) - the group we have been exploiting for the particle multiplets. This was a suggestion made by B. Sakita and W. Rühl early this year.

For reasons which I cannot go into here my colleagues Drs. R. Delbourgo, J. Strathdee and I (and independently Sakita, Wali, Bég and Pais in USA) preferred to work with a larger (inhomogeneous) non-compact group U(6, 6). Our ignorance of mathematical literature in January 1965 when this work was done was so abysmal, and our procedure was so completely motivated by the dynamical requirements of physics that once we had fixed on the type of non-compactness we needed we gave it a new name; we called the group $\tilde{U}(12)$, U(12) because it fell in the sequence of U(1), U(2), U(3) and U(6); ~ Twiddle to display its non-compact character. The group U(6, 6) contains SL(6, C) as a subgroup, its maximal compact sub-group being U(6) x U(6). It represented a symmetry higher than the SL(6, C).

At this stage one must remember one is not just trying to reproduce the multiplet structure of elementary particles. The intention is deeper - the intent is to make a go at the dynamics.

Again it would take me too far from my present topic to discuss what we achieved. Suffice it to say, we succeeded brilliantly in parts. In the exuberant days of January and February when we were investigating what is called the three point function, we felt we had solved the whole of nuclear physics. When we came next to the four point function, the naive application of the symmetry ideas gave some predictions which agreed with experiment and others which did not.

One could trace the reasons for the successes with the three point function and the failure with the four point quite readily. The inhomogeneous U(6, 6) group contains 143 translations and 143 rotations. The physical space-time allows for just 4 translations. The remainder were the translations in the internal spaces. The inevitable, the honest corollary of marrying space-time invariance with internal symmetries, should be to treat all the translations at the same level. Even we were not prepared to do so. We approximated to the physical situation by breaking the exact invariance in selecting a sub-set of translations; We could trace back in this approximation procedure, and the residual symmetries it left, the seeds of our successes and also our failures.

Are approximate internal symmetries something one should feel ashamed about? At the present stage of the subject's development there is only one answer: No. As I empahsised earlier, apart from the conservation of charge and conservation of Poincaré spin, no other internal symmetry is exact. You may recall very early in my lecture I said the proton and the neutron are two components of the same physical entity, the nucleon, to the extent that one can neglect electro-magnetism. Recall that this (approximate - approximate to one part in a hundred) identity of the proton and the neutron was at the heart of the isotopic spin concept. The entire classification based on the isotopic group- and, per se, the unitary group - was approximate. With the inhomogeneous U(6, 6) the idealisation is carried still further; the symmetry is exact if one considers all the 143 translations.

The physical situation with just 4 translations is an approximation to it. The moral was; one must learn to sharpen one's mathematical approximation technique for application to the four point function.

I have described so far one aspect of the use of non-compact groups in recent physical theory. There is a completely different aspect which is also currently being developed. Recall that the major problem with the non-compact groups is the unitarity of their representations. All finite dimensional representations of such groups are non-unitary; it's only the infinite-dimensional representations which are unitary. If to each component of a unitary multiplet there corresponds a distinct particle or a distinct energy level, clearly the infinite-dimensional representations are not too pleasant to work with. This is because one does not always have in physics an infinity of particles or an infinity of energy levels to correspond to the infinite dimensionality of the representation. Following Dirac and Wigner one has tended to escape this dilemma for spin-multiplets by postulating invariance for an inhomogeneous group structure. The introduction of translations then allowed us to use the finite dimensional representations, the unitarity of the norm being restored through the translations in accordance with ideas of induced representations.

But there are some situations in physics where we do indeed possess an infinity of energy levels and one might perhaps use the infinite-dimensional unitary representations without involving the inhomogeneous group. One of the most famous of these cases is the case of the Hydrogen atom - the very first system to be studied in quantum theory. This case is so instructive that I shall give it in some detail. The Hamiltonian for the hydrogen atom in a $\frac{1}{r}$ potential has the form

$$H = \frac{p^2}{2m} - {e^2}/{r}$$

where, as usual, we postulate that $\underline{r}$ and $\underline{p}$ are conjugate operators following the standard rules of quantum mechanics;

i.e. $\underline{r} = x_i, \quad \underline{p} = p_i$

$$[x_i, p_j] = i\hbar\, \delta_{ij}$$

The angular-momentum operator has the conventional definition:-

$$\underline{J} = \underline{r} \times \underline{p}$$

Define now the so-called Lenz-vector

$$\underline{A} = \left(\frac{1}{2e^2 m}\right) \left\{(\underline{L} \times \underline{p} - \underline{p} \times \underline{L})\right\} + \frac{\underline{r}}{r}$$

It has been known since 1936, with the work of Hulthen, Bargmann, Fock and others that the operators $\underline{J}$ and the operators $\underline{M} = \frac{1}{2\sqrt{H}} \underline{A}$ close on the Algebra of the compact group SO(4). Notice the very curious manner in which the Hamiltonian makes its appearance.

The representations of the relevant SO(4) are given in terms of two numbers (k, l) where

$$K = \tfrac{1}{2} (\underline{J} + \underline{M})$$

$$\underline{L} = \tfrac{1}{2} (\underline{J} + \underline{M})$$

There is one further restriction $\underline{K}^2 - \underline{L}^2 = 0$ which follows from the equations of motion. The hydrogen energy levels are therefore among the infinite sequence of SO(4) representations with $k = \ell$:-

$$(0, 0);\ (\tfrac{1}{2}, \tfrac{1}{2})\ (1, 1);\ (3/2,\ 3/2);\ \ldots\ldots$$

As I remarked earlier this is a very special problem, a very special Hamiltonian and very special group structure which has emerged.

Where do non-compact groups come into this? Quite recently Barut, Budini and Fronsdal and Gell-Mann, Ne'eman and Dothan have made the remark that one can introduce four additional non-compact generators N_α which when commuted with $\underline{J}$ and $\underline{M}$ close on the Algebra of SO(1, 4). Further,one representation of this non-compact group consists of the entire infinite sequence

$$(0,0),\ (\tfrac{1}{2},\tfrac{1}{2}).\ \ (1,1),\ 3/2,\ 3/2), \ldots\ldots$$

mentioned above. For this very special problem then the non-compact group SO(1, 4) is the correct dynamical group. Perhaps there are other cases where non-compact groups might likewise solve completely the dynamics of the system.

Now no progress is possible in the consideration of non-compact groups and the application of the ideas sketched above till at least one knows the sequences of unitary representations of these groups. This is the prime essential. In this respect it is crucial to remember one thing. As you may see in all my recital, the physicist revels in discrete numbers; it is therefore the so-called degenerate series which are usually of interest to him. Further his emphasis is always on the Algebra, on the diagonalisation of the generators and usually not with the group or the function space it acts on. After the representation theory is available will come the heart-breaking task of reducing products of representations. Even in our wildest of hopes we cannot believe that mathematicians will lift a finger to help us solve this problem. But perhaps we shall be pleasantly deceived !